Ratio $\mathcal{R}(D_s)$ for $B_s \to D_s \ell \nu_\ell$ by using the QCD light-cone sum rules within the framework of heavy quark effective field theory

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In this paper, we study the $B_s \to D_s$ transition form factors by using the light-cone sum rules within the framework of heavy quark effective field theory. We adopt a chiral current correlation function to do the calculation, the resultant transition form factors $f_+^{B_s \to D_s}(q^2)$ and $f_0^{B_s \to D_s}(q^2)$ are dominated by the contribution of D_s-meson leading-twist distribution amplitude, while the contributions from less certain D_s-meson twist-3 distribution amplitudes are greatly suppressed. At the largest recoil point, we obtain $f_{+,0}^{B_s \to D_s}(0) = 0.533_{-0.094}^{+0.112}$. By further extrapolating the transition form factors into all the physically allowable q^2 region with the help of the z-series parametrization approach, we calculate the branching fractions $\mathcal{B}(B_s \to D_s \ell' \nu_{\ell'})$ with $(\ell' = e, \mu)$ and $\mathcal{B}(B_s \to D_s \tau \nu_\tau)$, which gives $\mathcal{R}(D_s) = 0.334 \pm 0.017$.

I. INTRODUCTION

It is one of the most attractive research topics in the field of high energy physics to accurately test the standard model (SM) and to search new physics effects beyond the SM. The $B \to D^{(*)}$ semileptonic decay provides such an example. The ratio $\mathcal{R}(D^{(*)}) = \mathcal{B}(B \to D^{(*)} \tau \bar{\nu}_\tau)/\mathcal{B}(B \to D^{(*)} \ell' \bar{\nu}_{\ell'})$ with $(\ell' = e, \mu)$ has been measured by various groups, e.g., the *BABAR* Collaboration firstly reported $\mathcal{R}^{\exp}(D) = 0.440 \pm 0.058 \pm 0.042$ and $\mathcal{R}^{\exp}(D^*) = 0.332 \pm 0.024 \pm 0.018$ [1,2], the BELLE Collaboration subsequently given $\mathcal{R}^{\exp}(D) = 0.375 \pm 0.064 \pm 0.026$ and $\mathcal{R}^{\exp}(D^*) = 0.293 \pm 0.038 \pm 0.015$ in year 2015 [3], $\mathcal{R}^{\exp}(D^*) = 0.270 \pm 0.035_{-0.025}^{+0.028}$ in year 2016 [4,5], and $\mathcal{R}^{\exp}(D) = 0.307 \pm 0.037 \pm 0.016$ and $\mathcal{R}^{\exp}(D^*) = 0.283 \pm 0.018 \pm 0.014$ in year 2019 [6], and the LHCb

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Collaboration reported $\mathcal{R}^{\exp}(D^*) = 0.336 \pm 0.027 \pm 0.030$ in year 2015 [7] and $\mathcal{R}^{\exp}(D^*) = 0.283 \pm 0.018 \pm 0.014$ in year 2017 [8,9]. The Heavy Flavor Average Group (HFLAG) gave the weighted average of those measurements, i.e., $\mathcal{R}^{\exp}(D) = 0.339 \pm 0.026 \pm 0.014$ and $\mathcal{R}^{\exp}(D^*) = 0.295 \pm 0.010 \pm 0.010$ [10], where they also gave the averages of theoretical predictions $\mathcal{R}^{\text{th}}(D) = 0.298 \pm 0.003$ and $\mathcal{R}^{\text{th}}(D^*) = 0.252 \pm 0.005$ from Refs. [11–13]. Those theoretical values are consistent with other predictions calculated using various approaches, such as the heavy quark effective theory (HQET) [14–16], the lattice QCD (LQCD) [17–19], the light-cone sum rules (LCSR) [20,21]. Since the theoretical predictions are generally smaller than the measured ones, this difference may indicate new physics beyond the SM [22–25].

The LHCb collaboration has measured the branching fraction $\mathcal{B}(B_s^0 \to D_s^- \mu^+ \nu_\mu) = (2.49 \pm 0.12 \pm 0.14 \pm 0.16) \times 10^{-2}$ [26] and gave the ratio of the branching fractions $\mathcal{B}(B_s^0 \to D_s^- \mu^+ \nu_\mu)$ and $\mathcal{B}(B^0 \to D^- \mu^+ \nu_\mu)$, i.e., $\mathcal{R} = 1.09 \pm 0.05 \pm 0.06 \pm 0.05$. This indicates $B_s \to D_s \ell \nu_\ell$ could behave closely to $B \to D \ell \nu_\ell$. Therefore, it is meaningful to make a detailed study on the similar ratio $\mathcal{R}(D_s)$.

At present, there is still no published data on the ratio $\mathcal{R}(D_s)$, while many theoretical studies on it have been done in Refs. [27–35]. As the key components of calculating the ratio $\mathcal{R}(D_s)$, the $B_s \to D_s$ transition form factors (TFFs) $f_+^{B_s \to D_s}(q^2)$ and $f_0^{B_s \to D_s}(q^2)$ have been studied under various approaches, e.g., the QCD sum rules (QCDSR)

[36], the constituent quark model (CQM) [37], the light-cone sum rules (LCSR) [38,39], the Bethe-Salpeter equation (BSE) [40], and the lattice QCD (LQCD) [31–34]. In addition, there are other discussions and analysis on the properties of $B_s \to D_s$ TFFs [41]. Different approaches are applicable in various energy scale regions, for example, the LCSR is applicable in the largest low and intermediate q^2-region; and in the present paper, as the same as the previous treatment of $B \to \pi$ TFFs [42], we will adopt the LCSR approach within the framework of heavy quark effective field theory (HQEFT) [43–48] to calculate the $B_s \to D_s$ TFFs.

