Theoretical analysis of the leptonic decays $B \to \ell\ell\ell\bar{\nu}_\ell$: Identical leptons in the final state

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We study the effects of identical leptons in the final state of the $B^+ \to \ell^+\ell^-\ell^+\bar{\nu}_\ell$ decay. The amplitude of the process is described by the same form factors as the amplitude of the $B \to \ell\ell\ell'\bar{\nu}'_\ell$ decay for nonidentical leptons in the final state. However, the differential distributions are strongly different, as the $B^+ \to \ell^+\ell^-\ell^+\bar{\nu}_\ell$ amplitude contains both the direct (M_a) and exchange (M_b) diagrams. We calculate a number of the differential distributions. In particular, we propose an interesting observable that can be readily measured experimentally: the differential distribution over the invariant mass of the pair of leptons of the same charge, $\ell^+\ell^+$. The good news is that the interference between M_a and M_b, $d\mathcal{B}_{ab}$ is found to be at the level of less than 1% in all considered differential distributions and therefore can be neglected in the full kinematical region of this decay.

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I. INTRODUCTION

This paper extends our recent analysis [1] of the $B \to lll'\nu'$ decay ($l' \neq l$) to the case of identical leptons in the final state ($l' = l$). Such reactions are being studied experimentally [2–5], thus requiring a proper theoretical understanding. So far, there have been a few theoretical papers [6–10] where B decays into two lepton pairs have been studied.

The $B \to \gamma^* l'\nu'$ amplitude (see Fig. 1) may be parametrized via Lorentz-invariant form factors as follows:

$$\begin{aligned} &T_{\alpha\nu}(q, q'|p) \\ &\quad = i\int dx e^{iqx}\langle 0|T\{j^{\text{e.m.}}_\alpha(x), \bar{u}(0)\mathcal{O}_\nu b(0)\}|\bar{B}_u(p)\rangle \\ &\quad = \sum_i L^{(i)}_{\alpha\nu}(q, q')F_i(q'^2, q^2) + \ldots, \qquad p = q + q', \end{aligned} \tag{1.1}$$

where q' is the momentum of the weak $b \to u$ current and q is the momentum of the electromagnetic current. In Eq. (1.1), $\mathcal{O}_\nu = \gamma_\nu, \gamma_\nu\gamma_5$ and $j^{\text{e.m.}}_\alpha$ is the conserved electromagnetic current,

$$j^{\text{e.m.}}_\alpha(0) = eQ_b\bar{b}(0)\gamma_\alpha b(0) + eQ_u\bar{u}(0)\gamma_\alpha u(0). \tag{1.2}$$

The quantities $L^{(i)}_{\alpha\nu}(q, q')$ represent the transverse Lorentz structures, $q^\alpha L^{(i)}_{\alpha\nu}(q, q') = 0$, and the dots stand for the longitudinal part which is constrained by the conservation of the electromagnetic current ($\partial_\alpha j^{\text{e.m.}}_\alpha = 0$) and the equal-time commutation relations.

The form factors $F_i(q'^2, q^2)$ are complicated functions of the two variables q'^2 and q^2; the general properties of these objects in QCD have been studied recently in Ref. [11]. Notably, gauge invariance provides essential constraints on some of the form factors describing the transition of the B meson into a real photon, i.e., at $q^2 = 0$ [12–15].

In the past, theoretical analyses focused on a family of similar reactions, namely, the $B \to \gamma l^+l^-$ and $B \to \gamma l\nu$ decays (see, e.g., Refs. [16–26]); these processes are described by the same form factors as four-lepton B decays, but are evaluated at a zero value of one of the momenta squared. The corresponding form factors depend on one variable, q'^2, where q' the momentum of the weak current; for instance, for radiative leptonic decays $B \to \gamma l'\nu'$, one needs the form factors $F_i(q'^2, q^2 = 0)$.

The four-lepton decay of interest, $B \to l^+l^-l'\nu'$, requires the form factors $F_i(q'^2, q^2)$ for $0 < q^2, q'^2 < M_B^2$. The dependence of the form factors on the variable q'^2 can be predicted reasonably well: there are no hadron resonances in the full decay region $0 < q'^2 < M_B^2$, and the q'^2 dependence of the form factors is determined to a large extent by the influence of the beauty mesons with the

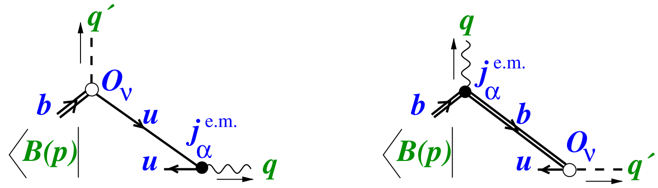

FIG. 1. Feynman diagrams describing the amplitude (1.1).

appropriate quantum numbers; all of these mesons are heavier than the B meson and therefore lie beyond the physical decay region of the variable q'^2. The calculation of the q^2 dependence of the form factors is a much more difficult task: light vector mesons $V = \rho^0, \omega, \ldots$ lie in the physical decay region and should be properly taken into account. At q^2 in the region of light vector-meson resonances, the form factors cannot be obtained directly in perturbative QCD (pQCD) [11]. Here considerations based on the explicit account of these light vector resonances—including their finite width effects—are mandatory; the resonance contributions of interest may be unambiguously expressed via the weak $B \to V$ form factors. Then, at $q^2 = 0$, gauge invariance constrains the values of the form factors. These features allow us to calculate the form factors $F_i(q'^2, q^2)$ in the region $0 < q^2 \leq 1$–$2\ \mathrm{GeV}^2$ which dominates the four-meson decay rates and obtain consistent predictions for the latter.

