Next to soft corrections to Drell-Yan and Higgs boson productions

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We present a framework that resums threshold enhanced large logarithms to all orders in perturbation theory for the production of a pair of leptons in the Drell-Yan process and of the Higgs boson in gluon fusion as well as in bottom quark annihilation. We restrict ourselves to contributions from diagonal partonic channels. These logarithms include the distributions $((1-z)^{-1}\ln^i(1-z))_+$ resulting from soft plus virtual (SV) and the logarithms $\ln^i(1-z)$ from next-to-SV contributions. We use collinear factorization and renormalization group invariance to achieve this. The former allows one to define a soft-collinear (SC) function that encapsulates soft and collinear dynamics of the perturbative results to all orders in the strong coupling constant. The logarithmic structure of these results is governed by universal infrared anomalous dimensions and process-dependent functions of the Sudakov differential equation that the SC satisfies. The solution to the differential equation is obtained by proposing an all-order ansatz in dimensional regularization, owing to several state-of-the-art perturbative results available to third order. The z space solutions thus obtained provide an integral representation to sum up large logarithms originating from both soft and collinear configurations, conveniently in Mellin N space. We show that in N space, the tower of logarithms $a_s^n/N^\alpha \ln^{2n-\alpha}(N), a_s^n/N^\alpha \ln^{2n-1-\alpha}(N) \cdots$ for $\alpha = 0, 1$ is summed to all orders in a_s.

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I. INTRODUCTION

Precision studies in the context of the Large Hadron Collider (LHC) play an important role to decipher the experimental data to understand the physics at extremely small length scales. The tests [1] of the Standard Model (SM) of high energy physics at the LHC with unprecedented accuracy can provide indirect clues to unravel physics beyond SM (BSM). Accurate measurements of SM observables such as the productions of lepton pairs, vector bosons such as photons, Zs and Ws, top quarks, and Higgs bosons are underway. From the theory side, the predictions for these observables are available, taking into account various higher order quantum effects. Both in the electroweak sector of SM and in quantum chromodynamics (QCD), the observables are computed in power series expansion of their coupling constants, viz., $e, g_{\rm EW}$ in SM and g_s in QCD. To name a few, the inclusive cross sections for deep inelastic scattering (DIS) and Higgs boson production in hadron colliders are known to third order in QCD, see [2,3] and [4–6] respectively and for invariant mass distribution up to third order in QCD see [7–9], for complete list see [8,10–28] for Higgs production in gluon fusion and [7–9,19–21,25,29–37] for Drell-Yan production.

The LHC is the hadronic machine, and even electroweak induced processes get large quantum corrections resulting from strong interaction. QCD is the theory of strong interactions and provides a framework to compute these corrections. The measurements and predictions from QCD have reached the level that demands the inclusion of electroweak effects (EW). The EW corrections to hadronic observables are hard to compute at higher orders due to the presence of heavy particles such as Ws, Zs, and tops in the loops. The results of higher order quantum effects from QCD and EW theory provide a theoretical laboratory to understand both ultraviolet (UV) and infrared (IR) structures of the underlying quantum field theory (QFT) and also to demonstrate the universal structure. For IR, see [38–41] (see [42,43] for a QFT with mixed gauge groups). This is due to certain factorization properties of scattering amplitudes in UV and IR regions. The consequence of the factorization is the renormalization group (RG) invariance which demonstrates the structure of logarithms of the renormalization scale μ_R from UV and of the factorization scale μ_F from IR to all orders in perturbation theory. The renormalization scale separates the UV divergent part from the finite part of Green's function or on-shell amplitudes, quantifying the arbitrariness in the finite part. While the parameters of the renormalized version of the

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theory are functions of the renormalization scale, the physical observables are expected to be independent of this scale. This is the consequence of renormalization group invariance. The anomalous dimensions of the RG equations govern the structure of the logarithms of the renormalization scale in the perturbation theory to all orders. As the UV sector, the infrared sectors of both SM and QCD are also very rich. Massless gauge fields such as photons in QED and gluons in QCD and light matter particles at high energies give soft and collinear divergences, collectively called IR divergences, in scattering amplitudes. The IR divergences are shown to factorize from on-shell amplitudes and from certain cross sections, respectively, in a process independent way at an arbitrary factorization scale. The resulting IR renormalization group equations are governed by IR anomalous dimensions. The IR renormalization group equations are peculiar in the sense that the resulting evolution is controlled not only by the factorization scale but also by the energy scale(s) in the amplitude or in the scattering process. Unlike the UV divergences that are removed by appropriate renormalization constants, the IR divergences do not require any such renormalization procedure as they add up to zero for infrared safe observables thanks to the Kinoshita—Lee—Nauenberg (KLN) theorem [44,45]. The structure of the resulting IR logarithms at every order in the perturbation theory is governed by the IR anomalous dimensions. Hence, most of the logarithms present at higher orders are due to UV and IR divergences present at the intermediate stages of the computations. The logarithms of renormalization and factorization scales present in the perturbative expansions often play an important role to estimate the error that results due to the truncation of the perturbative series. The less the dependence is on these scales, the more the reliability of the truncated results. Note that there are also logarithms that are functions of physical scales or the corresponding scaling variables in the observables. In certain kinematical regions, these logarithms that are present at every order can be large enough to spoil the reliability of the truncated perturbative series. Since the structure of these logarithms at every order is controlled by anomalous dimensions of IR renormalization group equations, they can be systematically summed up to all orders. This procedure is called resummation. There are classic examples in QCD. For example, the threshold logarithms of the kind

$$\mathcal{D}_i(z) = \left(\frac{\ln^i(1-z)}{1-z}\right)_+ \tag{1}$$

are present in the perturbative results of the inclusive cross section in deep inelastic scattering and of the invariant mass distribution of a pair of leptons in the Drell-Yan (DY) process. Here the subscript $+$ means that $\mathcal{D}_i(z)$ is a plus distribution. For DIS, the scaling variable is $z = -q^2/2p.q$ and $z = M^2_{l^+l^-}/\hat{s}$ for DY. The momentum transfer from lepton to parton with momentum p in DIS is denoted by q and the invariants $\hat{s}$ and $M^2_{l^+l^-}$ are the center of mass energy of incoming partons and the invariant mass of final state leptons in DY. The distributions $\mathcal{D}_i(z)$ are often called threshold logarithms as they dominate in the threshold region, namely z approaches 1. In this limit, the entire energy of the incoming particles in the scattering event goes into producing a set of hard particles along with an infinite number of soft gluons each carrying almost zero momentum. In particular, the logarithms of the form $\ln^i(1-z)/(1-z)$ result from the processes involving real radiations of soft gluons and collinear particles. While these contributions are ill-defined in four spacetime dimensions in the limit $z \to 1$, the inclusion of pure virtual contributions gives distributions $\mathcal{D}_i(z)$ and $\delta(1-z)$. The terms that constitute these distributions and $\delta(1-z)$ are called soft plus virtual (SV) contributions. The SV results in QCD are available for numerous observables in hadron colliders. For SV results up to third order, see [19–21,36,37,46–49]. These logarithms in the perturbative results when convoluted with appropriate parton distribution functions to obtain the hadronic cross section not only can dominate over other contributions but also can give large contributions at every order. The presence of these large corrections at every order spoil the reliability of the predictions from the truncated series. The seminal works by Sterman [50] and Catani and Trentedue [51] provide resolution to this problem through reorganization of the perturbative series called threshold resummation, for its applications to various inclusive processes (see [52–58] for Higgs production in gluon fusion, see [59,60] for bottom quark annihilation, and see [37,53,61–63] for DY). Since z space results involve convolutions of these distributions, the Mellin space approach using the conjugate variable N is used for resummation. In the Mellin space, large logarithms of the kind $\mathcal{D}_i(z)$ become functions of $\ln^{j+1} N$, $j \leq i$ with $\mathcal{O}(1/N)$ suppressed terms in the corresponding N space threshold limit, namely $N \to \infty$. Threshold resummation allows one to resum $\omega = 2a_s(\mu_R^2)\beta_0 \ln N$ terms to all orders in ω and then to organize the resulting perturbative result in powers of coupling constant $a_s(\mu_R^2) = g_s^2(\mu_R^2)/16\pi^2$, where g_s is the strong coupling constant. Here, β_0 is the leading coefficient of the QCD beta function. If $\mathcal{O}_N$ is an observable in Mellin N space, with N being the conjugate variable to z of the observable $\mathcal{O}(z)$ in z space, then the resummation of threshold logarithms gives

$$\ln \mathcal{O}_N = \ln N g_1^{\mathcal{O}}(\omega) + \sum_{i=0}^{\infty} a_s^i(\mu_R^2) g_{i+2}^{\mathcal{O}}(\omega) + \ln g_0^{\mathcal{O}}(a_s(\mu_R^2)), \tag{2}$$

where $g_0^{\mathcal{O}}(a_s(\mu_R^2))$ is N independent and is given by

$$g_0^{\mathcal{O}}(a_s(\mu_R^2)) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) g_{0i}^{\mathcal{O}}. \tag{3}$$

The inclusion of more and more terms in (2) predicts the leading logarithms (LL), next to leading (NLL) logarithms, etc., of $\mathcal{O}$ to all orders in a_s. The functions $g_i^{\mathcal{O}}(\omega)$ are functions of process independent universal IR anomalous dimensions while $g_0^{\mathcal{O}}$ depend on the hard process. For inclusive reactions such as DIS, the invariant mass distribution of lepton pairs in DY, Higgs boson productions in various channels, all the ingredients to perform the resummation of threshold logarithms in N space up to third order [next to next to next to leading logarithmic (N^{3}LL) accuracy] are available.

While the resummed results provide reliable predictions that can be compared against the experimental data, it is important to find out the role of subleading terms, namely $\ln^i(1-z), i = 0, 1, \ldots$. We call them by next to SV (NSV) contributions. In addition, to understand the role of NSV terms, the question of whether these terms can also be resummed systematically to all orders exactly as the leading SV terms are resummed remains unanswered. These questions have already been addressed in great detail, and remarkable progress has been made in recent times leading to a better understanding of NSV terms. For example, applying diagrammatic techniques and using factorization properties or through physical evolution equations, several interesting results on both fixed order and resummed predictions for NSV terms are available for the production of a colorless state in hadron colliders. See [3,23,64–74] for more details. In this paper, exploiting mass factorization and renormalization group invariance and using the Sudakov $K+G$ equation we make an attempt to provide an all order result both in z space and in N space, which can predict NSV terms of diagonal channels in DY and Higgs boson production to all orders in perturbation theory.

II. NEXT TO SV IN z SPACE

In the following, we study the inclusive cross sections for the production of a pair of leptons in DY and the production of a single scalar Higgs boson in gluon fusion and in bottom quark annihilation. Let us denote the corresponding inclusive cross sections generically by $\sigma(q^2,\tau)$. In the QCD improved parton model, σ is written in terms of parton level coefficient functions (CF) denoted by $\Delta_{ab}(q^2,\mu_R^2,\mu_F^2,z)$ convoluted with appropriate parton distribution functions (PDFs), $f_c(x_i,\mu_F^2)$, of incoming partons:

$$\sigma(q^2,\tau) = \sigma_0(\mu_R^2)\sum_{ab}\int dx_1 \int dx_2 f_a(x_1,\mu_F^2) f_b(x_2,\mu_F^2) \times \Delta_{ab}(q^2,\mu_R^2,\mu_F^2,z), \tag{4}$$

where σ_0 is the born level cross section. The scaling variable τ is defined by $\tau = q^2/S$, S is the hadronic center of mass energy. For DY, $q^2 = M_{l^+l^-}^2$, the invariant mass of the final state leptons, and $q^2 = m_H^2$ for the Higgs boson productions, with m_H being the mass of the Higgs boson. The subscripts a, b in Δ_{ab} and c in f_c collectively denote the type of parton (quark, antiquark, and gluon), their flavor, etc. The scaling variable x_i is the momentum fraction of the incoming partons. In the CF, $z = q^2/\hat{s}$ is the partonic scaling variable and $\hat{s}$ is the partonic center of mass energy and is related to hadronic S by $\hat{s} = x_1 x_2 S$ which implies $z = \tau/(x_1 x_2)$. The scale μ_F is factorization scale which results from mass factorization, and the scale μ_R is the renormalization scale which results from UV renormalization of the theory. Both σ_0 and Δ_{ab} depend on the renormalization scale; however, their product is independent of the scale if we include Δ_{ab} to all orders in perturbation theory.

The partonic cross section is computable order by order in QCD perturbation theory. Beyond leading order, one encounters UV, soft, and collinear divergences at the intermediate stages of the computation. If we use dimensional regularization to regulate all these divergences, the partonic cross sections depend on the spacetime dimension $n = 4+\epsilon$ and the divergences show up as poles in ϵ. The UV divergences are removed by QCD renormalization constants in a modified minimal subtraction ($\overline{MS}$) scheme. The soft divergences from the gluons and the collinear divergence resulting from final state partons cancel independently when we perform the sum over all the degenerate states. Since the hadronic observables under study are infrared safe, these partonic cross sections are factorizable in terms of collinear singular Altarelli-Parisi (AP) [75] kernels Γ_{ab} and finite CFs at an arbitrary factorization scale μ_F. The factorized formula that relates the collinear finite CFs Δ_{ab} and the parton level subprocesses is given by

$$\frac{1}{z}\hat{\sigma}_{ab}(q^2,z,\epsilon) = \sigma_0(\mu_R^2)\sum_{a'b'}\Gamma_{aa'}^T(z,\mu_F^2,\epsilon) \otimes (\Delta_{a'b'}(q^2,\mu_R^2,\mu_F^2,z,\epsilon)) \otimes \Gamma_{b'b}(z,\mu_F^2,\epsilon). \tag{5}$$

These kernels are then absorbed into the bare PDFs to define collinear finite PDFs. Note that the singular AP kernels do not depend on the type of partonic reaction but depend only on the type of partons in addition to the scaling variable z and scale μ_F. The symbol $\otimes$ refers to convolution, which is defined for functions, $f_i(x_i), i = 1, 2, \ldots, n$, as

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z) = \prod_{i=1}^{n}\left(\int dx_i f_i(x_i)\right) \times \delta(z - x_1 x_2 \cdots x_n). \tag{6}$$

The partonic cross section in perturbation theory in QCD can be expressed in powers of unrenormalized strong coupling constant $\hat{a}_s$:

$$\hat{\sigma}_{ab}(q^2,z,\epsilon)=\sum_{i=0}^{\infty}\hat{a}_s^{i+\alpha}\hat{\sigma}_{ab}^{(i)}(q^2,\mu_R^2,z,\epsilon), \tag{7}$$

where the value of α depends on the process under study. Since the aim of this paper is to investigate the structure of NSV terms in diagonal channels, we will restrict ourselves to $\Delta_{q\bar{q}}$ for DY, $\Delta_{b\bar{b}}$ for Higgs boson production in bottom quark annihilation, and Δ_{gg} for Higgs boson production in gluon fusion, throughout the paper unless stated otherwise. We call these CFs collectively by $\Delta_{c\bar{c}}$ with $c\bar{c}=q\bar{q},b\bar{b},gg$.

Before we proceed further with the diagonal channels, let us study the structure of mass factorized results (5) for both diagonal and off-diagonal channels in the threshold limit. In particular, we would like to find out which are the terms that survive if we want to retain only SV and/or NSV terms when we perform threshold expansion. We begin with the mass factorization formula for a diagonal channel. We will show that to retain only SV and NSV terms in $\Delta_{c\bar{c}}$ using the mass factorized result, it will be sufficient to keep only those components of AP kernels Γ_{ab}'s and of $\hat{\sigma}_{ab}$'s or Δ_{ab}'s that upon convolution give SV and/or NSV terms. For definiteness, let us look at the mass factorized Drell-Yan result:

$$\begin{aligned}\frac{\hat{\sigma}_{q\bar{q}}}{z\sigma_0}=&\Gamma_{qq}^T\otimes\Delta_{qq}\otimes\Gamma_{q\bar{q}}+\Gamma_{qq}^T\otimes\Delta_{qg}\otimes\Gamma_{g\bar{q}}+\Gamma_{qq}^T\otimes\Delta_{q\bar{q}}\otimes\Gamma_{\bar{q}\bar{q}}\\&+\Gamma_{qg}^T\otimes\Delta_{gq}\otimes\Gamma_{q\bar{q}}+\Gamma_{qg}^T\otimes\Delta_{gg}\otimes\Gamma_{g\bar{q}}+\Gamma_{qg}^T\otimes\Delta_{g\bar{q}}\otimes\Gamma_{\bar{q}\bar{q}}\\&+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}q}\otimes\Gamma_{q\bar{q}}+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}g}\otimes\Gamma_{g\bar{q}}+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}\bar{q}}\otimes\Gamma_{\bar{q}\bar{q}}.\end{aligned} \tag{8}$$

Here, we either have convolutions with terms involving only diagonal terms, such as $\Gamma_{qq}^T\otimes\Delta_{q\bar{q}}\otimes\Gamma_{\bar{q}\,\bar{q}}$, or with terms involving one diagonal and a pair of nondiagonal terms, for example, $\Gamma_{qq}^T\otimes\Delta_{qg}\otimes\Gamma_{g\bar{q}}$. The former gives SV plus NSV terms upon convolutions while the latter will give only beyond the NSV terms. And the diagonal Γ_{cc}'s also contain convolutions with only diagonal AP splitting functions, P_{cc}, or one diagonal and a pair of nondiagonal AP splitting functions $P_{ab}, a\neq b$. We drop those terms in diagonal Γ_{cc}'s that contain a pair of nondiagonal P_{ab}'s, as they contribute to beyond NSV accuracy. This results in

$$\frac{\hat{\sigma}_{q\bar{q}}^{\rm sv+nsv}}{z\sigma_0}=\Gamma_{qq}^T\otimes\Delta_{q\bar{q}}^{\rm sv+nsv}\otimes\Gamma_{\bar{q}\,\bar{q}}. \tag{9}$$

A similar argument will go through for $\hat{\sigma}_{b\bar{b}}$ and $\hat{\sigma}_{gg}$ as well. This allows us to write the mass factorized result given in (5) in terms of only diagonal terms $\hat{\sigma}_{c\bar{c}}$, $\Delta_{c\bar{c}}$, and AP kernels Γ_{cc}, and the sum over ab is dropped. Hence, dropping beyond NSV terms and restricting to only diagonal terms result (5) in taking the simple form

$$\begin{aligned}\Delta_{c\bar{c}}^{\rm sv+nsv}(q^2,\mu_R^2,\mu_F^2,z,\epsilon)=\sigma_0^{-1}(\mu_R^2)\bigg((\Gamma^T)_{cc}^{-1}(z,\mu_F^2,\epsilon)\\\otimes\frac{1}{z}\hat{\sigma}_{c\bar{c}}^{\rm sv+nsv}(q^2,z,\epsilon)\\\otimes(\Gamma)_{\bar{c}\,\bar{c}}^{-1}(z,\mu_F^2,\epsilon)\bigg).\end{aligned} \tag{10}$$

In summary, since our main focus here is on SV and NSV terms resulting from quark initiated processes for DY and gluon or bottom quark initiated processes for Higgs boson production, we can safely drop contributions from non-diagonal partonic channels in the mass factorized result of $\Delta_{c\bar{c}}$. In addition, gluon-gluon initiated channels which start contributing at NNLO onwards for DY and quark-antiquark initiated channels for Higgs boson production are also dropped as they do not contribute to NSV of $\Delta_{c\bar{c}}$.

Turning our attention to off-diagonal terms, for instance $\hat{\sigma}_{qg}$, we find

$$\begin{aligned}\frac{\hat{\sigma}_{qg}}{z\sigma_0}=&\Gamma_{qq}^T\otimes\Delta_{qq}\otimes\Gamma_{qg}+\Gamma_{qq}^T\otimes\Delta_{qg}\otimes\Gamma_{gg}+\Gamma_{qq}^T\otimes\Delta_{q\bar{q}}\otimes\Gamma_{\bar{q}g}\\&+\Gamma_{qg}^T\otimes\Delta_{gq}\otimes\Gamma_{qg}+\Gamma_{qg}^T\otimes\Delta_{gg}\otimes\Gamma_{gg}+\Gamma_{qg}^T\otimes\Delta_{g\bar{q}}\otimes\Gamma_{\bar{q}g}\\&+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}q}\otimes\Gamma_{qg}+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}g}\otimes\Gamma_{gg}+\Gamma_{q\bar{q}}^T\otimes\Delta_{\bar{q}\bar{q}}\otimes\Gamma_{\bar{q}g}.\end{aligned} \tag{11}$$

As in the case of diagonal channels, the mass factorization for the off-diagonal ones also contains both diagonal and off-diagonal terms from Δ_{ab} and AP kernels, in different combinations. As expected, in the above result, we find no single term that can give a pure SV contribution. This is because every term contains at least one off-diagonal term. Recall, this is not the case for $\hat{\sigma}_{q\bar{q}}$. Hence, the mass factorized result for the off-diagonal channel starts with NSV and beyond, where the former comes from terms containing at least two diagonal terms either from Δ_{ab} or Γ_{ab}. Since we are interested only in NSV terms, we drop terms that contain more than two off-diagonal terms in the mass factorization formula to obtain

$$\frac{\hat{\sigma}_{qg}^{\rm sv+nsv}}{z\sigma_0}=\Gamma_{qq}^T\otimes\Delta_{q\bar{q}}^{\rm sv+nsv}\otimes\Gamma_{\bar{q}g}+\Gamma_{qq}^T\otimes\Delta_{qg}^{\rm sv+nsv}\otimes\Gamma_{gg}. \tag{12}$$

Note that the off-diagonal Δ_{qg} receives a contribution from $\hat{\sigma}_{qg}$ as well as from $\Delta_{q\bar{q}}$ unlike the diagonal $\Delta_{q\bar{q}}$ which receives only from a single $\hat{\sigma}_{q\bar{q}}$.

This analysis using the mass factorization formula and threshold expansion, which is valid to all orders in perturbation theory, demonstrates a simple structure for the diagonal $\Delta_{c\bar{c}}$; namely it contains only one kind of term that comprises diagonal kernels and $\hat{\sigma}_{c\bar{c}}$. On the other hand, in the off-diagonal channel, we have two kinds of terms containing diagonal and off-diagonal Δ_{ab}'s which mix under factorization. As we will see in the following, due to the simple structure in the diagonal channels, we can study the all-order structure of NSV logarithms using certain homogeneous differential equations. However, the off-diagonal ones pose challenges to such a study due to inhomogeneous terms present in the corresponding differential equation. Hence, in the following, we will focus only on diagonal partonic channels.

Beyond leading order, the partonic channels that contribute to $\hat{\sigma}^{(i)}_{c\bar{c}}$ can be broadly classified into two classes, namely those containing no partonic final state/no emission and the ones with at least one partonic final state. The former ones are called form factor (FF) contributions while the latter ones are called real emission contributions. In FFs, the entire partonic center of mass energy goes into producing a pair of leptons in DY or a Higgs boson in Higgs boson production, while in real emission processes, the initial state energy is shared among all the final state particles. Let us denote FF of DY by $\hat{F}_q$ and FF of Higgs boson productions by $\hat{F}_b$, $\hat{F}_g$, respectively.