In our previous work [49], we have calculated the TFF $f_+^{B_s \to D_s}(q^2)$ within the LCSR approach, where the chiral current correlation function (correlator) has been adopted. This way, the contributions from the D_s-meson less known twist-3 light-cone distribution amplitudes (LCDAs) are disappeared, and $f_+^{B_s \to D_s}(q^2)$ is expressed as a convolution of the D_s-meson leading-twist LCDA $\phi_{2;D_s}(x,\mu)$ with the hard part. The chiral current correlator is not suitable for calculating the TFF $f_0^{B_s \to D_s}(q^2)$ [21,50].[1] The LCSRs in the framework of HQEFT can solve this embarrassment [42]. The HQEFT separates the nonperturbative long-distance terms from the short-distance dynamics via a systematic way, and then the long-distance terms can be decreased to a series over the wave functions or transition form factors. It has been pointed out that by choosing a proper chiral correlator, as will be adopted in this paper, one can suppress the uncertainties from the high-twist LCDAs and achieve a more accurate LCSR prediction for the $B_s \to D_s$ TFFs.

The remaining parts of the paper are organized as follows. In Sec. II, we present the calculation technologies for the two TFFs of the $B_s \to D_s \ell \bar{\nu}_\ell$ semileptonic decays by using the light-cone sum rules within the framework of HQEFT. In Sec. III, we present our numerical results and discussions. Section IV is reserved for a summary.

II. CALCULATION TECHNOLOGY

A. $B_s \to D_s$ transition matrix element

For the $B_s \to D_s l\bar{\nu}_l$ decays, the transition matrix element can be parametrized as follows:

$$
\begin{aligned}
&\langle D_s(p)|\bar{c}\gamma_\mu b|B_s(p+q)\rangle \\
&= 2f_+^{B_s \to D_s}(q^2)p_\mu + [f_+^{B_s \to D_s}(q^2) + f_-^{B_s \to D_s}(q^2)]q_\mu
\end{aligned} \quad (1)
$$

and

---

[1]One may turn to traditional choice of correlator to do the calculation [38], and the contributions proportional to the D_s-meson twist-3 LCDAs needs to be considered. However the twist-3 LCDAs $\phi_{3;D_s}^p(x,\mu)$ and $\phi_{3;D_s}^\sigma(x,\mu)$ are still less known.

$$
f_0^{B_s \to D_s}(q^2) = f_+^{B_s \to D_s}(q^2) + \frac{q^2}{m_{B_s}^2 - m_{D_s}^2} f_-^{B_s \to D_s}(q^2), \quad (2)
$$

where p is the momentum of the D_s-meson and $(p+q)$ is the momentum of B_s-meson. At the maximum recoil point, we have $f_+^{B_s \to D_s}(0) = f_0^{B_s \to D_s}(0)$. The transition matrix element can be expanded as $1/m_b$-power series within the framework of HQEFT. Based on the heavy quark symmetry, the transition matrix element of heavy quark in the effective theory is parametrized as [46–48]:

$$
\begin{aligned}
\langle D_s(p)|\bar{c}\gamma_\mu b|B_s(p+q)\rangle &= \frac{\sqrt{m_{B_s}}}{\sqrt{\bar{\Lambda}_{B_s}}}\langle D_s(p)|\bar{u}\gamma_\mu b_v^+|B_{sv}\rangle \\
&= -\frac{\sqrt{m_{B_s}}}{\sqrt{\bar{\Lambda}_{B_s}}}\text{Tr}[D_s(v,p)\gamma_\mu \mathcal{M}_v], \quad (3)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\Lambda}_{B_s} &= m_{B_s} - m_b, \\
D_s(v,p) &= \gamma^5[A(v \cdot p) + \hat{p}B(v \cdot p)], \\
\mathcal{M}_v &= -\sqrt{\bar{\Lambda}}(1 + \slashed{v})\gamma^5/2, \quad (4)
\end{aligned}
$$

where b_v^+ is the effective b-quark field and v is the B_s-meson velocity, $\hat{p}^\mu = p^\mu/(v \cdot p)$. $A(v \cdot p)$ and $B(v \cdot p)$ are leading-order heavy flavor-spin independent coefficient functions. $\bar{\Lambda} = \lim_{m_b \to \infty}\bar{\Lambda}_{B_s}$, which is the heavy flavor independent binding energy that reflects the effects of light degrees of freedom in the heavy hadron. Using those formulas, we obtain the $B_s \to D_s \ell \bar{\nu}_\ell$ TFFs $f_\pm(q^2)$, which are

$$
f_\pm^{B_s \to D_s}(q^2) = \frac{\sqrt{\bar{\Lambda}}}{\sqrt{m_{B_s}\bar{\Lambda}_{B_s}}}\left[A(y) \pm \frac{m_{B_s}}{y}B(y)\right] + \cdots, \quad (5)
$$

with

$$
y = v \cdot p = (m_{B_s}^2 + m_{D_s}^2 - q^2)/(2m_{B_s}), \quad (6)
$$

where "$\cdots$" denotes the higher-order $\mathcal{O}(1/m_b)$ contributions that will not be taken into consideration here.

B. Light-cone sum rule for $f_{+,0}^{B_s \to D_s}(q^2)$

To derive the sum rules of the two leading order heavy flavor-spin independent coefficient functions $A(y)$ and $B(y)$, we construct the following correlator:

$$
F_\mu(p,q) = i\int d^4x e^{iq \cdot x}\langle D_s(p)|T\{j_n(x), j_0^\dagger(0)\}|0\rangle, \quad (7)
$$

where the currents

$$j_n(x) = \bar{c}(x)\gamma_\mu(1+\gamma_5)b(x) \qquad (8)$$

$$j_0^\dagger(0) = \bar{b}(0)i(1+\gamma_5)s(0) \qquad (9)$$

Following the standard procedure of LCSR approach, we first deal with the hadronic representation for the correlation function. One can insert a complete series of the intermediate hadronic states in the correlator (13) in the physical q^2-region and isolate the pole term of the lowest pseudoscalar state from the hadronic representation. Then the correlator $F_\mu(p,q)$ becomes:

$$
\begin{aligned}
&F_\mu^{\text{Had}}(p,q) \\
&= \frac{\langle D_s(p)|\bar{c}\gamma_\mu b|B_s\rangle\langle B_s|\bar{b}i\gamma_5 s|0\rangle}{m_{B_s}^2 - (p+q)^2} \\
&\quad + \sum_{B_s^H} \frac{\langle D_s(p)|\bar{c}\gamma_\mu(1+\gamma_5)b|B_s^H\rangle\langle B_s^H|\bar{b}i(1+\gamma_5)s|0\rangle}{m_{B_s^H}^2 - (p+q)^2}.
\end{aligned}
$$
$$(10)$$

