Study of the four-body decays $B_S^0 \to \pi\pi\pi\pi$ in the perturbative QCD approach

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In this work we analyze the CP-averaged branching ratios and direct CP-violating asymmetries of the four-body decays $B_S \to \pi\pi\pi\pi$ decay from the S-wave resonances, $f_0(980)$ and $f_0(500)$ and P-wave resonances, $\rho(770)$ by introducing the S-wave and P-wave $\pi\pi$ distribution amplitudes within the framework of the perturbative QCD approach. We also calculate branching ratios of the two-body decays $B_S^0 \to \rho^0\rho^0$, $B_S^0 \to \rho^+\rho^-$ from the corresponding quasi-two-body decays models and compare our results with those obtained previously using the perturbative QCD approach, the QCD factorization approach, and the factorization-assisted topological amplitude approach. It is found that the predictions are consistent with present data within errors. The branching ratios of our calculations for the four-body decays $B_S \to \pi\pi\pi\pi$ are at the order of the 10^{-7}. For the CP-violating asymmetries, we found that CP-violating asymmetry can be enhanced, largely by the $\rho - \omega$ mixing resonances, when $\pi\pi$ pairs masses are in the vicinity of ω resonance.

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I. INTRODUCTION

CP-violating asymmetries which are associated with weak phase from the Cabibbo-Kobayashi-Maskawa (CKM) matrix and CP-averaged branching ratios have attracted a great deal of attention [1,2], since they are regard as offering the most important opportunity to testing the standard model and searching for new physics beyond standard model. Experimental data provides several CP-violation processes in K- and B-meson decays processes and frameworks for the two-body decays of the B meson, with vector and scalar final states, have been developed in recent decades [3–7]. Compared with two-body decays, the multibody decays of B meson are more interesting due to their more complicated processes.

Experimentally, the four-body decays of B meson with certain two-body invariant mass regions which are shown in Fig. 1 have been collected by LHCb [8–14], Belle [15,16], *BABAR* [17,18], and other Collaborations. Generally, it is not easy to calculate the dynamics of these decays; however, it can be simplified by employing the factorization theorems. Several factorization approaches, such as QCD factorizations (QCDF) [19–27], the soft collinear effective theory (SCET) [28–30], factorization-assisted topological amplitude approach (FAT) [31], and the perturbative QCD (PQCD) factorization approach [32–43] are used to investigate these decays. Compared with other approaches, the PQCD factorization which is based on the K_T factorization theorem is more appropriate to find out the four-body decays of B meson [44–46].

In the PQCD factorization framework, we usually use a factorization scale of about $1/b$ to separate the perturbative area from the nonperturbative area, where b is the conjugate variable obtained by Fourier transformation of the transverse momentum of the quark in the meson. The nonperturbative part below the $1/b$ energy scale will be included in wave functions that are universal and irrelevant to the process; however, the part above the $1/b$ energy scale depends on differential decay channels, and the numerical calculations of Feynman diagrams is carried out by using the perturbation theory. For the four-body decay $B_S^0 \to \pi\pi\pi\pi$, the amplitude can be written as [47,48]

$$\mathcal{A} = \mathcal{H} \otimes \Phi_B \otimes \Phi_{h_1h_2} \otimes \Phi_{h_3h_4}, \tag{1}$$

here $\mathcal{H}$ is hard decay kernel that can be perturbatively calculated, Φ_B is the wave function of B messon, $\Phi_{h_1h_2}$, and $\Phi_{h_3h_4}$ are wave functions of $\pi\pi$ pairs (which can be regarded as the nonperturbative part), and the differential wave functions can be found in the following.

In this work, we focus on the study of the quasi-two-body decays $B_S^0 \to N_1N_2 \to \pi\pi\pi\pi$ in which the $\pi\pi$ pair is selected in the low-invariant mass range (<1100 MeV) [9] and can arise from S wave and P wave contributions with

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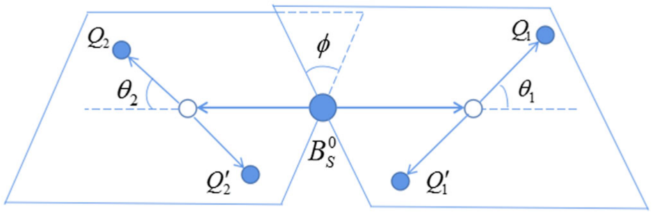

FIG. 1. Helicity angles of $(\pi^+\pi^-)(\pi^+\pi^-)$ decays, θ is defined as the polar angle of π^+ in $\pi^+\pi^-$ intermediate states and ϕ represents the angle among two $\pi\pi$ pairs in the rest frame of B meson.

the vector resonances ρ^0, ω, and the scalar resonances $f_0(980)$, $f_0(500)$. The strong interactions between S-wave contributions, P-wave contributions, and the final-state pion pairs can not be ignored, so we discuss them by introducing timelike form factors F_π. For the resonances ρ^0, we adopt the Gounaris-Sakurai model; the Breit-Wigner (BW) model is used for resonances ω. $f_0(980)$ is parametrized by the Flatté model and $f_0(500)$ is modeled by the Breit-Wigner function [49–51]. The vector or scalar resonance models of the pion pair have been borrowed for the study of quasi-two-body B-meson decays and the range of invariant mass in $\pi\pi$ pair varies from 300 MeV to 1100 MeV [9,48]. Besides, the four-body decays mainly cover six helicity amplitudes A_h with $h = VV(3), VS, SV$, and SS. The P wave amplitudes in which two pion-pair resonances form a vector meson correspond to $h = VV$. For the decays of $B \to VV$, the amplitude can be defined as three invariant helicity components; A_0 for longitudinally-polarized vector mesons and $A_\parallel$, $A_\perp$ for transversely-polarized vector mesons [52,53]. $h = VS$ and $h = SV$ refer to S-wave or a P-wave amplitudes from N_1 or N_2 and $h = SS$ is the amplitude that arises from S-wave amplitudes in which two pion-pair resonances form a scalar meson.

After the introduction above, we shall present theoretical framework in the PQCD factorization approach for four-body decays of the B meson in Sec. II, the S-wave and P-wave function of $\pi\pi$ pairs in Sec. III and Sec. IV. We parametrize the decay amplitude and direct CP asymmetries of the considered decay modes in Sec. V. In Sec. VI, the numerical results and analysis about the two-body and four-body decays are collected, and finally, we give a short summary in Sec. VII. The factorization formulas for the decay amplitudes are organized in the Appendix.