A relatively simple case of different lepton flavors $l \neq l'$ was considered in our recent paper [1]. In that case, one can easily calculate the differential distribution in q^2 (where q is the momentum of the l^+l^- pair) as well as in q'^2 (where q' is the momentum of the $l'\nu'$ pair); the angular variables do not enter the form factors and as a result all angular integrations may be calculated explicitly, yielding explicit forms for the differential distributions in q^2 and q'^2.

This paper focuses on the case of identical leptons in the final state $l = l'$. The amplitude is described by the same form factors as in the case $l \neq l'$, so we use the model for these form factors constructed in Ref. [1]. However, a specific feature of the case of identical leptons is the appearance of exchange diagrams. For such diagrams, the variables q^2 and q'^2 that determine the form factors do not coincide with the momenta of the l^+l^- and $l^+\nu$ pairs in the final state, and thus the angular variables appear explicitly in the form factors. As a result, the contribution of the exchange diagrams cannot be obtained as an explicit analytic expression and a numerical evaluation of the phase-space integrals is necessary. Here we provide all necessary details for the theoretical description of this reaction and report numerical predictions for a number of the differential distributions.

We propose an interesting kinematical variable: the differential distribution in the momentum of a pair of same-charge leptons (i.e., the l^+l^+ lepton in the case of the $B^+ \to l^+l^-l^+\nu$ decay and the μ^+e^+ pair in the case of the $B^+ \to \mu^+\mu^-e^+\nu_e$ or $B^+ \to e^+e^-\mu^+\nu_\mu$ decay). This distribution can be measured experimentally in a straightforward way and we obtain predictions for this differential distribution.

II. $B^- \to l^+l^-l'^-\bar{\nu}'$ FORM FACTORS

The amplitude of the $B \to ll l'\nu'$ transition for $l' \neq l$ may be parametrized as (see also Ref. [27])

$$A(B \to ll l'\nu') = ie^2 \frac{G_F}{\sqrt{2}} V_{ub} \cdot \bar{l}\gamma_\alpha l \cdot \bar{l}'\gamma_\nu(1-\gamma_5)\nu' \times \frac{1}{q^2}\left\{ (g_{\alpha\nu} q'q - q'_\alpha q_\nu)\frac{F_{1A}}{M_B} + q'_\alpha q_\nu \frac{F_{2A}}{M_B} + q'_\alpha q'_\nu \frac{F'_{2A}}{M_B} + i\epsilon_{\nu\alpha q' q}\frac{F_V}{M_B}\right\}, \tag{2.1}$$

where the form factors satisfy the constraints

$$F_{2A}(q'^2, q^2 = 0) = 0, \tag{2.2}$$

$$F'_{2A}(q'^2, q^2 = 0) = \frac{2Q_B f_B M_B}{M_B^2 - q'^2}. \tag{2.3}$$

Explicit formulas for the differential distributions in the case $l' \neq l$ were derived in Ref. [1]; we do not repeat these formulas here, but rather refer the reader to Ref. [1].

The same form factors parametrize the amplitude for the case $l' \neq l$; however, one has to take into account the contribution of the lepton exchange diagrams in which the variables q^2 and q'^2 have a complicated relationship with the momenta of the final lepton pairs. The details are given in the next section. We now recall the essential features of our model of the form factors as developed in Ref. [1].

(i) The contribution of the form factor $F'_{2A}(q^2, q'^2)$ can be neglected in the case $l = l'$, so in what follows we neglect its contribution.

(ii) For the form factors $F_{1A,2A,V}(q'^2, q^2)$, we use single-subtracted dispersion representations in q^2. This allows us to take into account all constraints coming from gauge invariance and from the known behavior in the large-energy limit of QCD [17].

(iii) We assume that the spectral densities are saturated by light vector-meson resonances ρ^0 and ω in the q^2 channel. Since these resonances emerge in the physical region of the B decay of interest, we take into account the q^2-dependent finite widths of these resonances [28]. In the end, we come to the following expressions for the form factors:

$$F_{1A}(q'^2,q^2)=F_A(q'^2)-\frac{Q_Bf_BM_B}{q'q}-q^2\sum_{V=\rho^0,\omega}\left(\frac{1}{M_V^2}\frac{2M_B(M_B+M_V)}{M_B^2-M_V^2-q'^2}\frac{M_Vf_V}{M_V^2-q^2-i\Gamma_V(q^2)M_V}A_1^{B\to V}(q'^2)\right), \tag{2.4}$$

$$F_{2A}(q'^2,q^2)=-q^2M_B\sum_{V=\rho^0,\omega}\frac{1}{M_V^2}\frac{2M_Vf_V}{M_V^2-q^2-i\Gamma_V(q^2)M_V}\left[\frac{M_B+M_V}{M_B^2-M_V^2-q'^2}A_1^{B\to V}(q'^2)-\frac{A_2^{B\to V}(q'^2)}{(M_B+M_V)}\right]$$
$$+Q_Bf_B\left(\frac{2M_B}{M_B^2-q'^2}-\frac{2M_B}{M_B^2-q'^2-q^2}\right), \tag{2.5}$$

$$F_V(q'^2,q^2)=F_V(q'^2)-q^2M_B\sum_{V=\rho^0,\omega}\left(\frac{1}{M_V^2}\frac{M_Vf_V}{M_V^2-q^2-i\Gamma_V(q^2)M_V}\frac{2V^{B\to V}(q'^2)}{M_B+M_V}\right). \tag{2.6}$$

