# Twist-3 gluon fragmentation contribution to hyperon polarization in semi-inclusive deep inelastic scattering

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We derive the twist-3 gluon fragmentation function (FF) contribution to the transversely polarized hyperon production in semi-inclusive deep inelastic scattering, $ep \to e\Lambda^{\uparrow}X$, in the leading order (LO) with respect to the QCD coupling in the framework of the collinear twist-3 factorization. Together with the known result for the contribution from the twist-3 distribution in the proton and the twist-3 quark FFs for the hyperon, this completes the LO cross section for this process. The constraint relations among the twist-3 FFs are taken into account. The formula is relevant to large-$P_T$ hyperon production in the future electron-ion-collider experiment.

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## I. INTRODUCTION

In a recent paper [1], three of the present authors studied the transverse polarization of hyperons produced in semi-inclusive deep inelastic scattering, $ep \to e\Lambda^{\uparrow}X$. For large-$P_T$ hyperon production, this process can be analyzed in the framework of the collinear factorization, in which the polarization appears as a twist-3 observable in the absence of a leading twist-2 effect. For $ep \to e\Lambda^{\uparrow}X$, the responsible twist-3 effects are (i) the twist-3 distribution functions (DFs) in the initial proton combined with the twist-2 transversity fragmentation function (FF) for $\Lambda$ and (ii) the twist-3 FFs for the polarized hyperon combined with the twist-2 unpolarized parton DFs in the proton. The twist-3 FFs in (ii) are chiral even, and both (a) quark and (b) gluon types of twist-3 FFs contribute. In Ref. [1], the twist-3 polarized cross section for $ep \to e\Lambda^{\uparrow}X$ from the above (i) and (ii)(a) was derived in the leading order (LO) with respect to the QCD coupling constant. As a sequel to Ref. [1], we will derive in this paper the LO cross section from (ii)(b), which completes the LO twist-3 cross section for this process. Since the gluons are ample in the collision environment and the twist-3 quark and gluon FFs mix under renormalization, the effect of (ii)(b) could be as important as (ii)(a). We also remind that the twist-3 fragmentation effect is important to understand the single transverse-spin asymmetry in $p^{\uparrow}p \to \pi X$ [2,3], which shows a similar rising asymmetry at large $x_F$ as the polarization in $pp \to \Lambda^{\uparrow}X$. Our present study has a direct relevance to the hyperon polarization phenomenon in the future electron-ion-collider (EIC) experiment.

Here, we make some remarks on the phenomenological use of the twist-3 cross section. As we will see, it contains several unknown nonperturbative functions, the determination of which requires a global analysis of data for various processes such as $ep \to e\Lambda^{\uparrow}X$, $e^+e^- \to \Lambda^{\uparrow}X$, and $pp \to \Lambda^{\uparrow}X$, etc., combined with an appropriate modeling of those functions. We also recall that in the small-$P_T$ region the transverse-momentum-dependent (TMD) factorization holds for $ep \to e\Lambda^{\uparrow}X$ and $e^+e^- \to \Lambda^{\uparrow}X$, and we anticipate that the two frameworks match in the intermediate region of $P_T$[1] as for the case of $p^{\uparrow}p \to \ell^+\ell^-X$ [4] and $ep^{\uparrow} \to e\pi X$ [5–7]. Information on the TMD functions obtained from the analysis of those small-$P_T$ data will also help to constrain the twist-3 functions owing to the relations between the TMD functions and the twist-3 functions [8,9]. In this connection, we mention the recent data on $e^+e^- \to \Lambda^{\uparrow}X$ at Belle [10] and the phenomenological analyses of the data

[1]A study on this matching will be reported elsewhere.

in terms of the TMD factorization [11–14]. These studies will be useful to analyze the EIC data at large $P_T$ in terms of the twist-3 cross section derived in this work.

The formalism of calculating the twist-3 gluon FFs contribution is very complicated and was completed only recently for a similar process in the $pp$ collision, $pp \to \Lambda^\uparrow X$ [15]. Here, we apply the method to $ep \to e\Lambda^\uparrow X$. Since the kinematics for this process was described in Ref. [1] and the method is in parallel to the case for $pp \to \Lambda^\uparrow X$ [15], our presentation in this paper will be brief, referring to those papers for the details.

The remainder of this paper is organized as follows: In Sec. II, we introduce the twist-3 gluon FFs relevant in our study. In Sec. III, we briefly describe the formalism for calculating the twist-3 gluon FF contribution to $ep \to e\Lambda^\uparrow X$ and present the LO cross section. Section IV is devoted to a brief summary.