In the effective theory of heavy quark, the hadronic representation (10) can be expanded in powers of $1/m_b$. Taking the transition matrix element (3) into consideration and neglecting the contributions from higher $1/m_b$ order, we can further write the hadronic representation as:

$$
\begin{aligned}
F_\mu^{\text{Had}}(p,q) &= 2F\frac{A(y)v^\mu + B(y)\hat{p}^\mu}{2\bar{\Lambda}_{B_s} - 2v\cdot k} \\
&\quad + \int_{s_0}^\infty ds\frac{\rho(y,s)}{s-2v\cdot k} + \text{Subtractions},
\end{aligned}
$$
$$(11)$$

with the matrix element [51]

$$\langle B_s|\bar{b}_v^+ i\gamma_5 d|0\rangle = \frac{i}{2}F\text{Tr}[\gamma_5\mathcal{M}_v], \qquad (12)$$

where F is the leading-order decay constant of the B_s-meson [52,53]. k is the residual momentum of the heavy hadronic. Using the ansatz of the quark-hadron duality the spectral density $\rho(y,s)$ can be obtained [54,55].

On the other hand, we apply the operator product expansion (OPE) to the correlator in the deep Euclidean region. The correlator (13) can be explicitly written as

$$
\begin{aligned}
F_\mu(p,q) &= i\int d^4x e^{i(q-m_b)\cdot x}\langle D_s(p)|T\{\bar{c}(x)\gamma_\mu(1+\gamma_5)b_v^+(x), \\
&\quad \bar{b}_v^+(0)i(1+\gamma_5)s(0)\}|0\rangle.
\end{aligned}
$$
$$(13)$$

Using the B-meson heavy-quark propagator $S(x,v) = (1+\slashed{v})\times\int_0^\infty dt\delta(x-vt)/2$ [52], the correlator can be expanded as a complex power series over the D_s-meson LCDAs. Due to the chiral suppressions, it is noted that the main contribution to the correlator comes from the

leading-twist LCDA, and the contributions from all the twist-3 LCDAs are exactly zero.

Through the dispersion relation, the OPE in deep Euclidean region and the hadron expression in physical region can be matched. And by further applying the Borel transformation to suppress the contributions from power-suppressed terms,[2] the LCSRs for the coefficient functions $A(y)$ and $B(y)$ are

$$A(y) = -\frac{f_{D_s}}{2F}\int_0^{s_0^{B_s}} ds e^{(2\bar{\Lambda}_{B_s}-s)/T}\frac{1}{y^2}\frac{\partial}{\partial u}g_2(u)\Big|_{u=1-\frac{s}{2y}}, \qquad (14)$$

$$
\begin{aligned}
B(y) &= -\frac{f_{D_s}}{2F}\int_0^{s_0^{B_s}} ds e^{(2\bar{\Lambda}_{B_s}-s)/T}\Bigg[-\phi_{2;D_s}(u) \\
&\quad + \left(\frac{1}{y}\frac{\partial}{\partial u}\right)^2 g_1(u) - \frac{1}{y^2}\frac{\partial}{\partial u}g_2(u)\Bigg]\Bigg|_{u=1-\frac{s}{2y}}, \qquad (15)
\end{aligned}
$$

where T is the Borel parameter and $s_0^{B_s}$ is the continuum threshold, g_1 and g_2 are twist-four LCDAs. Since the contributions from the twist-4 LCDAs are only several percent, so we shall directly adopt the light pseudoscale ones to do the calculations, whose explicit forms can be found in Refs. [56]. Substituting them into Eqs. (2), (5), we obtain

$$
\begin{aligned}
f_+^{B_s\to D_s}(q^2) &= -\frac{f_{D_s}\sqrt{\bar{\Lambda}}}{2F\sqrt{m_{B_s}\bar{\Lambda}_{B_s}}}\int_0^{s_0^{B_s}} ds e^{(2\bar{\Lambda}_{B_s}-s)/T} \\
&\quad \times\Bigg\{\frac{1}{y^2}\frac{\partial}{\partial u}g_2(u) + \frac{m_{B_s}}{y}\Bigg[-\phi_{2;D_s}(u) \\
&\quad + \left(\frac{1}{y}\frac{\partial}{\partial u}\right)^2 g_1(u) - \frac{1}{y^2}\frac{\partial}{\partial u}g_2(u)\Bigg]\Bigg\}\Bigg|_{u=1-\frac{s}{2y}}, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
f_0^{B_s\to D_s}(q^2) &= -\frac{f_{D_s}\sqrt{\bar{\Lambda}}}{2F\sqrt{m_{B_s}\bar{\Lambda}_{B_s}}}\int_0^{s_0^{B_s}} ds e^{(2\bar{\Lambda}_{B_s}-s)/T} \\
&\quad \times\Bigg\{\left(1+\frac{q^2}{m_{B_s}^2-m_{D_s}^2}\right)\frac{1}{y^2}\frac{\partial}{\partial u}g_2(u) \\
&\quad + \left(1-\frac{q^2}{m_{B_s}^2-m_{D_s}^2}\right)\frac{m_{B_s}}{y}\Bigg[-\phi_{2;D_s}(u) \\
&\quad + \left(\frac{1}{y}\frac{\partial}{\partial u}\right)^2 g_1(u) - \frac{1}{y^2}\frac{\partial}{\partial u}g_2(u)\Bigg]\Bigg\}\Bigg|_{u=1-\frac{s}{2y}}. \qquad (17)
\end{aligned}
$$

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[2]More specifically, the continuum contributions are suppressed with the exponential factor $e^{-s/T}$; and the higher twist contributions are generally suppressed by the powers of Borel parameter T, which for the present case are transferred as the suppression of the powers of y through the distribution integral over u.