(iv) The form factors $F_A(q'^2)$ and $F_V(q'^2)$ describe the $B\to\gamma l'\nu'$ transition; they emerge as subtraction terms at $q^2=0$ in the q^2-dispersion representations for the form factors $F_{1A,V}(q'^2,q^2)$. The form factors $F_A(q'^2)$ and $F_V(q'^2)$ are equal to each other at the leading order of the double $1/E_\gamma$ $(2M_BE_\gamma=M_B^2-q'^2)$ and $1/M_B$ expansions in QCD [17] but differ at the subleading orders [19,20,22]:

$$F_A(q'^2)=-\frac{Q_uf_BM_B}{2E_\gamma\lambda_B}+\frac{Q_bf_BM_B}{2E_\gamma m_b}+O(Q_uf_BM_B/E_\gamma^2), \tag{2.7}$$

$$F_V(q'^2)=-\frac{Q_uf_BM_B}{2E_\gamma\lambda_B}-\frac{Q_bf_BM_B}{2E_\gamma m_b}+O(Q_uf_BM_B/E_\gamma^2). \tag{2.8}$$

The magnitude of the form factors $F_A(q'^2)$ and $F_V(q'^2)$ is determined to a large extent by the parameter λ_B, the inverse moment of the B-meson light-cone distribution amplitude ϕ_B [17]. Taking into account a large uncertainty in the present knowledge of the parameter λ_B [18–21,26,29], we use the monopole forms (2.7) and (2.8) in the full kinematically allowed region of q'^2 and allow the variation of λ_B in the range $\lambda_B(1\text{ GeV})=(0.5\pm0.15)$ GeV.

(v) The contributions of the light vector mesons $V=\rho^0,\omega$ to the form factors $F_{1A,2A,V}(q'^2,q^2)$ are *unambiguous* (cf. Ref. [9]) and are expressed via the form factors $A_1^{B\to V}(q'^2)$, $A_2^{B\to V}(q'^2)$, and $V^{B\to V}(q'^2)$ describing the weak decay $B\to V$. In spite of many efforts to calculate these form factors in the broad kinematical decay region $0<q'^2<M_B^2$, our knowledge of these quantities is not very accurate; see, e.g., Refs. [30–33]. For our calculations we use the results from Ref. [30] and assign to them a 10% uncertainty. The uncertainties in these form factors, along with the uncertainty in the parameter λ_B, is the second main source of the uncertainty in the theoretical predictions for $B\to l^+l^-l'\nu'$ decays.

The results presented in the next section are obtained for our form factor model described in full detail in Sec. 5 of Ref. [1] and for the parameter $\lambda_B=0.65$.

III. THE DECAY $B^+\to\mu^+\mu^-\mu^+\bar\nu_\mu$

The case of two identical leptons is technically more involved than the case of all different leptons, considered in Ref. [1]. The reason is that an additional contribution from the interchange of the two μ^+ leptons arises.

The first diagram in Fig. 2, $M_a(k_1,k_2,k_3,k_4)$, is the same as for the B decay into nonidentical leptons (e.g., $B^+\to\mu^+\mu^-e^+\nu_e$). The second diagram is obtained from the first one by permutation of two final identical leptons: $M_b(k_1,k_2,k_3,k_4)=M_a(k_3,k_2,k_1,k_4)$. The total amplitude for the case of two identical leptons in the final state reads

$$M_{\rm tot}=\frac{1}{\sqrt2}(M_a-M_b)\quad\text{and}$$
$$|M_{\rm tot}|^2=\frac12(|M_a|^2+|M_b|^2-2\,\mathrm{Re}(M_aM_b^*)). \tag{3.1}$$

The factor $1/\sqrt2$ in the amplitude corresponds to the factor $1/2$ in the phase space for the case of two identical leptons. Thus, we use the expression for the phase space without the factor $1/2$ corresponding to identical particles in the final state.

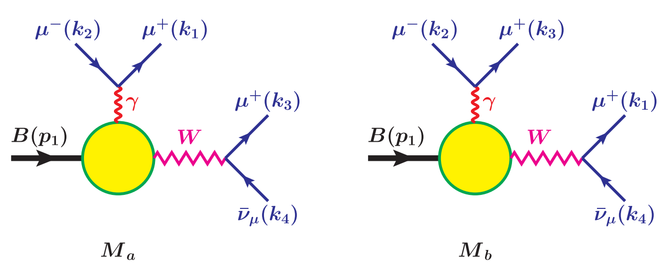

FIG. 2. Two diagrams describing the B decay into $\mu^+\mu^-\mu^+\bar\nu_\mu$.