## II. TWIST-3 GLUON FRAGMENTATION FUNCTIONS

### A. Three types of twist-3 gluon FFs and $q\bar{q}g$ FFs

Here, we list the twist-3 gluon FFs for the spin-1/2 hyperon which are necessary to derive the twist-3 cross section for $ep \to e\Lambda^\uparrow X$ [9,15]. They are classified into the intrinsic, kinematical, and dynamical FFs. First, the intrinsic gluon FFs are defined as the light-cone correlators of the gluon's field strength $F_a^{\mu\nu}$ with color index $a$ [9,16]:

$$\begin{aligned}\hat{\Gamma}^{\alpha\beta}(z) &= \frac{1}{N^2-1}\sum_X\int\frac{d\lambda}{2\pi}\mathrm{e}^{-i\lambda/z}\langle 0|([\infty w,0]F^{w\beta}(0))_a|hX\rangle \\ &\quad\times\langle hX|(F^{w\alpha}(\lambda w)[\lambda w,\infty w])_a|0\rangle \\ &= -g_\perp^{\alpha\beta}\hat{G}(z) - i\epsilon^{P_h w\alpha\beta}(S\cdot w)\Delta\hat{G}(z) \\ &\quad + M_h\epsilon^{P_h w S_\perp\{\alpha}w^{\beta\}}\Delta\hat{G}_{3\bar{T}}(z) \\ &\quad + iM_h\epsilon^{\alpha\beta w S_\perp}\Delta\hat{G}_{3T}(z) + \cdots,\end{aligned} \tag{1}$$

where $P_h$ is the four-momentum of the hyperon with its mass $M_h$. $P_h^\mu$ can be regarded as lightlike in the twist-3 accuracy, and $w^\mu$ is another lightlike vector satisfying $P_h\cdot w = 1$. $S^\mu$ is the spin vector of the hyperon normalized as $S^2 = -M_h^2$ and can be decomposed as $S^\mu = (S\cdot w)P_h^\mu + (S\cdot P_h)w^\mu + M_h S_\perp^\mu$ with the transverse spin vector $S_\perp^\mu$ ($S_\perp^2 = -1$). $g_\perp^{\alpha\beta} \equiv g^{\alpha\beta} - P_h^\alpha w^\beta - P_h^\beta w^\alpha$, $N = 3$ is the number of colors for $SU(N)$, and the ellipsis denotes twist-4 or higher. $|h\rangle$ denotes the hyperon state. $[\lambda w, \mu w] \equiv \mathcal{P}\exp\left[ig\int_\mu^\lambda d\tau w\cdot A(\tau w)\right]$ is the gauge-link operator which guarantees gauge invariance of the correlation function. We use the convention for the Levi-Civita symbol as $\epsilon^{0123} = 1$. The shorthand notation $\epsilon^{P_h w\alpha\beta} \equiv \epsilon^{\mu\nu\alpha\beta}P_{h\mu}w_\nu$, etc., is used, and $\{\alpha\beta\}$ denotes the symmetrization of Lorentz indices. $\hat{G}(z)$ and $\Delta\hat{G}(z)$ are twist-2 unpolarized and helicity FFs, respectively, and $\Delta\hat{G}_{3\bar{T}}(z)$ and $\Delta\hat{G}_{3T}(z)$ are intrinsic twist-3 FFs. All FFs in Eq. (1) are defined to be real and have a support on $0 < z < 1$. $\Delta\hat{G}_{3\bar{T}}(z)$ is naïvely $T$ odd and contributes to the hyperon polarization.

Second, the kinematical gluon FFs are defined from the derivative of the correlation functions for the intrinsic one:

$$\begin{aligned}\hat{\Gamma}_\partial^{\alpha\beta\gamma}(z) &= \frac{1}{N^2-1}\sum_X\int\frac{d\lambda}{2\pi}\mathrm{e}^{-i\lambda/z}\langle 0|([\infty w,0]F^{w\beta}(0))_a|hX\rangle \\ &\quad\times\langle hX|(F^{w\alpha}(\lambda w)[\lambda w,\infty w])_a|0\rangle\overleftarrow{\partial}^\gamma \\ &= -i\frac{M_h}{2}g_\perp^{\alpha\beta}\epsilon^{P_h w S_\perp\gamma}\hat{G}_T^{(1)}(z) + \frac{M_h}{2}\epsilon^{P_h w\alpha\beta}S_\perp^\gamma\Delta\hat{G}_T^{(1)}(z) \\ &\quad - i\frac{M_h}{8}(\epsilon^{P_h w S_\perp\{\alpha}g_\perp^{\beta\}\gamma} + \epsilon^{P_h w\gamma\{\alpha}S_\perp^{\beta\}})\Delta\hat{H}_T^{(1)}(z) \\ &\quad + \cdots,\end{aligned} \tag{2}$$

where

$$\begin{aligned}&F^{w\alpha}(\lambda w)[\lambda w,\infty w]|0\rangle\overleftarrow{\partial}^\gamma \\ &\quad\equiv \lim_{\xi\to 0}\frac{d}{d\xi_\gamma}F^{w\alpha}(\lambda w+\xi)[\lambda w+\xi,\infty w+\xi]|0\rangle.\end{aligned} \tag{3}$$

There are three twist-3 gluonic kinematical FFs, $\hat{G}_T^{(1)}(z)$, $\Delta\hat{G}_T^{(1)}(z)$, and $\Delta\hat{H}_T^{(1)}(z)$, which are real functions and have a support on $0 < z < 1$. Among them, $\hat{G}_T^{(1)}(z)$ and $\Delta\hat{H}_T^{(1)}(z)$ are naïvely $T$ odd contributing to the hyperon polarization, while $\Delta\hat{G}_T^{(1)}(z)$ is naïvely $T$ even. They can also be written as the $k_T^2/M_h^2$ moment of the TMD FFs [16].