The Borel parameter T and the continuum threshold $s_0^{B_s}$ shall be fixed such that the resulting TFFs do not depend too much on the precise values of those parameters. In addition, the continuum contribution, which is the part of the dispersive integral from $s_0^{B_s}$ to ∞ that is subtracted from both sides of the equation, should not be too large.

III. NUMERICAL ANALYSIS

A. Input parameters

To determine the TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ of the exclusive process $B_s \to D_s \ell \bar{\nu}_\ell$, we take [46,57]

$$m_{B_s} = 5.367 \pm 0.00014 \text{ GeV},$$
$$m_{D_s} = 1.968 \pm 0.00007 \text{ GeV},$$
$$f_{D_s} = 0.256 \pm 0.0042 \text{ GeV},$$
$$F = 0.30 \pm 0.04 \text{ GeV}^{3/2}.$$

For the D_s-meson leading-twist LCDA $\phi_{2;D_s}(u, \mu)$ in the LCSRs of TFFs (16) and (17), we adopt the light-cone harmonic oscillator (LCHO) model based on the Brodsky-Huang-Lepage (BHL) prescription suggested in Ref. [49], which has a better endpoint behavior and is derived from the approximate bound state solution in the quark model for meson in the rest frame and the connection between the equal-time wave function in the rest frame and the light-cone wave function by equating the off-shell propagator in the two frames. The LCHO model of $\phi_{2;D_s}(u, \mu)$ reads

$$\phi_{2;D_s}(u, \mu) = \frac{\sqrt{6} A_{D_s} \beta_{D_s}^2}{\pi^2 f_{D_s}} u\bar{u} \varphi_{2;D_s}(u) \exp\left[-\frac{\hat{m}_c^2 u + \hat{m}_s^2 \bar{u}}{8\beta_{D_s}^2 u\bar{u}}\right]$$
$$\times \left\{1 - \exp\left[-\frac{\mu^2}{8\beta_{D_s}^2 u\bar{u}}\right]\right\}, \quad (18)$$

where $\bar{u} = 1 - u$ and $\varphi_{2;D_s}(u) = 1 + \sum_{n=1}^4 B_n^{D_s} \times C_n^{3/2}(\xi)$ with $\xi = u - \bar{u}$. $\hat{m}_c$ and $\hat{m}_s$ are c- and s-constituent quark masses, whose values are taken as $\hat{m}_c \simeq 1.5$ GeV and $\hat{m}_s \simeq 0.5$ GeV. A_{D_s}, β_{D_s}, $B_1^{D_s}$, $B_2^{D_s}$, $B_3^{D_s}$, and $B_4^{D_s}$ are model parameters, whose initial values at the scale $\mu = 2$ GeV have been given in Ref. [49]. For the present process, the typical factorization scale $\mu = (m_{B_s}^2 - m_b^2)^{1/2} = 3$ GeV. The input parameters at the scale $\mu = 3$ GeV can be

TABLE I. The D_s-meson leading-twist LCDA parameters at the scale $\mu = 3$ GeV. The values in the second row are for the central values, and the values in third and fourth rows are uncertainties.

$A_{D_s}(\text{GeV}^{-1})$	$B_1^{D_s}$	$B_2^{D_s}$	$B_3^{D_s}$	$B_4^{D_s}$	$\beta_{D_s}(\text{GeV})$
1.246	−0.214	−0.167	0.055	0.005	5.521
11.001	−0.165	0.014	−0.004	0.003	1.046
1.184	−0.189	−0.163	0.047	0.008	6.970

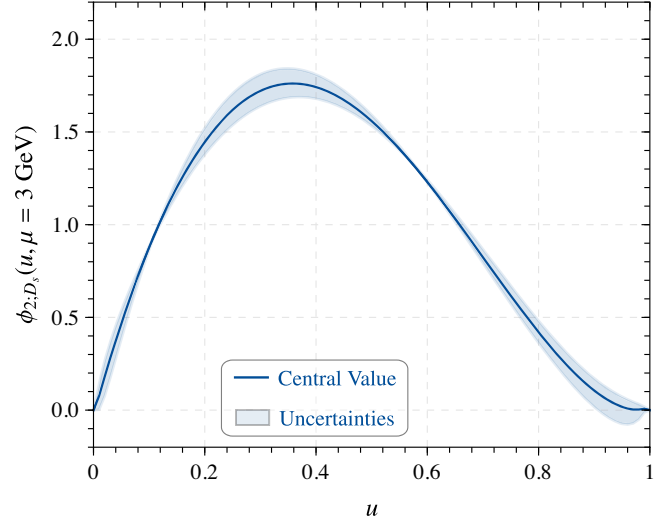FIG. 1. The behavior of LCDA $\phi_{2;D_s}(u, \mu = 3 \text{ GeV})$, the shaded area indicates LCDA's uncertainties.

achieved by using the conventional one-loop evolution equation [58], and these values are given in Table I. Figure 1 shows the behavior of the D_s-meson leading-twist LCDA $\phi_{2;D_s}(u, \mu = 3 \text{ GeV})$ with the typical values exhibited in Table I, where the solid line is the central value and the shaded band shows its uncertainty given in Table I.