II. THEORETICAL FRAMEWORK

For the four-body $B_S^0 \to \pi\pi\pi\pi$ decay, the weak effective Hamiltonian is given by [54]

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}}\left\{V_{ub}^* V_{uX}[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] - V_{tb}^* V_{tX}\left[\sum_{i=3}^{10} C_i(\mu)O_i(\mu)\right]\right\}. \tag{2}$$

Here $X = (d, s)$, G_F is Fermi coupling constant, $V_{ub}^* V_{uX}$ and $V_{tb}^* V_{tX}$ are CKM factors, C_i are Wilson coefficients. O_i are four-quark operators, and which can be written as

$$\begin{aligned}
O_1 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) u_\beta \bar{u}_\beta \gamma^\mu (1-\gamma_5) X_\alpha, \\
O_2 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) u_\alpha \bar{u}_\beta \gamma^\mu (1-\gamma_5) X_\beta, \\
O_3 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\alpha \sum_{X'} \overline{X'}_\beta \gamma^\mu (1-\gamma_5) X'_\beta, \\
O_4 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\beta \sum_{X'} \overline{X'}_\beta \gamma^\mu (1-\gamma_5) X'_\alpha, \\
O_5 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\alpha \sum_{X'} \overline{X'}_\beta \gamma^\mu (1+\gamma_5) X'_\beta, \\
O_6 &= \bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\beta \sum_{X'} \overline{X'}_\beta \gamma^\mu (1+\gamma_5) X'_\alpha, \\
O_7 &= \frac{3}{2}\bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\alpha \sum_{X'} e_{X'} \overline{X'}_\beta \gamma^\mu (1+\gamma_5) X'_\beta, \\
O_8 &= \frac{3}{2}\bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\beta \sum_{X'} e_{X'} \overline{X'}_\beta \gamma^\mu (1+\gamma_5) X'_\alpha, \\
O_9 &= \frac{3}{2}\bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\alpha \sum_{X'} e_{X'} \overline{X'}_\beta \gamma^\mu (1-\gamma_5) X'_\beta, \\
O_{10} &= \frac{3}{2}\bar{b}_\alpha \gamma_\mu (1-\gamma_5) X_\beta \sum_{X'} e_{X'} \overline{X'}_\beta \gamma^\mu (1-\gamma_5) X'_\alpha,
\end{aligned} \tag{3}$$

where α and β are color indices, $X' = u, d, s, c$ or b quarks. O_1 and O_2 are tree operators, $O_i (i = 3, \ldots, 10)$ are penguin operators, in which $O_i (i = 7, \ldots, 10)$ are the operators from electroweak penguin diagrams.

The light cone coordinate system is used in the B-meson rest frame, and the system is expressed as

$$p^+ = \frac{p^0 + p^3}{\sqrt{2}}, \qquad p^- = \frac{p^0 - p^3}{\sqrt{2}}, \qquad p_\top = (p^1, p^2), \tag{4}$$

through the following relational formula

$$\begin{aligned}
p^2 &= 2p^+p^- - p_\top^2, \\
p_1 \cdot p_2 &= p_1^+ p_2^- + p_1^- p_2^+ - p_{1\top} \cdot p_{2\top}.
\end{aligned} \tag{5}$$

For the $B(p_B) \to N_1(p)N_2(q) \to Q_1(q_1)Q_1'(q_1') \times Q_2(q_2)Q_2'(q_2')$ decay, we choose the B-meson mass M_B, $p_B = p + q$, $p = q_1 + q_1'$, $q = q_2 + q_2'$, and let N_1 and N_2 be intermediate states moving along with the direction of $n = (1, 0, 0_\top)$ and $v = (0, 1, 0_\top)$, respectively, and the Feynman diagrams have been described in Fig. 2. So we define the intermediate states and quark momentum as [47,48,55]

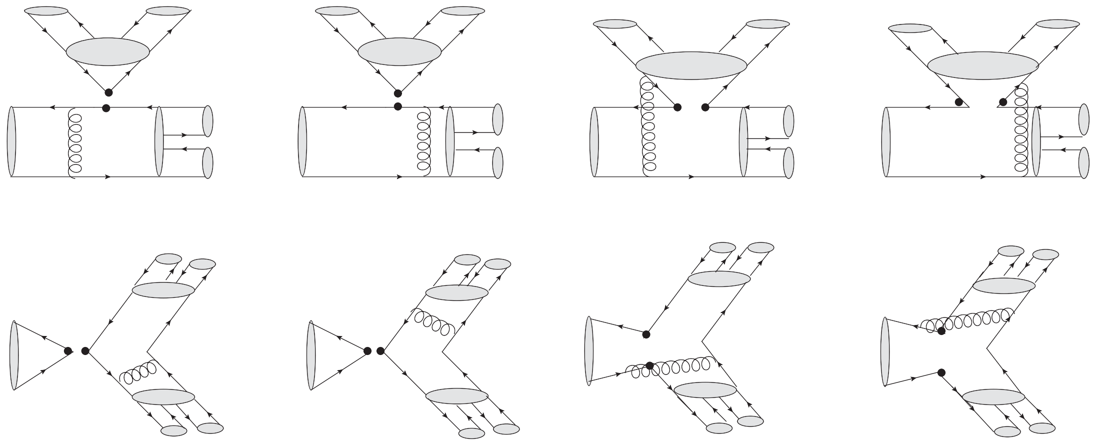

FIG. 2. The lowest order Feynman diagrams for the $B_S \to N_1 N_2 \to \pi\pi\pi\pi$ decays.

$$p_B = \frac{M_B}{\sqrt{2}}(1, 1, 0_\top), \qquad k_B = \left(0, \frac{M_B}{\sqrt{2}} x_B, k_{B\top}\right),$$
$$p = \frac{M_B}{\sqrt{2}}(g^+, g^-, 0_\top), \qquad k_p = \left(\frac{M_B}{\sqrt{2}} x_1 g^+, 0, k_{1\top}\right),$$
$$q = \frac{M_B}{\sqrt{2}}(f^-, f^+, 0_\top), \qquad k_q = \left(0, \frac{M_B}{\sqrt{2}} x_2 f^+, k_{2\top}\right). \quad (6)$$

The above factors are

$$g^\pm = \frac{1}{2}[1 + \eta_1 - \eta_2 \pm \sqrt{(1 + \eta_1 - \eta_2)^2 - 4\eta_1}],$$
$$f^\pm = \frac{1}{2}[1 - \eta_1 + \eta_2 \pm \sqrt{(1 + \eta_1 - \eta_2)^2 - 4\eta_1}], \quad (7)$$

where $\eta_{1,2} = \omega_{1,2}^2/M^2$ are mass ratios, and the invariant mass $\omega_{1,2}^2$ and their momentum p, q satisfy the relation $\omega_1^2 = p^2$ and $\omega_2^2 = q^2$. x_i, $i = B, 1, 2$ indicate momentum fractions inside the meson and they run between 0–1. We also study P-wave pairs by introducing corresponding longitudinal-polarization vectors, and the vectors can be written as

$$\epsilon_p = \frac{1}{\sqrt{2\eta_1}}(g^+, -g^-, 0_\top), \qquad \epsilon_q = \frac{1}{\sqrt{2\eta_2}}(-f^-, f^+, 0_\top), \quad (8)$$

with $\epsilon_p^2 = \epsilon_q^2 = -1$ and $\epsilon_p \cdot p = \epsilon_q \cdot q = 0$.