Third, the dynamical gluon FFs are defined from the three-gluon correlators. Contraction of color indices with two structure constants for color $SU(N)$, i.e., $-if_{abc}$ and $d_{abc}$, yields two types of FFs [9,17–19]:

$$\begin{aligned}\hat{\Gamma}_{FA}^{\alpha\beta\gamma}\left(\frac{1}{z_1},\frac{1}{z_2}\right) &= \frac{-if_{abc}}{N^2-1}\sum_X\iint\frac{d\lambda}{2\pi}\frac{d\mu}{2\pi}\mathrm{e}^{-i\lambda/z_1}\mathrm{e}^{-i\mu(1/z_2-1/z_1)}\langle 0|F_b^{w\beta}(0)|hX\rangle\langle hX|F_a^{w\alpha}(\lambda w)gF_c^{w\gamma}(\mu w)|0\rangle \\ &= -M_h\left(\hat{N}_1\left(\frac{1}{z_1},\frac{1}{z_2}\right)g_\perp^{\alpha\gamma}\epsilon^{P_h w S_\perp\beta} + \hat{N}_2\left(\frac{1}{z_1},\frac{1}{z_2}\right)g_\perp^{\beta\gamma}\epsilon^{P_h w S_\perp\alpha} - \hat{N}_2\left(\frac{1}{z_2}-\frac{1}{z_1},\frac{1}{z_2}\right)g_\perp^{\alpha\beta}\epsilon^{P_h w S_\perp\gamma}\right),\end{aligned} \tag{4}$$

$$\hat{\Gamma}_{FS}^{\alpha\beta\gamma}\left(\frac{1}{z_1},\frac{1}{z_2}\right) = \frac{d_{abc}}{N^2-1}\sum_X \iint \frac{d\lambda}{2\pi}\frac{d\mu}{2\pi} e^{-i\lambda/z_1} e^{-i\mu(1/z_2-1/z_1)} \langle 0|F_b^{w\beta}(0)|hX\rangle\langle hX|F_a^{w\alpha}(\lambda w) g F_c^{w\gamma}(\mu w)|0\rangle$$
$$= -M_h\left(\hat{O}_1\left(\frac{1}{z_1},\frac{1}{z_2}\right) g_\perp^{\alpha\gamma}\epsilon^{P_h w S_\perp \beta} + \hat{O}_2\left(\frac{1}{z_1},\frac{1}{z_2}\right) g_\perp^{\beta\gamma}\epsilon^{P_h w S_\perp \alpha} + \hat{O}_2\left(\frac{1}{z_2}-\frac{1}{z_1},\frac{1}{z_2}\right) g_\perp^{\alpha\beta}\epsilon^{P_h w S_\perp \gamma}\right), \quad (5)$$

where the gauge-link operators are suppressed for simplicity. There are four purely gluonic dynamical FFs, $\hat{N}_{1,2}(\frac{1}{z_1},\frac{1}{z_2})$ and $\hat{O}_{1,2}(\frac{1}{z_1}.\frac{1}{z_2})$, which are complex functions and have a support on $1/z_2 > 1$ and $1/z_2 > 1/z_1 > 0$. Their real parts are naïvely $T$ even, while their imaginary parts are naïvely $T$ odd. $\hat{N}_1(\frac{1}{z_1},\frac{1}{z_2})$ and $\hat{O}_1(\frac{1}{z_1},\frac{1}{z_2})$ satisfy the symmetry relations

$$\hat{N}_1\left(\frac{1}{z_1},\frac{1}{z_2}\right) = -\hat{N}_1\left(\frac{1}{z_2}-\frac{1}{z_1},\frac{1}{z_2}\right),$$
$$\hat{O}_1\left(\frac{1}{z_1},\frac{1}{z_2}\right) = \hat{O}_1\left(\frac{1}{z_2}-\frac{1}{z_1},\frac{1}{z_2}\right). \quad (6)$$

Finally, we introduce other dynamical FFs defined from the quark-antiquark-gluon correlators [9], which are necessary for the derivation of the twist-3 cross section for $ep \to e\Lambda^\uparrow X$:

$$\tilde{\Delta}_{ij}^{\alpha}\left(\frac{1}{z_1},\frac{1}{z_2}\right) = \frac{1}{N}\sum_X \iint \frac{d\lambda}{2\pi}\frac{d\mu}{2\pi} e^{-i\lambda/z_1} e^{-i\mu(1/z_2-1/z_1)}$$
$$\times \langle 0|gF_a^{w\alpha}(\mu w)|hX\rangle\langle hX|\bar{\psi}_j(\lambda w)t^a\psi_i(0)|0\rangle$$
$$= M_h\left(\epsilon^{\alpha P_h w S_\perp}(\not{P}_h)_{ij}\tilde{D}_{FT}\left(\frac{1}{z_1},\frac{1}{z_2}\right) + iS_\perp^\alpha(\gamma_5\not{P}_h)_{ij}\tilde{G}_{FT}\left(\frac{1}{z_1},\frac{1}{z_2}\right)\right), \quad (7)$$