Our next step is to factor out the square of the UV renormalized FF ($Z_{UV,c}\hat{F}_c$) with $c = q, \bar{q}, b, g$ from the partonic channels $\hat{\sigma}_{c\bar{c}}$. Here the $Z_{UV,c}$ is an overall renormalization constant that is required for Higgs boson production from gluon fusion and bottom quark annihilation. We call the resulting one the soft-collinear function, given by

$$\mathcal{S}_c(\hat{a}_s, \mu^2, q^2, z, \epsilon) = (\sigma_0(\mu_R^2))^{-1}(Z_{UV,c}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon))^{-2} \times |\hat{F}_c(\hat{a}_s, \mu^2, Q^2, \epsilon)|^{-2} \times \delta(1-z) \otimes \hat{\sigma}^{\mathrm{sv+nsv}}_{c\bar{c}}(q^2, z, \epsilon), \tag{13}$$

where $\hat{a}_s$ is the bare strong coupling constant, $Q^2 = -q^2$. Note that $\mathcal{S}_c$ does not depend on μ_R^2, and hence, $\mathcal{S}_c$ is RG invariant. The function $\mathcal{S}_c$ is computable in perturbation theory in powers of $\hat{a}_s$, and later in Sec. II A 1 we discuss its perturbative structure and also how several of its coefficients can be determined from the fixed order results. Since we have restricted ourselves to SV + NSV contributions to $\Delta_{c\bar{c}}$, that is, those resulting from the phase space region in the limit $z \to 1$, we keep only those terms that are proportional to distributions $\delta(1-z)$, $\mathcal{D}_i(z)$, and NSV terms of the kind of $\ln^i(1-z)$ with $i = 0, 1, \ldots$ and drop the rest of the terms resulting from the convolutions. Substituting for $\hat{\sigma}_{c\bar{c}}$ from (13) in terms of $\mathcal{S}^c$, in (10) and keeping only the diagonal terms in AP kernels, we find

$$\begin{aligned}\Delta_c(q^2, \mu_R^2, \mu_F^2, z) &= \Delta^{\mathrm{sv+nsv}}_{c\bar{c}}(q^2, \mu_R^2, \mu_F^2, z) \\ &= (Z_{UV,c}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon))^2 |\hat{F}_c(\hat{a}_s, \mu^2, Q^2, \epsilon)|^2 \delta(1-z) \otimes (\Gamma^T)^{-1}_{cc}(z, \mu_F^2, \epsilon) \\ &\quad \otimes \mathcal{S}_c(\hat{a}_s, \mu^2, q^2, z, \epsilon) \otimes \Gamma^{-1}_{\bar{c}\bar{c}}(z, \mu_F^2, \epsilon).\end{aligned} \tag{14}$$

The decomposition formula for $\Delta^{\mathrm{sv+nsv}}_{c\bar{c}}$ given in (14), is the first step toward obtaining the all order perturbative structure, which we are going to unravel in the subsequent section. It is to be noted that owing to the simplification in the mass factorized formula, given in (10), we obtain the above all order decomposition formula. It provides the pathway to study the partonic CFs in terms of certain building blocks, namely the form factor $\hat{F}_c$, overall renormalization constants $Z_{UV,c}$, the soft-collinear function $\mathcal{S}_c$, and the AP splitting kernels Γ_{cc}, which conspire among themselves in such a way to lead to a structure for Δ_c in terms of certain anomalous dimensions, as well as universal and process dependent coefficients. In the next subsection, using differential equations that each of these building blocks satisfies, we obtain an all-order structure for $\Delta^{\mathrm{sv+nsv}}_{c\bar{c}}$.

A. Next to SV formalism

In this section we discuss the formalism which accounts for both SV and NSV corrections to Δ_c owing to the decomposition formula given in (14). We study the underlying evolution equations corresponding to each of the building blocks, namely $\{\hat{F}_c, Z_{UV,c}, \Gamma_{cc}, S_c\}$, with respect to the renormalization and factorization scales and also the energy scale of the process under study. Following this, we derive the perturbative structure of each of the components and thereby present the analytic structure of the partonic CF.

In the master formula, Eq. (14), the form factor for the DY process is the matrix element of vector current $\bar{\psi}_q \gamma_\mu \psi_q$ between on-shell quark states, and for the Higgs boson production in gluon fusion (bottom quark annihilation), it is the matrix element of $G^a_{\mu\nu}G^{\mu\nu a}$ ($\bar{\psi}_b \psi_b$) between on-shell gluon (bottom quark) states. Here ψ_c is the c-type quark field operator and $G_{\mu\nu a}$ is the gluon field strength operator with a being the $SU(N_c)$ gauge group index in the adjoint representation. These FFs are known in QCD up to third order in perturbation theory [76–88]. The evolution equation for the overall renormalization constant with respect to the renormalization scale reads as

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z_{UV,c}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2) \gamma^c_{i-1}, \tag{15}$$

where γ^c_i is the UV anomalous dimension. For the vector current, the UV anomalous dimension is zero to all orders in QCD while for the Higgs boson productions, γ^c_i's are nonzero. For $c = b$, see [89], and for $c = g$, it is expressed in terms of QCD beta function coefficients to all orders [90].

Perturbative results of FF in renormalizable quantum field theory demonstrate rich structure; in particular, one finds that they satisfy certain differential equations. The simplest one is the RG equation that FFs satisfy, namely $\mu_R^2 \frac{d\hat{F}_c}{d\mu_R^2} = 0$, using which we can predict the logarithms resulting from the UV sector, i.e., the logarithms of the form $\ln^k(\mu_R^2), k = 1, \ldots,$ at every order in perturbation theory. In addition, these FFs satisfy the Sudakov differential equation [46,91–97] which is used to study their IR structure in terms of certain IR anomalous dimensions such as cusp A^c, collinear B^c, and soft f^c anomalous dimensions. In dimensional regularization, the equation takes the following form:

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}_c(\hat{a}_s, Q^2, \mu^2, \epsilon) = \frac{1}{2}\left[K^c\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) + G^c\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right)\right], \quad (16)$$

where $Q^2 = -q^2$. The above equation is called the $K + G$ equation. The unrenormalized FFs contain both UV and IR divergences. The latter results from soft gluons and massless partons which give soft and collinear divergences, respectively. UV divergences go away after UV renormalization. The IR divergences of the FFs can be shown to factorize. The divergence of FFs are such that the factorized IR divergent part is q^2 dependent. The consequence of these facts is that the right-hand side of the differential equation can be expressed in terms of two functions K^c and G^c in such a way that K^c accounts for all the poles in ϵ, whereas G^c is a finite term in the limit $\epsilon \to 0$. The RG invariance of FFs implies, in the limit $\epsilon \to 0$,

$$\mu_R^2 \frac{d}{d\mu_R^2} K^c\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -\mu_R^2 \frac{d}{d\mu_R^2} G^c\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = -A^c(a_s(\mu_R^2)). \quad (17)$$

The solutions to (17) are given in [20,46]. Substituting these solutions in (16) one can find the structure of FF in terms of IR anomalous dimensions A^c (cusp), B^c (collinear), and f^c (soft) as well as the process dependent quantities ($g_j^{c,k}$). A more elaborate discussion on the structure of FF can be found in [46]. The IR anomalous dimensions are known to three loops in QCD (see [28,78,79,84,98–101]) and for beyond three loops, see [87].

The fact that the initial state collinear divergences in parton level cross sections factorizes in terms of AP kernels $\Gamma_{ab}(z, \mu_F^2, \epsilon)$ implies the RG evolution equation with respect to the scale μ_F:

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma_{ab}(z, \mu_F^2, \epsilon) = \frac{1}{2} \sum_{a'=q,\bar{q},g} P_{aa'}(z, a_s(\mu_F^2)) \otimes \Gamma_{a'b}(z, \mu_F^2, \epsilon), \quad a, b = q, \bar{q}, g. \quad (18)$$

Since we are interested only in diagonal Altarelli-Parisi kernels for our analysis, the corresponding AP splitting functions $P_{cc}(z, \mu_F^2)$ are expanded around $z = 1$, and all those terms that do not contribute to SV + NSV are dropped. The AP splitting functions near $z = 1$ take the following form:

$$P_{cc}(z, a_s(\mu_F^2)) = 2B^c(a_s(\mu_F^2))\delta(1 - z) + P'_{cc}(z, a_s(\mu_F^2)), \quad (19)$$

where

$$P'_{cc}(z, a_s(\mu_F^2)) = 2[A^c(a_s(\mu_F^2))\mathcal{D}_0(z) + C^c(a_s(\mu_F^2))\ln(1 - z) + D^c(a_s(\mu_F^2))] + \mathcal{O}((1 - z)). \quad (20)$$

In the rest of the paper, we drop the terms in P'_{cc} proportional to $\mathcal{O}((1 - z))$ for our study. The constants C^c and D^c can be obtained from the splitting functions P'_{cc} which are known to three loops in QCD [100,101] (see [3,100–108] for the lower order ones). Similar to the cusp and the collinear anomalous dimensions, the constants C^c and D^c are also expanded in powers of $a_s(\mu_F^2)$ as

$$X^c(a_s(\mu_F^2)) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) X_i^c, \quad \forall \; X^c = \{C^c, D^c\}, \quad (21)$$

where C_i^c and D_i^c to third order are available in [100,101].

1. The soft-collinear function

Our next task is to study the soft-collinear function, $\mathcal{S}_c$, in detail. Equation (13) can be used to compute this function order by order in QCD perturbation theory. The $\mathcal{S}_c$ should contain right IR divergences to cancel those resulting from FF and AP kernels to give IR finite Δ_c. The IR structure of $\mathcal{S}_c$ in the SV limit was studied in [20,46] using a differential equation analogous to (16) supplemented with RG invariance. It was found that this function demonstrates a rich infrared structure in the SV approximation. Further, it provides a suitable framework to obtain the SV contribution order by order in perturbation theory. Since the function $\mathcal{S}_c$ obtained in [20,46] is an all order result in z space which allows one to write the integral representation suitable for studying resummation in Mellin N space. In the following, we proceed along this direction to study NSV contributions in z space to all orders in perturbation theory and to provide an integral representation that can be used for performing Mellin N space resummation. Using (14) and the $K + G$ equation of FFs, Eq. (16), one can set up an evolution equation for the functions $\mathcal{S}_c$. In other words, we can easily show that $\mathcal{S}_c$ satisfies the $K+G$ type of differential equation of the form

$$q^2 \frac{d}{dq^2}\mathcal{S}_c(\hat{a}_s, q^2, \mu^2, \epsilon, z) = \Gamma_{\mathcal{S}_c}(\hat{a}_s, q^2, \mu^2, \epsilon, z) \otimes \mathcal{S}_c(\hat{a}_s, q^2, \mu^2, \epsilon, z), \tag{22}$$

where

$$\Gamma_{\mathcal{S}_c} = \left[\bar{K}^c\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon, z\right) + \bar{G}^c\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon, z\right)\right], \tag{23}$$

where $\Gamma_{\mathcal{S}_c}$ in the above equation is written as a sum of $\bar{K}^c$ which accounts for all the divergent terms and $\bar{G}^c$, the finite function of (z,ϵ). The scale μ_s signifies the arbitrariness in separating the divergent part from the remaining finite terms. In consequence to the above differential equation (22), the soft-collinear function, $\mathcal{S}_c$, admits an exponential solution given by

$$S_c(\hat{a}_s, q^2, \mu^2, \epsilon, z) = \mathcal{C}\exp\left(\int_0^{q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\mathcal{S}_c}(\hat{a}_s, \lambda^2, \mu^2, \epsilon, z)\right) = \mathcal{C}\exp(2\Phi^c(\hat{a}_s, q^2, \mu^2, \epsilon, z)), \tag{24}$$

where the initial condition $\mathcal{S}_c(q^2, q^2=0, \mu^2, \epsilon, z) = \delta(1-z)$ is used. The exponent Φ^c gets only contribution from $c\bar{c}$ initiated processes containing at least one real radiation. The symbol "$\mathcal{C}$" refers to convolution. For instance, $\mathcal{C}$ acting on any exponential of a function has the following expansion:

$$\mathcal{C}e^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}(f\otimes f)(z) + \cdots. \tag{25}$$

In addition, $\mathcal{S}^c$'s satisfy the renormalization group equation, namely $\mu_s^2 \frac{dS^c}{d\mu_s^2} = 0$, which implies

$$\mu_s^2\frac{d}{d\mu_s^2}\bar{K}^c(a_s(\mu_s^2), z) = -\mu_s^2\frac{d}{d\mu_s^2}\bar{G}^c(a_s(\mu_s^2), z) = -\bar{A}^c(a_s(\mu_s^2))\delta(1-z), \tag{26}$$

where $\bar{A}^c$ is analogous to the cusp anomalous dimension that appears in the $K+G$ equation of FFs. The perturbative solution to (22) can be obtained by integrating the differential equation after substituting the fixed order solutions of RGs for $\bar{K}^c$ and $\bar{G}^c$. We propose an all order ansatz for the solution Φ^c which takes the general form

$$\Phi^c(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2(1-z)^2}{\mu^2 z}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \times\left(\frac{i\epsilon}{1-z}\right)\hat{\phi}_c^{(i)}(z,\epsilon), \tag{27}$$

where $S_\epsilon = \exp\left(\frac{\epsilon}{2}[\gamma_E - \ln(4\pi)]\right)$ with γ_E being the Euler Mascheroni constant. The form of the solution given in (27) is inspired by the result for the production of a pair of leptons in the quark-antiquark channel or Higgs boson in gluon fusion at next to leading order in a_s. A separate section (see Sec. III) is devoted to justify this form. The term $\left(\frac{q^2(1-z)^2}{\mu^2 z}\right)^{\frac{\epsilon}{2}}$ in the parentheses results from two body phase space while $\hat{\phi}_c(z,\epsilon)/(1-z)$ comes from the square of the matrix elements for corresponding amplitudes. In general, the term $q^2(1-z)^2/z$ inside the parentheses is the hard scale in the problem, and it controls the evolution of Φ^c at every order. The function $\hat{\phi}_c^{(i)}(z,\epsilon)$ is regular as $z\to 0$ but contains poles in ϵ. We have factored out $1/(1-z)$ explicitly so that it generates all the distributions $\mathcal{D}_j$ and $\delta(1-z)$ and NSV terms $\ln^k(1-z), k=0,\ldots,$ when combined with the factor $((1-z)^2)^{i\epsilon/2}$ and $\hat{\phi}_c^{(i)}(z,\epsilon)$ at each order in $\hat{a}_s$. Note that the term $z^{-i\epsilon/2}$ inside the parentheses does not give distributions $\mathcal{D}_j$ and $\delta(1-z)$; however, they can contribute to NSV terms $\ln^j(1-z), j=0,1,\ldots,$ when we expand around $z=1$. In addition, the terms proportional to $(1-z)$ in $\hat{\phi}_c$ near $z=1$ also give NSV terms for Φ^c. Although the form of solution for Φ^c is good enough to study NSV terms, we rewrite this in a convenient form which separates SV terms from the NSV in Φ^c. Hence, we decompose Φ^c as $\Phi^c = \Phi_A^c + \Phi_B^c$ in such a way that Φ_A^c contains only SV terms and the remaining Φ_B^c contains next to soft-virtual terms in the limit $z\to 1$. The distribution Φ_A^c satisfies the $K+G$ equation given in Eq. (35) of [46]; also see [20] for details. The solution for Φ_A^c in powers of $\hat{a}_s$ in dimensional regularization is given in [46]. It is given by

$$\Phi_A^c(\hat{a}_s, q^2, \mu^2, \epsilon, z) = \sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \times\left(\frac{i\epsilon}{1-z}\right)\hat{\phi}_{\rm SV}^{c(i)}(\epsilon), \tag{28}$$

where

$$\hat{\phi}_{\rm SV}^{c(i)}(\epsilon) = \frac{1}{i\epsilon}[\bar{K}^{c(i)}(\epsilon) + \bar{G}_{\rm SV}^{c(i)}(\epsilon)]. \tag{29}$$

The constants $\bar{K}^{c(i)}(\epsilon)$ and $\bar{G}_{\rm SV}^{c(i)}(\epsilon)$ are known to third order in perturbation theory [20,21,25,36,46]. For the reader's convenience, we enlist the results of $\bar{K}^{c(i)}(\epsilon)$ and $\bar{G}_{\rm SV}^{c(i)}(\epsilon)$ in Appendix B. After substituting these perturbative constants one can get the perturbative structure of the SV coefficients as

$$\hat{\phi}_{\rm SV}^{c(1)}(\epsilon) = \frac{1}{\epsilon^2}(2A_1^c) + \frac{1}{\epsilon}(\bar{\mathcal{G}}_1^c(\epsilon)), \qquad \hat{\phi}_{SV}^{c(2)}(\epsilon) = \frac{1}{\epsilon^3}(-\beta_0 A_1^c) + \frac{1}{\epsilon^2}\left(\frac{1}{2}A_2^c - \beta_0 \bar{\mathcal{G}}_1^c(\epsilon)\right) + \frac{1}{2\epsilon}\bar{\mathcal{G}}_2^c(\epsilon),$$
$$\hat{\phi}_{SV}^{c(3)}(\epsilon) = \frac{1}{\epsilon^4}\left(\frac{8}{9}\beta_0^2 A_1^c\right) + \frac{1}{\epsilon^3}\left(-\frac{2}{9}\beta_1 A_1^c - \frac{8}{9}\beta_0 A_2^c - \frac{4}{3}\beta_0^2 \bar{\mathcal{G}}_1^c(\epsilon)\right) + \frac{1}{\epsilon^2}\left(\frac{2}{9}A_3^c - \frac{1}{3}\beta_1 \bar{\mathcal{G}}_1^c(\epsilon) - \frac{4}{3}\beta_0 \bar{\mathcal{G}}_2^c(\epsilon)\right) + \frac{1}{\epsilon}\left(\frac{1}{3}\bar{\mathcal{G}}_3^c(\epsilon)\right),$$
$$\hat{\phi}_{SV}^{c(4)}(\epsilon) = \frac{1}{\epsilon^5}(-\beta_0^3 A_1^c) + \frac{1}{\epsilon^4}\left(\frac{2}{3}\beta_0\beta_1 A_1^c + \frac{3}{2}\beta_0^2 A_2^c - 2\beta_0^3 \bar{\mathcal{G}}_1^c(\epsilon)\right) - \frac{1}{\epsilon^3}\left(\frac{1}{12}\beta_2 A_1^c - \frac{1}{4}\beta_1 A_2^c - \frac{3}{4}\beta_0 A_3^c + \frac{4}{3}\beta_0\beta_1 \bar{\mathcal{G}}_1^c(\epsilon) + 3\beta_0^2 \bar{\mathcal{G}}_2^c(\epsilon)\right)$$
$$+ \frac{1}{\epsilon^2}\left(\frac{1}{8}A_4^c - \frac{1}{6}\beta_2 \bar{\mathcal{G}}_1^c(\epsilon) - \frac{1}{2}\beta_1 \bar{\mathcal{G}}_2^c(\epsilon) - \frac{3}{2}\beta_0 \bar{\mathcal{G}}_3^c(\epsilon)\right) + \frac{1}{\epsilon}\left(\frac{1}{4}\bar{\mathcal{G}}_4^c(\epsilon)\right). \tag{30}$$

The integral representation for Φ_A^c is given in [20] and is reproduced here for completeness:

$$\Phi_A^c(\hat{a}_s, \mu^2, q^2, z, \epsilon) = \left(\frac{1}{1-z}\left\{\int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} A^c(a_s(\lambda^2)) + \bar{G}_{\rm SV}^c(a_s(q^2(1-z)^2), \epsilon)\right\}\right)_+$$
$$+ \delta(1-z)\sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \hat{\phi}_{\rm SV}^{c(i)}(\epsilon) + \frac{1}{(1-z)_+}\sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{\mu_F^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \bar{K}^{c(i)}(\epsilon). \tag{31}$$

Having all the information about the SV coefficients, let us now study in detail the structure of Φ_B^c using Eq. (22). Subtracting out the $K + G$ equation for the SV part Φ_A^c from (28), we find that Φ_B^c satisfies

$$q^2 \frac{d}{dq^2}\Phi_B^c(q^2, z, \epsilon) = \frac{1}{2}\left[G_L^c\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon, z\right)\right], \tag{32}$$

where $G_L^c = \bar{G}^c - \bar{G}_{\rm SV}^c$,

$$G_L^c\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) = \sum_{i=1}^{\infty} a_s^i(q^2(1-z)^2)\mathcal{G}_{L,i}^c(z, \epsilon). \tag{33}$$

The NSV part of the solution that satisfies (32) takes the following form:

$$\Phi_B^c(\hat{a}_s, \mu^2, q^2, z, \epsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \hat{\varphi}_c^{(i)}(z, \epsilon), \tag{34}$$

where the perturbative expansion of the NSV coefficient $\hat{\varphi}_c^{(i)}(z, \epsilon)$ reads as

$$\hat{\varphi}_c^{(1)}(z, \epsilon) = \frac{1}{\epsilon}\mathcal{G}_{L,1}^c(z, \epsilon), \qquad \hat{\varphi}_c^{(2)}(z, \epsilon) = \frac{1}{\epsilon^2}(-\beta_0 \mathcal{G}_{L,1}^c(z, \epsilon)) + \frac{1}{2\epsilon}\mathcal{G}_{L,2}^c(z, \epsilon),$$
$$\hat{\varphi}_c^{(3)}(z, \epsilon) = \frac{1}{\epsilon^3}\left(\frac{4}{3}\beta_0^2 \mathcal{G}_{L,1}^c(z, \epsilon)\right) + \frac{1}{\epsilon^2}\left(-\frac{1}{3}\beta_1 \mathcal{G}_{L,1}^c(z, \epsilon) - \frac{4}{3}\beta_0 \mathcal{G}_{L,2}^c(z, \epsilon)\right) + \frac{1}{3\epsilon}\mathcal{G}_{L,3}^c(z, \epsilon),$$
$$\hat{\varphi}_c^{(4)}(z, \epsilon) = \frac{1}{\epsilon^4}(-2\beta_0^3 \mathcal{G}_{L,1}^c(z, \epsilon)) + \frac{1}{\epsilon^3}\left(\frac{4}{3}\beta_0\beta_1 \mathcal{G}_{L,1}^c(z, \epsilon) + 3\beta_0^2 \mathcal{G}_{L,2}^c(z, \epsilon)\right)$$
$$+ \frac{1}{\epsilon^2}\left(-\frac{1}{6}\beta_2 \mathcal{G}_{L,1}^c(z, \epsilon) - \frac{1}{2}\beta_1 \mathcal{G}_{L,2}^c(z, \epsilon) - \frac{3}{2}\beta_0 \mathcal{G}_{L,3}^c(z, \epsilon)\right) + \frac{1}{4\epsilon}\mathcal{G}_{L,4}^c(z, \epsilon). \tag{35}$$

The ϵ expansion of the renormalized NSV quantities $\mathcal{G}_{L,i}^c(z, \epsilon)$ can be further decomposed as

$$\mathcal{G}_{L,i}^c(z, \epsilon) = L_i^c(z) + \bar{\chi}_{L,i}^c(z) + \sum_{j=1}^{\infty} \epsilon^j \mathcal{G}_{L,i}^{c,(j)}(z), \tag{36}$$

with

$$\bar{\chi}^c_{L,i}(z) = \bar{\chi}^c_i|_{(\bar{\mathcal{G}}^{c,(j)}_i \to \mathcal{G}^{c,(j)}_{L,i}(z))}, \tag{37}$$

where $\bar{\chi}^c_i$ is given in (B6). Unlike the SV renormalized coefficients $\bar{\mathcal{G}}^{c,(j)}_i$, the NSV coefficients $\mathcal{G}^{c,(j)}_{L,i}(z)$ in the above equations are parametrized in terms of $\ln^k(1-z), k = 0, 1, \ldots,$ and all the terms that vanish as $z \to 1$ are dropped,

$$\mathcal{G}^{c,(j)}_{L,i}(z) = \sum_{k=0}^{i+j-1} \mathcal{G}^{c,(j,k)}_{L,i} \ln^k(1-z). \tag{38}$$

The highest power of the $\ln(1-z)$ at every order depends on the order of the perturbation, namely the power of a_s and also the power of ϵ at each order in a_s. We determine this highest power by studying results for the bare partonic cross sections $\hat{\sigma}_{c\bar{c}}$ at higher orders in $\hat{a}_s$, expanded in powers of ϵ to high accuracy. Alternatively, we can use the known mass factorized results for $\Delta_{c\bar{c}}$ to obtain this power. In the former approach, we used the results for $\hat{\sigma}_{c\bar{c}}$, computed up to second order in $\hat{a}_s$, i.e., $i = 1, 2$ with ϵ expanded up to third power for $i = 1$ and first power for $i = 2$. In the case of $\Delta_{c\bar{c}}$, we used the known results up to third order in a_s to obtain the highest power of logarithms. Extrapolating the findings from these two fixed order results to all orders in $\hat{a}_s$ and ϵ, we obtain the highest power for $\ln(1-z)$ to be $i + j - 1$. We devote a separate subsection (see Sec. III B) to elaborate on this peculiar structure of the logarithms.