Considering the final state meson q_1, q_1' and q_2, q_2', we decompose them as

$$q_1 = \left(\frac{M_B}{\sqrt{2}} g^+ \left(\zeta_1 + \frac{(r_1 - r_1')}{2\eta_1}\right), \quad \frac{M_B}{\sqrt{2}} g^- \left(1 - \zeta_1 + \frac{(r_1 - r_1')}{2\eta_1}\right), p_\top\right),$$
$$q_1' = \left(\frac{M_B}{\sqrt{2}} g^+ \left(1 - \zeta_1 - \frac{(r_1 - r_1')}{2\eta_1}\right), \quad \frac{M_B}{\sqrt{2}} g^- \left(\zeta_1 - \frac{(r_1 - r_1')}{2\eta_1}\right), -p_\top\right),$$
$$q_2 = \left(\frac{M_B}{\sqrt{2}} f^- \left(1 - \zeta_2 + \frac{(r_2 - r_2')}{2\eta_2}\right), \quad \frac{M_B}{\sqrt{2}} f^+ \left(\zeta_2 + \frac{(r_2 - r_2')}{2\eta_2}\right), q_\top\right),$$
$$q_2' = \left(\frac{M_B}{\sqrt{2}} f^- \left(\zeta_2 - \frac{(r_2 - r_2')}{2\eta_2}\right), \quad \frac{M_B}{\sqrt{2}} f^+ \left(1 - \zeta_2 - \frac{(r_2 - r_2')}{2\eta_2}\right), -q_\top\right), \quad (9)$$

where the mass ratios $r_i = \frac{M_i^2}{M_B^2}, r_i' = \frac{M_i^{(\prime)2}}{M_B^2}$. By introducing variables ζ_i ($i = 1, 2$), we can derive the meson momentum fractions

$$\frac{q_1^+}{p^+} = \zeta_1 + \frac{(r_1 - r_1')}{2\eta_1}, \qquad \frac{q_2^-}{q^-} = \zeta_2 + \frac{(r_2 - r_2')}{2\eta_2}. \quad (10)$$

The transverse momenta are given by

$$p_\top^2 = \zeta_1(1 - \zeta_1)\omega_1^2 + \frac{(m_1^2 - m_1^{(\prime)2})^2}{4\omega_1^2} - \frac{(m_1^2 + m_1^{(\prime)2})}{2},$$
$$q_\top^2 = \zeta_2(1 - \zeta_2)\omega_2^2 + \frac{(m_2^2 - m_2^{(\prime)2})^2}{4\omega_2^2} - \frac{(m_2^2 + m_2^{(\prime)2})}{2}. \quad (11)$$

One can make the above formula simple by introducing

$$\alpha_i = \frac{(r_i - r_i')^2}{4\eta_i} - \frac{(r_i + r_i')}{2\eta_i}. \tag{12}$$

Then we can deduce the following relationships for ζ_i and polar angle θ_i [56].

$$2\zeta_i - 1 = \sqrt{1+4\alpha_i}\cos\theta_i, \tag{13}$$

$$\zeta_i \in \left[\frac{1-\sqrt{1+4\alpha_i}}{2}, \frac{1+\sqrt{1+4\alpha_i}}{2}\right]. \tag{14}$$

In the PQCD approach the wave functions are treated as nonperturbative inputs. For B_X ($X = u,\ s,\ d$), the wave function can be expressed as [57,58].

$$\Phi_B = \frac{i}{\sqrt{2N_c}}(\not{p}_B + M_B)\gamma_5\phi_B(x_B, b_B), \tag{15}$$

where $N_c = 3$ is the number of colors, and the distribution amplitude ϕ_B can be chosen as [59,60]

$$\phi_B(x_B, b_B) = N_B {x_B}^2(1-x_B)^2 \exp\left[-\frac{M_B^2 {x_B}^2}{2\omega_B^2} - \frac{1}{2}(\omega_B b_B)^2\right], \tag{16}$$

with the normalization

$$\int_0^1 dx\phi_B(x, b=0) = \frac{f_B}{2\sqrt{2N_c}}, \tag{17}$$

where $N_B = 91.784$ GeV is the normalization constant and f_B is the decay constant. For B_s^0 meson, we use the shape parameter $\omega_{B_s} = 0.48 \pm 0.048$ GeV [61].

At the same time, for the two-meson distribution amplitudes, we will discuss the S wave and P wave via intermediate resonances $f_0(980)$, $f_0(500)$, $\rho(770)$, and $\omega(782)$ which are listed in Table I. The corresponding timelike form factors of them are collected below [48].

III. *S* WAVE FUNCTION

For the quasi-two-body decays $B_S^0 \to N_1 N_2 \to \pi\pi\pi\pi$, we proceed it mainly via quasi-two-body channels, which contain S wave and P wave pion-pair resonant state. Similar to previous Ref [51], S-wave two-pion distribution amplitudes are written as

$$\Phi_{\pi\pi}^S = \frac{1}{\sqrt{2N_c}}[\not{p}\phi_S^0(x,\omega) + \omega\phi_S^s(x,\omega) + \omega(\not{n}\not{v} - 1)\phi_S^t(x,\omega)], \tag{18}$$

with the Gegenbauer coefficient a_S, and the twist-2 and twist-3 light cone distribution amplitudes $\phi_S^0(x,\omega)$, $\phi_S^s(x,\omega)$, $\phi_S^t(x,\omega)$.

$$\begin{aligned}
\phi_S^0(x,\omega) &= \frac{9F_S(\omega^2)}{\sqrt{2N_c}} a_S x(1-x)(1-2x), \\
\phi_S^s(x,\omega) &= \frac{F_S(\omega^2)}{2\sqrt{2N_c}}, \\
\phi_S^t(x,\omega) &= \frac{F_S(\omega^2)}{2\sqrt{2N_c}}(1-2x).
\end{aligned} \tag{19}$$