where $t^a$ is the generators of $SU(N)$ and the spinor indices $i$ and $j$ are shown explicitly. These two functions $\tilde{D}_{FT}(\frac{1}{z_1},\frac{1}{z_2})$ and $\tilde{G}_{FT}(\frac{1}{z_1},\frac{1}{z_2})$ are complex functions and have a support on $1/z_1 > 0, 1/z_2 < 0$ and $1/z_1 - 1/z_2 > 1$. Their real parts are naïvely $T$ even, while the imaginary parts are naïvely $T$ odd.

### B. Constraint relations among twist-3 gluon FFs

The gluon FFs introduced above are not independent but are subject to the QCD equation-of-motion (EOM) relations and the Lorentz invariance relations (LIRs). The complete set of those relations was derived in Ref. [9]. Here, we quote those relations which are useful to simplify the twist-3 cross section for $ep \to e\Lambda^\uparrow X$. The relevant EOM relation allows us to express the intrinsic FF in terms of the kinematical and dynamical FFs as

$$\frac{1}{z}\Delta\hat{G}_{3\bar{T}}(z) = -\mathrm{Im}\tilde{D}_{FT}(z) + \frac{1}{2}(\hat{G}_T^{(1)}(z) + \Delta\hat{H}_T^{(1)}(z))$$
$$+ \int d\left(\frac{1}{z'}\right)\frac{1}{1/z-1/z'}\mathrm{Im}\left(2\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right) - \hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\right), \quad (8)$$

where $\tilde{D}_{FT}(z)$ is defined as

$$\tilde{D}_{FT}(z) \equiv \frac{2}{C_F}\int_0^{1/z} d\left(\frac{1}{z'}\right)\tilde{D}_{FT}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right),$$
$$\text{with} \quad C_F = \frac{N^2-1}{2N}. \quad (9)$$

Other relations derived from the LIRs and the EOM relations represent the derivative of the kinematical FFs in terms of other FFs as

$$\frac{1}{z}\frac{\partial\hat{G}_T^{(1)}(z)}{\partial(1/z)} = -2(\mathrm{Im}\tilde{D}_{FT}(z) - \hat{G}_T^{(1)}(z)) + 4\int d\left(\frac{1}{z'}\right)\frac{1}{1/z-1/z'}\mathrm{Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right) - \hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\right)$$
$$+ 2\int d\left(\frac{1}{z'}\right)\frac{1/z}{(1/z-1/z')^2}\mathrm{Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right) - 2\hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\right) \quad (10)$$

and

$$\frac{1}{z}\frac{\partial\Delta\hat{H}_T^{(1)}(z)}{\partial(1/z)} = -4(\mathrm{Im}\tilde{D}_{FT}(z) - \Delta\hat{H}_T^{(1)}(z)) + 8\int d\left(\frac{1}{z'}\right)\frac{1}{1/z-1/z'}\mathrm{Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right)\right)$$
$$+ 4\int d\left(\frac{1}{z'}\right)\frac{1/z}{(1/z-1/z')^2}\mathrm{Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right)\right). \quad (11)$$

The relations (8), (10), and (11) show that the purely gluonic twist-3 FFs are related to the quark-antiquark-gluon FFs, which implies the contribution to $ep \to e\Lambda^\uparrow X$ from the latter needs to be considered together. It has been shown that the above three relations (8), (10), and (11) are crucial to guarantee the frame independence of the cross section for $pp \to \Lambda^\uparrow X$. Using these relations, we will express the cross section in terms of $\hat{G}_T^{(1)}$, $\Delta\hat{H}_T^{(1)}$, $\mathrm{Im}\hat{N}_{1,2}$, $\mathrm{Im}\hat{O}_{1,2}$, $\mathrm{Im}\tilde{D}_{FT}$, and $\mathrm{Im}\tilde{G}_{FT}$ [see Eq. (54) below], which gives the most concise expression for the cross section. We also note that, in principle, the twist-3 kinematical FFs, $\hat{G}_T^{(1)}$ and $\Delta\hat{H}_T^{(1)}$, can be also eliminated in terms of the twist-3 dynamical FFs [see Eqs. (74) and (75) in Ref. [9]].