B. The $B_s \to D_s$ TFFs

Next, we calculate the $B_s \to D_s$ TFFs $f_+^{B_s \to D_s}(q^2)$ and $f_0^{B_s \to D_s}(q^2)$ by using the LCSRs (16) and (17) in the q^2-region when the LCSR approach is applicable, i.e., $0 < q^2 < 7 \text{ GeV}^2$. For the purpose, we first determine the continuum threshold $s_0^{B_s}$ and the Borel parameter T. Within the framework of HQEFT, the continuum threshold $s_0^{B_s} \equiv 2\bar{\Lambda}_{B_s^*} = 2(m_{B_s^*} - m_b)$ with B_s^* being the B_s-meson first excited state [52],[3] and we take $s_0^{B_s} = 3.85 \pm 0.15$ GeV. To determine the required Borel window, as suggested in Refs. [52,53], we require the TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ to be as stable as possible within corresponding Borel windows. Figure 2 shows the TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ versus the Borel parameter T at several typical squared momenta transfer, in which the solid, the dashed, the dot-dashed and the dotted lines are for $q^2 = 0, 3, 5, 7 \text{ GeV}^2$, respectively. One can find that, the TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ are stable for a large T, e.g., the uncertainty caused by T is less than 5% when $T \geq 10$ GeV for all those q^2 values. Therefore, we take the Borel window as 10 GeV $< T <$ 20 GeV.

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[3]Under the traditional LCSR treatment, we have taken the continuum threshold to be $38 \pm 1 \text{ GeV}^2$ [49], which can be understood as a value near the squared mass of the first excited state of B_s-meson and is consistent with our present choice of $s_0^{B_s}$.
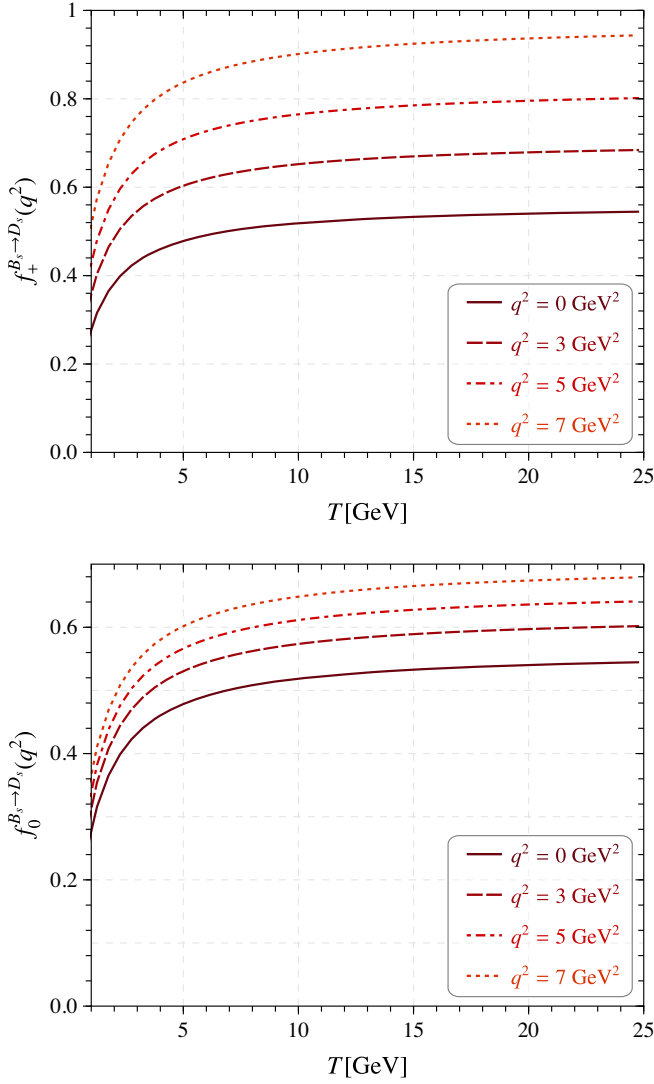
FIG. 2. The TFFs $f_+^{B_s \to D_s}(q^2)$ and $f_0^{B_s \to D_s}(q^2)$ versus the Borel parameter T, where several typical q^2 are adopted and all input parameters have been set as their central values.

At the maximum recoil point $q^2 = 0$, we have

$$f_{+,0}^{B_s \to D_s}(0) = 0.533_{-0.063}^{+0.082}|_{\phi_{2;D_s}} {}_{-0.014}^{+0.007}|_T {}_{-0.063}^{+0.065}|_{s_0^{B_s}}$$
$${}_{-0.037}^{+0.027}|_F {}_{-0.004}^{+0.004}|_{m_b}. \qquad (19)$$

There are also errors caused by the uncertainties of m_{B_s} and m_{D_s}, which are negligibly small. It is found that the uncertainties of the D_s-meson leading-twist LCDA $\phi_{2;D_s}$ and the continuum threshold $s_0^{B_s}$ are main errors of $f_{+,0}^{B_s \to D_s}(0)$. By adding all those errors in quadrature, we obtain $f_{+,0}^{B_s \to D_s}(0) = 0.533_{-0.128}^{+0.160}$. We present the theoretical predictions of $f_{+,0}^{B_s \to D_s}(q^2)$ at the maximum recoil point $q^2 = 0$ in Table II, where the predictions under the pQCD approach [27], the pQCD + LQCD approach [28], the

TABLE II. Theoretical predictions of the TFF $f_{+,0}^{B_s \to D_s}(q^2)$ at the maximum recoil point under various approaches.

Methods	$f_{+,0}^{B_s \to D_s}(0)$
This work (HQEFT)	$0.533_{-0.094}^{+0.112}$
pQCD [27]	$0.55_{-0.12}^{+0.15}$
pQCD+LQCD [28]	0.52 ± 0.10
RQM [29]	0.74 ± 0.02
LQCD [31]	0.656 ± 0.031
LQCD [32]	0.661 ± 0.042
QCDSR [36]	0.7 ± 0.1
QCDSR [59]	0.24
LCSR [38]	$0.43_{-0.08}^{+0.09}$
BSE [40]	$0.57_{-0.03}^{+0.02}$

LQCD approach [31,32], the QCD SR approach [36,59], the LCSR approach [38], the BSE approach [40] and the RQM approach [29] are also presented. Our present prediction of $f_{+,0}^{B_s \to D_s}(0)$ is in good agreement with the values calculated with the pQCD prediction [27], the pQCD + LQCD prediction [28] and the BSE prediction [40].