Similar to the SV case, the NSV function Φ^c_B can be written in an integral form using (32) and the perturbative structure given in (35) as

$$\begin{aligned}
&\Phi^c_B(\hat{a}_s, \mu^2, q^2, z, \epsilon) \\
&= \int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} L^c(a_s(\lambda^2), z) \\
&\quad + \varphi_{f,c}(a_s(q^2(1-z)^2), z, \epsilon)|_{\epsilon=0} + \varphi_{s,c}(a_s(\mu_F^2), z, \epsilon).
\end{aligned} \tag{39}$$

Here, the first line is completely finite as $\epsilon \to 0$ while the second line, $\varphi_{s,c}$, is divergent. The fact that Φ^c_B is RG invariant implies that $\varphi_{s,c}$ satisfies the renormalization group equation:

$$\mu_F^2 \frac{d}{d\mu_F^2} \varphi_{s,c}(a_s(\mu_F^2), z) = L^c(a_s(\mu_F^2), z). \tag{40}$$

Further the Δ_c in (14) is finite at every order in a_s in the limit $\epsilon \to 0$ allows us to determine the coefficients L^c in terms of the NSV coefficients C^c and D^c in splitting kernels, given in (19). We find at each order in perturbative expansion

$$L^c(a_s(\mu_F^2), z) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) L^i_c(z) \tag{41}$$

with $L^c_i(z) = C^c_i \ln(1-z) + D^c_i$, where the coefficients C^c_i and D^c_i are related to those of cusp A^c_i and collinear B^c_i anomalous dimensions in the following way up to third order [101,109]:

$$\begin{aligned}
&D^c_1 = -A^c_1, \qquad D^c_2 = -A^c_2 + A^c_1(B^c_1 - \beta_0), \\
&D^c_3 = -A^c_3 - A^c_1(-B^c_2 + \beta_1) - A^c_2(-B^c_1 + \beta_0), \\
&C^c_1 = 0, \qquad C^c_2 = (A^c_1)^2, \qquad C^c_3 = 2A^c_1 A^c_2.
\end{aligned} \tag{42}$$

Having fixed the divergent part of Φ^c_B completely, we turn to the structure of the finite piece $\varphi_{f,c}$. We first expand them in powers of renormalized coupling a_s,

$$\begin{aligned}
&\varphi_{f,c}(a_s(q^2(1-z)^2), z) \\
&\quad = \sum_{i=1}^{\infty} a_s^i(q^2(1-z)^2) \sum_{k=0}^{i} \varphi^{(k)}_{c,i} \ln^k(1-z),
\end{aligned} \tag{43}$$

where the highest power of $\ln(1-z)$ is in accord with the same in Eq. (38). We will discuss more on this structure in Sec. III B. The coefficients $\varphi^{(k)}_{c,i}$ can be expressed in terms of their unrenormalized counterpart $\mathcal{G}^{c,(j,k)}_{L,i}$'s in (38) as

$$\begin{aligned}
\varphi^{(k)}_{c,1} &= \mathcal{G}^{c,(1,k)}_{L,1}, \quad k = 0, 1, \\
\varphi^{(k)}_{c,2} &= \left(\frac{1}{2}\mathcal{G}^{c,(1,k)}_{L,2} + \beta_0 \mathcal{G}^{c,(2,k)}_{L,1}\right), \quad k = 0, 1, 2, \\
\varphi^{(k)}_{c,3} &= \left(\frac{1}{3}\mathcal{G}^{c,(1,k)}_{L,3} + \frac{2}{3}\beta_1 \mathcal{G}^{c,(2,k)}_{L,1} + \frac{2}{3}\beta_0 \mathcal{G}^{c,(2,k)}_{L,2} + \frac{4}{3}\beta_0^2 \mathcal{G}^{c,(3,k)}_{L,1}\right), \quad k = 0, 1, 2, 3, \\
\varphi^{(k)}_{c,4} &= \left(\frac{1}{4}\mathcal{G}^{c,(1,k)}_{L,4} + \frac{1}{2}\beta_2 \mathcal{G}^{c,(2,k)}_{L,1} + \frac{1}{2}\beta_1 \mathcal{G}^{c,(2,k)}_{L,2} + \frac{1}{2}\beta_0 \mathcal{G}^{c,(2,k)}_{L,3} + 2\beta_0\beta_1 \mathcal{G}^{c,(3,k)}_{L,1} + \beta_0^2 \mathcal{G}^{c,(3,k)}_{L,2} + 2\beta_0^3 \mathcal{G}^{c,(4,k)}_{L,1}\right), \quad k = 0, 1, 2, 3, 4,
\end{aligned} \tag{44}$$

where $\mathcal{G}_{L,1}^{c,(2,3)}$, $\mathcal{G}_{L,1}^{c,(2,4)}$, $\mathcal{G}_{L,2}^{c,(2,4)}$, $\mathcal{G}_{L,1}^{c,(3,4)}$ are all zero. The structure of divergent and finite pieces of Φ_B^c allows us to determine the coefficients $\mathcal{G}_{L,i}^{c,(j,k)}$ and $\varphi_{c,i}^{(k)}$, and we postpone the discussion on this to the next section.

So far, we have discussed the logarithmic structure of the building blocks of mass factorized CFs within the framework of perturbation theory. We used respective first order differential equations satisfied by each of them as given in (15), (16), (18), and (24). We found that each of them admits the solution which is of the exponential form whose exponents are controlled by process independent anomalous dimensions as well as process dependent coefficients. Substituting these solutions for the building blocks, we obtain

$$\Delta_{c\bar{c}}^{\mathrm{sv+nsv}}(q^2,\mu_R^2,\mu_F^2,z), = \mathcal{C}\exp(\Psi^c(q^2,\mu_R^2,\mu_F^2,z,\epsilon))|_{\epsilon=0}, \tag{45}$$

where Ψ^c is a finite function in the limit $\epsilon \to 0$ and is given by

$$\begin{aligned}\Psi^c(q^2,\mu_R^2,\mu_F^2,z,\epsilon) = {}& (\ln(Z_{UV,c}(\hat{a}_s,\mu^2,\mu_R^2,\epsilon))^2 \\ &+ \ln|\hat{F}_c(\hat{a}_s,\mu^2,Q^2,\epsilon)|^2)\delta(1-z) \\ &+ 2\Phi^c(\hat{a}_s,\mu^2,q^2,z,\epsilon) \\ &- 2\mathcal{C}\ln\Gamma_{cc}(\hat{a}_s,\mu^2,\mu_F^2,z,\epsilon).\end{aligned} \tag{46}$$

This all order result is the master formula which can be used for obtaining SV + NSV contributions to Δ_c order by order in perturbation theory provided various functions that appear in Eq. (46) are known to the desired accuracy. In particular, it can predict certain SV and NSV terms to all orders in a_s in terms of lower order terms. We elaborate this in more detail in Sec. IV. In the above formula, we keep the entire FF and overall renormalization constant as they are proportional to only $\delta(1-z)$. However, in the functions Φ^c and $\ln\Gamma_{cc}$, we keep only SV and NSV terms.

Before we conclude this subsection, we discuss the general structure of the renormalization group equation corresponding to $\mathcal{S}_c$ resulting from infrared singularities originating from soft and collinear emissions. IR singularities in $\mathcal{S}_c$ are found to be factorizable; i.e., we can write $\mathcal{S}_c(q^2,z) = Z_c(q^2,\mu_s^2,z)\otimes\mathcal{S}_{c,\mathrm{fin}}(q^2,\mu_s^2,z)$ with μ_s being the IR factorization scale. This is a consequence $\bar{K}^c+\bar{G}^c$ decomposition, valid to all orders in perturbation theory. Here, Z_c contains all the IR singularities of $\mathcal{S}_c$ in terms of poles in ϵ and $\mathcal{S}_{c,\mathrm{fin}}$ is IR finite in the limit $\epsilon\to 0$. We can relate Z_c to $\bar{K}^c$ and $\bar{G}^c$ through $\bar{K}^c = d\log Z_c/d\log(q^2)$ and $\bar{G}^c = d\log\mathcal{S}_{c,\mathrm{fin}}/d\log(q^2)$, respectively. The complete singular structure of Z_c can be obtained by solving the renormalization group equation

$$\mu_s^2\frac{dZ_c(\mu_s^2,q^2,z,\epsilon)}{d\mu_s^2} = \gamma_{\mathcal{S},c}(\mu_s^2,q^2,z,\epsilon)\otimes Z_c(\mu_s^2,q^2,z), \tag{47}$$

where $\gamma_{\mathcal{S},c}$ takes the remarkable structure $\xi_1(\mu_s^2,z)\times\log(q^2/\mu_s^2)+\xi_2(\mu_s^2,z)$ to all orders in perturbation theory. This structure follows from the fact that Z_c has to contain the right infrared poles to cancel against those from form factor and AP kernels leaving Δ_c finite. The latter gives

$$\gamma_{\mathcal{S},c} = \left(A^c(\mu_s^2)\log\left(\frac{q^2}{\mu_s^2}\right) - \frac{f^c(\mu_s^2)}{2}\right)\delta(1-z) + P'_{cc}(\mu_s). \tag{48}$$

Note that the anomalous dimensions A^c and f^c control the renormalization group equation (RGE) of the SV parts, namely $\delta(1-z)$ and $1/(1-z)_+$, whereas the RGE of NSV parts is governed through the collinear anomalous dimensions C^c and D^c. This suggests that Z_c can be further decomposed into Z_c^A and Z_c^B. Here, Z_c^A contains the singularities in the SV part arising from pure soft modes, and Z_c^B accounts for those in the NSV part resulting from soft and collinear modes. In other words, the soft-collinear function $\mathcal{S}_c$ can be factorized into two exponential functions with exponents Φ_A^c and Φ_B^c, and each is governed by its own renormalization group equation in terms of an independent set of anomalous dimensions. In conclusion, we have presented a formula, given in (46), which gives the analytical structure of the partonic CF in terms of the anomalous dimensions and SV and NSV coefficients.

2. Results for NSV coefficients

In this subsection, we evaluate explicit expressions for the NSV coefficients, introduced earlier, by comparing against the state-of-the-art results of CFs and their building blocks such as FF and AP kernels. At every order a_s^i, the coefficients $\mathcal{G}_{L,i}^{c,(j,k)}$ for various values of (j,k) can be determined using (14) and (46) known to order a_s^i expanded in a double series expansion of $\epsilon^j\ln^k(1-z)$. In order to do this we use the available information up to two loop level to obtain $\mathcal{G}_{L,i}^{c,(j,k)}$ for $i=1$, 2 for all the allowed values of (j,k).

We find that unlike the SV coefficients $\bar{\mathcal{G}}_i^{c,j}$ [see (B7)], the quark and gluon NSV coefficients $\mathcal{G}_{L,i}^{c,(j,k)}$ do not satisfy the maximal non-Abelian relation beyond one loop. Recall that $\bar{\mathcal{G}}_i^{c,(j)}$ satisfy $\bar{\mathcal{G}}_i^{q,(j)} = (C_F/C_A)\bar{\mathcal{G}}_i^{g,(j)}$, confirmed up to third order in a_s as shown in [20,46].

Third order contributions to Δ_c for DY became available very recently in [9], and for the Higgs boson productions in gluon fusion as well as in bottom quark annihilation the third order results were presented in [4–6]. The analytical results for FFs, over all renormalization constants, the functions Φ_A^c and $\Gamma_{c\bar{c}}$ are all available up to third order in the literature. Using these results, we can in principle extract the relevant coefficients $\mathcal{G}_{L,i}^{q,(j,k)}$ to third order. In the absence of analytical results for second order corrections to Δ_q for positive powers of ϵ, we cannot determine the coefficients $\mathcal{G}_{L,i}^{q,(j,k)}$ at the third order.

However, the combination of these coefficients, namely $\varphi_{f,c}$, given in (44), can be extracted for $c = q$ (DY) and for $c = b$ ($b\bar{b}H$) and $c = g$ (ggH) up to third order using the available results to third order. We find for the DY

$$
\begin{aligned}
\varphi_{q,1}^{(0)} &= 4C_F, \qquad \varphi_{q,1}^{(1)} = 0,\\
\varphi_{q,2}^{(0)} &= C_F C_A\left(\frac{1402}{27} - 28\zeta_3 - \frac{112}{3}\zeta_2\right) + C_F^2(-32\zeta_2) + n_f C_F\left(-\frac{328}{27} + \frac{16}{3}\zeta_2\right),\\
\varphi_{q,2}^{(1)} &= 10C_F C_A - 10C_F^2, \qquad \varphi_{q,2}^{(2)} = -4C_F^2,\\
\varphi_{q,3}^{(0)} &= C_F C_A^2\left(\frac{727211}{729} + 192\zeta_5 - \frac{29876}{27}\zeta_3 - \frac{82868}{81}\zeta_2 + \frac{176}{3}\zeta_2\zeta_3 + 120\zeta_2^2\right)\\
&\quad + C_F^2 C_A\left(-\frac{5143}{27} - \frac{2180}{9}\zeta_3 - \frac{11584}{27}\zeta_2 + \frac{2272}{15}\zeta_2^2\right) + C_F^3\left(23 + 48\zeta_3 - \frac{32}{3}\zeta_2 - \frac{448}{15}\zeta_2^2\right)\\
&\quad + n_f C_F C_A\left(-\frac{155902}{729} + \frac{1292}{9}\zeta_3 + \frac{26312}{81}\zeta_2 - \frac{368}{15}\zeta_2^2\right)\\
&\quad + n_f C_F^2\left(-\frac{1309}{9} + \frac{496}{3}\zeta_3 + \frac{2536}{27}\zeta_2 + \frac{32}{5}\zeta_2^2\right) + n_f^2 C_F\left(\frac{12656}{729} - \frac{160}{27}\zeta_3 - \frac{704}{27}\zeta_2\right),\\
\varphi_{q,3}^{(1)} &= C_F C_A^2\left(\frac{244}{9} + 24\zeta_3 - \frac{8}{9}\zeta_2\right) + C_F^2 C_A\left(-\frac{18436}{81} + \frac{544}{3}\zeta_3 + \frac{964}{9}\zeta_2\right)\\
&\quad + C_F^3\left(-\frac{64}{3} - 64\zeta_3 + \frac{80}{3}\zeta_2\right) + n_f C_F C_A\left(-\frac{256}{9} - \frac{28}{9}\zeta_2\right) + n_f C_F^2\left(\frac{3952}{81} - \frac{160}{9}\zeta_2\right),\\
\varphi_{q,3}^{(2)} &= C_F C_A^2\left(34 - \frac{10}{3}\zeta_2\right) + C_F^2 C_A\left(-96 + \frac{52}{3}\zeta_2\right) + C_F^3\left(\frac{16}{3}\right) + n_f C_F C_A\left(-\frac{10}{3}\right) + n_f C_F^2\left(\frac{40}{3}\right),\\
\varphi_{q,3}^{(3)} &= C_F^2 C_A\left(-\frac{176}{27}\right) + n_f C_F^2\left(\frac{32}{27}\right),
\end{aligned}
\tag{49}
$$

and for the Higgs boson production

$$
\begin{aligned}
\varphi_{g,1}^{(0)} &= 4C_A,\\
\varphi_{g,1}^{(1)} &= 0,\\
\varphi_{g,2}^{(0)} &= C_A^2\left(\frac{1306}{27} - 28\zeta_3 - \frac{208}{3}\zeta_2\right) + n_f C_A\left(-\frac{196}{27} + \frac{16}{3}\zeta_2\right),\\
\varphi_{g,2}^{(1)} &= C_A^2\left(\frac{2}{3}\right) + n_f C_A\left(-\frac{2}{3}\right),\\
\varphi_{g,2}^{(2)} &= -4C_A^2,\\
\varphi_{g,3}^{(0)} &= C_A^3\left(\frac{563231}{729} + 192\zeta_5 - \frac{34292}{27}\zeta_3 - \frac{113600}{81}\zeta_2 + \frac{176}{3}\zeta_2\zeta_3 + \frac{3488}{15}\zeta_2^2\right)\\
&\quad + n_f C_A^2\left(-\frac{117778}{729} + \frac{1888}{9}\zeta_3 + \frac{26780}{81}\zeta_2 - \frac{232}{15}\zeta_2^2\right)\\
&\quad + n_f C_F C_A\left(-\frac{2653}{27} + \frac{616}{9}\zeta_3 + \frac{40}{3}\zeta_2 + \frac{32}{5}\zeta_2^2\right) + n_f^2 C_A\left(\frac{1568}{729} - \frac{160}{27}\zeta_3 - \frac{152}{9}\zeta_2\right),
\end{aligned}
$$

$$\varphi_{g,3}^{(1)} = C_A^3\left(-\frac{18988}{81}+\frac{448}{3}\zeta_3+\frac{1280}{9}\zeta_2\right)+n_fC_A^2\left(\frac{1528}{81}-8\zeta_3-\frac{248}{9}\zeta_2\right)$$
$$+n_fC_FC_A\left(4-\frac{8}{3}\zeta_2\right)+n_f^2C_A\left(\frac{56}{27}\right),$$
$$\varphi_{g,3}^{(2)} = C_A^3\left(-\frac{1432}{27}+\frac{40}{3}\zeta_2\right)+n_fC_A^2\left(\frac{164}{27}+\frac{2}{3}\zeta_2\right)+n_f^2C_A\left(\frac{8}{27}\right),$$
$$\varphi_{g,3}^{(3)} = C_A^3\left(-\frac{176}{27}\right)+n_fC_A^2\left(\frac{32}{27}\right). \quad (50)$$

While the NSV functions Φ_B^c for quarks and gluons are not related, they are found to be universal up to second order in the sense that they do not depend on the hard process. For example, to second order in a_s, Φ_B^q of DY is found to be identical to that of Higgs boson production in bottom quark annihilation [110]. In addition, we find that they agree with that of graviton (G) production in quark annihilation processes [111–116]. In terms of $\varphi_{q,i}^{(k)}$ it translates to

$$\varphi_{q,i}^{(k)}\Big|_{q+\bar{q}\to l^+l^-+X} = \varphi_{q,i}^{(k)}\Big|_{b+\bar{b}\to H+X} = \varphi_{q,i}^{(k)}\Big|_{q+\bar{q}\to G+X} \qquad i=1,2, k=0,i. \quad (51)$$

Similarly, to second order in a_s, Φ_B^g from Higgs boson production in gluon fusion is found to be identical to that of graviton production in the gluon fusion channel and pseudoscalar Higgs boson production [18,117–121] in gluon fusion. That is,

$$\varphi_{g,i}^{(k)}\Big|_{g+g\to H+X} = \varphi_{g,i}^{(k)}\Big|_{g+g\to A+X} = \varphi_{g,i}^{(k)}\Big|_{g+g\to G+X} \qquad i=1,2, k=0,i. \quad (52)$$

However, the universality breaks at third order; namely we find that the $\varphi_{b,3}^{(k)}$ for $k=0$, 1 differs from that of DY production while for $k=2$, 3 they agree:

$$\varphi_{b,3}^{(0)} = \varphi_{q,3}^{(0)} - 16C_AC_F(C_A-2C_F),$$
$$\varphi_{b,3}^{(1)} = \varphi_{q,3}^{(1)} + 8C_AC_F(C_A-2C_F),$$
$$\varphi_{b,3}^{(k)} = \varphi_{q,3}^{(k)}, \quad k=2,3. \quad (53)$$

The origin of this violation for $k=0$, 1 at third order, which has been evaluated using the state-of-the-art results [4–6,9], needs to be understood within the framework of factorization.

III. MORE ON THE SOFT-COLLINEAR FUNCTION, Φ_B^c

A. On the form of the solution

In this section, we discuss in detail the peculiar structure of SV and NSV solutions given in (28) and (34), respectively, that satisfy the $K+G$ equation. Both of them contain divergent as well as finite terms at every order. For example, the SV part of the solution, Φ_A^c, contains the right soft and collinear divergences proportional to distributions $\delta(1-z)$ and $\mathcal{D}_0(z)$ to cancel those from the FF entirely and from the AP kernels partially and the z dependent finite terms to correctly reproduce all the distributions in the SV part of CFs Δ_c. The NSV part, Φ_B^c, removes the remaining collinear divergences of the AP kernels. The finite part of it when combined with the SV counterpart of Φ_A^c contributes to next to SV terms to CFs Δ_c. As we mentioned in the previous section, the z dependence of the solution is inspired from the structure of various contributions that constitute the next to leading order contributions to a variety of inclusive reactions, namely production of a pair of leptons in quark-antiquark annihilation, a Higgs boson in gluon fusion or in bottom quark annihilation at hadron colliders. In addition, the renormalization group equation, Eq. (40), brings in an additional z dependent logarithmic structure through the anomalous dimensions $C^c(a_s)$ and $D^c(a_s)$.

Note that the solution given in (27) is organized in such a way that the term Φ_A^c contains only leading contributions, namely the distributions such as $\delta(1-z)$ and $\mathcal{D}_j(z)$, the so-called SV terms, and the term Φ_B^c, the subleading terms, i.e., the next to SV logarithms $\ln^k(1-z), k=0,1,\ldots$. Even though Φ_A^c does not contain next to SV terms, they contribute to next to SV terms to Δ_c, when the exponential is expanded in powers of a_s. Not only do distributions result from the convolutions of two or more distributions, they also give next to SV logarithms. In addition, the convolution of distributions with next to SV terms in turn give pure NSV logarithms. Hence, the leading solution Φ_A^c plays an important role for generating next to SV terms for the CFs Δ_c at every order in perturbation theory.

The solution Φ_A^c [see (28)] at every order in $\hat{a}_s$ is found to factorize into the z dependent piece, $((1-z)^m)^{i\epsilon/2}\frac{1}{1-z}$ with $m=2$, and the z independent coefficients $\hat{\phi}_{\rm SV}^{c(i)}(\epsilon)$. The peculiarity of this solution is that we can retain the independence of $\hat{\phi}_{\rm SV}^{c(i)}(\epsilon)$ with respect to the variable z at every order in $\hat{a}_s$, thanks to the presence of the factor $((1-z)^m)^{i\epsilon/2}\frac{1}{(1-z)}$ which not only ensures the finiteness of the SV part of CFs Δ_c but also gives the right distributions at every order. The factor m takes the value $m=2$ for DY and Higgs productions as observed in (28) and the origin of it can be traced to the number of external legs that require mass factorization [20]. It was observed in [20,122] that the parameter m takes the value $m=1$ for the SV part of the solutions to CFs of structure functions of DIS and of semi-inclusive annihilation (SIA) of hadron production and the reason is that only one of the external legs requires mass factorization. The uniqueness of the structure of $\hat{\phi}_{\rm SV}^{c(i)}$ may be attributed to the fact that the entire z dependence of the solution factorizes at every order as $((1-z)^m)^{i\epsilon/2}\frac{1}{1-z}$, leaving $\hat{\phi}_{\rm SV}^{c(i)}(\epsilon)$ z independent.

As the SV part, the NSV part of the solution is also determined by demanding that it should contain the right divergences to cancel those present in AP kernels. The structure of the finite part of the solution is determined by (39), which when combined with the SV part of the solution, reproduces the correct NSV terms for Δ_c. The perturbative structure of higher order results allows only certain powers of logarithms at every order in perturbation theory thanks to the inherent transcendentality structure of Feynman integrals that appear at every order in a_s and in ϵ in the dimensionally regularized theory. We find that the coefficients $\varphi_c^{(i)}(z,\epsilon)$ are consistent with this expectation. In addition, the solution demonstrates an interesting structure that deserves a mention.

Recall that the first order differential equation for soft-collinear function $\mathcal{S}_c$ gives the solution $\mathcal{C}\exp(2\Phi^c)$. We applied the boundary condition $\mathcal{S}_c(q^2=0,z)=\delta(1-z)$ as we use dimensional regularization. Although it is an evolution equation with respect to q^2, the solution captures its dependence on both q^2 as well as z at every order in a_s. This is because the differential equation is valid for all z near threshold (SV + NSV). Given the boundary condition, the exponential of the solution is unique and the explicit dependence on q^2 and z is controlled by the kernels $\bar{K}^c$ and $\bar{G}^c$. The latter are extracted from the explicit perturbative results on Δ_C, $\hat{F}_c$, and $\Gamma_{c\bar{c}}$ available in the literature. We found that in the SV part of the exponent Φ_A^c, the complete z dependence can be factored out through $(1-z)^{i\epsilon}$ at every order a_s. However, this is not possible for the NSV part, Φ_B^c. The explicit results obtained through third order in a_s suggest the following all order structure for the Φ_B^c in terms of $\hat{a}_s,\epsilon$, and $\log(1-z)$:

$$\Phi_B^c(\hat{a}_s,\mu^2,q^2,z,\epsilon)=\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2}{\mu^2}\right)^{i\frac{\epsilon}{2}}S_\epsilon^i \times\sum_{j=-i}^{\infty}\sum_{k=0}^{i+j}\hat{\Phi}_k^{c,(i,j)}\epsilon^j\log^k(1-z). \quad (54)$$

Note that respective expansion coefficients $\hat{\Phi}_k^{c,(i,j)}$ can be uniquely determined from the fixed order results. The upper limit in the summation over k is the generalization based on the extrapolation of fixed order results. The justification for this extrapolation is discussed later in this section.