## III. TWIST-3 GLUON FF CONTRIBUTION TO $ep \to e\Lambda^\uparrow X$

### A. Kinematics

Here, we briefly summarize the kinematics for the process [1],

$$e(\ell) + p(p) \to e(\ell') + \Lambda^\uparrow(P_h, S_\perp) + X, \qquad (12)$$

where $\ell$, $\ell'$, $p$, and $P_h$ are the momenta of each particle and $S_\perp$ is the transverse spin vector for $\Lambda$. With the virtual photon's momentum $q = \ell - \ell'$, we introduce the five Lorentz invariants as

$$S_{ep} \equiv (p+\ell)^2 \simeq 2p\cdot\ell, \qquad Q^2 \equiv -q^2,$$
$$x_{bj} \equiv \frac{Q^2}{2p\cdot q}, \qquad z_f \equiv \frac{p\cdot P_h}{p\cdot q}, \qquad q_T \equiv \sqrt{-q_t^2}, \qquad (13)$$

where

$$q_t^\mu \equiv q^\mu - \frac{P_h\cdot q}{p\cdot P_h}p^\mu - \frac{p\cdot q}{p\cdot P_h}P_h^\mu \qquad (14)$$

is a spacelike momentum satisfying $q_t\cdot p = q_t\cdot P_h = 0$. As in Ref. [1], we work in the hadron frame [20] (see Fig. 1), where $p^\mu$ and $q^\mu$ are collinear and take the following form:

$$p^\mu = \frac{Q}{2x_{bj}}(1,0,0,1), \qquad (15)$$

$$q^\mu = (0,0,0,-Q). \qquad (16)$$

Defining the azimuthal angles for the hadron plane and the lepton plane as $\chi$ and $\phi$, respectively, as shown in Fig. 1, $P_h^\mu$ and $\ell^\mu$ can be written as

$$P_h^\mu = \frac{z_f Q}{2}\left(1+\frac{q_T^2}{Q^2}, \frac{2q_T}{Q}\cos\chi, \frac{2q_T}{Q}\sin\chi, -1+\frac{q_T^2}{Q^2}\right), \qquad (17)$$

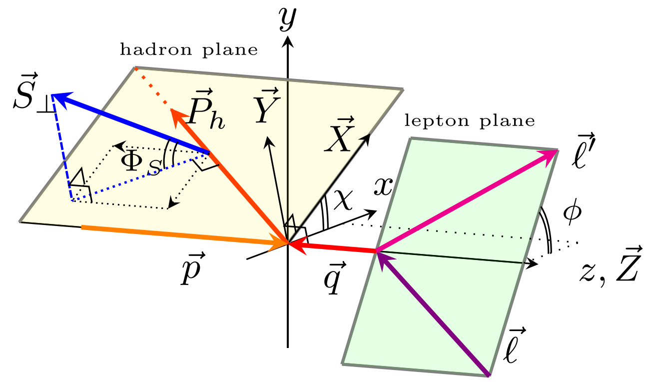


FIG. 1. Hadron frame and the transverse spin vector $\vec{S}_\perp$. To make clear the convention for $\Phi_S$, rotate the $Z$ and $X$ axes around the $Y$ axis by $\theta$ (polar angle of $\vec{P}_h$) so that the new $Z$ axis becomes parallel to $\vec{P}_h$. $\Phi_S$ is defined to be the azimuthal angle of $\vec{S}_\perp$ around $\vec{P}_h$ measured from the new $X$ axis, just like $\phi$ and $\chi$ are measured from the $x$ axis around the $z$ axis.

$$\ell^\mu = \frac{Q}{2}(\cosh\psi, \sinh\psi\cos\phi, \sinh\psi\sin\phi, -1), \qquad (18)$$

where $\psi$ is defined by

$$\cosh\psi \equiv \frac{2x_{bj}S_{ep}}{Q^2} - 1. \qquad (19)$$

With this parametrization, the transverse momentum of the hyperon $P_{hT}$ is given by $P_{hT} = z_f q_T$.

For the calculation of the cross section, we introduce four axes by

$$T^\mu \equiv \frac{1}{Q}(q^\mu + 2x_{bj}p^\mu) = (1,0,0,0),$$
$$Z^\mu \equiv -\frac{q^\mu}{Q} = (0,0,0,1),$$
$$X^\mu \equiv \frac{1}{q_T}\left[\frac{P_h^\mu}{z_f} - q^\mu - \left(1+\frac{q_T^2}{Q^2}\right)x_{bj}p^\mu\right] = (0,\cos\chi,\sin\chi,0),$$
$$Y^\mu \equiv \epsilon^{\mu\nu\rho\sigma}Z_\nu T_\rho X_\sigma = (0,-\sin\chi,\cos\chi,0), \qquad (20)$$

where the actual form in the hadron frame is given after the last equality in each equation. The final hyperon resides in the $XZ$ plane, and the transverse spin vector of the hyperon can be written as

$$S_\perp^\mu = \cos\theta\cos\Phi_S X^\mu + \sin\Phi_S Y^\mu - \sin\theta\cos\Phi_S Z^\mu, \qquad (21)$$

where $\theta$ is the polar angle of $\vec{P}_h$ as measured from the $Z$ axis and $\Phi_S$ is the azimuthal angle of $\vec{S}_\perp$ around $\vec{P}_h$ as measured from the $XZ$ plane. From Eq. (17), the polar angle $\theta$ is written as

$$\cos\theta = \frac{P_{hz}}{|\vec{P}_h|} = \frac{q_T^2 - Q^2}{q_T^2 + Q^2}, \tag{22}$$

$$\sin\theta = \frac{P_{hT}}{|\vec{P}_h|} = \frac{2q_T Q}{q_T^2 + Q^2}. \tag{23}$$