A. Differential distribution over the momentum of the $\mu^+\mu^-$ pair

In a theoretical consideration, one can calculate the branching fraction and one-dimensional differential distribution for two kinematical variables $q_{12}^2 = (k_1 + k_2)^2$ (momentum of one of the $\mu^+\mu^-$ pairs) and $q_{34}^2 = (k_3 + k_4)^2$ (momentum of the $\mu^+\nu$ pair). For diagram M_a, $q_{12} = q$ and $q_{34} = q'$, so that the angular variables do not enter the form factors; the angular integrals may be taken analytically. For diagram M_b, the photon momentum q and the weak-vertex momentum q' do not coincide with q_{12} and q_{34}, so that the angular variables appear in the arguments of the form factors; all angular integrals should be taken numerically. Obviously, the contributions to the branching fraction coming from $|M_a|^2$ and $|M_b|^2$ are identically the same due to the symmetry $k_1 \leftrightarrow k_3$ of the phase-space measure. But verifying this property is a nontrivial check for numerical evaluation of the five-dimensional integrals. Figure 3 shows the q_{12}^2-differential distributions. The differential distributions over the variable q_{23}^2 (where q_{23} is the momentum of another $\mu^+\mu^-$ pair that may be isolated in the amplitude) is the same because of the symmetry of the amplitude: the replacement $k_1 \to k_3$ leads to the replacement $M_a \to M_b$ and vice versa.

B. Differential distribution over the momentum of the $\mu^+\nu_\mu$ pair

In a theoretical consideration, we can also calculate the differential distribution over $q_{34}^2 = (k_3 + k_4)^2$ (momentum of the $\mu^+\nu$ pair). These distributions are shown in Fig. 4. Obviously, the mixed term may be neglected in the full range of q_{34}^2.

C. Differential distribution over the momentum of the $\mu^+\mu^+$ pair

An interesting observable that can be readily measured experimentally is the differential distribution over the momentum of the $\mu^+\mu^+$ pair. Unlike the $\mu^+\mu^-$ distributions, one has only one pair of same-charge leptons in each event. The process is described by the same two diagrams in Fig. 2 but one has to calculate the distribution over the variable $q_{13}^2 = (k_1 + k_3)^2$. The contributions $d\mathcal{B}_{aa}(q_{13}^2)$ and $d\mathcal{B}_{bb}(q_{13}^2)$ are equal to each other and coincide with the

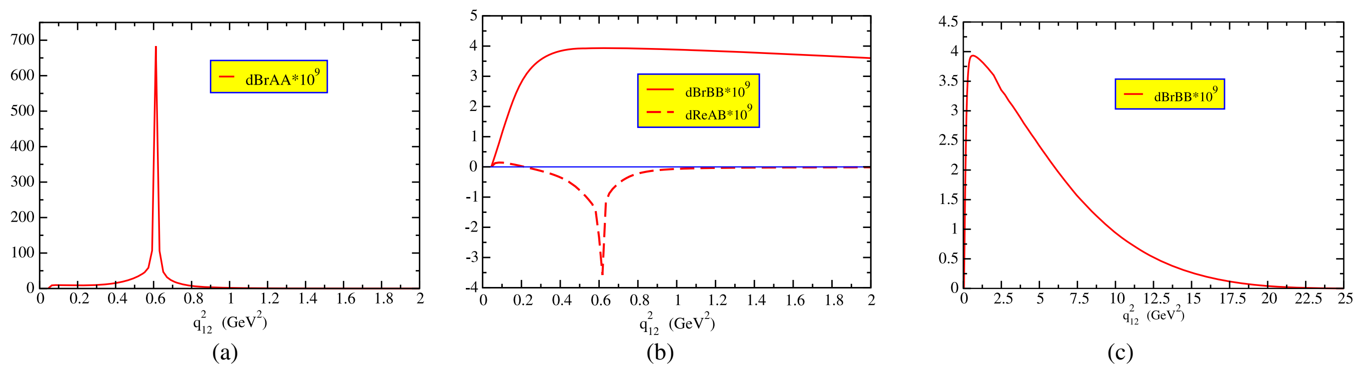

FIG. 3. Differential distributions in units of 10^{-9}: (a) $d\mathcal{B}_{aa}(q_{12}^2)$ at $0 < q_{12}^2(\mathrm{GeV}^2) < 2$; (b) $d\mathcal{B}_{ab}(q_{12}^2)$ vs $d\mathcal{B}_{bb}(q_{12}^2)$ at $0 < q_{12}^2(\mathrm{GeV}^2) < 2$; (c) $d\mathcal{B}_{bb}(q_{12}^2)$ in the full range $4m_\mu^2 < q_{12}^2 < (M_B - m_\mu)^2$.