$F_S(\omega^2)$ is timelike form factor. For a narrow resonance, we consider Breit-Wigner line shape to describe it, such as $f_0(500)$ and the $d\bar{d}$ component in the S-wave amplitude

$$F_S(\omega^2) = \frac{c m_{f_0(500)}^2}{m_{f_0(500)}^2 - \omega^2 - i m_{f_0(500)}\Gamma_{f_0(500)}(\omega^2)}, \tag{20}$$

with

$$\Gamma_S(\omega^2) = \Gamma_S \frac{m}{\omega}\sqrt{\frac{\omega^2 - 4m_\pi^2}{m_S^2 - 4m_\pi^2}} F_R^2. \tag{21}$$

For the resonance $f_0(980)$, although the timelike scalar form factor of $f_0(980)$, $f_0(1500)$, $f_0(1790)$ are $s\bar{s}$ components, (different from the last two resonances) the mass of the KK system in $f_0(980)$ is around 0.98 GeV. So we replace the Breit-Wigner formula with the Flatté model, which has been motivated by Refs. [49,62] and works well.

$$\begin{aligned}
F_S(\omega^2) = &\frac{c_1 m_{f_0(980)}^2 e^{i\theta_1}}{m_{f_0(980)}^2 - \omega^2 - i m_{f_0(980)}(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}\rho_{KK})} \\
&+ \frac{c_2 m_{f_0(1500)}^2 e^{i\theta_2}}{m_{f_0(1500)}^2 - \omega^2 - i m_{f_0(1500)}\Gamma_{f_0(1500)}(\omega^2)} \\
&+ \frac{c_3 m_{f_0(1790)}^2 e^{i\theta_3}}{m_{f_0(1790)}^2 - \omega^2 - i m_{f_0(1790)}\Gamma_{f_0(1790)}(\omega^2)},
\end{aligned} \tag{22}$$

where c_i and θ_i ($i = 1,\ 2,\ 3$) are tunable parameters. $c_1 = 0.9$, $c_2 = 0.106$, $c_3 = 0.066$. $g_{\pi\pi} = 0.167$ GeV, and $g_{KK} = 3.47 g_{\pi\pi}$ are coupling constants [63–66]. $\rho_{\pi\pi}$ and ρ_{KK} are phase space that can be expressed as

TABLE I. The widths, masses and decay models of intermediate states in our framework.

Resonance	Mass[MeV]	Width[MeV]	Model	J^P
$f_0(980)$	990 ± 20	65 ± 45	Flatté	0^+
$f_0(500)$	471 ± 21	534 ± 53	BW	0^+
$\rho(770)$	775.26 ± 0.25	149.1 ± 0.8	GS	1^-
$\omega(782)$	782.65 ± 0.12	8.49 ± 0.08	BW	1^-

$$\rho_{\pi\pi} = \frac{2}{3}\sqrt{1-\frac{4m_{\pi^\pm}^2}{\omega^2}} + \frac{1}{3}\sqrt{1-\frac{4m_{\pi^0}^2}{\omega^2}},$$
$$\rho_{KK} = \frac{1}{2}\sqrt{1-\frac{4m_{K^\pm}^2}{\omega^2}} + \frac{1}{2}\sqrt{1-\frac{4m_{K^0}^2}{\omega^2}}. \tag{23}$$

IV. P WAVE FUNCTION

The P-wave resonant states are associated with longitudinal and transverse polarizations. The relevant distribution amplitudes and timelike scalar form factors can be obtained from Ref. [67]

$$\Phi^L_{P(\pi\pi)} = \frac{1}{\sqrt{2N_c}}\left[\omega \not\epsilon_p \phi_P^0(x,\omega) + \omega\phi_P^s(x,\omega) + \frac{\not p_1 \not p_2 - \not p_2 \not p_1}{\omega(2\zeta-1)}\phi_P^t(x,\omega)\right](2\zeta-1),$$
$$\Phi^T_{P(\pi\pi)} = \frac{1}{\sqrt{2N_c}}\left[\gamma_5 \not\epsilon_T \not p \phi_P^T(x,\omega) + \omega\gamma_5\not\epsilon_T\phi_P^a(x,\omega) + i\omega\frac{\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\epsilon_{T\nu}p_\rho n_{-\sigma}}{p\cdot n-}\phi_P^v(x,\omega)\right]\sqrt{\zeta(1-\zeta)+\alpha}, \tag{24}$$

with

$$\phi_P^0(x,\omega) = \frac{3F_P^{\parallel}(\omega^2)}{\sqrt{2N_c}}x(1-x)\left[1+a_V^0\frac{3}{2}(5(1-2x)^2-1)\right],$$
$$\phi_P^s(x,\omega) = \frac{3F_P^{\perp}(\omega^2)}{2\sqrt{2N_c}}(1-2x)[1+a_V^s(10x^2-10x+1)],$$
$$\phi_P^t(x,\omega) = \frac{3F_P^{\perp}(\omega^2)}{2\sqrt{2N_c}}(1-2x)^2\left[1+a_V^t\frac{3}{2}(5(1-2x)^2-1)\right],$$
$$\phi_P^T(x,\omega) = \frac{3F_P^{\perp}(\omega^2)}{2\sqrt{2N_c}}x(1-x)\left[1+a_V^T\frac{3}{2}(5(1-2x)^2-1)\right],$$
$$\phi_P^a(x,\omega) = \frac{3F_P^{\parallel}(\omega^2)}{4\sqrt{2N_c}}(1-2x)[1+a_V^a(10x^2-10x+1)],$$
$$\phi_P^v(x,\omega) = \frac{3F_P^{\parallel}(\omega^2)}{8\sqrt{2N_c}}[1+(1-2x)^2]+a_V^v[3(2x-1)^2-1]. \tag{25}$$

Here $\phi_P^0(x,\omega)$ and $\phi_P^T(x,\omega)$ are twist-2 distribution amplitudes, $\phi_P^s(x,\omega)$, $\phi_P^t(x,\omega)$, $\phi_P^a(x,\omega)$ and $\phi_P^v(x,\omega)$ are twist-3 distribution amplitudes, which are associated with the longitudinal and transverse polarization. $a_V^{0,s,t}$ and $a_V^{T,a,v}$ are Gegenbauer moments, which are determined in Ref [48]. For the timelike factors $F_P^{\parallel}$ and $F_P^{\perp}$, we postulate the approximation $F_P^{\perp} = (f_V^T/f_V)F_P^{\parallel}$.