With the kinematical variables defined above, the polarized differential cross section for Eq. (12), $\sigma \equiv \sigma(p, \ell, \ell', P_h, S_\perp)$, takes the following form:

$$\frac{d^6\sigma}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} = \frac{\alpha_{em}^2}{128\pi^4 x_{bj}^2 S_{ep}^2 Q^2} z_f L^{\rho\sigma}(\ell, \ell') \times W_{\rho\sigma}(p, q, P_h), \tag{24}$$

where $\alpha_{em} = e^2/(4\pi)$ is the QED coupling constant, $L^{\rho\sigma} = 2(\ell^\rho \ell'^\sigma + \ell^\sigma \ell'^\rho) - Q^2 g^{\rho\sigma}$ is the leptonic tensor, and $W_{\rho\sigma}$ is the hadronic tensor. Although there are two azimuthal angles, $\phi$ and $\chi$, the cross section depends on the relative angle $\varphi \equiv \phi - \chi$ only. Therefore, it can be expressed in terms of $S_{ep}$, $Q^2$, $x_{bj}$, $z_f$, $q_T^2$, $\varphi$, and $\Phi_S$.

### B. Hadronic tensor

We now calculate the twist-3 gluon FF contribution to Eq. (24) following the formalism developed for $pp \to \Lambda^\uparrow X$ [15]. It occurs as a nonpole contribution from the hard part as in the case of other twist-3 fragmentation contributions in $ep^\uparrow \to e\pi X$ [21] and $pp \to \Lambda^\uparrow X$ [15,22]. We first factorize the twist-2 unpolarized quark DFs $f_1(x)$ from the hadronic tensor $W_{\rho\sigma}(p, q, P_h)$:

$$W_{\rho\sigma}(p, q, P_h) = \int \frac{dx}{x} f_1(x) w_{\rho\sigma}(xp, q, P_h), \tag{25}$$

where $x$ is the momentum fraction of the quark in the proton and we have omitted the factor associated with the quark's fractional electric charge as well as summation over quark flavors. Up to twist 3, $w_{\rho\sigma}$ receives contribution from the two-gluon, three-gluon, and quark-antiquark-gluon correlation functions corresponding to Figs. 2(a)–2(e):

$$w_{\rho\sigma} \equiv w_{\rho\sigma}^{(a)} + w_{\rho\sigma}^{(b)} + w_{\rho\sigma}^{(c)} + w_{\rho\sigma}^{(d)} + w_{\rho\sigma}^{(e)}, \tag{26}$$

where each term can be written as

$$w_{\rho\sigma}^{(a)} = \int \frac{d^4k}{(2\pi)^4} \Gamma_{ab}^{(0)\mu\nu}(k) S_{\mu\nu,\rho\sigma}^{ab}(k), \tag{27}$$

$$w_{\rho\sigma}^{(b)} = \frac{1}{2} \iint \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \Gamma_{\mathrm{L}abc}^{(1)\mu\nu\lambda}(k, k') S_{\mu\nu\lambda,\rho\sigma}^{\mathrm{L}abc}(k, k'), \tag{28}$$

$$w_{\rho\sigma}^{(c)} = \frac{1}{2} \iint \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \Gamma_{\mathrm{R}abc}^{(1)\mu\nu\lambda}(k, k') S_{\mu\nu\lambda,\rho\sigma}^{\mathrm{R}abc}(k, k'), \tag{29}$$