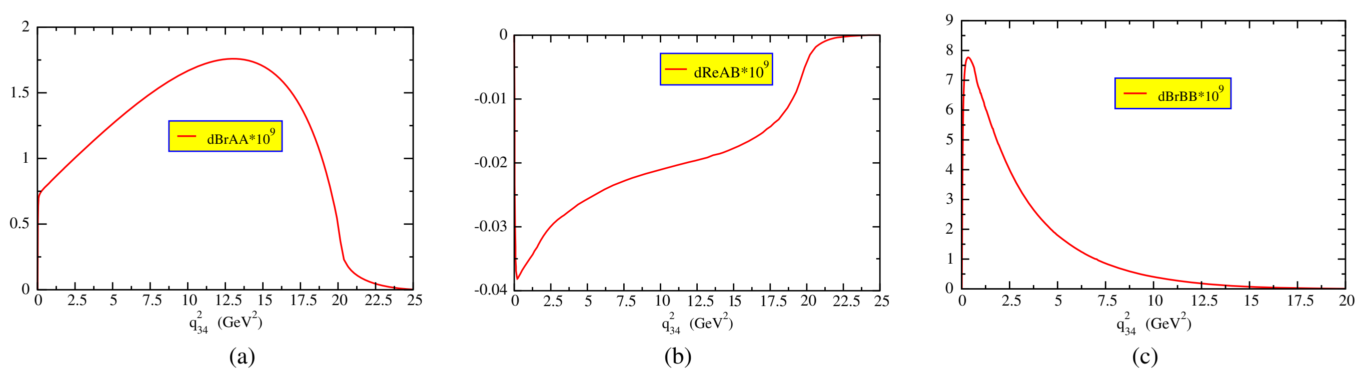

FIG. 4. Differential distributions in the full range $m_\mu^2 < q_{34}^2 < (M_B - 2m_\mu)^2$ (in units of 10^{-9}): (a) $d\mathcal{B}_{aa}(q_{34}^2)$, (b) $d\mathcal{B}_{ab}(q_{34}^2)$, and (c) $d\mathcal{B}_{bb}(q_{34}^2)$.

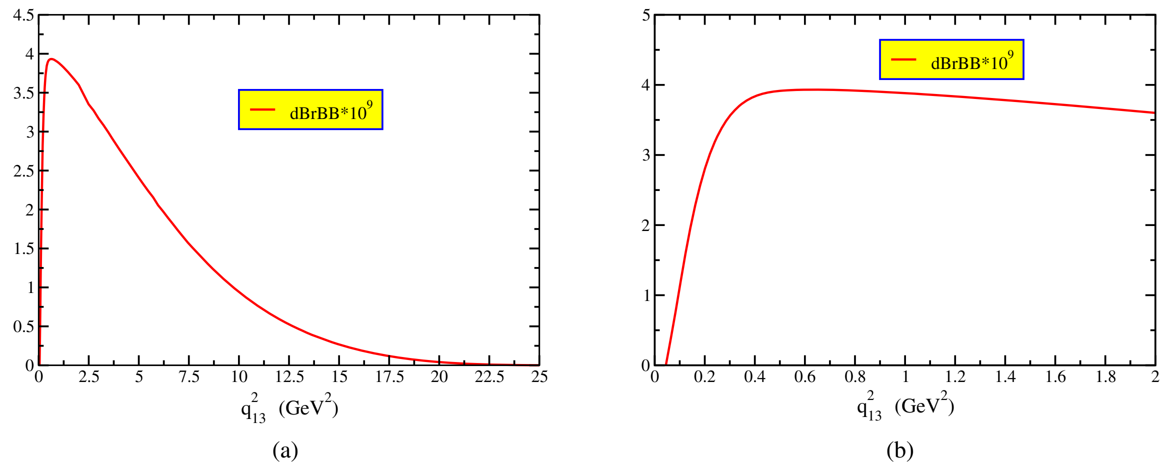

FIG. 5. Differential distribution (in units of 10^{-9}) $d\mathcal{B}(q_{13}^2)$ over the momentum of the same-charge lepton pair $\mu^+\mu^+$ ($q_{13} = k_1 + k_3$): (a) the full range $4m_\mu^2 < q_{13}^2 < (M_B - m_\mu)^2$; (b) the range $0 < q_{13}^2 < 2$ GeV^2.

distribution $d\mathcal{B}_{bb}(q_{12}^2)$ discussed above. Obviously, the mixed $d\mathcal{B}_{ab}(q_{13}^2)$ term can be safely neglected similar to the case of the distribution in the l^+l^- momentum q_{12}^2 considered above: (i) the integral $\int d\mathcal{B}_{bb}(q_{13}^2)dq_{13}^2$ comprises only 1% of $\int d\mathcal{B}_{aa}(q_{13}^2)dq_{13}^2 = \int d\mathcal{B}_{bb}(q_{13}^2)dq_{13}^2$; (ii) the distribution $d\mathcal{B}_{ab}(q_{13}^2)$ contains no resonances and is therefore smeared over the full kinematical q_{13}^2 range as a small addition to $d\mathcal{B}_{aa}(q_{13}^2) = d\mathcal{B}_{bb}(q_{13}^2)$ at the level of 1%. Figure 5 shows our predictions for $d\mathcal{B}(q_{13}^2)$.