We take the $\rho-\omega$ interference and the excited states into account for the form factor,

$$F^{\parallel}(\omega^2) = \left[GS_\rho(s,m_\rho,\Gamma_\rho)\frac{1+c_\omega \mathrm{BW}_\omega(s,m_\omega,\Gamma_\omega)}{1+c_\omega} + \sum_i c_i \mathrm{GS}_i(s_i,m_i,\Gamma_i)\right]\left[1+\sum_i c_i\right]^{-1}, \tag{26}$$

where $i=\rho(1450),\rho(1700),\rho(2254)$, and $s=m^2(\pi\pi)$ is the pion-pair invariant mass square. For the ω resonant state, we adopt the BW model, however, for ρ resonant state, the Gounaris-Sakurai model based on the Breit-Wigner model is used. These models can be found in Refs. [39,50,68].

$$\mathrm{GS}_\rho(s,m_\rho,\Gamma_\rho) = \frac{m_\rho^2[1+d(m_\rho)\Gamma_\rho/m_\rho]}{m_\rho^2 - s + f(s,m_\rho,\Gamma_\rho) - im_\rho\Gamma(s,m_\rho,\Gamma_\rho)}, \tag{27}$$

where

$$\Gamma(s,m_\rho,\Gamma_\rho) = \Gamma_\rho\frac{s}{m_\rho^2}\left(\frac{\beta_\pi(s)}{\beta_\pi(m_\rho^2)}\right)^3,$$
$$dm = \frac{3}{\pi}\frac{m_\pi^2}{k^2(m^2)}\ln\left(\frac{m+2k(m^2)}{2m_\pi}\right) + \frac{m}{2\pi k(m^2)} - \frac{m_\pi^2 m}{\pi k^3(m^2)},$$
$$f(s,m,\Gamma) = \frac{\Gamma m^2}{k^3(m^2)}[k^2(s)[h(s)-h(m^2)] + (m^2-s)k^2(m^2)h'(m^2)], \tag{28}$$

with

$$k(s) = \frac{1}{2}\sqrt{s}\beta_\pi(s),$$
$$h(s) = \frac{2}{\pi}\frac{k(s)}{\sqrt{s}}\ln\left(\frac{\sqrt{s}+2k(s)}{2m_\pi}\right),$$
$$\beta_\pi(s) = \sqrt{1-4\frac{m_\pi^2}{s}}. \tag{29}$$

V. THE DECAY AMPLITUDES

The rate for the $B_S^0 \to N_1N_2 \to \pi\pi\pi\pi$ decay in the B_S^0 meson rest frame can be described as [69–71]

$$\frac{d\mathcal{B}}{d\Omega} = \frac{\tau_{B_S^0}k(\omega_1)k(\omega_2)k(\omega_1,\omega_2)}{16(2\pi)^6 M_{B_S^0}^2}|\mathcal{A}|^2, \tag{30}$$

with $k(\omega) = \frac{\sqrt{\lambda(\omega^2, m_{h_1}^2, m_{h_2}^2)}}{2\omega}$ and $k(\omega_1,\omega_2) = \frac{\sqrt{[M_B^2-(\omega_1+\omega_2)^2][M_B^2-(\omega_1-\omega_2)^2]}}{2M_B}$ in the pion-pair center-of-mass system. $\tau_{B_s^0}$ is lifetime, Ω stands for θ_1, θ_2, ϕ, ω_1, ω_2, the Källén function $\lambda(a,b,c)=a^2+b^2+c^2-2(ab+ac+bc)$.

The six helicity amplitudes are involved in four body decay of the B meson and the relation between total amplitude and other components can be found in Refs [47,48,55]. Then branching ratio from Eq. (30) is replaced by the following

$$\mathcal{B}_h=\frac{\tau_B}{4(2\pi)^6m_B^2}\frac{2\pi}{9}C_h\int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1,\omega_2)|A_h|^2, \tag{31}$$

with

$$C_h=\begin{cases}(1+4\alpha_1)(1+4\alpha_2) & h=0,\parallel,\perp\\ 3(1+4\alpha_{1,2}) & h=VS,SV\\ 9 & h=SS.\end{cases}$$

Here integrations over ζ_1, ζ_2 and ϕ is defined in C_h.

For the CP-averaged branching ratio, we adopt the same definition as in Ref [5]. $\bar{\mathcal{B}}_h$ is branching ratio of the charge-conjugate channel for $B_S^0\to N_1N_2\to\pi\pi\pi\pi$ and other expressions can be found in Refs [72,73].

We define

$$\mathcal{B}_h^{\rm avg}=\frac{1}{2}(\mathcal{B}_h+\bar{\mathcal{B}}_h), \tag{32}$$

and $f_{0,\parallel,\perp}$ is the polarization fraction corresponding P-wave amplitudes,

$$f_{0,\parallel,\perp}=\frac{\mathcal{B}_{0,\parallel,\perp}}{\mathcal{B}_{\rm total}}, \tag{33}$$

with $\mathcal{B}_{\rm total}=\mathcal{B}_0+\mathcal{B}_\parallel+\mathcal{B}_\perp$.

VI. NUMERICAL RESULTS AND DISCUSSIONS

Based on the framework above, we start our calculations by introducing input parameters which are listed in Table II, covering the mass of the involved mesons(in GeV), the lifetime of B_s meson, the decay constants of B_s^0, ρ, ω mesons and Wolfenstein parameters [3]. The Gegenbauer moments are listed in Table III [48].

In Table IV we present the CP-averaged branching ratios of the $B_S^0\to V_1V_2\to\pi\pi\pi\pi$ decays in PQCD approach (here V stands for the vector resonance). The first main uncertainty of these results comes from the QCD scale

TABLE II. The input parameters of the $B_S^0\to N_1N_2\to\pi\pi\pi\pi$ decay.

Masses of the involved mesons	$M_{B_s^0}=5.367$ GeV	$M_{\pi^\pm}=0.140$ GeV	
	$m_b=4.8$ GeV	$m_c=1.27$ GeV	$m_{\pi^0}=0.135$ GeV
	$m_{f_0(980)}=0.99\pm0.02$ GeV	$m_{f_0(500)}=0.50$ GeV	
	$m_{\rho(770)}=0.775\pm0.02$ GeV	$m_{\omega(782)}=0.78265$ GeV	
Decay widths	$\Gamma_{\rho^0(770)}=0.1491$ GeV	$\Gamma_{\omega(782)}=8.49\times10^{-3}$ GeV	
Decay constants	$f_{B_s^0}=0.24\pm0.02$ GeV	$f_\rho=0.216\pm0.003$ GeV	$f_\rho^T=0.184$ GeV
	$f_\omega=0.187\pm0.005$ GeV	$f_\omega^T=0.151\pm0.009$ GeV	
Lifetime of meson	$\tau_{B_s^0}=1.512$ ps		
Wolfenstein parameters	$\lambda=0.22650$	$A=0.790$	
	$\bar\rho=0.141$	$\bar\eta=0.357$	

TABLE III. The Gegenbauer moments are collected from Ref. [48].