As mentioned above, the LCSRs (16) and (17) for TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ are only reliable in low and intermediate regions, i.e., $0 < q^2 < 7$ GeV2. To estimate the total decay width of the semileptonic decay $B_s \to D_s \ell \bar{\nu}_\ell$, we extrapolate the TFFs to the whole physically allowable q^2-region, $0 < q^2 < (m_{B_s} - m_{D_s})^2 = 11.50$ GeV^2, via the z-series parametrization [60,61]:

$$f_+^{B_s \to D_s}(q^2) = \frac{f_+^{B_s \to D_s}(0)}{1 - q^2/m_{B_s^*}^2} \left\{ 1 + \sum_{k=1}^{N-1} b_k \left[z(q^2)^k - z(0)^k \right. \right.$$
$$\left. \left. - (-1)^{N-k} \frac{k}{N} \left(z(q^2)^N - z(0)^N \right) \right] \right\}, \qquad (20)$$

$$f_0^{B_s \to D_s}(q^2) = f_0^{B_s \to D_s}(0) \left\{ 1 + \sum_{k=1}^{N-1} b_k (z(q^2)^k - z(0)^k) \right\}.$$
$$(21)$$

where

$$z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}},$$
$$t_0 = t_+ (1 - \sqrt{1 - t_+/t_-}),$$
$$t_\pm = (m_{B_s} \pm m_{D_s})^2.$$

By fitting the values of the TFFs in low and intermediate regions calculated via the LCSRs (16) and (17), the coefficients b_1, b_2 and b_3 in extrapolation formula (20) and (21) can be determined, and which have been exhibited in Table III. The quality-of-fit is defined as:

TABLE III. The fitted parameters and the quality-of-fit for the extrapolated TFFs $f^{B_s \to D_s}_{+,0}(q^2)$.

$f^{B_s \to D_s}_0(q^2)$	b_1	b_2	b_3	(Δ)
0.533	-2.378	-19.414	196.397	0.006%
$0.533^{+0.112}$	-3.586	-15.843	236.929	0.006%
$0.533_{-0.094}$	-4.646	-12.450	281.244	0.006%
$f^{B_s \to D_s}_+(q^2)$	b_1	b_2	b_3	(Δ)
0.533	-4.389	-0.641	123.375	0.006%
$0.533^{+0.112}$	-4.888	-1.687	165.880	0.008%
$0.533_{-0.094}$	-5.459	-4.242	209.787	0.006%

$$\Delta = \frac{\sum_t |f^i - f^{\text{fit}}|}{\sum_t |f^i|} \times 100\%, \quad t \in \left\{0, \frac{1}{2}, \dots, \frac{23}{2}, 12\right\} \text{GeV}^2. \tag{22}$$

The coefficients b_i are determined such that the quality-of-fit (Δ) is no more than 1%. The Δ values for the central, the upper, and lower TFFs are shown in Table III. These quality-of-fits are much smaller than 1%, indicating that our present extrapolations are of high accuracy. We present the extrapolated TFFs $f^{B_s \to D_s}_{+,0}(q^2)$ in Fig. 3, where the shaded hands are theoretical uncertainties from all the mentioned error sources. For comparison, we present the results of the pQCD + LQCD approach [28], the pQCD approach [28], the RQM approach [30] and the LQCD approach [31].

C. The $B_s \to D_s \ell \bar{\nu}_\ell$ branching fractions and the ratio $\mathcal{R}(D_s)$

The branching fraction of the semileptonic decay $B_s \to D_s \ell \bar{\nu}_\ell$ is defined as

$$\mathcal{B}(B_s \to D_s \ell \bar{\nu}_\ell) = \tau_{B_s} \times \int_0^{(m_{B_s} - m_{D_s})^2} dq^2 \frac{d\Gamma(B_s \to D_s \ell \bar{\nu}_\ell)}{dq^2}, \tag{23}$$

where $q^2_{\max} = (m_{B_s} - m_{D_s})^2$ and τ_{B_s} is the B_s-meson lifetime. Here the differential decay widths is

$$\frac{d\Gamma(B_s \to D_s \ell \bar{\nu}_\ell)}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_s}^3} \left(1 - \frac{m_\ell^2}{q^2}\right)^2 \left[\left(1 + \frac{m_\ell^2}{2q^2}\right)\right.$$
$$\times \lambda^{\frac{3}{2}}(q^2) |f^{B_s \to D_s}_+(q^2)|^2$$
$$+ \frac{3m_\ell^2}{2q^2}(m_{B_s}^2 - m_{D_s}^2)^2$$
$$\left. \times \lambda^{\frac{1}{2}}(q^2) |f^{B_s \to D_s}_0(q^2)|^2 \right], \tag{24}$$