In the following we represent the above solution in two different forms; both give the same expansion coefficients $\hat{\Phi}_k^{c,(i,j)}$ if we expand them in powers of ϵ. The first is the generalization of the form given in (34), and the second one is to demonstrate that these logarithms $\log(1-z)$ in the solution (54) originate from soft and collinear configurations. We find that the $K+G$ equation allows us to construct not just one solution but a form of solutions, a minimal form, satisfying the right divergent structure as well as the dependence on $\ln^k(1-z), k=0,1,\ldots$:

1. *Form I*

We begin with a form parametrized in terms of α:

$$\Phi_{B,\alpha}^c=\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2(1-z)^\alpha}{\mu^2}\right)^{\frac{i\epsilon}{2}}S_\epsilon^i\varphi_{c,\alpha}^{(i)}(z,\epsilon). \quad (55)$$

For any arbitrary choice of α, expansion coefficients can be determined by comparing against the (54) so that the predictions from the solutions $\Phi_{B,\alpha}^c$ are unaffected. The reason for the independence of the choice of α on the prediction is due to the explicit z-dependence of the coefficients $\varphi_{c,\alpha}^{(i)}(z,\epsilon)$ that we allow at every order in $\hat{a}_s$ and in ϵ. Note that in the above expression, if we first insert $1=(1-z)^{-\alpha'}(1-z)^{\alpha'}$ and define

$$(1-z)^{\alpha'}\varphi_{c,\alpha}^{(i)}(z,\epsilon)=\varphi_{c,\alpha-\alpha'}^{(i)}(z,\epsilon), \quad (56)$$

then we obtain $\Phi_{B,\alpha}^c=\Phi_{B,\alpha-\alpha'}^c$ for any α,α'. Hence any variation of α in the factor $(1-z)^{i\alpha\epsilon}$ can always be compensated by suitably adjusting the z independent coefficients of $\ln(1-z)$ terms in $\varphi_{c,\alpha}^{(i)}(z,\epsilon)$ at every order in $\hat{a}_s$ and in ϵ. The reason for this is the invariance of the solution under certain "gaugelike" transformations on both $(1-z)^{i\alpha\epsilon}$ and $\varphi_{c,f,\alpha}(z,\epsilon)$ at every order in $\hat{a}_s$. Note that the logarithmic structure of $\varphi_{c,\alpha}^{(i)}(z,\epsilon)$ plays an important role. Because of this invariance, these transformations affect neither the divergent structure nor the finite parts of $\Phi_{B,\alpha}^c$. We find that the invariance can be realized through the renormalization group equation of the strong coupling constant. To end, the solution given in Eq. (55) takes the following integral form:

$$\Phi^c_{B,\alpha} = \int_{\mu_F^2}^{q^2(1-z)^\alpha} \frac{d\lambda^2}{\lambda^2} L^c(a_s(\lambda^2), z) + \varphi_{f,c,\alpha}(a_s(q^2(1-z)^\alpha), z, \epsilon)|_{\epsilon=0} + \varphi_{s,c}(a_s(\mu_F^2), z, \epsilon). \tag{57}$$

The finite part $\varphi_{f,c,\alpha}$ can be expanded as

$$\varphi_{f,c,\alpha}(a_s(q_z^2(1-z)^\alpha), z) = \sum_{i=1}^\infty a_s^i(q_{z_\alpha}) \sum_{k=0}^i \varphi^{(k)}_{c,\alpha,i} L_z^k, \tag{58}$$

with $L_z^k = \ln^k(1-z), q_{z_\alpha} = q^2(1-z)^\alpha$. The fact that the predictions are insensitive to α relates the coefficient $\varphi^{(k)}_{c,\alpha,i}$ to $\varphi^{(k)}_{c,i}$, the solution corresponding to $\alpha = 2$, through

$$\begin{aligned}
&\varphi^{(0)}_{c,\alpha,1} = \varphi^{(0)}_{c,1}, \qquad \varphi^{(1)}_{c,\alpha,1} = -D_1^c\bar{\alpha} + \varphi^{(1)}_{c,1}, \qquad \varphi^{(0)}_{c,\alpha,2} = \varphi^{(0)}_{c,2},\\
&\varphi^{(1)}_{c,\alpha,2} = -\bar{\alpha}(D_2^c - \beta_0\varphi^{(0)}_{c,1}) + \varphi^{(1)}_{c,2}, \qquad \varphi^{(2)}_{c,\alpha,2} = -\frac{1}{2}\bar{\alpha}^2\beta_0 D_1^c - \bar{\alpha}(C_2^c - \beta_0\varphi^{(1)}_{c,1}) + \varphi^{(2)}_{c,2},\\
&\varphi^{(0)}_{c,\alpha,3} = \varphi^{(0)}_{c,3}, \qquad \varphi^{(1)}_{c,\alpha,3} = -\bar{\alpha}(D_3^c - \beta_1\varphi^{(0)}_{c,1} - 2\beta_0\varphi^{(0)}_{c,2}) + \varphi^{(1)}_{c,3},\\
&\varphi^{(2)}_{c,\alpha,3} = -\bar{\alpha}^2\left(\frac{1}{2}\beta_1 D_1^c + \beta_0 D_2^c - {\beta_0}^2\varphi^{(0)}_{c,1}\right) - \bar{\alpha}(C_3^c\bar{\alpha} - \beta_1\varphi^{(1)}_{c,1} - 2\beta_0\varphi^{(1)}_{c,2}) + \varphi^{(2)}_{c,3},\\
&\varphi^{(3)}_{c,\alpha,3} = {\beta_0}^2\left(-\frac{1}{3}D_1^c\bar{\alpha}^3 + \bar{\alpha}^2\varphi^{(1)}_{c,1}\right) + \beta_0\bar{\alpha}(-C_2^c\bar{\alpha} + 2\varphi^{(2)}_{c,2}) + \varphi^{(3)}_{c,3},\\
&\varphi^{(0)}_{c,\alpha,4} = \varphi^{(0)}_{c,4}, \qquad \varphi^{(1)}_{c,\alpha,4} = -D_4^c\bar{\alpha} + \beta_2\bar{\alpha}\varphi^{(0)}_{c,1} + 2\beta_1\bar{\alpha}\varphi^{(0)}_{c,2} + 3\beta_0\bar{\alpha}\varphi^{(0)}_{c,3} + \varphi^{(1)}_{c,4},\\
&\varphi^{(2)}_{c,\alpha,4} = -C_4^c\bar{\alpha} - \frac{1}{2}\beta_2 D_1^c\bar{\alpha}^2 - \beta_1 D_2^c\bar{\alpha}^2 - \frac{3}{2}\beta_0 D_3^c\bar{\alpha}^2 + \frac{5}{2}\beta_0\beta_1\bar{\alpha}^2\varphi^{(0)}_{c,1}\\
&\qquad + \beta_2\bar{\alpha}\varphi^{(1)}_{c,1} + 3{\beta_0}^2\bar{\alpha}^2\varphi^{(0)}_{c,2} + 2\beta_1\bar{\alpha}\varphi^{(1)}_{c,2} + 3\beta_0\bar{\alpha}\varphi^{(1)}_{c,3} + \varphi^{(2)}_{c,4},\\
&\varphi^{(3)}_{c,\alpha,4} = {\beta_0}^3\bar{\alpha}^3\varphi^{(0)}_{c,1} + {\beta_0}^2\bar{\alpha}^2(-D_2^c\bar{\alpha} + 3\varphi^{(1)}_{c,2}) - \frac{1}{6}\beta_1\bar{\alpha}(6C_2^c\bar{\alpha} + 5\beta_0\bar{\alpha}(D_1^c\bar{\alpha} - 3\varphi^{(1)}_{c,1})\\
&\qquad - 12\varphi^{(2)}_{c,2}) - \frac{3}{2}\beta_0\bar{\alpha}(C_3^c\bar{\alpha} - 2\varphi^{(2)}_{c,3}) + \varphi^{(3)}_{c,4},\\
&\varphi^{(4)}_{c,\alpha,4} = {\beta_0}^3\left(-\frac{1}{4}D_1^c\bar{\alpha}^4 + \bar{\alpha}^3\varphi^{(1)}_{c,1}\right) + {\beta_0}^2\bar{\alpha}^2(-C_2^c\bar{\alpha} + 3\varphi^{(2)}_{c,2}) + 3\beta_0\bar{\alpha}\varphi^{(3)}_{c,3} + \varphi^{(4)}_{c,4},
\end{aligned} \tag{59}$$

where $\bar{\alpha} = \alpha - 2$. The above relations are the transformations for $\varphi^{(k)}_{c,\alpha,i}$ that are required to compensate the contributions resulting from the change in the exponent of $(1-z)$ from $i\epsilon$ to $i\alpha\epsilon$. This invariance property with respect to the parameter α makes the solution a peculiar one compared to the SV counterpart.

2. Form II

We point out that the form of solutions parametrized by α is not the only one that satisfies the $K + G$ equation. For example, if we do not restrict z-dependence in $\varphi_c^{(i)}$, we can obtain a different form of solution. Then for such a solution, we need to add more terms on the right-hand side of (55) in such a way that all the requirements are fulfilled. In other words, if we assume the following form for the solution:

$$\tilde{\Phi}^c_B = \sum_{i=1}^\infty \hat{a}_s^i \sum_{\alpha=2}^{2i} \left(\frac{q^2(1-z)^{\frac{\alpha}{i}}}{\mu^2}\right)^{\frac{i\epsilon}{2}} S_\epsilon^i \tilde{\varphi}^{(i)}_{c,\alpha}(\epsilon) \tag{60}$$

with various $\tilde{\varphi}^{(i)}_{c,\alpha}(\epsilon)$'s to contain right divergent as well as finite terms which when we sum them up over α's, we can obtain Δ_c that agrees with the known result. In the following we explain this using an example that can provide the justification for the proposed solution. We use [123] for this purpose. In [123] inclusive production of the Higgs boson was computed using the method of threshold expansion up to third order in a_s in dimensional regularization. For the diagonal channel, $\hat{\sigma}_{gg}$, the results to third order show remarkable structure in terms of z and ϵ, namely the factorization of terms of the form $(1-z)^\epsilon$ and functions that depend only on ϵ. Generalizing this structure to ith order in a_s, one obtains the factorization of the form

$\sum_{\alpha=2}^{2i}(1-z)^{\alpha\epsilon/2}\chi_i^\alpha(\epsilon)$. The factor $(1-z)^{\alpha\epsilon/2}$ originates from soft and collinear configurations of partons. The corresponding soft and collinear scales are given by $(q^2(1-z))^{\alpha\epsilon/2}$, and hence one can conclude that the threshold expansion beyond SV approximation contains multiple scales parametrized by α. From the explicit computations one finds that every collinear parton gives $(1-z)^{\epsilon/2}$ and the soft parton gives $(1-z)^{\epsilon}$.[1] Pure virtual contributions to born amplitude give $\delta(1-z)$, and the hard part from the real emissions gives terms proportional to $(1-z)^\eta, \eta \geq 0$. For a given process, we can determine the values of α by studying the number of soft and collinear configurations. This way we can find out the allowed values of α for every process at every order in a_s. The highest power of α at a given order is determined by the number of allowed soft and collinear configurations in that order. The values of α extracted from results known to third order can be used to extrapolate the upper limit on α at ith order in a_s and it turns out to be $2i$. The coefficients of the scales $\chi_i^\alpha(\epsilon)$ can be expanded in powers of ϵ. The singularity structure in ϵ is completely determined by the finiteness of the mass factorized result. Note that the remarkable multiscale structure of the fixed order results [123] for the cross sections confirms the structure of $\tilde{\Phi}_B^c$ given above.

The fact that the exponent Φ_B^c (60) is identical to the exponent in (34) if we expand them around $\epsilon = 0$ implies

$$(1-z)^{i\epsilon}\hat{\varphi}_c^i(z,\epsilon) = \sum_{\alpha=2}^{2i}(1-z)^{\frac{\alpha\epsilon}{2}}\tilde{\varphi}_{c,\alpha}^i(\epsilon). \tag{61}$$

In the following we explain how the parameter α counts the soft and collinear modes. Let us begin with one loop ($i=1$) where we have $\alpha = 2$ and the corresponding soft scale is $(q^2(1-z)^2)^{\frac{\epsilon}{2}}$. At two loops ($i=2$), we have $\alpha = 2$, 3, 4 and the corresponding scales are $(q^4(1-z)^2)^{\frac{\epsilon}{2}}$, $(q^4(1-z)^3)^{\frac{\epsilon}{2}}$, and $(q^4(1-z)^4)^{\frac{\epsilon}{2}}$, respectively. Note that the first scale results from two collinear modes each with the scale $q^2(1-z)$, and the second one arises from the combination of soft and collinear modes each with the scales $q^2(1-z)$ and $q^2(1-z)^2$, respectively. The last one is from a combination of two soft modes with the scale $q^2(1-z)^2$ each. The explicit results on Φ_B^c up to third order suggest that the expansion coefficients vanish for $\alpha > i+j$ for all i, j.

While these two forms of solutions may look different in the structure, both of them give identical predictions to all orders for CFs, and in addition, it is easy to relate the coefficients of these solutions by finite transformations. Hence, they are equivalent. In the present paper, we use the form-I solution with the choice $\alpha = 2$ in (55) so that the solution resembles the SV part. Thanks to the invariance property of the solution, the different choices for α neither alter the qualitative behavior nor the quantitative predictions for Δ_c to all orders. For example, an alternate choice, say $\alpha = 1$, can only change the coefficients of $\ln^k(1-z)$ in the $\varphi_{f,c}$ without affecting the all order structure and the predictions for Δ_c. With our choice of $\alpha = 2$, the all order solution, equivalently integral representation resembles that of the SV part. We will see later that this choice will allow us to study N space resummation for both SV and NSV terms with single order one term, namely $\omega = 2a_s\beta_0 \ln N$.

[1] We thank Claude Duhr for explaining this point to us.

B. On the logarithmic structure

In the last section, we derived the z space result that can correctly predict certain SV and NSV terms to all orders from the knowledge of previous orders. This was possible due to a peculiar logarithm structure of the solution to the $K+G$ equation at every order in $\hat{a}_s$ and ϵ; see (38). In this subsection, we present an explicit result for Φ^c, $c = b$ to second order in perturbation theory in order to explain the structure of SV and NSV logarithms at a given order in $\hat{a}_s$ with an accuracy of ϵ^n. We have used the inclusive cross section for the production of the Higgs boson in bottom quark annihilation for this purpose. The conclusions remain unchanged as long as color neutral production in diagonal channels are considered. To order $\hat{a}_s^2$, the inclusive cross section for the production of the Higgs boson in bottom quark annihilation receives contributions from (a) pure real emissions

$$b+\bar{b} \to H+g, \qquad b+\bar{b} \to H+g+g,$$
$$b+\bar{b} \to H+b+\bar{b}, \qquad b+\bar{b} \to H+q+\bar{q},$$

(b) pure virtual corrections through one and two loop corrections to leading order $b+\bar{b} \to H$, and (c) interference of pure real emission process $b+\bar{b} \to H+g$ with the loop corrected process $b+\bar{b} \to H+g$. Here, q refers light quarks leaving t- and b-quarks. We compute these parton level subprocesses using the standard Feynman diagram approach. Beyond the leading order in strong coupling, all these subprocesses develop UV and IR divergences, and they are regulated in dimensional regularization. As we encounter a large number of Feynman diagrams, we use QGRAF to generate them and an in-house FORM routine to perform all the symbolic manipulations, e.g., for Dirac, $SU(N_c)$ color, and Lorentz algebra. We use the integration-by-parts (IBP) identities through a *Mathematica* based package, LiteRed, to reduce Feynman integrals to a minimum set of master integrals. In addition, for real emission and real-virtual processes the method of reverse unitarity is used along with IBP identities to reduce the resulting phase-space integrals to a set of a few master integrals. The master integrals for the virtual processes can be found in [88,124] and for the real emission in [124] up to the desired accuracy in ϵ. While individual subprocesses contain UV, soft, and collinear divergences, after renormalizing the strong coupling constant $\hat{a}_s$ and the Yukawa coupling λ, the sum becomes UV finite. In addition, the soft and final state collinear divergences cancel in real and virtual subprocesses, leaving only initial state collinear divergences in $\hat{\sigma}_{b\bar{b}}$.

Since we are interested only in those terms that are proportional to distributions and NSV logarithms $\ln^k(1-z)$, we expand [2] $\hat{\sigma}_{b\bar{b}}$ around $z=1$ and drop those terms that vanish when $z\to 1$. In order to extract Φ^c from the latter, we follow (13), where the virtual contributions are factored out from $\hat{\sigma}_{c\bar{c}}$, giving rise to the function $\mathcal{S}_c$. Owing to (22), $\mathcal{S}_b$ has an exponential structure

$$\mathcal{S}_b(z,q^2,\epsilon) = \mathcal{C}\exp\left(2\Phi^b(z,q^2,\epsilon)\right), \tag{62}$$

where $\Phi^b = \Phi^b_A + \Phi^b_B$. Expanding Φ^b_B in powers of $\hat{a}_s$ as

$$\begin{aligned}\Phi^b_B(\hat{a}_s,\mu^2,q^2,z,\epsilon) &= \sum_{i=1}^\infty \hat{a}_s^i\left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S^i_\epsilon \varphi_b^{(i)}(z,\epsilon)\\ &= \sum_{i=1}^\infty \hat{a}_s^i\left(\frac{q^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S^i_\epsilon \hat{\Phi}^{(i)}_{\mathrm{NSV},b}(z,\epsilon),\end{aligned} \tag{63}$$

and using explicit results for $\hat{\sigma}^{\mathrm{sv+nsv}}_{b\bar{b}}$, $Z_{UV,b}$, and $\hat{F}$, we obtain $\hat{\Phi}^{(i)}_{\mathrm{NSV},b}$ for $i=1,2$ in powers of ϵ. They are given by

$$\begin{aligned}
\hat{\Phi}^{(1)}_{\mathrm{NSV},b} &= C_F\Bigg\{\frac{1}{\epsilon}(-8) + (-8L_z+4) + \epsilon(-4L_z^2 + +4L_z + 3\zeta_2) + \epsilon^2\left(-\frac{4}{3}L_z^3 + 2L_z^2 + 3\zeta_2 L_z - \left(\frac{7}{3}\zeta_3 + \frac{3}{2}\zeta_2\right)\right)\\
&\quad + \epsilon^3\left(-\frac{1}{3}L_z^4 + \frac{2}{3}L_z^3 + \frac{3}{2}\zeta_2 L_z^2 - \left(\frac{7}{3}\zeta_3 + \frac{3}{2}\zeta_2\right)L_z + \left(\frac{7}{6}\zeta_3 + \frac{3}{16}\zeta_2^2\right)\right)\Bigg\},\\
\hat{\Phi}^{(2)}_{\mathrm{NSV},b} &= C_F C_A\Bigg\{\frac{1}{\epsilon^2}\left(\frac{88}{3}\right) + \frac{1}{\epsilon}\left(\frac{176}{3}L_z + 8\zeta_2 - \frac{664}{9}\right) + \left(\frac{176}{3}L_z^2 + \left(16\zeta_2 - \frac{1238}{9}\right)L_z + \frac{1402}{27} - 28\zeta_3 - \frac{178}{3}\zeta_2\right)\\
&\quad + \epsilon\left(\frac{352}{9}L_z^3 + \left(16\zeta_2 - \frac{2341}{18}\right)L_z^2 + \left(\frac{2750}{27} - 56\zeta_3 - \frac{356}{3}\zeta_2\right)L_z + \frac{934}{9}\zeta_3 - \frac{4021}{81} + \frac{982}{9}\zeta_2 - 4\zeta_2^2\right)\Bigg\}\\
&\quad + C_F^2\Bigg\{\frac{1}{\epsilon}(16L_z + 12) + (28L_z^2 + 14L_z - 32\zeta_2) + \epsilon\left(\frac{74}{3}L_z^3 + \frac{13}{2}L_z^2 + (6-76\zeta_2)L_z - 8 + 48\zeta_3 - \zeta_2\right)\Bigg\}\\
&\quad + C_F n_f\Bigg\{\frac{1}{\epsilon^2}\left(\frac{-16}{3}\right) + \frac{1}{\epsilon}\left(\frac{-32}{3}L_z + \frac{112}{9}\right) + \left(\frac{-32}{3}L_z^2 + \left(\frac{224}{9}\right)L_z + \frac{28}{3}\zeta_2 - \frac{328}{27}\right)\\
&\quad + \epsilon\left(\frac{-64}{9}L_z^3 + \frac{224}{9}L_z^2 + \left(\frac{56}{3}\zeta_2 - \frac{656}{27}\right)L_z + \frac{1030}{81} - \frac{124}{9}\zeta_3 - \frac{196}{9}\zeta_2\right)\Bigg\}.
\end{aligned} \tag{64}$$

As can be seen from the above results, at order $\hat{a}_s$, the leading pole in ϵ is of order one, it is two at $\hat{a}_s^2$, and the increment of one unit for the leading poles is expected to continue with the order of perturbation. However, the pole structure for $\hat{\sigma}_{b\bar{b}}$ shows an increment of two units. In addition, at every order in $\hat{a}_s$, for a given color factor, the combination of ϵ and the leading logarithm shows uniform transcendentality weight. In other words, if we assign n_ϵ weight for ϵ^{-n_ϵ} and n_L for $\ln^{n_L}(1-z)$, then the highest weight at every order in ϵ shows uniform transcendentality $w = n_\epsilon + n_L$. For instance, at one loop, we find $w=1$ at every order of ϵ and at two loops it is two ($w=2$). This clearly explains that the highest power of $\ln(1-z)$ at every order in ϵ is constrained by the order of $\hat{a}_s$ and the accuracy in ϵ and is found to be $i+j$ for the term $\hat{a}_s^i\epsilon^j$. This translates to $i+j-1$ for $\mathcal{G}^{(j)}_{L,i}$ in (38) as the latter is the coefficient of ϵ^{j-1}. This exercise provides an explanation for the logarithmic structure given in (38), in particular the upper limit of the summation. This logarithmic structure determines the structure of $\varphi_{f,c}$ given in (43). In Appendix C, we present $\mathcal{G}^{c,(j,k)}_{L,i}$ up to second order in $\hat{a}_s$ with $i=1,2$. We add that the inclusive results for Higgs production in gluon fusion as well as production of the pair of leptons in quark-antiquark annihilation also show exactly same logarithmic structure. Beyond second order, explicit results for $\hat{\sigma}_{ab}$ are not available in the literature. However, results for Δ_c, $\hat{F}_c$, and $\Gamma_{c\bar{c}}$ to third order have become available in recent times, and they can be used to determine $\hat{\Phi}^{(3)}_{\mathrm{NSV},c}$ for $c=q,b,g$ up to the accuracy ϵ^0. We find that the logarithmic structure at a given accuracy in ε is consistent with our expectation based on uniform transcendentality.

Precisely because of the peculiar logarithmic structure of the exponents, namely an increment by one unit, we get logarithms in CFs with an increment of two units. It is easy to understand this structure if we observe that when we expand the exponents containing $\mathcal{D}_k$ and $\ln^k(1-z)$ to obtain CFs, the resulting convolutions between various

[2]We thank Claude Duhr for helping us with the expansion of Harmonic Polylogs [125].

orders in a_s will be of the form $\mathcal{D}_k \otimes \mathcal{D}_l$ and/or $\mathcal{D}_k \otimes \ln^l(1-z)$ which will result in leading distributions $\mathcal{D}_{k+l+1}$ and leading NSV logarithms $\ln^{k+l+1}(1-z)$.

IV. ALL ORDER PREDICTIONS FOR Δ_c

In this section, we discuss the predictive power of the master formula (14). In other words, given $Z_{UV,c}, \hat{F}_c, \Phi^c$, and the Γ_{cc} up to a certain order in perturbation theory, we show that the master formula can predict certain SV and NSV terms to all orders in perturbation theory. The reason for the predictions of certain SV and NSV logarithms in CFs to all orders from the knowledge of data available from the first few orders is that near threshold, the building blocks (form factor $\hat{F}_c$, renormalization constant Z_c, soft collinear function $\mathcal{S}_c$, and AP kernel Γ_{cc}) that constitute CFs satisfy respective first order (homogeneous) differential equations whose solutions turn out to be exponential in form with the unit boundary conditions in dimensional regularization. Because of this exponential form for the CFs, the knowledge of (or the data on) the exponent at lower orders in a_s will be sufficient to predict certain SV and NSV logarithms of CFs to all order upon expanding the exponential. The partonic coefficient function Δ_c can be expanded order by order in perturbation theory in powers of $a_s(\mu_R^2)$ as

$$\Delta_c(q^2,\mu_R^2,\mu_F^2,z) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\Delta_c^{(i)}(q^2,\mu_R^2,\mu_F^2,z), \quad (65)$$

where the coefficient $\Delta_c^{(i)}$ can be obtained by first expanding the exponential given in (46) in powers of $a_s(\mu_R^2)$ and then performing all the resulting convolutions in z space. Note that $\Delta_c^{(0)} = \delta(1-z)$. We have dropped all those terms that are of order $\mathcal{O}((1-z)^\alpha), \alpha > 0$. Finally, we write the following decomposition:

$$\Delta_c^{(i)}(q^2,\mu_R^2,\mu_F^2,z) = \Delta_c^{\mathrm{SV},(i)}(q^2,\mu_R^2,\mu_F^2,z) + \Delta_c^{\mathrm{NSV},(i)}(q^2,\mu_R^2,\mu_F^2,z). \quad (66)$$

Here $\Delta_c^{\mathrm{SV},(i)}$ contains only SV terms, such as the distributions $\mathcal{D}_i (i=0,1,\ldots)$ and $\delta(1-z)$ and next to SV terms; i.e., the logarithms $\ln^i(1-z) (i=0,1,\ldots)$ are embedded within $\Delta_c^{\mathrm{NSV},(i)}$. Now given the distribution function Φ^c, up to a certain order in a_s, there are several SV and NSV logarithms that can be predicted to all orders in a_s. For example, we observe that if Ψ^c is known at leading order in a_s, we can predict all the leading distributions $\mathcal{D}_i$ and leading NSV terms $\ln^i(1-z)$ to all orders in a_s. In the following, we elaborate on this by comparing our predictions with the available N^{3}LO results and also predict N^{4}LO and some higher order results for a few observables.