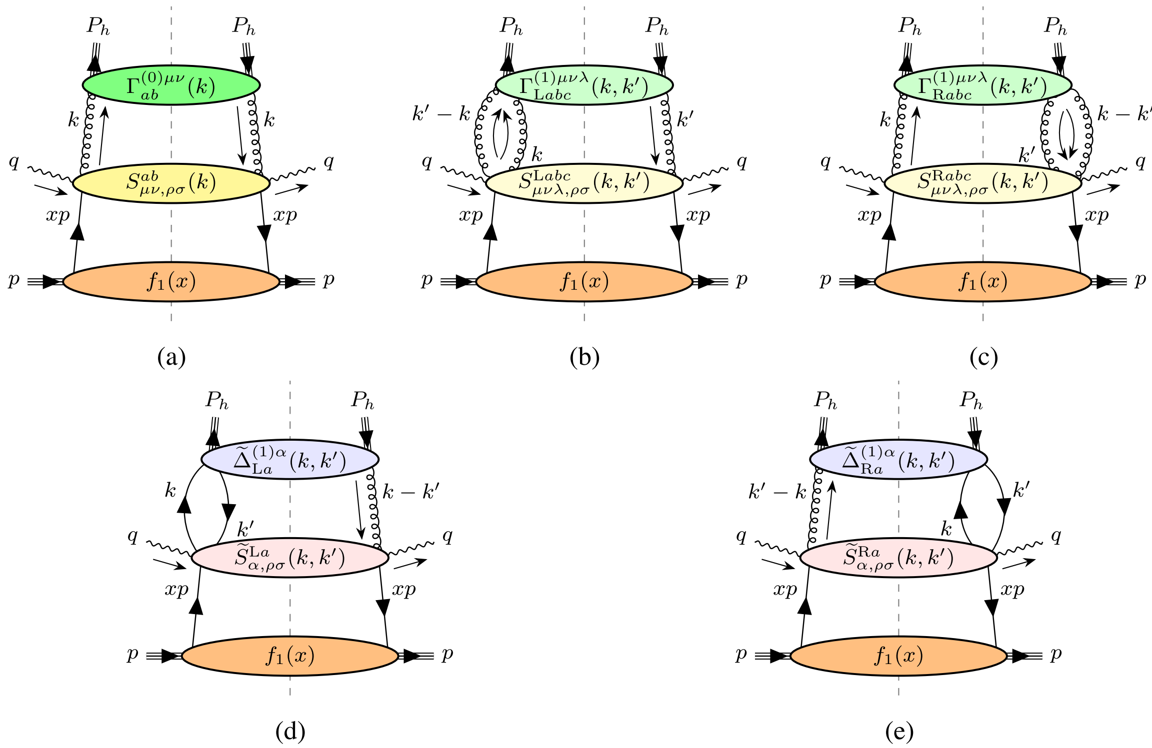


FIG. 2. Cut diagrams for the twist-3 gluon fragmentation contribution to $ep \to e\Lambda^\uparrow X$. In each diagram, the lower blob represents the unpolarized quark distribution, the middle one represents the partonic hard cross section, and the upper one represents the fragmentation matrix elements for the final hyperon. The diagrams (a) through (e) correspond to each term in (26).

$$w^{(\mathrm{d})}_{\rho\sigma} = \mathrm{Tr}\iint \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\tilde{\Delta}^{(1)\alpha}_{\mathrm{L}a}(k,k')\tilde{S}^{\mathrm{L}a}_{\alpha,\rho\sigma}(k,k'), \quad (30)$$

$$w^{(\mathrm{e})}_{\rho\sigma} = \mathrm{Tr}\iint \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\tilde{\Delta}^{(1)\alpha}_{\mathrm{R}a}(k,k')\tilde{S}^{\mathrm{R}a}_{\alpha,\rho\sigma}(k,k'). \quad (31)$$

Here, $S^{ab}_{\mu\nu,\rho\sigma}(k)$, $S^{\mathrm{L(R)}abc}_{\mu\nu\lambda,\rho\sigma}(k,k')$, and $\tilde{S}^{\mathrm{L(R)}a}_{\alpha,\rho\sigma}(k,k')$ represent the partonic hard parts with $k$ and $k'$ the momenta of partons fragmenting into the final hyperon, and the dependence on $q$ is suppressed for simplicity. $\Gamma^{(0)\mu\nu}_{ab}$, $\Gamma^{(1)\mu\nu\lambda}_{\mathrm{L(R)}abc}$, and $\tilde{\Delta}^{(1)\alpha}_{\mathrm{L(R)}a}$ denote the fragmentation matrix elements defined, respectively, as

$$\Gamma^{(0)\mu\nu}_{ab}(k) = \sum_X\int d^4\xi e^{-ik\cdot\xi}\langle 0|A^\nu_b(0)|hX\rangle\langle hX|A^\mu_a(\xi)|0\rangle, \quad (32)$$

$$\Gamma^{(1)\mu\nu\lambda}_{\mathrm{L}abc}(k,k') = \sum_X\iint d^4\xi d^4\eta e^{-ik\cdot\xi}e^{-i(k'-k)\cdot\eta}\langle 0|A^\nu_b(0)|hX\rangle \times\langle hX|A^\mu_a(\xi)gA^\lambda_c(\eta)|0\rangle, \quad (33)$$

$$\Gamma^{(1)\mu\nu\lambda}_{\mathrm{R}abc}(k,k') = \sum_X\iint d^4\xi d^4\eta e^{-ik\cdot\xi}e^{-i(k'-k)\cdot\eta} \times\langle 0|A^\nu_b(0)gA^\lambda_c(\eta)|hX\rangle\langle hX|A^\mu_a(\xi)|0\rangle, \quad (34)$$

$$\tilde{\Delta}^{(1)\alpha}_{\mathrm{L}a,ij}(k,k') = \sum_X\iint d^4\xi d^4\eta e^{-ik\cdot\xi}e^{-i(k'-k)\cdot\eta}\langle 0|gA^\alpha_a(\eta)|hX\rangle \times\langle hX|\psi_i(0)\overline{\psi}_j(\xi)|0\rangle, \quad (35)$$

$$\tilde{\Delta}^{(1)\alpha}_{\mathrm{R}a,ij}(k,k') = \sum_X\iint d^4\xi d^4\eta e^{-ik\cdot\xi}e^{-i(k'-k)\cdot\eta} \times\langle 0|\psi_i(0)\bar{\psi}_j(\xi)|hX\rangle\langle hX|gA^\alpha_a(\eta)|0\rangle. \quad (36)$$