where $\lambda(q^2) = (m_{B_s}^2 + m_{D_s}^2 - q^2)^2 - 4m_{B_s}^2 m_{D_s}^2$, which is the phase-space factor. $|V_{cb}|$, G_F and m_ℓ are CKM matrix
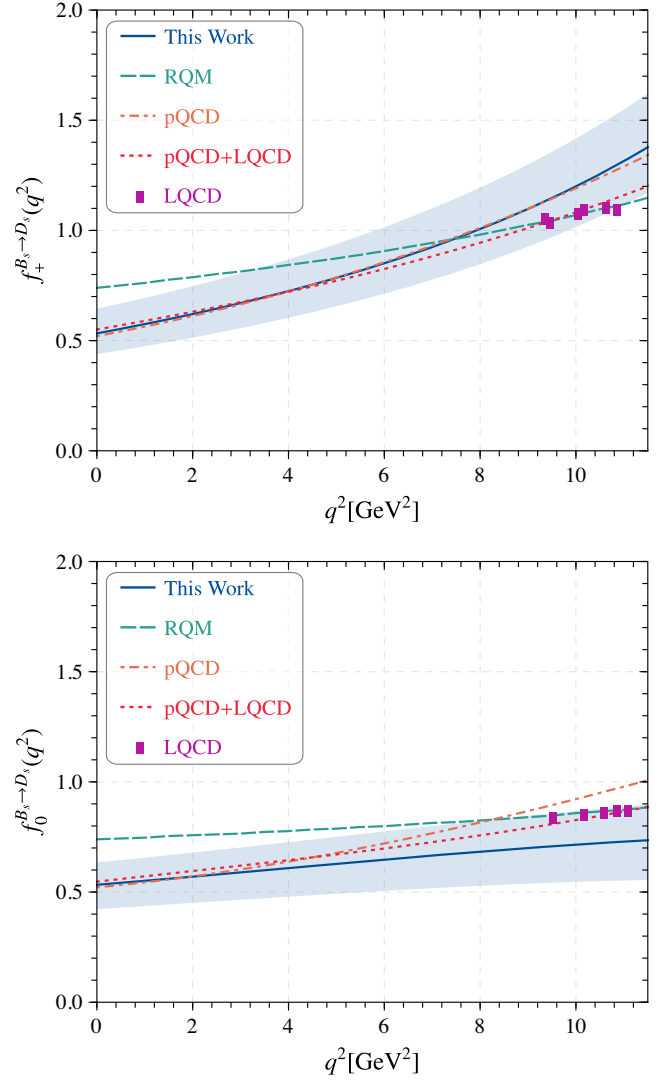

FIG. 3. The extrapolated TFFs $f^{B_s \to D_s}_{+,0}(q^2)$ versus q^2. The solid line are central values and the shaded bands are corresponding uncertainties. As a comparison, the predictions under the pQCD + LQCD approach [28], the pQCD approach [28], and the RQM approach [30] and the LQCD approach [31] are also presented.

element, the Fermi-coupling constant and the lepton mass, respectively, and we take [57]: $\tau_{B_s} = (1.510 \pm 0.004) \times 10^{-12}$ s, $|V_{cb}| = (40.5 \pm 1.5) \times 10^{-3}$, $G_F = 1.1663787(6) \times 10^{-5}$ GeV^{-2}, and τ-lepton mass $m_\tau = 1.776 \pm 0.00012$ GeV. For lepton $\ell' = e$ or μ, its mass is negligible, and then the above differential decay width can be simplified as

$$\frac{d\Gamma(B_s \to D_s \ell' \bar{\nu}_{\ell'})}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_s}^3} \lambda^{3/2}(q^2) |f^{B_s \to D_s}_+(q^2)|^2, \tag{25}$$

where $f^{B_s \to D_s}_0(q^2)$ has zero contribution due to the usual chiral suppression.

We present the differential decay widths of $B_s \to D_s \tau \bar{\nu}_\tau$ and $B_s \to D_s \ell' \bar{\nu}_{\ell'}$ in Fig. 4, in which the solid lines are for the central choices of all the input parameters, and the shaded bands are uncertainties by adding all the errors caused by the error sources such as $f_{+,0}^{B_s \to D_s}(q^2)$, m_{B_s}, m_{D_s}, $|V_{cb}|$, G_F and m_ℓ, etc., in quadrature. In addition, the predictions under the RQM approach [30] and the LQCD approach [34] are also given. One may observe that our prediction of $d\Gamma(B_s \to D_s \tau \bar{\nu}_\tau)/dq^2$ is consistent with the LQCD and RQM predictions in Refs. [30,34]; And for $d\Gamma(B_s \to D_s \ell' \bar{\nu}_{\ell'})/dq^2$, our prediction agrees with the LQCD and RQM predictions [30,34] in larger q^2 region, but is smaller than those predictions in lower q^2 region.

We present the branching fractions $\mathcal{B}(B_s \to D_s \ell' \bar{\nu}_{\ell'})$ and $\mathcal{B}(B_s \to D_s \tau \bar{\nu}_\tau)$ in Table IV, where the predictions under

TABLE IV. Theoretical predictions of the branching fractions $\mathcal{B}(B_s \to D_s \ell' \bar{\nu}_{\ell'})$ and $\mathcal{B}(B_s \to D_s \tau \bar{\nu}_\tau)$ (in unit: 10^{-2}).

Methods	$\mathcal{B}(B_s \to D_s \ell' \bar{\nu}_{\ell'})$	$\mathcal{B}(B_s \to D_s \tau \bar{\nu}_\tau)$
This work (HQEFT)	$1.817^{+0.802}_{-0.571}$	$0.606^{+0.266}_{-0.211}$
pQCD [28]	$1.97^{+0.89}_{-0.51}$	$0.72^{+0.32}_{-0.23}$
pQCD + LQCD [28]	$1.84^{+0.77}_{-0.51}$	$0.63^{+0.17}_{-0.13}$
pQCD [27]	$2.13^{+1.12}_{-0.77}$	$0.84^{+0.38}_{-0.28}$
RQM [30]	2.1 ± 0.2	0.62 ± 0.05
RQM [29]	$2.54^{+0.28}_{-0.27}$	$0.695^{+0.085}_{-0.075}$
CQM [37]	2.73–3.00	...
QCDSR [36]	2.46 ± 0.38	...
QCDSR [59]	2.8–3.5	...
LCSR [49]	$2.03^{+0.35}_{-0.49}$	...
LCSR [38]	$1.0^{+0.4}_{-0.3}$	$0.33^{+0.14}_{-0.11}$
LQCD [33]	2.013–2.469	0.619–0.724
BSE [40]	1.4–1.7	0.47–0.55

various approaches are also presented as a comparison. It is noted that our present predictions are consistent with most of the previous predictions within errors. Especially, our prediction of $\mathcal{B}(B_s \to D_s \ell' \bar{\nu}_{\ell'})$ is in good agreement with the pQCD prediction of Refs. [27,28] and the pQCD + LQCD approach [28], and our prediction of $\mathcal{B}(B_s \to D_s \tau \bar{\nu}_\tau)$ is in good agreement with the pQCD + LQCD prediction [28] and the RQM predictions of Refs. [29,30].