Given Ψ^c at order a_s, by expanding the master formula (14) in powers of a strong coupling constant, we obtain the leading SV terms $(\mathcal{D}_3,\mathcal{D}_2)$, $(\mathcal{D}_5,\mathcal{D}_4),\ldots,(\mathcal{D}_{2i-1},\mathcal{D}_{2i-2})$ and the leading NSV terms $\ln^3(1-z), \ln^5(1-z),\ldots,\ln^{2i-1}(1-z)$ at $a_s^2, a_s^3,\ldots,a_s^i$, respectively, for all i. Since C_1^c is identically zero, $\ln^{2i}(1-z)$ terms do not contribute for all i. Hence, we predict

$$\begin{aligned}\Delta_c^{\mathrm{NSV}} &= a_s\Delta_c^{\mathrm{NSV}(1)} + a_s^2[-128C_i^2L_z^3 + \mathcal{O}(L_z^2)] \\ &\quad + a_s^3[-512C_i^3L_z^5 + \mathcal{O}(L_z^4)] \\ &\quad + a_s^4\left[-\frac{4096}{3}C_i^4L_z^7 + \mathcal{O}(L_z^6)\right] + \mathcal{O}(a_s^5). \quad (67)\end{aligned}$$

Here we write $\ln^i(1-z) \equiv L_z^i$ for brevity. Also $C_i = C_F$ for $c = \{q,b\}$, i.e., for DY and Higgs production through bottom quark annihilation. And for Higgs production through gluon fusion, i.e., $c = g$, we have $C_i = C_A$. Thus with the knowledge of one loop anomalous dimensions $\{C_1^c, D_1^c, A_1^c, B_1^c, f_1^c\}$ and one-loop $\varphi_{c,1}^{(k)}$, we predicted the above NSV logarithms and the known NNLO and N^{3}LO results [4–6] for DY and Higgs boson productions confirm this. Note that these predictions will be unaffected if we include the second order result for Ψ^c simply because the leading logarithm at ε^j accuracy is $2+j$, and hence at ε^0 order the highest logarithm is $\log^2(1-z)$ which will only contribute to the subleading contribution at a_s^2. Similarly the prediction at third order will be unaffected by the third order result for Ψ^c and so on.

Similarly from Ψ^c to order a_s^2, we can predict the tower consisting of $(\mathcal{D}_3,\mathcal{D}_2)$, $(\mathcal{D}_5,\mathcal{D}_4)$, …, $(\mathcal{D}_{2i-3},\mathcal{D}_{2i-4})$ and of $L_z^4, L_z^6,\ldots,L_z^{2i-2}$ at $a_s^3, a_s^4,\ldots,a_s^i$, respectively, for all i. For the DY and Higgs production in bottom quark annihilation, our prediction reads as

$$\begin{aligned}\Delta_{q(b)}^{\mathrm{NSV}} &= a_s\Delta_{q(b)}^{\mathrm{NSV}(1)} + a_s^2\Delta_{q(b)}^{\mathrm{NSV}(2)} + a_s^3\left[-512C_F^3L_z^5 + \left(\frac{7040}{9}C_F^2C_A - \frac{1280}{9}n_fC_F^2 + 1728C_F^3\right)L_z^4 + \mathcal{O}(L_z^3)\right] \\ &\quad + a_s^4\left[-\frac{4096}{3}C_F^4L_z^7 + \left(\frac{39424}{9}C_F^3C_A + \frac{19712}{3}C_F^4 - \frac{7168}{9}n_fC_F^3\right)L_z^6 + \mathcal{O}(L_z^5)\right] + \mathcal{O}(a_s^5) \quad (68)\end{aligned}$$

and for the Higgs production in gluon fusion

$$\Delta_g^{\rm NSV} = a_s\Delta_g^{{\rm NSV}(1)} + a_s^2\Delta_g^{{\rm NSV}(2)} + a_s^3\left[-512C_A^3L_z^5 + \left(\frac{22592}{9}C_A^3 - \frac{1280}{9}n_fC_A^2\right)L_z^4 + \mathcal{O}(L_z^3)\right] + a_s^4\left[-\frac{4096}{3}C_A^4L_z^7 + \left(\frac{98560}{9}C_A^4 - \frac{7168}{9}n_fC_A^3\right)L_z^6 + \mathcal{O}(L_z^5)\right] + \mathcal{O}(a_s^5). \tag{69}$$

Our predictions for $L_z^i, i = 5,4$ agree with those obtained by explicit computation [6,123]. For the comparison purpose, we have presented the logarithms only up to order a_s^4; however, the master formula can predict such logarithms to all orders in a_s. Note that even though the L_z^4 term is absent at the second order in Ψ^c at the accuracy ε^0, we can predict this term simply because of convolutions between $\mathcal{D}_l$ and L_z^m from first and second order terms in Ψ^c.

Thanks to [6,9,123], the third order results are now available for all these processes allowing us to determine $\varphi_{f,c}$ for $c = q,\ b,\ g$ till third order. Using this, we can predict a tower of $(\mathcal{D}_3,\mathcal{D}_2),(\mathcal{D}_5,\mathcal{D}_4),\ldots,(\mathcal{D}_{2i-5},\mathcal{D}_{2i-6})$ and of $L_z^5,\ldots,L_z^{2i-3}$ at $a_s^4, a_s^5,\ldots,a_s^i$, respectively, for all i. In the following for the illustrative purpose, we have presented the NSV terms L_z till seventh order in a_s. For DY, we find

$$\begin{aligned}\Delta_q^{\rm NSV} ={}& a_s\Delta_q^{{\rm NSV}(1)} + a_s^2\Delta_q^{{\rm NSV}(2)} + a_s^3\Delta_q^{{\rm NSV}(3)} + \boldsymbol{a_s^4}\left[\left\{-\frac{4096}{3}C_F^4\right\}L_z^7 + \left\{\frac{39424}{9}C_F^3C_A + \frac{19712}{3}C_F^4 - \frac{7168}{9}n_fC_F^3\right\}L_z^6\right.\\
&+\left\{-\frac{123904}{27}C_F^2C_A^2 - \left(\frac{805376}{27} - 3072\zeta_2\right)C_F^3C_A + (9088 + 20480\zeta_2)C_F^4 + \frac{45056}{27}n_fC_F^2C_A\right.\\
&\left.\left.+\frac{139520}{27}n_fC_F^3 - \frac{4096}{27}n_f^2C_F^2\right\}L_z^5 + \mathcal{O}(L_z^4)\right] + \boldsymbol{a_s^5}\left[\left\{-\frac{8192}{3}C_F^5\right\}L_z^9\right.\\
&+\left\{\frac{51200}{3}C_F^5 - \frac{8192}{3}C_F^4n_f + \frac{45056}{3}C_F^4C_A\right\}L_z^8 + \left\{\left(\frac{72704}{3} + \frac{229376}{3}\zeta_2\right)C_F^5 - \left(\frac{1120256}{9} - \frac{32768}{3}\zeta_2\right)C_F^4C_A\right.\\
&\left.\left.-\frac{81920}{81}C_F^3n_f^2 + \frac{194560}{9}C_F^4n_f + \frac{901120}{81}C_F^3C_An_f - \frac{2478080}{81}C_F^3C_A^2\right\}L_z^7 + \mathcal{O}(L_z^6)\right]\\
&+\boldsymbol{a_s^6}\left[\left\{-\frac{65536}{15}C_F^6\right\}L_z^{11} + \left\{\frac{167936}{5}C_F^6 - \frac{180224}{27}C_F^5n_f + \frac{991232}{27}C_F^5C_A\right\}L_z^{10} + \left\{\left(\frac{145408}{3} + 196608\zeta_2\right)C_F^6\right.\right.\\
&\left.+\frac{5054464}{81}C_F^5n_f - \frac{327680}{81}C_F^4n_f^2 - \left(\frac{28997632}{81} - \frac{81920}{3}\zeta_2\right)C_F^5C_A + \frac{3604480}{81}C_F^4C_An_f - \frac{9912320}{81}C_F^4C_A^2\right\}L_z^9\\
&\left.+\mathcal{O}(L_z^8)\right] + \boldsymbol{a_s^7}\left[\left\{-\frac{262144}{45}C_F^7\right\}L_z^{13} + \left\{\frac{2392064}{45}C_F^7 - \frac{1703936}{135}C_F^6n_f + \frac{9371648}{135}C_F^6C_A\right\}L_z^{12}\right.\\
&+\left\{\left(\frac{1163264}{15} + \frac{5767168}{15}\zeta_2\right)C_F^7 + \frac{55115776}{405}C_F^6n_f - \left(\frac{315080704}{405} - \frac{262144}{5}\zeta_2\right)C_F^6C_A\right.\\
&\left.\left.-\frac{917504}{81}C_F^5n_f^2 + \frac{10092544}{81}C_F^5C_An_f - \frac{27754496}{81}C_F^5C_A^2\right\}L_z^{11} + \mathcal{O}(L_z^{10})\right] + \mathcal{O}(a_s^8);\end{aligned} \tag{70}$$

for the Higgs production in bottom quark annihilation,

$$\begin{aligned}\Delta_b^{\rm NSV} ={}& a_s\Delta_b^{{\rm NSV}(1)} + a_s^2\Delta_b^{{\rm NSV}(2)} + a_s^3\Delta_b^{{\rm NSV}(3)} + \boldsymbol{a_s^4}[\Delta_q^{{\rm NSV}(4)} - 6144C_F^4L_z^5 + \mathcal{O}(L_z^4)] + \boldsymbol{a_s^5}[\Delta_q^{{\rm NSV}(5)} - 16384C_F^5L_z^7 + \mathcal{O}(L_z^6)]\\
&+\boldsymbol{a_s^6}[\Delta_q^{{\rm NSV}(6)} - 32768C_F^6L_z^9 + \mathcal{O}(L_z^8)] + \boldsymbol{a_s^7}\left[\Delta_q^{{\rm NSV}(7)} - \frac{262144}{5}C_F^7L_z^{11} + \mathcal{O}(L_z^{10})\right] + \mathcal{O}(a_s^8);\end{aligned} \tag{71}$$

and for the Higgs production in gluon fusion,

$$
\begin{aligned}
\Delta_g^{\rm NSV} = {}& a_s\Delta_g^{{\rm NSV}(1)} + a_s^2\Delta_g^{{\rm NSV}(2)} + a_s^3\Delta_g^{{\rm NSV}(3)} + \boldsymbol{a_s^4}\bigg[\bigg\{-\frac{4096}{3}C_A^4\bigg\}L_z^7 + \bigg\{\frac{98560}{9}C_A^4 - \frac{7168}{9}n_fC_A^3\bigg\}L_z^6 \\
&+\bigg\{\bigg(-\frac{298240}{9}+23552\zeta_2\bigg)C_A^4 + \frac{174208}{27}n_fC_A^3 - \frac{4096}{27}n_f^2C_A^2\bigg\}L_z^5 + \mathcal{O}(L_z^4)\bigg] + \boldsymbol{a_s^5}\bigg[\bigg\{-\frac{8192}{3}C_A^5\bigg\}L_z^9 \\
&+\bigg\{\frac{96256}{3}C_A^5 - \frac{8192}{3}C_A^4n_f\bigg\}L_z^8 + \bigg\{\bigg(-\frac{12283904}{81}+\frac{262144}{3}\zeta_2\bigg)C_A^5 + \frac{2569216}{81}C_A^4n_f - \frac{81920}{81}n_f^2C_A^3\bigg\}L_z^7 \\
&+\mathcal{O}(L_z^6)\bigg] + \boldsymbol{a_s^6}\bigg[\bigg\{-\frac{65536}{15}C_A^6\bigg\}L_z^{11} + \bigg\{\frac{9490432}{135}C_A^6 - \frac{180224}{27}C_A^5n_f\bigg\}L_z^{10} + \bigg\{\bigg(\frac{671744}{3}\zeta_2 - \frac{4261888}{9}\bigg)C_A^6 \\
&+\frac{8493056}{81}C_A^5n_f - \frac{327680}{81}n_f^2C_A^4\bigg\}L_z^9 + \mathcal{O}(L_z^8)\bigg] + \boldsymbol{a_s^7}\bigg[\bigg\{-\frac{262144}{45}C_A^7\bigg\}L_z^{13} + \bigg\{\frac{3309568}{27}C_A^7 - \frac{1703936}{135}C_A^6n_f\bigg\}L_z^{12} \\
&+\bigg\{\bigg(-\frac{449429504}{405}+\frac{1310720}{3}\zeta_2\bigg)C_A^7 + \frac{11583488}{45}C_A^6n_f - \frac{917504}{81}n_f^2C_A^5\bigg\}L_z^{11} + \mathcal{O}(L_z^{10})\bigg] + \mathcal{O}(a_s^8). \qquad (72)
\end{aligned}
$$

Our predictions for L_z^7, L_z^6, and L_z^5 terms at fourth order for Δ_c agree with those of [3,23,28,71] predicted using physical evolution equations. As can be seen from (70)–(72), given the third order results, our master formula can predict three highest logarithms for fifth order onwards in a_s. For instance, at a_s^5, we can predict L_z^9, L^8, L_z^7. Again, these predictions are from the lower order results. Generalizing this, if we know Ψ^c up to nth order, we can predict $(\mathcal{D}_{2i-2n+1}, \mathcal{D}_{2i-2n})$ and L_z^{2i-n} at every order in a_s^i for all i. Table I is devoted to summarizing the predictions from the master formula for any given order of a_s. We also present the explicit structure of Δ_c till four loop in Appendix E as well as in the Supplementary Material [126].

The predictive power of the master formula to all orders in a_s in terms of distributions and $\ln(1-z)$ terms in Δ_c is due to the all order structure of the exponent Ψ^c, and this can be further exploited to resum them. We devote a separate section for this. So far, we have compared our higher predictions for SV and NSV logarithms obtained using the lower order results against those available in the literature and found that our all order master formula correctly predicts these logarithms. For example, from the knowledge of the second order result for Ψ^c, we can correctly predict $\ln^5(1-z)$ and $\ln^4(1-z)$ terms at third order. Even though this second order information is not sufficient to predict the lower order NSV logarithms, namely $\ln^k(1-z)$ for $k = 3, 2, 1, 0$ at a_s^3 level, we observe that our predictions for these logarithms agree with the known results for several color factors.

In Table II we compare our predictions for $\ln^3(1-z)$ terms at the third order, which are obtained using Ψ^c considered till a_s^2, against the known results for the DY production, Higgs productions in bottom quark annihilation, and gluon fusion. As can be seen from the table, the master formula correctly predicts the results for many color factors. For instance, for DY, the predictions for color factors $C_F^3, C_Fn_f^2, C_AC_Fn_f$, and $C_A^2C_F$ are matching with the exact results. However, for the other color factors, certain third order information is required, which is represented as χ_i which when taken into account will reproduce the exact $\ln^3(1-z)$ terms at third order.

TABLE I. Towers of distributions ($\mathcal{D}_i$) and NSV logarithms [$\ln^i(1-z)$] that can be predicted for Δ_c using (14). Here $\Psi_c^{(i)}$ and $\Delta_c^{(i)}$ denote Ψ_c and Δ_c at order a_s^i, respectively. Also the symbol L_z^i denotes $\ln^i(1-z)$.

Given				Predictions		
$\Psi_c^{(1)}$	$\Psi_c^{(2)}$	$\Psi_c^{(3)}$	$\Psi_c^{(n)}$	$\Delta_c^{(2)}$	$\Delta_c^{(3)}$	$\Delta_c^{(i)}$
$\mathcal{D}_0$, $\mathcal{D}_1$, δ L_z^1, L_z^0				$\mathcal{D}_3$, $\mathcal{D}_2$ L_z^3	$\mathcal{D}_5$, $\mathcal{D}_4$ L_z^5	$\mathcal{D}_{(2i-1)}$, $\mathcal{D}_{(2i-2)}$ $L_z^{(2i-1)}$
	$\mathcal{D}_0$, $\mathcal{D}_1$, δ L_z^2, L_z^1, L_z^0				$\mathcal{D}_3$, $\mathcal{D}_2$ L_z^4	$\mathcal{D}_{(2i-3)}$, $\mathcal{D}_{(2i-4)}$ $L_z^{(2i-2)}$
		$\mathcal{D}_0$, $\mathcal{D}_1$, δ $L_z^3, \ldots, L_z^0$				$\mathcal{D}_{(2i-5)}$, $\mathcal{D}_{(2i-6)}$ $L_z^{(2i-3)}$
			$\mathcal{D}_0$, $\mathcal{D}_1$, δ $L_z^n, \ldots, L_z^0$			$\mathcal{D}_{(2i-(2n-1))}$, $\mathcal{D}_{(2i-2n)}$ $L_z^{(2i-n)}$

TABLE II. Comparison of $\ln^3(1-z)$ coefficients at the third order against exact results. The left column stands for the exact results, and the right column gives the respective contributions when Ψ^c is taken till two loop.

Color factors	$gg \to H$		Color factors	Drell-Yan (DY)		$b\bar{b} \to H$	
C_A^3	$\frac{-111008}{27} + 3584\zeta_2$	$\frac{-110656}{27} + 3584\zeta_2 + \chi_1$	C_F^3	$2272 + 3072\zeta_2$	$2272 + 3072\zeta_2$	$736 + 3072\zeta_2$	$736 + 3072\zeta_2$
$C_A^2 n_f$	$\frac{6560}{9}$	$\frac{19616}{27} + \chi_2$	$C_F^2 n_f$	$\frac{19456}{27}$	$\frac{6464}{9} + \chi_3$	$\frac{19456}{27}$	$\frac{6464}{9} + \chi_3$
$C_A n_f^2$	$\frac{-256}{27}$	$\frac{-256}{27}$	$C_A C_F^2$	$\frac{-111904}{27} + 512\zeta_2$	$\frac{-37184}{9} + 512\zeta_2 + \chi_4$	$\frac{-111904}{27} + 512\zeta_2$	$\frac{-37184}{9} + 512\zeta_2 + \chi_4$
			$C_F n_f^2$	$\frac{-256}{27}$	$\frac{-256}{27}$	$\frac{-256}{27}$	$\frac{-256}{27}$
			$C_A C_F n_f$	$\frac{2816}{27}$	$\frac{2816}{27}$	$\frac{2816}{27}$	$\frac{2816}{27}$
			$C_A^2 C_F$	$\frac{-7744}{27}$	$\frac{-7744}{27}$	$\frac{-7744}{27}$	$\frac{-7744}{27}$

V. RESUMMATION OF NEXT TO SV IN N SPACE

To study all order behavior of Δ_c in Mellin space, it is convenient to use the integral representations of both Φ_A^c and Φ_B^c given in (31) and (39), respectively. Substituting the solutions for $\hat{F}_c$ and renormalization constant $Z_{UV,c}$ and the $\ln\Gamma_{cc}$ along with the integral representations for Φ_A^c and Φ_B^c in (14), we find

$$\Delta_c(q^2,\mu_R^2,\mu_F^2,z) = C_0^c(q^2,\mu_R^2,\mu_F^2)\mathcal{C}\exp(2\Psi_\mathcal{D}^c(q^2,\mu_F^2,z)), \tag{73}$$

where

$$\Psi_\mathcal{D}^c(q^2,\mu_F^2,z) = \frac{1}{2}\int_{\mu_F^2}^{q^2(1-z)^2}\frac{d\lambda^2}{\lambda^2}P'_{cc}(a_s(\lambda^2),z) + \mathcal{Q}^c(a_s(q^2(1-z)^2),z), \tag{74}$$

with

$$\mathcal{Q}^c(a_s(q^2(1-z)^2),z) = \left(\frac{1}{1-z}\bar{G}_{\rm SV}^c(a_s(q^2(1-z)^2))\right)_+ + \varphi_{f,c}(a_s(q^2(1-z)^2),z). \tag{75}$$

The coefficient C_0^c is the z independent coefficient and is expanded in powers of $a_s(\mu_R^2)$ as

$$C_0^c(q^2,\mu_R^2,\mu_F^2) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)C_{0i}^c(q^2,\mu_R^2,\mu_F^2), \tag{76}$$

where the coefficients C_{0i}^c are presented in the ancillary files along with the arXiv submission. Also one can find C_0^c for DY and Higgs production in [37]. Equation (73) is our z space resummed result for Δ_c in integral representation that can be used to predict SV and NSV terms to all orders in perturbation theory in terms of universal anomalous dimensions, A^c, B^c, C^c, D^c, f^c, SV coefficients $\bar{\mathcal{G}}_i^{c,(j)}$, NSV coefficients $\mathcal{G}_{L,i}^{c,(j,k)}$, and process dependent C_{0i}^c. We have few comments in order. The next to SV corrections to various inclusive processes were studied in a series of papers [64–67,73,74,127], and a lot of progress has been made that leads to better understanding of the underlying physics. Our result has a close resemblance with the one which was conjectured in [64], and indeed there are few terms which are common in both the results. Our result, Eq. (74), differs from Eq. (31) in [64], in the upper limit of the integral, the presence of extra term $\varphi_{f,c}$, and the dependence on the variable z. These differences do not alter the SV predictions but will give NSV terms different from those obtained using Eq. (31) of [64].

The Mellin moment of Δ_c is now straightforward to compute using the integral representation given in (74). Note that Eq. (74) is suitable for obtaining only SV and NSV terms while the predictions using this formula beyond NSV terms such as those proportional to $\mathcal{O}((1-z)^n \ln^j(1-z)); n,j \geq 0$ in z space and terms of $\mathcal{O}(1/N^2)$ in N space will not be correct. Hence, we compute the Mellin moment of (73) in the appropriate limit of N such that the resulting expression in N space correctly predicts all the SV and NSV terms. The limit $z \to 1$ translates to $N \to \infty$, and if one is interested in including NSV terms, we need to keep $\mathcal{O}(1/N)$ corrections in the large N limit. The Mellin moment of Δ_c is given by

$$\Delta_{c,N}(q^2,\mu_R^2,\mu_F^2) = C_0(q^2,\mu_R^2,\mu_F^2)\exp\left(\Psi_N^c(q^2,\mu_F^2)\right), \tag{77}$$

where

$$\Psi_N^c(q^2,\mu_F^2) = 2\int_0^1 dz z^{N-1}\Psi_\mathcal{D}^c(q^2,\mu_F^2,z). \tag{78}$$

The computation of the Mellin moment in the large N limit which retains SV and NSV terms involves two major steps: 1. following [64] we replace $\int dz(z^{N-1}-1)/(1-z)$ and $\int dz z^{N-1}$ by a theta function $\theta(1-z-1/N)$ and apply the operators $\Gamma_A(N\frac{d}{dN})$ and $\Gamma_B\left(N\frac{d}{dN}\right)$ on the integrals, respectively; 2. we perform the integrals over λ^2 after expressing $a_s(\lambda^2)$ in terms of $a_s(\mu_R^2)$ obtained using the resummed solution to the RG equation of a_s in (A5). Step 1 makes sure that we retain only $\ln^j(N)$ and $(1/N)\ln^j(N)$ terms, and step 2 guarantees the resummation of $2\beta_0 a_s(\mu_R^2)\ln N$ terms to all orders and also the organization of the result in powers of $a_s(\mu_R^2)$. The details of the computation are described in Appendix A. The Mellin moment of the exponent takes the following form:

$$\Psi_N^c = \Psi_{\rm sv,N}^c + \Psi_{\rm nsv,N}^c \tag{79}$$

where we have split Ψ_N^c in such a way that all those terms that are functions of $\ln^j(N), j = 0, 1, \ldots$, are kept in $\Psi_{{\rm sv},N}^c$ and the remaining terms that are proportional to $(1/N)\ln^j(N), j = 0, 1, \ldots$, are contained in $\Psi_{{\rm nsv},N}^c$. Hence,

$$\Psi_{{\rm sv},N}^c = \ln(g_0^c(a_s(\mu_R^2))) + g_1^c(\omega)\ln N + \sum_{i=0}^{\infty} a_s^i(\mu_R^2) g_{i+2}^c(\omega), \tag{80}$$

where $g_i^c(\omega)$ are identical to those in [51,53,60] obtained from the resummed formula for SV terms. It is to be noted that $g_i^c(\omega)$ vanishes in the limit $\omega \to 0$. The coefficients $g_0^c(a_s)$ are expanded in powers of a_s as (see [53])

$$\ln(g_0^c(a_s(\mu_R^2))) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2) g_{0,i}^c. \tag{81}$$

We also provide $g_0^c(a_s(\mu_R^2))$ in the ancillary files along with the arXiv submission. The N independent coefficients C_0^c and g_0^c are related to the coefficients $\tilde{g}_0^c$ given in the paper [60,63] using the following relation:

$$\tilde{g}_0^c(q^2, \mu_R^2, \mu_F^2) = C_0^c(q^2, \mu_R^2, \mu_F^2) g_0^c(a_s(\mu_R^2)), \tag{82}$$

which can be expanded in terms of $a_s(\mu_R^2)$ as

$$\tilde{g}_0^c(a_s(\mu_R^2)) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\tilde{g}_{0,i}^c. \tag{83}$$

The function $\Psi_{{\rm nsv},N}^c$ is given by

$$\Psi_{{\rm nsv},N}^c = \frac{1}{N}\sum_{i=0}^{\infty} a_s^i(\mu_R^2)(\bar{g}_{i+1}^c(\omega) + h_i^c(\omega, N)), \tag{84}$$

with

$$h_i^c(\omega, N) = \sum_{k=0}^{i} h_{ik}^c(\omega)\ln^k(N). \tag{85}$$

where $\bar{g}_i^c(\omega)$ and $h_{ik}^c(\omega)$ are presented in Appendices G and F, respectively. We also provide these coefficients till four loop in the Supplementary Material [126]. We can see that in each coefficient, say $g_i^c(\omega), \bar{g}_i^c(\omega), h_{ik}^c(\omega)$ from the SV as well as the NSV, we are resumming in Mellin space the "order one" term ω to all orders in perturbation theory. This is the consequence of the argument in the coupling constant $a_s(q^2(1-z)^2)$ resulting from the integral over λ and the function $\mathcal{Q}^c$. The peculiarity of the series is that the SV $g_1^c(\omega)$ comes with $\ln N$, and hence it starts with a double logarithm. This extra $\ln N$ arises from the Mellin moment of the factor $1/(1-z)_+$ appearing in the exponent. Similarly for $\Psi_{{\rm nsv},N}^c$ we note that it is proportional to $1/N$ at every order as expected. Explicit $\ln N$ that appear with $h_{ik}^c(\omega)$ results from the explicit $\ln(1-x)$ appearing in the exponent. The sum containing $\bar{g}_i^c, i = 1, 2, \ldots$, results entirely from A^c coefficients of P'_{cc} and from the function $\bar{G}_{\rm SV}^c$ of (75). We find that none of the coefficients $\bar{g}_i^c(\omega)$ contain explicit $\ln N$. The second sum comes from C^c, D^c coefficients of P'_{cc} and from $\varphi_{f,c}$ and each term in this expansion contains explicit $\ln^k(N), k = 0, \ldots, i$. We find that the coefficient of h_{01}^c is proportional to C_1^c which is identically zero. Hence, at order a_s^0, there is no $(1/N)\ln N$ term.