$a_S=0.2\pm0.2$	$a_\rho^0=0.08\pm0.13$	$a_\rho^s=-0.23\pm0.24$	$a_\rho^t=-0.354\pm0.062$
$a_\rho^T=0.50\pm0.50$	$a_\rho^a=0.40\pm0.40$	$a_\rho^v=-0.50\pm0.50$	

TABLE IV. The CP-averaged branching ratios of the $B_S^0\to V_1V_2\to\pi\pi\pi\pi$ decay (in units of 10^{-8}), the errors come from QCD scale, hard scale and the Gegenbauer moments.

Components	$B_S^0\to(\rho^+\to)\pi^+\pi^0(\rho^-\to)\pi^-\pi^0$	$B_S^0\to(\rho^0\to)\pi^+\pi^-(\rho^0\to)\pi^+\pi^-$
B_0	$0.76^{+0.30+0.26+0.07}_{-0.21-0.20-0.03}$	$0.38^{+0.12+0.12+0.01}_{-0.10-0.12-0.01}$
$B_\parallel$	$0.04^{+0.03+0.02+0.02}_{-0.02-0.01-0.00}$	$0.02^{+0.01+0.01+0.01}_{-0.01-0.01-0.00}$
$B_\perp$	$0.10^{+0.40+0.10+0.03}_{-0.07-0.02-0.00}$	$0.01^{+0.04+0.01+0.00}_{-0.01-0.00-0.00}$
$B_{\rm total}$	$0.90^{+0.69+0.37+0.12}_{-0.30-0.23-0.04}$	$0.41^{+0.17+0.13+0.02}_{-0.13-0.12-0.01}$

$\Lambda_{\rm QCD} = 0.25 \pm 0.05$ GeV, the second error from hard scale t, which varies from $0.75t \sim 1.25t$, and the third error from the Gegenbauer moments a^0, a^s, a^t, and the moments a^T, a^a,a^v in transversely-polarized wave functions. Other errors such as the decay constants of the B_S^0 and the Wolfenstein parameters, are small and can be neglected. From Table IV, we can see that the prediction results for branching ratios of pure annihilation decays $B_S^0 \to \rho^0\rho^0 \to \pi\pi\pi\pi$ and $B_S^0 \to \rho^+\rho^- \to \pi\pi\pi\pi$, which all cover the two kinds of topological penguin diagrams contributions, are at the order of 10^{-8} with large uncertainties.

In order to compare the branching ratios with other approaches and the experimental results of two-body decays [3,7,31,61,74,75], we predict branching ratios of the corresponding two-body vector resonance by the following relation [Eq. (34)] and $\mathcal{B}(\rho^0 \to \pi\pi) = 1$.

As aforementioned, the relation of branching ratios between two body vector resonance and corresponding quasi-two-body decay in narrow-width approximation has been obtained as

$$\begin{aligned}&\mathcal{B}(B_s^0 \to \rho^0(\to \pi\pi)\rho^0(\to \pi\pi))\\ &\approx \mathcal{B}(B_s^0 \to \rho^0\rho^0) \times \mathcal{B}(\rho^0 \to \pi\pi) \times \mathcal{B}(\rho^0 \to \pi\pi). \quad (34)\end{aligned}$$

The CP-averaged branching ratio which we predicted for $B_S^0 \to \rho^0\rho^0$ is 0.41×10^{-8} while for $B_S^0 \to \rho^+\rho^-$ is 0.90×10^{-8}. Our branching ratios are in agreement with the results of FAT approaches and QCDF approaches within errors; however, they are lower than the results of previous PQCD [74,75]. The branching ratio of $B_S^0 \to \rho^0\rho^0$ is lower than the upper bound in experiment—all other annihilation decays have not been measured and are excepted to be confirmed by future experiments.

We now discuss the predictions for the polarization fraction of B_S^0 meson. For pure annihilation two-body decays, the contributions are dominated by the longitudinal polarization fraction f_0 and the fractions of these decays can reach to about 100%, which have been pointed out in previous predictions of two-body decays. It is found that our results are in agreement well with former fractions [74,75], so we predict the polarization fraction in four-body decays mainly in this paper.

As shown in the following, we can find that for decay $B_S^0 \to (\rho^+ \to)\pi^+\pi^0(\rho^- \to)\pi^-\pi^0$, the longitudinal polarization fraction f_0 is about 84.44%, and f_0 is about 92.68% for $B_S^0 \to (\rho^0 \to)\pi^+\pi^-(\rho^0 \to)\pi^+\pi^-$ decay. The uncertainties of results come from QCD scale and the hard scale. The results show that the transverse polarization cannot be ignored and can help to make significant contributions in pure annihilation decays.

TABLE V. The CP-averaged branching ratios of the $B_S^0 \to N_1N_2 \to \pi\pi\pi\pi$ decay, uncertainties of these results come from the shape parameter, hard scale and Gegenbauer moments.

Modes	$\mathcal{B}(10^{-8})$
$B_S^0 \to (f_0(980) \to)\pi^+\pi^-(f_0(980) \to)\pi^+\pi^-$	$11.75^{+2.55+0.04+0.00}_{-2.89-1.60-0.03}$
$B_S^0 \to (\rho^0 \to)\pi^+\pi^-(f_0(500) \to)\pi^+\pi^-$	$0.11^{+0.05+0.05+0.00}_{-0.02-0.01-0.00}$
$B_S^0 \to (f_0(980) \to)\pi^+\pi^-(f_0(980) \to)\pi^0\pi^0$	$5.88^{+1.28+0.02+0.00}_{-1.45-0.80-0.01}$
$B_S^0 \to (\rho^0 \to)\pi^+\pi^-(f_0(500) \to)\pi^0\pi^0$	$0.06^{+0.02+0.02+0.00}_{-0.01-0.01-0.00}$
$B_S^0 \to (f_0(980) \to)\pi^0\pi^0(f_0(980) \to)\pi^0\pi^0$	$2.94^{+0.64+0.01+0.00}_{-0.72-0.40-0.01}$

$$f_0 = \begin{cases} 84.44^{+0.24+8.20}_{-0.06-2.82}\% & B_S^0 \to (\rho^+ \to)\pi^+\pi^0(\rho^- \to)\pi^-\pi^0, \\ 92.68^{+1.60+3.18}_{-0.08-1.50}\% & B_S^0 \to (\rho^0 \to)\pi^+\pi^-(\rho^0 \to)\pi^+\pi^-. \end{cases} \quad (35)$$