D. Branching ratio of the $B^+ \to \mu^+\mu^-\mu^+\bar{\nu}_\mu$ decay

Table I presents our numerical results for the total branching ratio of the $B^+ \to \mu^+\mu^-\mu^+\bar{\nu}_\mu$ decay and the separate contributions coming from $|M_a|^2$, $|M_b|^2$ and the interference term $2 \times \mathrm{Re}(M_a M_b^*)$. We use the shorthand notations

$$\mathcal{B}_{aa} = \frac{\tau_B}{2M_B}\int d\Phi |M_a|^2, \qquad \mathcal{B}_{bb} = \frac{\tau_B}{2M_B}\int d\Phi |M_b|^2,$$
$$\mathcal{B}_{ab} = \frac{\tau_B}{2M_B}\int d\Phi 2\mathrm{Re}(M_a M_b^*), \tag{3.2}$$

$$\mathcal{B}_{\mathrm{tot}} = \frac{1}{2}(\mathcal{B}_{aa} + \mathcal{B}_{bb} - \mathcal{B}_{ab}), \tag{3.3}$$

TABLE I. Branching ratio of the $B^+ \to \mu^+\mu^-\mu^+\bar{\nu}_\mu$ decay. Separate contributions coming from $|M_a|^2$, $|M_b|^2$ and the interference term $2 \times \mathrm{Re}(M_a M_b^*)$ are also given. The $\mathcal{B}_{\mathrm{tot}}^{\mathrm{exp.\ cut}}$ is the result obtained by applying the LHCb event selection criterion (3.4). The results correspond to $\lambda_B = 0.65$.

Mode	$\frac{1}{2}(\mathcal{B}_{aa}+\mathcal{B}_{bb})$	$\frac{1}{2}\mathcal{B}_{ab}$	$\mathcal{B}_{\mathrm{tot}}$	$\mathcal{B}_{\mathrm{tot}}^{\mathrm{exp.\ cut}}$
$\mu^+\mu^-\mu^+\bar{\nu}_\mu$	2.80×10^{-8}	-2.26×10^{-10}	2.82×10^{-8}	2.73×10^{-8}

where the phase-space measure is given by Eqs. (A12) and (A13). One can see that the contribution of the interference term $2 \times \mathrm{Re}(M_a M_b^*)$ is negative and 2 orders of magnitude less than the contributions of $|M_a|^2$ and $|M_b|^2$. Thus, the interference term may be neglected. This is very good news as the calculation of the interference term is the most time-consuming part of the full calculation. We also provide the $\mathcal{B}_{\mathrm{tot}}^{\mathrm{exp.\ cut}}$ which is calculated making use of the LHCb [5] event selection criterion: in each event, one can form two $\mu^+\mu^-$ pairs; the events are selected on the basis of the criterion that the lowest of the two $\mu^+\mu^-$ mass combinations should be less than 0.98 GeV. In our calculation this corresponds to restricting the phase-space integration by the condition

$$\min\{(k_1 + k_2)^2, (k_3 + k_2)^2\} \le 0.96\ \mathrm{GeV}^2. \tag{3.4}$$

IV. DISCUSSION AND CONCLUSIONS

Making use of the model for the form factors of Ref. [1], we performed a detailed analysis of the exchange diagrams and interference effects that appear in the case with identical leptons in the final state. We calculated the differential distributions in various variables, namely, q_{12}^2 (the square of the invariant mass of one of the $\mu^+\mu^-$ pairs; Fig. 3), q_{34}^2 (the square of the invariant mass of one of the $\mu^+\nu_\mu$ pairs; Fig. 5), and q_{13}^2 (the square of the invariant mass of the $\mu^+\mu^+$ pair; Fig. 4). The latter differential distribution may be readily measured experimentally.

Our findings may be summarized as follows:

(i) For the differential distribution in q_{12}^2, $d\mathcal{B}_{aa}$ has a sharp resonance structure in the region of ρ and ω resonances. The distribution of $d\mathcal{B}_{bb}$ spreads over the full range of q_{12}^2 and exhibits no resonance structure. Nevertheless, the integrated differential rates $\mathcal{B}_{aa}$ and $\mathcal{B}_{bb}$ are equal to each other. The interference term

$d\mathcal{B}_{ab}$ contributes at less than the 1% level and may be safely neglected. Notably, the distribution $d\mathcal{B}(q_{12}^2)$ is fully determined by the resonances in all regions of q_{12}^2: in the region $0 < q_{12}^2 < 1$ GeV^2 via $d\mathcal{B}_{aa}$, and in the region 1 GeV$^2 < q_{12}^2$ via $d\mathcal{B}_{bb}$. Consequently, the perturbative tail of the form factors at $q_{12}^2 >$ 1–2 GeV^2 does not show up in the differential distributions for identical leptons in the final state at all. This makes an essential difference with the case of nonidentical leptons, where the region $q_{12}^2 \gg$ 1 GeV^2 is determined by the pQCD behavior of the form factors.

(ii) The differential distribution over the momentum of the $\mu^+\nu_\mu$ pair, q_{34}^2, has an interesting shape, different for $d\mathcal{B}_{aa}(q_{34}^2)$ and $d\mathcal{B}_{bb}(q_{34}^2)$, and a numerically negligible interference term $d\mathcal{B}_{ab}(q_{34}^2)$. This differential distribution is rather interesting theoretically but is unlikely to be experimentally measurable.