Summarizing, we find that in Mellin N space one obtains a compact expression for the exponent in terms of quantities that are functions of $\omega = 2a_s(\mu_R^2)\beta_0 \ln N$ as we use resummed a_s to perform the integral. In addition, the resummed a_s allows us to organize the N space perturbative expansion in such a way that ω is treated as order one at every order in $a_s(\mu_R^2)$. Both integral representation in z space and Mellin moment of the integral in N space contain exactly the same information and hence predict SV and NSV logarithms to all orders in perturbation theory. The all order structure is more transparent in N space compared to the z space result, and it is technically easy to use the resummed result in N space for any phenomenological studies.

Let us first consider $\Psi_{{\rm sv},N}^c$ given in (80). If we keep only $\tilde{g}_{0,0}$ and g_1 terms in (80) and expand the exponent in powers of $a_s = a_s(\mu_R^2)$, we can predict leading $a_s^i\ln^{2i}(N)$ terms for all $i > 1$. This happens because of the all order structure of Φ_A^c in z space. For example, if we know Φ_A^c to order a_s, we can predict the rest of the other terms of the form $a_s^i\mathcal{D}_{2i-1}(z)$ in Φ_A^c for all $i > 1$. If we further include $\tilde{g}_{0,1}$ and g_2 terms, then we can predict next to leading $a_s^i\ln^{2i-1}(N)$ terms for all $i > 2$. Again this is due to the fact that in z space, knowing Φ_A^c to second order one can predict $a_s^i\mathcal{D}_{2i-2}(z)$ terms for all $i > 3$. In general, the resummed result with terms $\tilde{g}_{0,0}^c, \ldots, \tilde{g}_{0,n-1}^c$ and $g_1^n, \ldots, g_n^c$ can predict $a_s^i\ln^{2i-n+1}(N)$ or $a_s^i\mathcal{D}_{2i-n}(z)$ terms for $i > n$.

The inclusion of subleading terms through $\exp(\Psi_{{\rm nsv},N}^c)$ gives additional $(1/N)\ln^j(N)$ terms in N space or $\ln^j(1-z)$ terms in z space. In perturbative QCD, $C_1^c = 0$, where $c = q, g$, and we use this in the rest of our analysis. As the $\Psi_{{\rm sv},N}^c$ exponent, $\Psi_{{\rm nsv},N}^c$ also organizes the perturbation theory by keeping $2a_s(\mu_R^2)\beta_0\ln N$ terms as

TABLE III. The all order predictions for $1/N$ coefficients of $\Delta_{c,N}$ for a given set of resummation coefficients $\{\tilde{g}^c_{0,i}, g^c_i(\omega), \bar{g}^c_i(\omega), h^c_i(\omega)\}$ at a given order. Here $L^i_N = \frac{1}{N}\ln^i(N)$.

Given		Predictions		
Logarithmic Accuracy	Resummed Exponents	$\Delta^{(2)}_{c,N}$	$\Delta^{(3)}_{c,N}$	$\Delta^{(i)}_{c,N}$
NSV-LL	$\tilde{g}^c_{0,0}, g^c_1, \bar{g}^c_1, h^c_0$	L^3_N	L^5_N	L^{2i-1}_N
NSV-NLL	$\tilde{g}^c_{0,1}, g^c_2, \bar{g}^c_2, h^c_1$		L^4_N	L^{2i-2}_N
NSV-N^{2}LL	$\tilde{g}^c_{0,2}, g^c_3, \bar{g}^c_3, h^c_2$			L^{2i-3}_N
NSV-N^{n}LL	$\tilde{g}^c_{0,n}, g^c_{n+1}, \bar{g}^c_{n+1}, h^c_n$			$L^{2i-(n+1)}_N$

order one at every order in a_s. However, these terms are suppressed by the $1/N$ factor at every order in a_s.

We find that if we keep $\{\tilde{g}^c_{0,0}, g^c_1\}$ in $\Psi^c_{\mathrm{sv},N}$ and $\{\bar{g}^c_1, h^c_0\}$ in $\Psi^c_{\mathrm{nsv},N}$ and drop the rest, one can predict $(a^i_s/N)\ln^{2i-1}(N)$ terms for CFs for all $i>1$. We call this tower of logarithms NSV-leading logarithm (NSV-LL). Similarly, knowing, along with the previous ones, $\{\tilde{g}^c_{0,1}, g^c_2\}$ in $\Psi^c_{\mathrm{sv},N}$ and $\{\bar{g}^c_2, h^c_1\}$ in $\Psi^c_{\mathrm{nsv},N}$, one can predict $(a^i_s/N)\ln^{2i-2}(N)$ for CFs for all $i>2$. This belongs to NSV-next-to-leading logarithm (NSV-NLL). In general, the resummed result with $\bar{g}^c_1, \ldots, \bar{g}^c_{n+1}$ and $h^c_0, \ldots, h^c_n$ in $\Psi^c_{\mathrm{nsv},N}$ along with $\tilde{g}^c_{0,0}, \ldots, \tilde{g}^c_{0,n}$ and $g^c_1, \ldots, g^c_{n+1}$ in $\Psi^c_{\mathrm{sv},N}$ can predict $(a^i_s/N)\ln^{2i-(n+1)}(N)$ for all $i>n$ in Mellin space N, and it is NSV-N^{n}LL. We summarize our findings in Table III.

We find that unlike SV resummed terms, which result from only $\mathcal{D}_0$ and $a_s(q^2(1-z)^2)$, the resummation of NSV terms is controlled in addition by $\ln(1-z)$ at each order in a_s as can be seen from (74). This logarithmic dependence in Φ^c_B at each order along with resummed $a_s(q^2(1-z)^2)$ allows one to reorganize order one terms differently from the SV case. Hence, the resulting NSV resummed result has a different logarithmic structure in terms of order one ω compared to that of SV.

A few remarks on the resummed result are in order in light of the previous section. Note that we considered a particular solution Φ^c_B that corresponds to the case $\alpha=2$ and summed up order one terms ω in Mellin N space using the resummed solution to RGE of a_s. While the SV part is insensitive to α, the NSV terms, namely the resummation exponents $h^c(\omega)$, depend on α ($\alpha=2$) through ω resulting from $a_s(q^2/N^\alpha)$ and the coefficients $\varphi^{(k)}_{c,\alpha,i}$. We had already seen how $\varphi^{(k)}_{c,\alpha,i}$ transforms with respect to α. The resummed result in the N space for arbitrary α will be a function of $a_s(q^2/N^\alpha)$. This will lead to the resummation of order one $\omega_\alpha = \alpha\beta_0 a_s(\mu_R^2)\ln N$ to all orders in a_s. Hence, the summation of order one ω_α terms with α dependent coefficients $\varphi_{c,\alpha,i}$ leads to a variety of resummed predictions each depending on the choice of α. However, the fixed order predictions for the CFs Δ_c will be unaffected, thanks to the invariance in the NSV solution. This invariance has allowed us to choose $\alpha=2$ to resum order one ω terms analogous to the SV counterpart.

There have been several attempts [69,70,72–74] in the past to understand the structure of NSV logarithms of inclusive cross sections and its all order structure, and in this context, we compare our prediction at the LL level for CF of DY, Δ^{LL}_q against that of [72]. Note that [72] contains NSV terms only to LL accuracy. In [72], within the framework of soft-collinear effective theory (SCET), the authors have obtained leading logarithmic terms at NSV for the quark-antiquark production channel of the DY process to all orders in a_s. This was achieved by extending the factorization properties of the cross section to the NSV level and using renormalization group equations of NSV operators and soft functions. Using our N space result in the LL approximation, that is for DY

$$\Delta^{\mathrm{LL}}_{c,N} = \tilde{g}^c_{0,0}\exp\left[\ln N g^c_1(\omega) + \frac{1}{N}\left(\bar{g}^c_1(\omega) + h^c_0(\omega, N)\right)\right]\Bigg|_{\mathrm{LL}}, \tag{86}$$

we obtain

$$\Delta^{\mathrm{LL}}_{c,N} = \exp\left[8C_F a_s\left(\ln^2 N + \frac{\ln N}{N}\right)\right], \tag{87}$$

where we have expanded the exponents in powers of a_s and kept only terms of $\mathcal{O}(1/N)$. The above N space result can be Mellin transformed to z space, and it reads as

$$\Delta^{\mathrm{LL}}_{c,N} = \Delta^{\mathrm{LL}}_{c,\mathrm{SV}} - 16C_F a_s \exp[8C_F a_s \ln^2(1-z)]\ln(1-z). \tag{88}$$

The above result agrees exactly with Eq. (4.2) of [72] for $\mu=Q$. Our result given in (79) contains terms that can in principle resum N^{n}LL, $n\geq 0$ provided the universal anomalous dimensions and process dependent coefficients are available to the desired accuracy in a_s. Hence, given three loop results, which are available for several observables, we can perform N^{2}LL resummation taking into account NSV logarithms.

VI. PHYSICAL EVOLUTION KERNEL

In the past, in [128], the scheme invariant approach through the physical evolution equation was explored to understand the structure of NSV terms for the coefficient functions of the DIS cross section. The physical evolution kernel that controls the evolution of the physical observables with respect to external scale q^2 is invariant under scheme transformations with respect to renormalization and factorization. This property can be exploited to understand certain universal structures of perturbative predictions. By suitably modifying the physical evolution kernel (PEK)

[128] with the help of scales in the strong coupling constant and using the renormalization group invariance, predictions at second and third orders for the CFs of DIS structure functions were made, given the known lower order results for CFs. Even though the predictions did not agree for some of the color factors, it was found that they were very close to the known results. Using the second order results for DIS, semi-inclusive e^+e^- annihilation, and DY, a striking observation was made by Moch and Vogt in [71] (and [23,28]) on the PEK, namely the enhancement of single logarithms at large z to all order in $1-z$. It was found that if one conjectures that it will hold true at every order in a_s, the structure of corresponding leading $\ln(1-z)$ terms in the kernel can be constrained. This allowed them to predict certain next to SV logarithms at higher orders in a_s which are in agreement with the known results up to third order.

Motivated by this approach, we use our formulation that describes next to SV logarithms in both z and N spaces to study the structure of the physical evolution equation and present our findings on the structure of leading logarithms in the PEK. For convenience we work in Mellin space. The Mellin moment of hadronic cross section $\sigma(q^2,\tau)$ is given by

$$\sigma_N(q^2) = \int_0^1 d\tau \tau^{N-1}\sigma(q^2,\tau). \tag{89}$$

The hadronic observable $\sigma(q^2,\tau)$ is renormalization scheme (RS) independent, namely it does not depend on the scheme in which CFs Δ_{ab} and the structure functions f_c are defined. The fact that f_c is independent of q^2, the first derivative of σ with respect to q^2 will not depend on f_c. Restricting ourselves to SV and NSV terms, we can define physical evolution kernel $\mathcal{K}^c$ by

$$\begin{aligned}\mathcal{K}^c(a_s(\mu_R^2),N) &= q^2\frac{d}{dq^2}\ln\left(\frac{\sigma_N(q^2)}{\sigma_0(q^2)}\right)\Big|_{\rm sv+nsv}\\ &= q^2\frac{d}{dq^2}\ln\Delta_{c,N}(q^2),\end{aligned} \tag{90}$$

which is independent of any renormalization scheme. The kernel $\mathcal{K}^c(a_s(\mu_R^2),N)$ can be computed order by order in perturbation theory using $\ln\Delta_{c,N}$,

$$\mathcal{K}^c(a_s(\mu_R^2),N) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2)\mathcal{K}_{i-1}^c(N). \tag{91}$$

As in [71], the leading $(1/N)\ln^i(N)$ terms at every order defined by $\mathcal{K}^c$,

$$\bar{\mathcal{K}}_i^c = \mathcal{K}_i^c|_{(1/N)\ln^i(N)}, \tag{92}$$

can be obtained. Using (77), we find that these terms can be obtained directly from $\Psi^c_{\mathrm{nsv},N}$ alone and are given by

$$\begin{aligned}
\bar{\mathcal{K}}_0^c &= A_1^c + 2D_1^c,\\
\bar{\mathcal{K}}_1^c &= 2A_1^c\beta_0 - 2C_2^c + 4\beta_0 D_1^c + 2\beta_0\varphi_{c,1}^{(1)},\\
\bar{\mathcal{K}}_2^c &= 4A_1^c\beta_0^2 - 8\beta_0 C_2^c + 8\beta_0^2 D_1^c + 8\beta_0^2\varphi_{c,1}^{(1)} - 4\beta_0\varphi_{c,2}^{(2)},\\
\bar{\mathcal{K}}_3 &= 8A_1^c\beta_0^3 - 24\beta_0^2 C_2^c + 16\beta_0^3 D_1^c + 24\beta_0^3\varphi_{c,1}^{(1)} - 24\beta_0^2\varphi_{c,2}^{(2)} + 6\beta_0\varphi_{c,3}^{(3)},\\
\bar{\mathcal{K}}_4^c &= 16A_1^c\beta_0^4 - 64\beta_0^3 C_2^c + 32\beta_0^4 D_1^c + 64\beta_0^4\varphi_{c,1}^{(1)} - 96\beta_0^3\varphi_{c,2}^{(2)} + 48\beta_0^2\varphi_{c,3}^{(3)} - 8\beta_0\varphi_{c,4}^{(4)}.
\end{aligned} \tag{93}$$

We find that the structure of $\bar{\mathcal{K}}_i^c$ resembles very much that of [71]. Interestingly, the leading logarithms at every order depend only on the universal anomalous dimensions A_1^c, D_1^c, and C_2^c, and the diagonal coefficients $\varphi_{c,k}^k$ with $k < i$, where i is the order of the perturbation. In addition, if we substitute the known values for these quantities in Eq. (93), we obtain

$$\begin{aligned}
\bar{\mathcal{K}}_1^c &= -8\beta_0 C_i - 32C_i^2,\\
\bar{\mathcal{K}}_2^c &= -16\beta_0^2 C_i - 112\beta_0 C_i^2,\\
\bar{\mathcal{K}}_3^c &= -32\beta_0^3 C_i - \frac{896}{3}\beta_0^2 C_i^2,\\
\bar{\mathcal{K}}_4^c &= -64\beta_0^4 C_i - \frac{2176}{3}\beta_0^3 C_i^2 - 8\beta_0\varphi_{c,4}^{(4)},
\end{aligned} \tag{94}$$

where $C_i = C_F$ for $c = q, b$ and $C_i = C_A$ for $c = g$.

The reason for the agreement of our predictions for PEK to third order with those of [71] is simply because of the $K+G$ equation that Φ^c satisfies. In fact, the $K+G$ equation is a partonic version of the physical evolution equation and the partonic PEK given by $\bar{K}^c + \bar{G}^c$. The logarithm structure of PEK is controlled by the upper limit i in the summation over the index k in (43). In N space, the highest power of corresponding $\ln N$ in the $1/N$ coefficient of $\mathcal{K}^c$ is in turn controlled by the upper limit on the summation in (38). Our predictions based on the inherent transcendentality structure of perturbative results are in complete agreement with the logarithmic structure of CFs or PEKs obtained from explicit results. Note that the structure of PEK (93) expressed in terms of A_1^c, C_2^c, D_1^c, and $\varphi_{c,i}^{(i)}$ is straightforward to understand from $K+G$

equations and renormalization group invariance. However, as was already noted in [71], the coefficient of the leading logarithms contains a peculiar structure containing only β_0^i and β_0^{i-1} at every order in a_s^i. In addition, if the structure continues to be true at every order, the coefficients $\varphi_{c,i}^{(i)}$ have to be proportional to β_0^{i-2} for every i, which can be tested when results beyond third order become available.

VII. CONCLUSIONS

Understanding the structure of threshold logarithms in inclusive reactions such as the production of a pair of leptons in the Drell-Yan process and of the Higgs boson in gluon annihilation as well as bottom quark annihilation is important because they not only dominate but also become large in certain kinematical regions spoiling the reliability of the perturbative predictions. The soft plus virtual contributions that dominate in the threshold region are well understood in terms of certain IR anomalous dimensions and process independent soft distributions. A systematic way of resumming SV logarithms to all orders exists in Mellin N space. While SV contributions dominate, the next to SV contributions are as important as SV for any precision studies and hence cannot be ignored. Next to SV terms also can give large contributions at every order, thereby spoiling the reliability of the perturbation series. The canonical resolution through resummation for the next to SV terms is unfortunately hard to achieve. In this article, we have studied the structure of next to SV logarithms in both z and N spaces for the diagonal partonic channels. Using IR factorization and UV renormalization group invariance, we show that both SV and next to SV contributions satisfy the Sudakov differential equation whose solution provides an all order perturbative result in the strong coupling constant. We show that like SV contributions, next to SV contributions also demonstrate IR structure in terms of certain infrared anomalous dimensions. However, NSV terms depend, in addition, on certain process dependent functions. The underlying universal IR structure of NSV terms can be further unraveled when results for a variety of inclusive reactions become available. In z space, we show that the next to SV contributions do exponentiate, allowing us to predict the corresponding next to SV logarithms to all orders. We find that the NSV part of the solution is invariant under gaugelike transformations, allowing us to construct a form of solutions, all giving identical fixed order predictions for NSV terms of CFs Δ_c. We show that the exponent in the z space has an integral representation which can be used to study these threshold logarithms in Mellin N space. We also show that the NSV logarithms in N space organize themselves exactly as the SV ones in such a way so as to keep $2a_s(\mu_R^2)\beta_0 \ln N$ as an order one term to all orders in $a_s(\mu_R^2)$. Unlike the SV part of the resummed result, the resummation coefficients for NSV terms are found to be controlled not only by process independent anomalous dimensions but also by process dependent $\varphi_{c,i}^{(k)}$.

The all order master formula that we obtain in z space demonstrates a perturbative structure which can predict certain SV and NSV logarithms to all orders in strong coupling constant a_s, given the lower order results. From the available results at a_s and at a_s^2 for the CFs, our predictions for third order NSV logarithms are in complete agreement with the known results available for a variety of inclusive reactions, namely DY production and Higgs productions in bottom quark annihilation and gluon fusion. Using the corresponding CFs that are known to third order, our formalism allows us to predict three leading NSV logarithms to all orders starting from fourth order, of which we reported here the results to order a_s^7. We have studied the logarithmic structure of the physical evolution kernel, in particular the leading logarithms, and found that they are controlled only by process independent anomalous dimensions β_0, A_1^c, C_2^c, D_1^c and diagonal coefficients $\varphi_{c,i}^{(i)}$ at every order a_s^i. We conclude by noting that the structure of NSV logarithms demonstrates a rich perturbative structure that needs to be explored further.

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APPENDIX A: DETAILS OF THE MELLIN MOMENT OF $\Psi^c_{\mathcal{D}}$

In this section, we evaluate the Mellin moment of $\Psi^c_{\mathcal{D}}$ in the following way. At first, following Eq. (78) we decompose Ψ^c_N into $\Sigma^c_{\mathrm{sv},N}$ and $\Sigma^c_{\mathrm{nsv},N}$. So, we begin with

$$\Sigma^c_{\mathrm{sv},N} = \int_0^1 dz \left(\frac{z^{N-1}-1}{1-z}\right)\left(\int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} 2A^c(a_s(\lambda^2)) + 2\bar{G}^c_{\mathrm{SV}}(a_s(q^2(1-z)^2))\right). \tag{A1}$$

We follow the method described in [64] to perform the Mellin moment. In the large N, keeping $\frac{1}{N}$ corrections, we replace

$$\int_0^1 dz(z^{N-1}-1) \to \Gamma_A\left(N\frac{d}{dN}\right)\int_0^1 dz\theta\left(1-z-\frac{1}{N}\right), \tag{A2}$$

where $\Gamma_A(N\frac{d}{dN})$ is given in Appendix D. We expand Γ_A in powers of Nd/dN and apply on the integral. We then make the appropriate change of variables and interchange of integrals to obtain

$$\Sigma^c_{\mathrm{sv},N} = -\int_{q^2/N^2}^{q^2}\frac{d\lambda^2}{\lambda^2}\left\{\left(\ln\frac{q^2}{\lambda^2 N^2}-2\gamma_1^A\right)A^c(a_s(\lambda^2)) + \bar{G}^c_{\mathrm{SV}}(a_s(\lambda^2)) + \lambda^2\frac{d}{d\lambda^2}\mathcal{F}^c_A(a_s(\lambda^2))\right\} \\ + \mathcal{F}^c_A(a_s(q^2)) - 2(\gamma_1^A+\ln N)\int_{\mu_F^2}^{q^2}\frac{d\lambda^2}{\lambda^2}A^c(a_s(\lambda^2)), \tag{A3}$$

where

$$\mathcal{F}^c_A(a_s(\lambda^2)) = -2\gamma_1^A\bar{G}^c_{\mathrm{SV}}(a_s(\lambda^2)) + 4\sum_{i=0}^{\infty}\gamma^A_{i+2}\left(-2\beta(a_s(\lambda^2))\frac{\partial}{\partial a_s(\lambda^2)}\right)^i\left\{A^c(a_s(\lambda^2)) \\ + \beta(a_s(\lambda^2))\frac{\partial}{\partial a_s(\lambda^2)}\bar{G}^c_{\mathrm{SV}}(a_s(\lambda^2))\right\}. \tag{A4}$$

Here $\beta(a_s(\lambda^2))$ is defined as $\beta(a_s(\lambda^2)) = -\sum_{i=0}^{\infty}\beta_i a_s^{i+2}(\lambda^2)$ (also see [129–131] for QCD). Replacing $a_s(\lambda^2)$ by

$$a_s(\lambda^2) = \left(\frac{a_s(\mu_R^2)}{l}\right)\left[1-\frac{a_s(\mu_R^2)}{l}\frac{\beta_1}{\beta_0}\ln l + \left(\frac{a_s(\mu_R^2)}{l}\right)^2\left(\frac{\beta_1^2}{\beta_0^2}(\ln^2 l - \ln l + l - 1) - \frac{\beta_2}{\beta_0}(l-1)\right) \\ + \left(\frac{a_s(\mu_R^2)}{l}\right)^3\left(\frac{\beta_1^3}{\beta_0^3}\left(2(1-l)\ln l + \frac{5}{2}\ln^2 l - \ln^3 l - \frac{1}{2} + l - \frac{1}{2}l^2\right) + \frac{\beta_3}{2\beta_0}(1-l^2) + \frac{\beta_1\beta_2}{\beta_0^2}(2l\ln l - 3\ln l - l(1-l))\right)\right], \tag{A5}$$

where $l = 1-\beta_0 a_s(\mu_R^2)\ln(\mu_R^2/\lambda^2)$ and performing the integrals over λ^2 we obtain the result. The entire result is decomposed into two parts. The ones proportional to $\frac{1}{N}$ are expressed in terms of $\bar{g}^c_i(\omega)$ given in Eq. (84). And the remaining part is embedded in Eq. (80).