Compared with the decays of double-vector resonant states, the decays of scalar resonances which have less experimental data, are more difficult to predict because of the large decay width in scalar resonances. In Table V we calculate the four -body decays of $B_S^0 \to (f_0(980) \to) \pi\pi(f_0(980) \to)\pi\pi$ and $B_S^0 \to (\rho^0 \to)\pi\pi(f_0(500) \to)\pi\pi$; we ignore the decay of $B_S^0 \to (f_0(500) \to)\pi\pi(f_0(500) \to)\pi\pi$ because of its lower branching ratio. Comparing Table V with Table IV, we find that the decay of $B_S^0 \to (f_0(980) \to) \pi^+\pi^-(f_0(980) \to)\pi^+\pi^-$ is the largest contribution in total branching ratio, because there is only pure annihilation contribution in decay of $B_S^0 \to \rho\rho \to \pi\pi\pi\pi$. This result have not been reported by experiments and are expected to be studied in future LHCb and Belle-II experiments.

For overall direct CP asymmetry, we define

$$A^{\rm dir} = \frac{\bar{\mathcal{B}}_{\rm total} - \mathcal{B}_{\rm total}}{\bar{\mathcal{B}}_{\rm total} + \mathcal{B}_{\rm total}}, \quad (36)$$

where

$$\mathcal{B}_{\rm total} = \mathcal{B}_0 + \mathcal{B}_\parallel + \mathcal{B}_\perp. \quad (37)$$

The direct CP asymmetry in each component can be defined as

$$A_h^{\rm dir} = \frac{\bar{\mathcal{B}}_h - \mathcal{B}_h}{\bar{\mathcal{B}}_h + \mathcal{B}_h}, \quad (38)$$

where $h = 0, \parallel, \perp$.

The CP-violating asymmetries are listed in Table VI. For $B_S^0 \to (f_0(980) \to)\pi^+\pi^-(f_0(980) \to)\pi^+\pi^-$ decay, because it is a pure penguin process with transition $b \to s s\bar{s}$, the result is small or even zero. For a pure annihilation-type decay process, the CP-violating asymmetries are also small. The results have been discussed in previous works [74,75]. However, we found that CP-violating

TABLE VI. The CP-violating asymmetries of the $B_S^0 \to N_1 N_2 \to \pi\pi\pi\pi$ decay, the errors come from QCD scale, the Gegenbauer moments and hard scale.

Asymmetries	$B_S^0 \to (\rho^+ \to)\pi^+\pi^0(\rho^- \to)\pi^-\pi^0$	$B_S^0 \to (\rho^0 \to)\pi^+\pi^-(\rho^0 \to)\pi^+\pi^-$
$A^{\rm dir}$	$(6.71^{+0.00+0.00+0.00}_{-3.42-4.71-5.43})\%$	$(7.22^{+0.25+1.84+1.93}_{-0.00-0.38-1.02})\%$
$A_0^{\rm dir}$	$(7.64^{+0.00+0.00+0.00}_{-3.62-3.66-6.32})\%$	$(7.68^{+0.00+0.00+1.77}_{-0.00-4.02-5.14})\%$
$A_\parallel^{\rm dir}$	$(0.02^{+1.28+6.31+6.50}_{-0.00-0.00-0.00})\%$	$(0.05^{+0.40+2.49+2.66}_{-0.00-0.00-0.00})\%$
$A_\perp^{\rm dir}$	$(1.66^{+0.00+0.00+5.55}_{-0.25-0.50-1.66})\%$	$(3.28^{+0.00+0.00+0.00}_{-0.50-0.99-3.28})\%$
Asymmetries	$B_S^0 \to (\rho^0 \to)\pi^+\pi^-(f_0(500) \to)\pi^+\pi^-$	$B_S^0 \to (f_0(500) \to)\pi^+\pi^-(f_0(500) \to)\pi^+\pi^-$
$A^{\rm dir}$	$(11.92^{+0.00+1.94+2.17}_{-0.70-1.43-3.68})\%$	$(0.26^{+0.46+1.42+12.34}_{-0.00-0.00-0.04})\%$
Asymmetries	$B_S^0 \to (f_0(980) \to)\pi^+\pi^-(f_0(980) \to)\pi^+\pi^-$	
$A^{\rm dir}$	0.00%	

TABLE VII. The CP-violating asymmetries of the $B_S^0 \to \rho^0(\omega)\rho^0(\omega) \to \pi^+\pi^-\pi^+\pi^-$ decay, the errors come from the Gegenbauer moments, hard scale and QCD scale.

Asymmetries	$B_S^0 \to \rho^0(\omega)\omega(\rho^0) \to \pi^+\pi^-\pi^+\pi^-$(this work)	$B_S^0 \to \rho^0(\omega)\rho^0(\omega) \to \pi^+\pi^-\pi^+\pi^-$([77])
$A^{\rm dir}$	$(27.30^{+8.04+8.38+8.60}_{-0.00-17.78-20.71})\%$	$(27.20^{+0.05+0.28+7.13}_{-0.15-0.31-6.11})\%$
$A_0^{\rm dir}$	$(37.02^{+0.00+8.30+14.25}_{-0.01-24.82-26.98})\%$	…
$A_\parallel^{\rm dir}$	$(0.14^{+3.01+3.08+3.55}_{-0.00-0.00-0.00})\%$	…
$A_\perp^{\rm dir}$	$(11.56^{+0.00+0.12+1.96}_{-0.66-0.92-10.22})\%$	…

asymmetry can be enhanced largely by the $\rho-\omega$ mixing resonances when $\pi\pi$ pairs masses are in the vicinity of the ω resonance [76], so it is important for us to study CP-violating asymmetry via ρ and ω resonances in three-body and four-body decays. The result of $B_S^0 \to \rho^0(\omega)\rho^0(\omega) \to \pi^+\pi^-\pi^+\pi^-$ are listed in Table VII. We also compare our prediction with previous result [77], as we can see, for the unpolarized CP-violating asymmetry, our result is in agreement with previous result; however, our result has big errors because of the different approach we adopted.