(iii) The differential distribution in q_{13}^2, the square of the invariant mass of the $\mu^+\mu^+$ pair, has a relatively flat nonresonant structure in the full range of q_{13}^2. The contributions of the M_a and M_b diagrams are equal to each other, $d\mathcal{B}_{aa}(q_{13}^2) = d\mathcal{B}_{bb}(q_{13}^2)$. The interference term $d\mathcal{B}_{ab}(q_{13}^2)$ is smeared over the full q_{13}^2 region as a minor positive addition at the level of less than 1% and may be safely neglected.

(iv) The good news is that the interference term between the direct diagram M_a and the exchanged diagram M_b provides a positive contribution at the level of less than 1% to the differential distribution in all regions of the kinematical variables and thus can be safely neglected. This greatly simplifies the calculation procedure as the interference AB term represents the most time-consuming part of the calculations.

(v) For $\mathcal{B}(B \to \mu^+\mu^-\mu^+\nu_\mu)$, taking into account all uncertainties, we confirm our result of Ref. [1]:

$$\begin{aligned} &\mathrm{Br}(B^+ \to \mu^+\mu^-\mu^+\bar{\nu}_\mu) \\ &\quad = (3.02^{+0.45}_{-0.25}|_{\lambda_b} \pm 0.62|_{\text{weak ffs}})10^{-8}. \end{aligned} \tag{4.1}$$

Applying the kinematical selection rule for the $\mu^+\mu^-$ pairs (3.4) (as done by the LHCb Collaboration [5]) leads to a small reduction at the level of 3% of our theoretical result (4.1).

In summary, we reinforce our previous finding that our theoretical estimate is only marginally compatible with the upper limits obtained by the LHCb Collaboration [5], $\mathrm{Br}(B^+ \to \mu^+\mu^-\mu^+\bar{\nu}_\mu) \leq 1.6 \times 10^{-8}$.

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APPENDIX: KINEMATICS OF THE B DECAY WITH FOUR LEPTONS IN THE FINAL STATE

We consider the reaction

$$B^+(p) \to \ell'^+(k_3) + \bar{\nu}_{\ell'}(k_4) + \ell^+(k_1) + \ell^-(k_2). \tag{A1}$$

The two planes of the final particles are shown in Fig. 6. The decay amplitude is described by five kinematical variables:

(1) $q_{34}^2 \equiv (k_3 + k_4)^2$ is the $\ell'^+\nu_{\ell'}$ invariant mass.
(2) $q_{12}^2 \equiv (k_1 + k_2)^2$ is the dilepton invariant mass.
(3) θ^* is the angle of the ℓ'^+ in the $\ell'^+\bar{\nu}_{\ell'}$ c.m. system with respect to the $\ell'^+\bar{\nu}_{\ell'}$ flight direction.
(4) θ is the angle of the ℓ^+ in the dilepton c.m. system with respect to the $\ell^+\ell^-$ flight direction.
(5) χ is the azimuthal angle between the $\ell'^+\nu_{\ell'}$ and dilepton planes.

All particles are on their mass shell:

$$\begin{aligned} &p^2 = M_B^2, \qquad k_3^2 = m_3^2 \equiv m_{\ell'}^2, \\ &k_4^2 = m_4^2 \equiv m_{\nu_{\ell'}}^2 = 0, \qquad k_1^2 = k_2^2 \equiv m_\ell^2. \end{aligned} \tag{A2}$$

We also introduce the mass notations for the momenta squared: $m_{12} \equiv \sqrt{q_{12}^2}$ and $m_{34} \equiv \sqrt{q_{34}^2}$.

The boosted 4-momenta from the $\ell^+\ell^-$ c.m. system to the B-meson rest frame are written as

$$q_{12}^\mu = (E_{12}, 0, 0, |\mathbf{k}|), \tag{A3}$$

$$\begin{aligned} k_1^\mu = \frac{1}{2}(&E_{12} + v|\mathbf{k}|\cos\theta, +vm_{12}\sin\theta\cos\chi, \\ &+ vm_{12}\sin\theta\sin\chi, |\mathbf{k}| + vE_{12}\cos\theta), \end{aligned} \tag{A4}$$

$$\begin{aligned} k_2^\mu = \frac{1}{2}(&E_{12} - v|\mathbf{k}|\cos\theta, -vm_{12}\sin\theta\cos\chi, \\ &- vm_{12}\sin\theta\sin\chi, |\mathbf{k}| - vE_{12}\cos\theta). \end{aligned} \tag{A5}$$

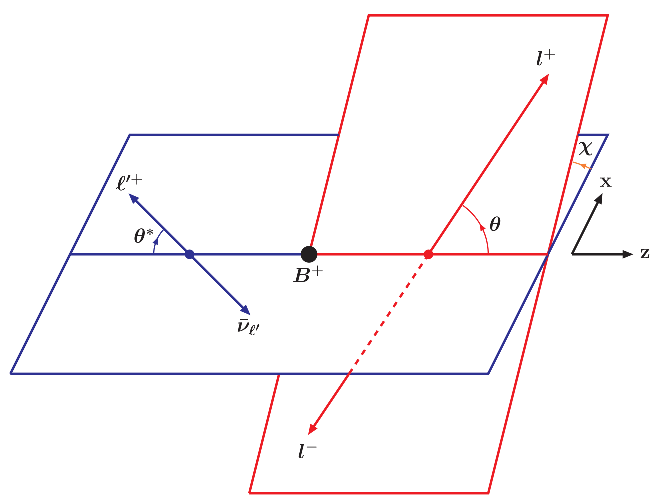

FIG. 6. Definition of the angles θ^*, θ, and χ in the decay of $B^+ \to \ell'^+ + \bar{\nu}_{\ell'} + \ell^+ + \ell^-$.