Similarly we define

$$\Sigma^c_{\mathrm{nsv},N} = 2\int_0^1 dz z^{N-1}\left\{\int_{\mu_F^2}^{q^2(1-z)^2}\frac{d\lambda^2}{\lambda^2}L^c(a_s(\lambda^2),z) + \varphi_{f,c}(a_s(q^2(1-z)^2),z)\right\}. \tag{A6}$$

Following [64], in the large N and keeping $\frac{1}{N}$ corrections, we replace

$$\int_0^1 dz z^{N-1} \to \Gamma_B\left(N\frac{d}{dN}\right)\int_0^1\frac{dz}{1-z}\theta\left(1-z-\frac{1}{N}\right), \tag{A7}$$

where $\Gamma_B\left(N\frac{d}{dN}\right)$ is given in Appendix D and we replace Nd/dN by

$$N\frac{d}{dN} = N\frac{\partial}{\partial N} - 2\beta(a_s(\lambda^2))\frac{\partial}{\partial a_s(\lambda^2)}, \tag{A8}$$

to deal with N appearing in the argument of $a_s(q^2/N^2)$ and also the explicit ones present in $\varphi_{f,c}$. After a little algebra, we obtain

$$\Sigma^c_{\mathrm{nsv},N} = -\frac{1}{N}\int_{\frac{q^2}{N^2}}^{q^2}\frac{d\lambda^2}{\lambda^2}\left\{\xi^c(a_s(\lambda^2),N) + \lambda^2\frac{d}{d\lambda^2}\mathcal{F}^c_B(a_s(\lambda^2),N)\right\} + \frac{1}{N}\mathcal{F}^c_B(a_s(q^2),N)$$
$$+\frac{1}{N}\int_{\mu_F^2}^{q^2}\frac{d\lambda^2}{\lambda^2}\xi^c(a_s(\lambda^2),N), \tag{A9}$$

where the functions ξ^c are defined as

$$\xi^c(a_s,N) = -2(-\gamma_1^B(D^c(a_s) - C^c(a_s)\ln N) + \gamma_2^B C^c(a_s)) \tag{A10}$$

and

$$\mathcal{F}^c_B(a_s(\lambda^2),N) = 2\gamma_1^B\varphi_{f,c}(a_s(\lambda^2),N) - 4\gamma_2^B\left(\lambda^2\frac{d}{d\lambda^2}\varphi_{f,c}(a_s(\lambda^2),N) + \tilde{\xi}^c(a_s(\lambda^2),N)\right)$$
$$+8(\gamma_3^B + \tilde{\gamma}^B)\left(\lambda^2\frac{d}{d\lambda^2}\left\{\lambda^2\frac{d}{d\lambda^2}\varphi_{f,c}(a_s(\lambda^2),N) + \tilde{\xi}^c(a_s(\lambda^2),N)\right\} + \frac{1}{2}C^c(a_s(\lambda^2))\right), \tag{A11}$$

where

$$\tilde{\xi}^c(a_s,N) = (D^c(a_s) - C^c(a_s)\ln N),$$
$$\varphi_{f,c}(a_s(\lambda^2),N) = \sum_{i=1}^{\infty}\sum_{k=0}^{i} a_s^i(\lambda^2)\varphi_{c,i}^{(k)}(-\ln N)^k, \qquad \tilde{\gamma}^B = \sum_{i=4}^{\infty}\gamma_i^B\left(N\frac{d}{dN}\right)^{i-3}. \tag{A12}$$

Using Eq. (A5), we perform λ^2 integrations to obtain the result in terms of $h^c_{ij}(\omega)$ given in Eq. (84).

APPENDIX B: PERTURBATIVE CONSTANT OF Φ^c_A

In this section, we present the SV coefficients $\bar{K}^{c(i)}(\epsilon)$ to fourth order:

$$\bar{K}^{c(1)}(\epsilon) = \frac{1}{\epsilon}\{2A_1^c\},$$
$$\bar{K}^{c(2)}(\epsilon) = \frac{1}{\epsilon^2}\{-2\beta_0 A_1^c\} + \frac{1}{\epsilon}\{A_2^c\},$$
$$\bar{K}^{c(3)}(\epsilon) = \frac{1}{\epsilon^3}\left\{\frac{8}{3}\beta_0^2 A_1^c\right\} + \frac{1}{\epsilon^2}\left\{-\frac{2}{3}\beta_1 A_1^c - \frac{8}{3}\beta_0 A_2^c\right\} + \frac{1}{\epsilon}\left\{\frac{2}{3}A_3^c\right\},$$
$$\bar{K}^{c(4)}(\epsilon) = \frac{1}{\epsilon^4}\{-4\beta_0^3 A_1^c\} + \frac{1}{\epsilon^3}\left\{\frac{8}{3}\beta_0\beta_1 A_1^c + 6\beta_0^2 A_2^c\right\} + \frac{1}{\epsilon^2}\left\{-\frac{1}{3}\beta_2 A_1^c - \beta_1 A_2^c - 3\beta_0 A_3^c\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}A_4^c\right\}, \tag{B1}$$

where A_i^c are the ith order cusp anomalous dimensions:

$$A^c(a_s(\mu_R^2)) = \sum_i a_s^i(\mu_R^2)A_i^c. \tag{B2}$$

The finite quantity $\bar{G}^{c(i)}_{\mathrm{SV}}(\epsilon)$ is related to its renormalized counterparts $\bar{\mathcal{G}}^c_i(\epsilon)$ in the following way:

$$\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \bar{G}^{c(i)}_{\mathrm{SV}}(\epsilon) = \sum_{i=1}^{\infty} a_s^i(q^2(1-z)^2)\bar{\mathcal{G}}^c_i(\epsilon), \tag{B3}$$

and we find

$$\begin{aligned}
\bar{G}^{c(i)}_{\rm SV}(\epsilon) &= \bar{\mathcal{G}}^c_1(\epsilon),\\
\bar{G}^{c(2)}_{\rm SV}(\epsilon) &= \frac{1}{\epsilon}(-2\beta_0\bar{\mathcal{G}}^c_1(\epsilon)) + \bar{\mathcal{G}}^c_2(\epsilon),\\
\bar{G}^{c(3)}_{\rm SV}(\epsilon) &= \frac{1}{\epsilon^2}(4\beta_0^2\bar{\mathcal{G}}^c_1(\epsilon)) + \frac{1}{\epsilon}(-\beta_1\bar{\mathcal{G}}^c_1(\epsilon) - 4\beta_0\bar{\mathcal{G}}^c_2(\epsilon)) + \bar{\mathcal{G}}^c_3(\epsilon),\\
\bar{G}^{c(4)}_{\rm SV}(\epsilon) &= \frac{1}{\epsilon^3}(-8\beta_0^3\bar{\mathcal{G}}^c_1(\epsilon)) + \frac{1}{\epsilon^2}\left(\frac{16}{3}\beta_0\beta_1\bar{\mathcal{G}}^c_1(\epsilon) + 12\beta_0^2\bar{\mathcal{G}}^c_2(\epsilon)\right)\\
&\quad + \frac{1}{\epsilon}\left(-\frac{2}{3}\beta_2\bar{\mathcal{G}}^c_1(\epsilon) - 2\beta_1\bar{\mathcal{G}}^c_2(\epsilon) - 6\beta_0\bar{\mathcal{G}}^c_3(\epsilon)\right) + \bar{\mathcal{G}}^c_4(\epsilon). \qquad \text{(B4)}
\end{aligned}$$

Through explicit determination of the quantity $\bar{\mathcal{G}}^I_i(\epsilon)$, it was found that it is dependent only on the initial partons and can be further decomposed as

$$\bar{\mathcal{G}}^c_i(\epsilon) = -f^c_i + \bar{\chi}^c_i + \sum_{j=1}^{\infty}\epsilon^j\bar{\mathcal{G}}^{c,(j)}_i, \qquad \text{(B5)}$$

where

$$\begin{aligned}
\bar{\chi}^c_1 &= 0,\\
\bar{\chi}^c_2 &= -2\beta_0\bar{\mathcal{G}}^{c,(1)}_1,\\
\bar{\chi}^c_3 &= -2\beta_1\bar{\mathcal{G}}^{c,(1)}_1 - 2\beta_0(\bar{\mathcal{G}}^{c,(1)}_2 + 2\beta_0\bar{\mathcal{G}}^{c,(2)}_1),\\
\bar{\chi}^c_4 &= -2\beta_2\bar{\mathcal{G}}^{c,(1)}_1 - 2\beta_1(\bar{\mathcal{G}}^{c,(1)}_2 + 4\beta_0\bar{\mathcal{G}}^{c,(2)}_1) - 2\beta_0(\bar{\mathcal{G}}^{c,(1)}_3 + 2\beta_0\bar{\mathcal{G}}^{c,(2)}_2 + 4\beta_0^2\bar{\mathcal{G}}^{c,(3)}_1). \qquad \text{(B6)}
\end{aligned}$$

The SV coefficients $\bar{\mathcal{G}}^{c,k}_i$ in Eq. (B5) are found to exhibit the Casimir scaling principle up to three loop. Hence, these coefficients for the Drell-Yan and Higgs production from gluon and bottom quark annihilation channels can be expressed together in the following way:

$$\begin{aligned}
\bar{\mathcal{G}}^{c,(1)}_1 &= C_R(-3\zeta_2), \qquad \bar{\mathcal{G}}^{c,(2)}_1 = C_R\left(\frac{7}{3}\zeta_3\right),\\
\bar{\mathcal{G}}^{c,(3)}_1 &= C_R\left(-\frac{3}{16}\zeta_2^2\right), \qquad \bar{\mathcal{G}}^{c,(4)}_1 = C_R\left(-\frac{7}{8}\zeta_2\zeta_3 + \frac{31}{20}\zeta_5\right),\\
\bar{\mathcal{G}}^{c,(1)}_2 &= C_RC_A\left(\frac{2428}{81} - \frac{469}{9}\zeta_2 + 4\zeta_2^2 - \frac{176}{3}\zeta_3\right) + C_Rn_f\left(-\frac{328}{81} + \frac{70}{9}\zeta_2 + \frac{32}{3}\zeta_3\right),\\
\bar{\mathcal{G}}^{c,(2)}_2 &= C_Rn_f\left(\frac{976}{243} - \frac{196}{27}\zeta_2 - \frac{1}{20}\zeta_2^2 - \frac{310}{27}\zeta_3\right) + C_RC_A\left(-\frac{7288}{243} + \frac{1414}{27}\zeta_2 + \frac{11}{40}\zeta_2^2 + \frac{2077}{27}\zeta_3 - \frac{203}{3}\zeta_2\zeta_3 + 43\zeta_5\right),\\
\bar{\mathcal{G}}^{c,(1)}_3 &= C_R{C_A}^2\left(\frac{152}{63}{\zeta_2}^3 + \frac{1964}{9}{\zeta_2}^2 + \frac{11000}{9}\zeta_2\zeta_3 - \frac{765127}{486}\zeta_2 + \frac{536}{3}{\zeta_3}^2 - \frac{59648}{27}\zeta_3 - \frac{1430}{3}\zeta_5 + \frac{7135981}{8748}\right)\\
&\quad + C_RC_An_f\left(-\frac{532}{9}{\zeta_2}^2 - \frac{1208}{9}\zeta_2\zeta_3 + \frac{105059}{243}\zeta_2 + \frac{45956}{81}\zeta_3 + \frac{148}{3}\zeta_5 - \frac{716509}{4374}\right) + C_RC_Fn_f\left(\frac{152}{15}{\zeta_2}^2 - 88\zeta_2\zeta_3\right.\\
&\quad \left. + \frac{605}{6}\zeta_2 + \frac{2536}{27}\zeta_3 + \frac{112}{3}\zeta_5 - \frac{42727}{324}\right) + C_R{n_f}^2\left(\frac{32}{9}{\zeta_2}^2 - \frac{1996}{81}\zeta_2 - \frac{2720}{81}\zeta_3 + \frac{11584}{2187}\right). \qquad \text{(B7)}
\end{aligned}$$

Here, $C_R = C_A$ for $c = g$ and $C_R = C_f$ for $c = q, b$, with $C_A \equiv N_c$ and $C_F \equiv \frac{N_c^2-1}{2N_c}$ the Casimirs of adjoint and fundamental representations. Also, $\bar{G}^c_{\rm SV}(a_s(q^2(1-z)^2), \epsilon)$ are related to the threshold exponent $\mathbf{D}^c(a_s(q^2(1-z)^2))$ via Eq. (46) of [20].

APPENDIX C: PERTURBATIVE CONSTANT OF Φ_B^c

In this appendix, we present the relations between the expansion coefficients $\varphi_{c,i}^{(k)}$ appearing in Eq. (43) and the coefficients $\mathcal{G}_{L,i}^{c,(j,k)}$:

$$\begin{aligned}
\varphi_{c,1}^{(k)} &= \mathcal{G}_{L,1}^{c,(1,k)}, \quad k=0,1,\\
\varphi_{c,2}^{(k)} &= \left(\frac{1}{2}\mathcal{G}_{L,2}^{c,(1,k)} + \beta_0 \mathcal{G}_{L,1}^{c,(2,k)}\right), \quad k=0,1,2,\\
\varphi_{c,3}^{(k)} &= \left(\frac{1}{3}\mathcal{G}_{L,3}^{c,(1,k)} + \frac{2}{3}\beta_1 \mathcal{G}_{L,1}^{c,(2,k)} + \frac{2}{3}\beta_0 \mathcal{G}_{L,2}^{c,(2,k)} + \frac{4}{3}\beta_0^2 \mathcal{G}_{L,1}^{c,(3,k)}\right), \quad k=0,1,2,3,\\
\varphi_{c,4}^{(k)} &= \left(\frac{1}{4}\mathcal{G}_{L,4}^{c,(1,k)} + \frac{1}{2}\beta_2 \mathcal{G}_{L,1}^{c,(2,k)} + \frac{1}{2}\beta_1 \mathcal{G}_{L,2}^{c,(2,k)} + \frac{1}{2}\beta_0 \mathcal{G}_{L,3}^{c,(2,k)} + 2\beta_0\beta_1 \mathcal{G}_{L,1}^{c,(3,k)} + \beta_0^2 \mathcal{G}_{L,2}^{c,(3,k)} + 2\beta_0^3 \mathcal{G}_{L,1}^{c,(4,k)}\right), \quad k=0,1,2,3,4,
\end{aligned} \tag{C1}$$

where $\mathcal{G}_{L,1}^{c,(2,3)}$, $\mathcal{G}_{L,1}^{c,(2,4)}$, $\mathcal{G}_{L,2}^{c,(2,4)}$, $\mathcal{G}_{L,1}^{c,(3,4)}$ are all zero. We also present the explicit results for $\mathcal{G}_{L,i}^{c,(j,k)}$ for bottom quark annihilation which is found to be the same as Drell-Yan till second order in $\hat{a}_s$:

$$\begin{aligned}
\mathcal{G}_{L,1}^{b,(1,0)} &= 4C_F, \qquad \mathcal{G}_{L,1}^{b,(2,0)} = 3C_F\zeta_2, \qquad \mathcal{G}_{L,1}^{b,(3,0)} = -C_F\left(\frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3\right),\\
\mathcal{G}_{L,2}^{b,(1,0)} &= C_A C_F\left(\frac{2804}{27} - \frac{290}{3}\zeta_2 - 56\zeta_3\right) + C_F n_f\left(-\frac{656}{27} + \frac{44}{3}\zeta_2\right) - 64C_F^2\zeta_2,\\
\mathcal{G}_{L,2}^{b,(1,1)} &= 20C_F(C_A - C_F), \qquad \mathcal{G}_{L,2}^{b,(1,2)} = -8C_F^2,
\end{aligned} \tag{C2}$$

and for Higgs boson production in gluon fusion:

$$\begin{aligned}
\mathcal{G}_{L,1}^{g,(1,0)} &= 4C_A, \qquad \mathcal{G}_{L,1}^{g,(2,0)} = 3C_A\zeta_2, \qquad \mathcal{G}_{L,1}^{g,(3,0)} = -C_A\left(\frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3\right),\\
\mathcal{G}_{L,2}^{g,(1,0)} &= C_A^2\left(\frac{2612}{27} - \frac{482}{3}\zeta_2 - 56\zeta_3\right) + C_A n_f\left(-\frac{392}{27} + \frac{44}{3}\zeta_2\right),\\
\mathcal{G}_{L,2}^{g,(1,1)} &= \frac{4}{3}C_A(C_A - n_f), \qquad \mathcal{G}_{L,2}^{g,(1,2)} = -8C_A^2,
\end{aligned} \tag{C3}$$

and the remaining coefficients up to second order are identically zero.

APPENDIX D: EXPANSION COEFFICIENTS OF $\Gamma_A(x)$ AND $\Gamma_B(x)$

In this appendix, we present the expansion coefficients of $\Gamma_A(x)$ and $\Gamma_B(x)$ used in Eqs. (A2) and (A7) of Appendix A. As in [64], the operators $\Gamma_A(x)$ and $\Gamma_B(x)$ are expanded in powers of x as

$$\Gamma_A(x) = \sum_{k=0} -\gamma_k^A x^k, \tag{D1}$$

where coefficients γ_k^A are given by [64]

$$\gamma_k^A = \frac{\Gamma_k(N)}{k!}(-1)^{k-1}. \tag{D2}$$

See Eq. (25) of [64] for the definition of $\Gamma_k(N)$. We find

$$
\begin{aligned}
\gamma_0^A &= 1, \qquad \gamma_1^A = \gamma_E - \frac{1}{2N},\\
\gamma_2^A &= \frac{1}{2}(\gamma_E^2+\zeta_2) - \frac{1}{2N}(1+\gamma_E),\\
\gamma_3^A &= \frac{1}{6}\gamma_E^3 + \frac{1}{2}(\gamma_E\zeta_2) + \frac{1}{3}\zeta_3 - \frac{1}{4N}(\gamma_E^2+2\gamma_E+\zeta_2),\\
\gamma_4^A &= \frac{1}{24}\gamma_E^4 + \frac{1}{4}(\gamma_E^2\zeta_2) + \frac{9}{40}\zeta_2^2 + \frac{1}{3}(\gamma_E\zeta_3) - \frac{1}{12N}(\gamma_E^3+3\gamma_E^2+3\zeta_2+3\gamma_E\zeta_2+2\zeta_3),\\
\gamma_5^A &= \frac{1}{120}\gamma_E^5 + \frac{1}{12}(\gamma_E^3\zeta_2) + \frac{1}{40}(9\gamma_E\zeta_2^2) + \frac{1}{6}(\gamma_E^2\zeta_3) + \frac{1}{6}(\zeta_2\zeta_3) + \frac{1}{5}\zeta_5 - \frac{1}{240N}(20\gamma_E^3+5\gamma_E^4+30\gamma_E^2\zeta_2+27\zeta_2^2\\
&\quad + 40\zeta_3 + 20\gamma_E(3\zeta_2+2\zeta_3)),\\
\gamma_6^A &= \frac{1}{720}\gamma_E^6 + \frac{1}{48}(\gamma_E^4\zeta_2) + \frac{9}{80}(\gamma_E^2\zeta_2^2) + \frac{61}{560}\zeta_2^3 + \frac{1}{18}(\gamma_E^3\zeta_3) + \frac{1}{6}(\gamma_E\zeta_2\zeta_3) + \frac{1}{18}\zeta_3^2 + \frac{1}{5}\gamma_E\zeta_5\\
&\quad - \frac{1}{240N}(5\gamma_E^4+\gamma_E^5+10\gamma_E^3\zeta_2+27\zeta_2^2+20\zeta_2\zeta_3+10\gamma_E^2(3\zeta_2+2\zeta_3)+\gamma_E(27\zeta_2^2+40\zeta_3)+24\zeta_5),\\
\gamma_7^A &= \frac{1}{5040}\gamma_E^7 + \frac{1}{240}(\gamma_E^5\zeta_2) + \frac{3}{80}(\gamma_E^3\zeta_2^2) + \frac{61}{560}(\gamma_E\zeta_2^3) + \frac{1}{72}(\gamma_E^4\zeta_3) + \frac{1}{12}(\gamma_E^2\zeta_2\zeta_3) + \frac{3}{40}(\zeta_2^2\zeta_3)\\
&\quad + \frac{1}{18}(\gamma_E\zeta_3^2) + \frac{1}{10}(\gamma_E^2\zeta_5) + \frac{1}{10}(\zeta_2\zeta_5) + \frac{1}{7}\zeta_7 - \frac{1}{10080N}(42\gamma_E^5+7\gamma_E^6+105\gamma_E^4\zeta_2+549\zeta_2^3+840\zeta_2\zeta_3\\
&\quad + 140\gamma_E^3(3\zeta_2+2\zeta_3)+21\gamma_E^2(27\zeta_2^2+40\zeta_3)+56(5\zeta_3^2+18\zeta_5)+42\gamma_E(27\zeta_2^2+20\zeta_2\zeta_3+24\zeta_5)),
\end{aligned}
\tag{D3}
$$

and similarly $\Gamma_B(x)$ is given by [64]

$$
\Gamma_B(x) = \sum_{k=1}\gamma_k^B x^k, \tag{D4}
$$

where γ_{k+1}^B are given by [64]

$$
\gamma_{k+1}^B = \frac{\Gamma^k(1)}{k!}(-1)^k. \tag{D5}
$$

Explicitly we find

$$
\begin{aligned}
\gamma_1^B &= 1, \qquad \gamma_2^B = \gamma_E, \qquad \gamma_3^B = \frac{1}{2}(\gamma_E^2+\zeta_2), \qquad \gamma_4^B = \frac{1}{6}\gamma_E^3 + \frac{1}{2}(\gamma_E\zeta_2) + \frac{1}{3}\zeta_3, \qquad \gamma_5^B = \frac{1}{24}\gamma_E^4 + \frac{1}{4}(\gamma_E^2\zeta_2) + \frac{9}{40}\zeta_2^2 + \frac{1}{3}(\gamma_E\zeta_3),\\
\gamma_6^B &= \frac{1}{120}\gamma_E^5 + \frac{1}{12}(\gamma_E^3\zeta_2) + \frac{1}{40}(9\gamma_E\zeta_2^2) + \frac{1}{6}(\gamma_E^2\zeta_3) + \frac{1}{6}(\zeta_2\zeta_3) + \frac{1}{5}\zeta_5,\\
\gamma_7^B &= \frac{1}{720}\gamma_E^6 + \frac{1}{48}(\gamma_E^4\zeta_2) + \frac{9}{80}(\gamma_E^2\zeta_2^2) + \frac{61}{560}\zeta_2^3 + \frac{1}{18}(\gamma_E^3\zeta_3) + \frac{1}{6}(\gamma_E\zeta_2\zeta_3) + \frac{1}{18}\zeta_3^2 + \frac{1}{5}\gamma_E\zeta_5,\\
\gamma_8^B &= \frac{1}{5040}\gamma_E^7 + \frac{1}{240}(\gamma_E^5\zeta_2) + \frac{3}{80}(\gamma_E^3\zeta_2^2) + \frac{61}{560}(\gamma_E\zeta_2^3) + \frac{1}{72}(\gamma_E^4\zeta_3) + \frac{1}{12}(\gamma_E^2\zeta_2\zeta_3) + \frac{3}{40}(\zeta_2^2\zeta_3) + \frac{1}{18}(\gamma_E\zeta_3^2)\\
&\quad + \frac{1}{10}(\gamma_E^2\zeta_5) + \frac{1}{10}(\zeta_2\zeta_5) + \frac{1}{7}\zeta_7.
\end{aligned}
\tag{D6}
$$