VII. SUMMARY

In this work we study the CP-averaged branching ratios and direct CP-violating asymmetries of the quasi-two-body decays $B_S^0 \to N_1 N_2 \to \pi\pi\pi\pi$ decay from the S-wave resonances, $f_0(980)$ and $f_0(500)$, and P-wave resonances, $\rho(770)$, by introducing the S-wave and P-wave $\pi\pi$ distribution amplitudes within the framework of the perturbative QCD approach. We also calculate branching ratios of the two-body decays $B_S^0 \to \rho^0\rho^0$, $B_S^0 \to \rho^+\rho^-$ from the corresponding quasi-two-body decays models and compare our results with those obtained in previous perturbative QCD approach, QCD factorization approach and FAT approach. The predictions are in agreement with present data within errors. For the CP-violating asymmetries, it is small in pure annihilation-type decay process; however, we found that CP-violating asymmetry can be enhanced largely by the $\rho-\omega$ mixing resonances when $\pi\pi$ pairs masses are in the vicinity of ω resonance.

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APPENDIX: FORMULAS FOR THE CALCULATION USED IN THE TEXT

In this section we list the decay amplitude for each considered decay mode of four-body B meson.

$$
\begin{aligned}
&A_h(B_s^0 \to (\rho^+ \to)\pi^+\pi^0(\rho^- \to)\pi^-\pi^0) \\
&= \frac{G_F}{\sqrt{2}}\left(V_{ub}^* V_{us}\left[\left(C_1 + \frac{1}{3}C_2\right)F_a^{ll,h} + C_2 M_a^{ll,h}\right]\right. \\
&\quad - 2V_{tb}^* V_{ts}\left[\left(2C_3 + \frac{2}{3}C_4 + 2C_5 + \frac{2}{3}C_6 + \frac{1}{2}C_7 + \frac{1}{6}C_8\right.\right. \\
&\quad \left. + \frac{1}{2}C_9 + \frac{1}{6}C_{10}\right)F_a^{ll,h} + \left(2C_4 + \frac{1}{2}C_{10}\right)M_a^{ll,h} \\
&\quad \left.\left. + \left(2C_6 + \frac{1}{2}C_8\right)M_a^{sp,h}\right]\right),
\end{aligned}
\tag{A1}
$$

$$\begin{aligned} A_h(B_s^0 \to (\rho^0 \to)\pi\pi(\rho^0 \to)\pi\pi) &= G_F\bigg(V_{ub}^* V_{us}\bigg[\bigg(C_1+\frac{1}{3}C_2\bigg)F_a^{ll,h}+C_2 M_a^{ll,h}\bigg] \\ &\quad - V_{tb}^* V_{ts}\bigg[\bigg(2C_3+\frac{2}{3}C_4+2C_5+\frac{2}{3}C_6+\frac{1}{2}C_7+\frac{1}{6}C_8 \\ &\quad +\frac{1}{2}C_9+\frac{1}{6}C_{10}\bigg)F_a^{ll,h}+\bigg(2C_4+\frac{1}{2}C_{10}\bigg)M_a^{ll,h} \\ &\quad +\bigg(2C_6+\frac{1}{2}C_8\bigg)M_a^{sp,h}\bigg]\bigg), \end{aligned} \tag{A2}$$

$$\begin{aligned} A_h(B_s^0 \to (\rho^0 \to)\pi\pi(\omega \to)\pi\pi) &= \frac{G_F}{\sqrt{2}}\bigg(V_{ub}^* V_{us}\bigg[\bigg(C_1+\frac{1}{3}C_2\bigg)F_a^{ll,h}+C_2 M_a^{ll,h}\bigg] \\ &\quad - V_{tb}^* V_{ts}\bigg[\bigg(\frac{3}{2}C_7+\frac{1}{2}C_8+\frac{3}{2}C_9+\frac{1}{2}C_{10}\bigg)F_a^{ll,h} \\ &\quad +\bigg(\frac{3}{2}C_{10}\bigg)M_a^{ll,h}+\bigg(\frac{3}{2}C_8\bigg)M_a^{sp,h}\bigg]\bigg). \end{aligned} \tag{A3}$$

Here $G_F = 1.16639\times 10^{-5}\ \mathrm{GeV}^{-2}$ is the Fermi coupling constant. For the double P-wave resonance, we decompose the decay amplitudes into three helicity components with $h = 0, \parallel, \perp$. $(V-A)\otimes(V-A)$, $(V-A)\otimes(V+A)$, $(S-P)\otimes(S+P)$ are defined as LL, LR, and SP. F_e and M_e refer to the factorizable or nonfactorizable emission diagrams, F_a and M_a refer to the factorizable or nonfactorizable annihilation diagrams.

$$\begin{aligned} A(B_s^0 \to (\rho^0 \to)\pi\pi(f_0(500) \to)\pi\pi) &= \frac{G_F}{\sqrt{2}}\bigg(V_{ub}^* V_{us}\bigg[\bigg(C_1+\frac{1}{3}C_2\bigg)F_a^{ll,vs}+C_2 M_a^{ll,vs}\bigg] \\ &\quad - V_{tb}^* V_{ts}\bigg[\bigg(\frac{3}{2}C_7+\frac{1}{6}C_8+\frac{3}{2}C_9+\frac{1}{2}C_{10}\bigg)F_a^{ll,vs} \\ &\quad +\bigg(\frac{3}{2}C_{10}\bigg)M_a^{ll,vs}+\bigg(\frac{3}{2}C_8\bigg)M_a^{sp,vs}\bigg]\bigg), \end{aligned} \tag{A4}$$

$$\begin{aligned} A(B_s^0 \to (f_0(980) \to)\pi\pi(f_0(980) \to)\pi\pi) &= -\sqrt{2}G_F V_{tb}^* V_{ts}\bigg[\bigg(\frac{4}{3}C_3+\frac{4}{3}C_4+C_5+\frac{1}{3}C_6-\frac{1}{2}C_7-\frac{1}{6}C_8-\frac{2}{3}C_9-\frac{2}{3}C_{10}\bigg)F_a^{ll,ss} \\ &\quad +\bigg(C_6+\frac{1}{3}C_5-\frac{1}{2}C_8-\frac{1}{6}C_7\bigg)(F_e^{sp,ss}+F_a^{sp,ss})+\bigg(C_3+C_4-\frac{1}{2}C_9-\frac{1}{2}C_{10}\bigg) \\ &\quad \times(M_e^{ll,ss}+M_a^{ll,ss})+\bigg(C_5-\frac{1}{2}C_7\bigg)(M_e^{lr,ss}+M_a^{lr,ss}) \\ &\quad +\bigg(C_6-\frac{1}{2}C_8\bigg)(M_e^{sp,ss}+M_a^{sp,ss})\bigg]. \end{aligned} \tag{A5}$$

Here a S-wave resonance is described as vs, and ss stands for the double S-wave resonance component. The explicit expressions for the factorizable contributions $F_{a,e}$ and the nonfactorizable contributions $M_{a,e}$ from Fig. 2 can be found in Refs [47,48,55].

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