Here $v=\lambda(q_{12}^2,m_l^2,m_l^2)/q_{12}^4=\sqrt{1-4m_l^2/q_{12}^2}$ [$\lambda(a,b,c)\equiv(a-b-c)^2-4bc$] and

$$|\mathbf{k}|=\frac{\lambda^{1/2}(M_B^2,q_{34}^2,q_{12}^2)}{2M_B},\qquad E_{12}=\frac{M_B^2-q_{23}^2+q_{12}^2}{2M_B},$$
$$E_{23}=\frac{M_B^2+q_{23}^2-q_{12}^2}{2M_B},\qquad E_{12}+E_{23}=M_B. \tag{A6}$$

The boosted momenta from the $\ell'\nu_{\ell'}$ c.m. system to the B-meson rest frame read

$$q_{34}^\mu=(E_{34},0,0,-|\mathbf{k}|), \tag{A7}$$

$$k_3^\mu=\frac{1}{E_{34}}(E_{34}E_3+|\mathbf{k}||\mathbf{k_3}|\cos\theta^*,+E_{34}|\mathbf{k_3}|\sin\theta^*,0,$$
$$-E_3|\mathbf{k}|-E_{34}|\mathbf{k_3}|\cos\theta^*), \tag{A8}$$

$$k_4^\mu=\frac{1}{E_{34}}(E_{34}E_4-|\mathbf{k}||\mathbf{k_3}|\cos\theta^*,-E_{34}|\mathbf{k_3}|\sin\theta^*,0,$$
$$-E_4|\mathbf{k}|+E_{34}|\mathbf{k_3}|\cos\theta^*), \tag{A9}$$

where

$$|\mathbf{k_3}|=\frac{\lambda^{1/2}(q_{34}^2,m_3^2,m_4^2)}{2m_{34}},\qquad E_3=\frac{q_{34}^2+m_3^2-m_4^2}{2m_{34}},$$
$$E_4=\frac{q_{34}^2-m_3^2+m_4^2}{2m_{34}},\qquad E_3+E_4=m_{34}. \tag{A10}$$

The differential decay rate is given by

$$d\Gamma(B\to\ell'\nu_{\ell'}\ell^+\ell^-)=\frac{1}{2m_1}|M(k_1,\dots,k_4)|^2d\Phi, \tag{A11}$$

$$d\Phi=\frac{1}{(2\pi)^8}\delta^{(4)}(p_1-k_3-k_4-k_1-k_2)\frac{d^3\vec{k}_3}{2k_3^0}\frac{d^3\vec{k}_4}{2k_4^0}\frac{d^3\vec{k}_1}{2k_1^0}\frac{d^3\vec{k}_2}{2k_2^0}, \tag{A12}$$

where $k_i^0=\sqrt{m_\ell^2+\vec{k}_i^2}$ for ($i=1$, 2) and $k_i^0=\sqrt{m_i^2+\vec{k}_i^2}$ for ($i=3$, 4).

The integration over the phase space may be reduced to the integration over the two kinematical variables k^2 and q^2 and three angles θ^*, θ, and χ. Then, the differential phase volume in Eq. (A12) is given by

$$d\Phi=\frac{v}{(4\pi)^6}\frac{|\mathbf{k}|}{M_B}\frac{|\mathbf{k_3}|}{E_{34}}dq_{12}2dq_{34}^2d\cos\theta^*d\cos\theta d\chi$$
$$0\le\theta^*,\theta\le\pi,\qquad 0\le\chi\le 2\pi. \tag{A13}$$

The kinematical constraints on the variables q_{12}^2 and q_{34}^2 come from the positivity of the λ functions $\lambda(q_{34}^2,m_{\ell'}^2,0)$, $\lambda(q_{12}^2,m_\ell^2,m_\ell^2)$, and $\lambda(M_B^2,q_{12}^2,q_{34}^2)$ and are

$$4m_\ell^2\le q_{12}^2,\qquad m_{\ell'}^2\le q_{34}^2,\ \sqrt{q_{12}^2}+\sqrt{q_{34}^2}\le M_B^2. \tag{A14}$$

To calculate the single differential distribution in q_{12}^2 or q_{34}^2, we have the following integration limits:

$$dq_{34}^2dq_{12}^2:\ m_{\ell'}^2\le q_{34}^2\le(M_B-2m_\ell)^2,$$
$$4m_\ell^2\le q_{12}^2\le\left(M_B-\sqrt{q_{34}^2}\right)^2, \tag{A15}$$

$$dq_{12}^2dq_{34}^2:\ 4m_\ell^2\le q_{12}^2\le(M_B-m_{\ell'})^2,$$
$$m_{\ell'}^2\le q_{34}^2\le\left(M_B-\sqrt{q_{12}^2}\right)^2. \tag{A16}$$

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