Combining Eqs. (23)–(25), we can obtain the ratio $\mathcal{R}(D_s)$

$$\mathcal{R}(D_s) = \frac{\int_{m_\tau^2}^{q_{\max}^2} d\Gamma(B_s \to D_s \tau \bar{\nu}_\tau)/dq^2}{\int_0^{q_{\max}^2} d\Gamma(B_s \to D_s l' \bar{\nu}_{l'})/dq^2}, \quad (26)$$

which leads to

$$\mathcal{R}(D_s) = 0.334 \pm 0.017. \quad (27)$$

We present the ratios under various approaches in Table V. And to be consistent with the above branching fractions,

TABLE V. The ratios $\mathcal{R}(D_s)$ under various approaches.

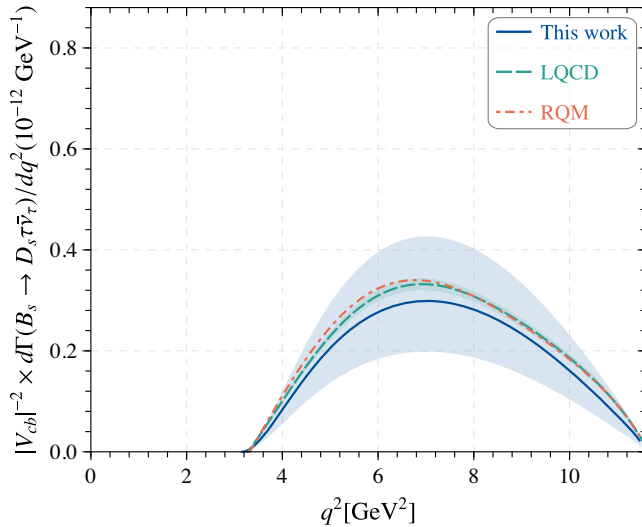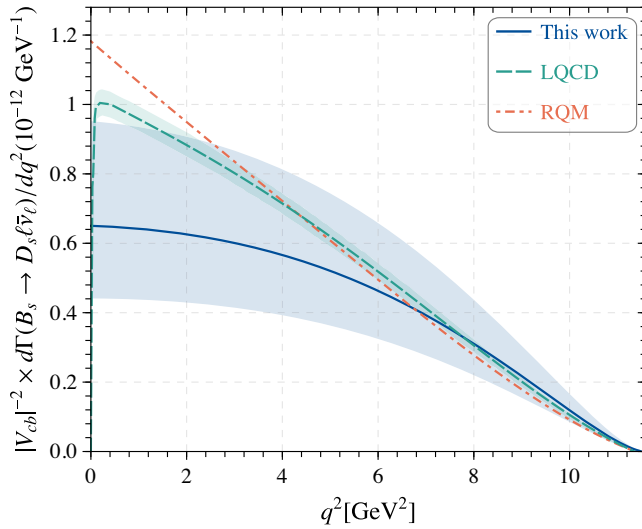| Methods	$\mathcal{R}(D_s)$
This work (HQEFT)	0.334 ± 0.017
pQCD [28]	$0.365^{+0.009}_{-0.012}$
pQCD+LQCD [28]	$0.341^{+0.024}_{-0.025}$
pQCD [27]	0.392 ± 0.022
RQM [30]	0.295
RQM [29]	$0.274^{+0.020}_{-0.019}$
LQCD [33]	$0.299^{+0.027}_{-0.022}$
LQCD [31]	0.314 ± 0.006
CCQM [35]	0.271 ± 0.069
LCSR [38]	0.33
LCSR [39]	0.2979 ± 0.0044

FIG. 4. The differential decay widths of $B_s \to D_s \ell \bar{\nu}_\ell$, where the uncertainties are squared averages of those from all the mentioned error sources. The predictions under the LQCD approach [34] and the RQM approach [30] are also presented.

our ratio $\mathcal{R}(D_s)$ is in good agreement with prediction under the pQCD + LQCD approach [28].

IV. SUMMARY

In the present paper, we make a detailed study on the TFFs of the semileptonic decay $B_s \to D_s \ell \nu_\ell$ under the LCSR approach within the framework of HQEFT. By using the chiral correlator, the TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ are dominated by the leading-twist contributions and the accuracy of the LCSR prediction is improved. At the maximum recoil point, we have $f_{+,0}^{B_s \to D_s}(0) = 0.533_{-0.094}^{+0.112}$. After applying the z-series extrapolation, we obtain the TFFs in the whole physical q^2-region. Figure 3 and Fig. 4 show the extrapolated TFFs $f_{+,0}^{B_s \to D_s}(q^2)$ and the differential decay widths of $B_s \to D_s \ell \bar{\nu}_\ell$, respectively. Furthermore, we derive the branching fractions $\mathcal{B}(B_s \to D_s \ell \bar{\nu}_\ell) = (1.817_{-0.571}^{+0.802}) \times 10^{-2}$ and $\mathcal{B}(B_s \to D_s \tau \bar{\nu}_\tau) = (6.061_{-2.114}^{+2.660}) \times 10^{-3}$. The resultant ratio $\mathcal{R}(D_s) = 0.334 \pm 0.017$ agrees well with the previous prediction under a combined approach of pQCD + LQCD [28]. This could be treated as a good example of showing the consistency of the TFFs under various approaches [62]. Analyzing the data in Table V, we can find that the predictions of $\mathcal{R}(D_s)$ through various methods are not in good agreement with each other, which needs more reasonable and accurate research in the future. At the same time, we also look forward to the experimental measurements of $\mathcal{R}(D_s)$, so as to test the theoretical prediction for $\mathcal{R}(D_s)$ within the framework of SM.

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