APPENDIX E: ANALYTICAL STRUCTURE OF NSV COEFFICIENTS OF $\Delta_{c\bar{c}}$ TILL FOUR LOOP

The partonic coefficient function given in Eq. (65) can be written as

$$
\Delta_c^{(i)}(q^2,\mu_R^2,\mu_F^2,z) = \Delta_c^{\mathrm{SV},(i)}(q^2,\mu_R^2,\mu_F^2,z) + \Delta_c^{\mathrm{NSV},(i)}(q^2,\mu_R^2,\mu_F^2,z), \tag{E1}
$$

where $\Delta_c^{\mathrm{SV},(i)}(q^2, \mu_R^2, \mu_F^2, z)$ can be found in [20,21,25,36,46]. Here we present $\Delta_c^{\mathrm{NSV},(i)}$ to fourth order where we set $\mu_R^2 = \mu_F^2 = q^2$ with the following expansion:

$$\Delta_c^{\mathrm{NSV},(i)}(z) = \sum_{k=0}^{2i} \Delta_c^{ik} \ln^k(1-z). \tag{E2}$$

The following results with the explicit dependence on μ_R and μ_F are provided in the ancillary files supplied with the arXiv submission. We also put $\Delta_c^{30}, \ldots, \Delta_c^{32}$ and $\Delta_c^{40}, \ldots, \Delta_c^{44}$ in the ancillary files as they were lengthy:

$$\begin{aligned}
\Delta_c^{10} &= 2\varphi_1^{c,(0)}, \quad \Delta_c^{11} = 2\varphi_1^{c,(1)} + 4D_1^c, \quad \Delta_c^{12} = 4C_1^c,\\
\Delta_c^{20} &= 2\varphi_{c,2}^{(0)} + 4\varphi_{c,1}^{(0)}(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}) + 4f_1^c\varphi_{c,1}^{(1)}\zeta_2 - 16f_1^cC_1^c\zeta_3 + 8A_1^c\varphi_{c,1}^{(1)}\zeta_3 + 6A_1^c\varphi_{c,1}^{(0)}\zeta_2 + 2(f_1^c)^2 - \frac{16}{5}A_1^cC_1^c\zeta_2^2 - 8(A_1^c)^2\zeta_2\\
&\quad + 8D_1^cf_1^c\zeta_2 + 16D_1^cA_1^c\zeta_3,\\
\Delta_c^{21} &= 2\varphi_{c,2}^{(1)} + 4\varphi_{c,1}^{(1)}(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}) - 4\varphi_{c,1}^{(0)}\beta_0 - 4f_1^c\varphi_{c,1}^{(0)} + 16f_1^cC_1^c\zeta_2 - 2A_1^c\varphi_{c,1}^{(1)}\zeta_2 + 4D_2^c - 8A_1^cf_1^c + 64A_1^cC_1^c\zeta_3 - 4D_1^cA_1^c\zeta_2\\
&\quad + 8D_1^c(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}),\\
\Delta_c^{22} &= 2\varphi_{c,2}^{(2)} - 4\varphi_{c,1}^{(1)}\beta_0 + 4C_2^c + C_1^c(8\bar{\mathcal{G}}_1^{c,(1)} + 8\tilde{G}_1^{c,1}) - 4f_1^c\varphi_{c,1}^{(1)} + 4A_1^c\varphi_{c,1}^{(0)} - 20A_1^cC_1^c\zeta_2 + 8(A_1^c)^2 - 8D_1^cf_1^c - 4D_1^c\beta_0,\\
\Delta_c^{23} &= -4C_1^c\beta_0 - 8f_1^cC_1^c + 4A_1^c\varphi_{c,1}^{(1)} + 8D_1^cA_1^c, \quad \Delta_c^{24} = 8A_1^cC_1^c,\\
\Delta_c^{33} &= 2\varphi_{c,3}^{(3)} - 8\beta_0(\varphi_{c,2}^{(2)} - \varphi_{c,1}^{(1)}\beta_0 + C_2^c) - 4C_1^c(\beta_1 + 6\beta_0\bar{\mathcal{G}}_1^{c,(1)} + 2\beta_0\tilde{G}_1^{c,1}) - 8(f_2^cC_1^c + f_1^cC_2^c) - 4f_1^c\varphi_{c,2}^{(2)} + 12f_1^c\varphi_{c,1}^{(1)}\beta_0\\
&\quad - 16f_1^cC_1^c(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}) + 4(f_1^c)^2\varphi_{c,1}^{(1)} + 4A_2^c\varphi_{c,1}^{(1)} + 4A_1^c\varphi_{c,2}^{(1)} + 8A_1^c\varphi_{c,1}^{(1)}(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}) - \frac{32}{3}A_1^c\varphi_{c,1}^{(0)}\beta_0 + 68A_1^cC_1^c\zeta_2\beta_0\\
&\quad - 8A_1^cf_1^c\varphi_{c,1}^{(0)} + 104A_1^cf_1^cC_1^c\zeta_2 - 16(A_1^c)^2\beta_0 - 20(A_1^c)^2\varphi_{c,1}^{(1)}\zeta_2 + 320(A_1^c)^2C_1^c\zeta_3 - 32(A_1^c)^2f_1^c + 16D_1^cf_1^c\beta_0 + 8D_1^c(f_1^c)^2\\
&\quad + 8D_1^cA_2^c + 16D_1^cA_1^c(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1}) - 40D_1^c(A_1^c)^2\zeta_2 + \frac{16}{3}D_1^c\beta_0^2 + 8D_2^cA_1^c,\\
\Delta_c^{34} &= \frac{16}{3}C_1^c\beta_0^2 + 16f_1^cC_1^c\beta_0 + 8C_1^c((f_1^c)^2 + A_2^c) + 16(A_1^c)^3 + 4A_1^c\varphi_{c,2}^{(2)} - \frac{32}{3}A_1^c\varphi_{c,1}^{(1)}\beta_0 + 8A_1^cC_2^c + 16A_1^cC_1^c(\bar{\mathcal{G}}_1^{c,(1)} + \tilde{G}_1^{c,1})\\
&\quad - 8A_1^cf_1^c\varphi_{c,1}^{(1)} + 4(A_1^c)^2\varphi_{c,1}^{(0)} - 72(A_1^c)^2C_1^c\zeta_2 - \frac{40}{3}D_1^cA_1^c\beta_0 - 16D_1^cA_1^cf_1^c,\\
\Delta_c^{35} &= -\frac{40}{3}A_1^cC_1^c\beta_0 - 16A_1^cf_1^cC_1^c + 4(A_1^c)^2\varphi_{c,1}^{(1)} + 8D_1^c(A_1^c)^2, \quad \Delta_c^{36} = 8(A_1^c)^2C_1^c,\\
\Delta_c^{45} &= -8C_1^c\beta_0^3 - \frac{88}{3}f_1^cC_1^c\beta_0^2 - 24(f_1^c)^2C_1^c\beta_0 - \frac{16}{3}(f_1^c)^3C_1^c - \frac{56}{3}A_2^cC_1^c\beta_0 - 16A_2^cf_1^cC_1^c + 4A_1^c\varphi_{c,3}^{(3)} - \frac{56}{3}A_1^c\varphi_{c,2}^{(2)}\beta_0 + 24A_1^c\varphi_{c,1}^{(1)}\beta_0^2\\
&\quad + \frac{64}{3}A_1^cD_1^c\beta_0^2 - \frac{64}{3}A_1^cC_2^c\beta_0 - \frac{1}{3}A_1^cC_1^c(40\beta_1 + 176\beta_0\bar{\mathcal{G}}_1^{c,(1)} + 80\beta_0g_1^{c,1}) - 16A_1^cf_2^cC_1^c - 8A_1^cf_1^c\varphi_{c,2}^{(2)} + \frac{88}{3}A_1^cf_1^c\varphi_{c,1}^{(1)}\beta_0\\
&\quad + \frac{128}{3}A_1^cf_1^cD_1^c\beta_0 - 16A_1^cf_1^cC_2^c - 32A_1^cf_1^cC_1^c(\bar{\mathcal{G}}_1^{c,(1)} + g_1^{c,1}) + 8A_1^c(f_1^c)^2\varphi_{c,1}^{(1)} + 16A_1^c(f_1^c)^2D_1^c + 8A_1^cA_2^c\varphi_{c,1}^{(1)} + 16A_1^cA_2^cD_1^c\\
&\quad + 4(A_1^c)^2\varphi_{c,2}^{(1)} + 8(A_1^c)^2\varphi_{c,1}^{(1)}(\bar{\mathcal{G}}_1^{c,(1)} + g_1^{c,1}) - \frac{40}{3}(A_1^c)^2\varphi_{c,1}^{(0)}\beta_0 + 8(A_1^c)^2D_2^c + 16(A_1^c)^2D_1^c(+\bar{\mathcal{G}}_1^{c,(1)} + g_1^{c,1}) + \frac{776}{3}(A_1^c)^2C_1^c\beta_0\zeta_2\\
&\quad - 8(A_1^c)^2f_1^c\varphi_{c,1}^{(0)} + 240(A_1^c)^2f_1^cC_1^c\zeta_2 - \frac{128}{3}(A_1^c)^3\beta_0 - 36(A_1^c)^3\varphi_{c,1}^{(1)}\zeta_2 - 72(A_1^c)^3D_1^c\zeta_2 + \frac{1792}{3}(A_1^c)^3C_1^c\zeta_3 - 48(A_1^c)^3f_1^c,\\
\Delta_c^{46} &= \frac{64}{3}A_1^cC_1^c\beta_0^2 + \frac{128}{3}A_1^cf_1^cC_1^c\beta_0 + 16A_1^c(f_1^c)^2C_1^c + 16A_1^cA_2^cC_1^c + 4(A_1^c)^2\varphi_{c,2}^{(2)} - \frac{40}{3}(A_1^c)^2\varphi_{c,1}^{(1)}\beta_0 - \frac{56}{3}(A_1^c)^2D_1^c\beta_0 + 8(A_1^c)^2C_2^c\\
&\quad + 16(A_1^c)^2C_1^c(\bar{\mathcal{G}}_1^{c,(1)} + g_1^{c,1}) - 8(A_1^c)^2f_1^c\varphi_{c,1}^{(1)} - 16(A_1^c)^2f_1^cD_1^c + \frac{8}{3}(A_1^c)^3\varphi_{c,1}^{(0)} - 104(A_1^c)^3C_1^c\zeta_2 + 16(A_1^c)^4,\\
\Delta_c^{47} &= -\frac{56}{3}(A_1^c)^2C_1^c\beta_0 - 16(A_1^c)^2f_1^cC_1^c + \frac{8}{3}(A_1^c)^3\varphi_{c,1}^{(1)} + \frac{16}{3}(A_1^c)^3D_1^c, \quad \Delta_c^{48} = \frac{16}{3}(A_1^c)^3C_1^c.
\end{aligned} \tag{E3}$$

The symbols $\bar{\mathcal{G}}_j^{c,(k)}$ and $g_j^{c,k}$ are also provided in the Supplementary Material [126].

APPENDIX F: EXPRESSIONS OF RESUMMATION CONSTANTS $h^c_{ij}(\omega)$

The resummation constants $h^c_{ij}(\omega)$ given in Eq. (85) are found to be as follows. Here $\bar{L}_\omega = \ln(1-\omega)$, $L_{qr} = \ln(\frac{q^2}{\mu_R^2})$, $L_{fr} = \ln(\frac{\mu_F^2}{\mu_R^2})$, and $\omega = 2\beta_0 as(\mu_R^2)\ln N$:

$$\begin{aligned}
h^c_{00}(\omega) &= \frac{2}{\beta_0}\bar{L}_\omega[-\gamma^B_2 C^c_1 + \gamma^B_1 D^c_1],\\
h^c_{01}(\omega) &= \frac{2}{\beta_0}\bar{L}_\omega[-\gamma^B_1 C^c_1],\\
h^c_{10}(\omega) &= \frac{1}{(1-\omega)}\Bigg[\frac{2\beta_1 D^c_1}{\beta_0^2}\{\gamma^B_1\omega + \gamma^B_1\bar{L}_\omega\} - \frac{2\beta_1}{\beta_0^2}C^c_1\{\gamma^B_2\omega + \gamma^B_2\bar{L}_\omega\} - 2\frac{D^c_2}{\beta_0}\gamma^B_1\omega + 2\frac{C^c_2}{\beta_0}\gamma^B_2\omega - 2\varphi^{(1)}_{c,1}\gamma^B_2 + 2\varphi^{(0)}_{c,1}\gamma^B_1\\
&\quad + 2D^c_1\{L_{qr}\gamma^B_1 - L_{fr}\gamma^B_1 + L_{fr}\gamma^B_1\omega - 2\gamma^B_2\} - 2C^c_1\{L_{qr}\gamma^B_2 - L_{fr}\gamma^B_2 + L_{fr}\gamma^B_2\omega - 4\gamma^B_3\}\Bigg],\\
h^c_{11}(\omega) &= \frac{1}{(1-\omega)}\Bigg[\frac{\beta_1}{\beta_0^2}C^c_1\{-2\gamma^B_1\omega - 2\gamma^B_1\bar{L}_\omega\} + 2\frac{C^c_2}{\beta_0}\gamma^B_1\omega - 2\varphi^{(1)}_{c,1}\gamma^B_1 + C^c_1\{-2L_{qr}\gamma^B_1 + 2L_{fr}\gamma^B_1 - 2L_{fr}\gamma^B_1\omega + 4\gamma^B_2\}\\
&\quad + \frac{\omega}{(1-\omega)}\Bigg\{\frac{\varphi^{(2)}_{c,2}}{\beta_0}\gamma^B_1\Bigg\}\Bigg],\\
h^c_{21}(\omega) &= \frac{1}{(1-\omega)^2}\Bigg[\frac{\beta_1^2 C^c_1}{\beta_0^3}\{-\gamma^B_1\omega^2 + \gamma^B_1\bar{L}^2_\omega\} + \frac{\beta_2 C^c_1}{\beta_0^2}\gamma^B_1\omega^2 + \frac{\beta_1 C^c_2}{\beta_0^2}\{-2\omega + \omega^2 - 2\bar{L}_\omega\}\gamma^B_1\\
&\quad + \frac{C^c_3}{\beta_0}\{2\gamma^B_1\omega - \gamma^B_1\omega^2\} + 2\frac{\beta_1\varphi^{(1)}_{c,1}}{\beta_0}\gamma^B_1\bar{L}_\omega + \frac{\beta_1 C^c_1}{\beta_0}\{2L_{qr}\gamma^B_1 - 4\gamma^B_2\}\bar{L}_\omega + 4\varphi^{(2)}_{c,2}\gamma^B_2\\
&\quad - 2\varphi^{(1)}_{c,2}\gamma^B_1 + C^c_2\{-2L_{qr}\gamma^B_1 + 2L_{fr}\gamma^B_1(1-\omega)^2 + 4\gamma^B_2\} + 2\beta_0\varphi^{(1)}_{c,1}\{L_{qr}\gamma^B_1 - 2\gamma^B_2\}\\
&\quad + \beta_0 C^c_1\{L^2_{qr}\gamma^B_1 - 4L_{qr}\gamma^B_2 - L^2_{fr}\gamma^B_1 + 2L^2_{fr}\gamma^B_1\omega + 8\gamma^B_3 - L^2_{fr}\gamma^B_1\omega^2\}\Bigg],\\
h^c_{22}(\omega) &= \frac{\omega}{(1-\omega)^3}\Bigg[\frac{-\varphi^{(3)}_{c,3}}{\beta_0}\gamma^B_1\Bigg],\\
h^c_{32}(\omega) &= \frac{1}{(1-\omega)^3}\Bigg[-4\gamma^B_1\bar{L}_\omega\Bigg\{\frac{\beta_1\varphi^{(2)}_{c,2}}{\beta_0}\Bigg\} - 6\varphi^{(3)}_{c,3}\gamma^B_2 + 2\varphi^{(2)}_{c,3}\gamma^B_1 - 4\beta_0\varphi^{(2)}_{c,2}\{L_{qr}\gamma^B_1 - 2\gamma^B_2\}\Bigg],\\
h^c_{33}(\omega) &= \frac{\omega}{(1-\omega)^4}\Bigg[\frac{\varphi^{(4)}_{c,4}}{\beta_0}\gamma^B_1\Bigg],\\
h^c_{42}(\omega) &= \frac{1}{(1-\omega)^4}\Bigg[\frac{2\beta_1^2}{\beta_0^2}\varphi^{(2)}_{c,2}\{3\bar{L}^2_w - 2\omega - 2\bar{L}_w\}\gamma^B_1 + \frac{4\beta_2}{\beta_0}\varphi^{(2)}_{c,2}\gamma^B_1\omega + \frac{18\beta_1}{\beta_0}\varphi^{(3)}_{c,3}\gamma^B_2\bar{L}_w + 24\varphi^{(4)}_{c,4}\gamma^B_3 - \frac{6\beta_1}{\beta_0}\varphi^{(2)}_{c,3}\gamma^B_1\bar{L}_w - 6\varphi^{(3)}_{c,4}\gamma^B_2\\
&\quad + 2\varphi^{(2)}_{c,4}\gamma^B_1 - 4\beta_1\varphi^{(2)}_{c,2}\{L_{qr}\gamma^B_1 - 3L_{qr}\gamma^B_1\bar{L}_w - 2\gamma^B_2 + 6\gamma^B_2\bar{L}_w\} + 18\beta_0\varphi^{(3)}_{c,3}\{L_{qr}\gamma^B_2 - 4\gamma^B_3\}\\
&\quad - 6\beta_0\varphi^{(2)}_{c,3}\{L_{qr}\gamma^B_1 - 2\gamma^B_2\} + 6\beta_0^2\varphi^{(2)}_{c,2}\{+L^2_{qr}\gamma^B_1 - 4L_{qr}\gamma^B_2 + 8\gamma^B_3\}\Bigg],\\
h^c_{43}(\omega) &= \frac{2}{(1-\omega)^4}\Bigg[\frac{3\beta_1}{\beta_0}\varphi^{(3)}_{c,3}\gamma^B_1\bar{L}_w + 4\varphi^{(4)}_{c,4}\gamma^B_2 - \varphi^{(3)}_{c,4}\gamma^B_1 + 3\beta_0\varphi^{(3)}_{c,3}\{L_{qr}\gamma^B_1 - 2\gamma^B_2\}\Bigg],\\
h^c_{44}(\omega) &= \frac{\omega}{(1-\omega)^5}\Bigg[\frac{-\varphi^{(5)}_{c,5}}{\beta_0}\gamma^B_1\Bigg].
\end{aligned} \tag{F1}$$

The above results along with the bigger ones $[h^c_{20}(\omega), h^c_{30}(\omega), h^c_{31}(\omega)]$ and $[h^c_{40}(\omega), h^c_{41}(\omega)]$ are all provided in the Supplementary Material [126].

APPENDIX G: EXPRESSIONS OF RESUMMATION CONSTANTS $\bar{g}_i^c(\omega)$

The resummation constants $\bar{g}_i^c(\omega)$ given in Eq. (84) are presented below. Here $\bar{L}_\omega = \ln(1-\omega)$, $L_{qr} = \ln(\frac{q^2}{\mu_R^2})$, $L_{fr} = \ln(\frac{\mu_F^2}{\mu_R^2})$, and $\omega = 2\beta_0 a_s(\mu_R^2)\ln N$. Also, $\mathbf{D}_i^c$ are the threshold exponents given in [46]:

$$
\begin{aligned}
\bar{g}_1^c(\omega) &= \frac{A_1^c}{\beta_0},\\
\bar{g}_2^c(\omega) &= \frac{1}{(1-\omega)}\left[\frac{\mathbf{D}_1^c}{2} - \frac{A_2^c}{\beta_0}\omega + \frac{A_1^c\beta_1}{\beta_0^2}(\omega+\bar{L}_w) - A_1^c(2+2\gamma_E - L_{qr} + L_{fr}(1-\omega))\right],\\
\bar{g}_3^c(\omega) &= \frac{1}{(1-\omega)^2}\Bigg[\mathbf{D}_2^c\left\{\frac{1}{2}\right\} + \mathbf{D}_1^c\left\{-\frac{\bar{L}_w}{2}\frac{\beta_1}{\beta_0} + \beta_0\left(1+\gamma_E - \frac{1}{2}L_{qr}\right)\right\} - \frac{A_3^c}{\beta_0}\left\{1-\frac{\omega}{2}\right\}\omega + A_2^c\Bigg\{+\frac{\beta_1}{\beta_0^2}\left(\omega - \frac{1}{2}\omega^2 + \bar{L}_w\right)\\
&\quad - (2+2\gamma_E - L_{qr} + L_{fr}(1-\omega)^2)\Bigg\} - \frac{\beta_2 A_1^c}{\beta_0^2}\frac{\omega^2}{2} + A_1^c\Bigg\{\frac{\beta_1^2}{2\beta_0^3}(\omega^2 - \bar{L}_w^2) + \frac{\beta_1}{\beta_0}(2+2\gamma_E - L_{qr})\bar{L}_w - 2\beta_0\Bigg(2\gamma_E + \gamma_E^2\\
&\quad + \zeta_2 - L_{qr} - L_{qr}\gamma_E + \frac{1}{4}L_{qr}^2 - \frac{1}{4}L_{fr}^2(1-\omega)^2\Bigg)\Bigg\}\Bigg],\\
\bar{g}_4^c(\omega) &= \frac{1}{(1-\omega)^3}\Bigg[\mathbf{D}_1^c\Bigg\{\frac{\beta_1^2}{2\beta_0^2}(-\omega - \bar{L}_w + \bar{L}_w^2) + \frac{\omega\beta_2}{2\beta_0} + \beta_1\left(1+\gamma_E - \frac{1}{2}L_{qr}\right)(1-2\bar{L}_w) + 2\beta_0^2\Bigg(2\gamma_E + \gamma_E^2 + \zeta_2 - L_{qr} - L_{qr}\gamma_E\\
&\quad + \frac{1}{4}L_{qr}^2\Bigg)\Bigg\} + \mathbf{D}_2^c\left\{-\frac{\beta_1}{\beta_0}\bar{L}_w + \beta_0(2+2\gamma_E - L_{qr})\right\} + \frac{1}{2}\mathbf{D}_3^c - \frac{A_4^c}{\beta_0}\omega\left\{1-\omega+\frac{1}{3}\omega^2\right\}\\
&\quad + A_3^c\left\{\frac{\beta_1}{\beta_0^2}\left(\omega - \omega^2 + \frac{1}{3}\omega^3 + \bar{L}_w\right) - 2 - 2\gamma_E + L_{qr} - L_{fr} + 3L_{fr}\left(1-\omega+\frac{\omega^2}{3}\right)\omega\right\} + A_2^c\Bigg\{\frac{\beta_1^2}{\beta_0^3}\left(\omega^2 - \frac{1}{3}\omega^3 - \bar{L}_w^2\right)\\
&\quad - \frac{\beta_2}{\beta_0^2}\left(1-\frac{1}{3}\omega\right)\omega^2 + 2\frac{\beta_1}{\beta_0}(2+2\gamma_E - L_{qr})\bar{L}_w - \beta_0\Bigg(8\gamma_E + 4\gamma_E^2 + 4\zeta_2 - 4L_{qr}(1+\gamma_E) + L_{qr}^2 - L_{fr}^2 + 3L_{fr}^2\\
&\quad \times\left(1-\omega+\frac{\omega^2}{3}\right)\omega\Bigg)\Bigg\} + A_1^c\Bigg\{-\frac{\beta_1^3}{\beta_0^4}\left(\frac{1}{2}\omega^2 - \frac{1}{3}\omega^3 + \bar{L}_w\omega + \frac{1}{2}\bar{L}_w^2 - \frac{1}{3}\bar{L}_w^3\right)\\
&\quad + \frac{\beta_1\beta_2}{\beta_0^3}\left(\omega - \frac{2}{3}\omega^2 + \bar{L}_w\right)\omega - \frac{\beta_3}{\beta_0^2}\omega^2\left(\frac{1}{2} - \frac{1}{3}\omega\right)\\
&\quad + 2\frac{\beta_1^2}{\beta_0^2}(\omega + \bar{L}_w - \bar{L}_w^2)\left(1+\gamma_E - \frac{L_{qr}}{2}\right) + \frac{\beta_2}{\beta_0}(-2-2\gamma_E + L_{qr})\omega + \beta_1\Bigg(-4\gamma_E - 2\gamma_E^2 - 2\zeta_2 + 2L_{qr}\\
&\quad - \frac{1}{2}L_{qr}^2 + 2L_{qr}\gamma_E\Bigg)(1-2\bar{L}_w) + \frac{\beta_1}{2}L_{fr}^2(1-\omega)^3 - \beta_0^2\Bigg(8\gamma_E^2 + \frac{8}{3}\gamma_E^3 + \frac{16}{3}\zeta_3 + 2(4\zeta_2 + L_{qr}^2)(1+\gamma_E)\\
&\quad - 4L_{qr}\gamma_E(2+\gamma_E) - 4L_{qr}\zeta_2 - \frac{1}{3}L_{qr}^3 + \frac{1}{3}L_{fr}^3(1-\omega)^3\Bigg)\Bigg\}\Bigg]. \qquad \text{(G1)}
\end{aligned}
$$

As before here we also provide the above results along with $\bar{g}_5^c(\omega)$ in the Supplementary Material [126].

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