The contribution with two parton lines in the left (right) of the cut in Figs. 2(b)–2(e) are characterized by the symbol L(R) in the hard parts and the fragmentation matrix elements. The superscripts (0) and (1) indicate the order of the gauge coupling $g$ corresponding, respectively, to the two-parton and three-parton correlation functions. The factor $1/2$ in Eqs. (28) and (29) takes into account the exchange symmetry in the corresponding matrix element. In Eqs. (30) and (31), the hard parts and the fragmentation matrix elements are matrices in both color and spinor spaces for the quark, and Tr indicates trace over both indices. The hard parts and the fragmentation matrix elements satisfy $\Gamma^{(1)\mu\nu\lambda}_{\mathrm{R}abc}(k,k') = \Gamma^{(1)\nu\mu\lambda}_{\mathrm{L}bac}(k',k)^*$, $\tilde{\Delta}^{(1)\alpha}_{\mathrm{R}a}(k,k')=\gamma^0\tilde{\Delta}^{(1)\alpha}_{\mathrm{L}a}(k',k)^\dagger\gamma^0$, $S^{\mathrm{R}abc}_{\mu\nu\lambda,\rho\sigma}(k,k')=S^{\mathrm{L}bac}_{\nu\mu\lambda,\sigma\rho}(k',k)^*$, and $\tilde{S}^{\mathrm{R}a}_{\alpha,\rho\sigma}(k,k') = \gamma^0\tilde{S}^{\mathrm{L}a}_{\alpha,\sigma\rho}(k',k)^\dagger\gamma^0$. We, thus, have

$$w_{\rho\sigma} = w^{(\mathrm{a})}_{\rho\sigma} + 2\mathrm{Re}w^{(\mathrm{b})}_{\rho\sigma} + 2\mathrm{Re}w^{(\mathrm{d})}_{\rho\sigma}. \quad (37)$$

To extract the twist-3 contribution to $ep\to e\Lambda^\uparrow X$, we apply the collinear expansion to the hard part, $S^{ab}_{\mu\nu,\rho\sigma}$, $S^{\mathrm{L}abc}_{\mu\nu\lambda,\rho\sigma}$, and $\tilde{S}^{\mathrm{L}a}_{\alpha,\rho\sigma}$, with respect to the momenta $k$ and $k'$ around $P_h/z$ and $P_h/z'$, respectively, taking into account of the following Ward-Takahashi (WT) identities [15]:

$$k^\mu S^{ab}_{\mu\nu,\rho\sigma}(k) = k^\nu S^{ab}_{\mu\nu,\rho\sigma}(k) = 0, \quad (38)$$

$$k^\mu S^{\mathrm{L}abc}_{\mu\nu\lambda,\rho\sigma}(k,k') = \frac{if^{abc}}{N^2-1}S_{\lambda\nu,\rho\sigma}(k'), \quad (39)$$

$$k'^\nu S^{\mathrm{L}abc}_{\mu\nu\lambda,\rho\sigma}(k,k') = 0, \quad (40)$$

$$(k'-k)^\lambda S^{\mathrm{L}abc}_{\mu\nu\lambda,\rho\sigma}(k,k') = \frac{-if^{abc}}{N^2-1}S_{\mu\nu,\rho\sigma}(k'), \quad (41)$$

$$(k-k')^\alpha\tilde{S}^{\mathrm{L}a}_{\alpha,\rho\sigma}(k,k') = 0, \quad (42)$$

where $S_{\mu\nu,\rho\sigma}(k)\equiv\delta^{ab}S^{ab}_{\mu\nu,\rho\sigma}(k)$. We note that, unlike the case for $pp\to\Lambda^\uparrow X$, no ghostlike terms appear in the WT identities (38)–(42) for the present case. This way, one obtains the hadronic tensor $w_{\rho\sigma}$ in terms of the gauge invariant FFs as [see Eq. (51) in Ref. [15] and Eq. (56) in Ref. [21]]

$$\begin{aligned} w_{\rho\sigma} &= \Omega^\alpha_\mu\Omega^\beta_\nu\int d\left(\frac{1}{z}\right)z^2\hat{\Gamma}^{\mu\nu}(z)S_{\alpha\beta,\rho\sigma}\left(\frac{1}{z}\right) - i\Omega^\alpha_\mu\Omega^\beta_\nu\Omega^\gamma{}_\lambda\int d\left(\frac{1}{z}\right)z^2\hat{\Gamma}^{\mu\nu\lambda}_\partial(z)\frac{\partial S_{\alpha\beta,\rho\sigma}(k)}{\partial k^\gamma}\bigg|_{k=P_h/z} \\ &+\mathrm{Re}\left[i\Omega^\alpha_\mu\Omega^\beta_\nu\Omega^\gamma{}_\lambda\iint d\left(\frac{1}{z}\right)d\left(\frac{1}{z'}\right)zz'\frac{1}{1/z-1/z'}\right. \\ &\times\left.\left\{-\frac{if^{abc}}{N}\hat{\Gamma}^{\mu\nu\lambda}_{FA}\left(\frac{1}{z'},\frac{1}{z}\right)+\frac{Nd^{abc}}{N^2-4}\hat{\Gamma}^{\mu\nu\lambda}_{FS}\left(\frac{1}{z'},\frac{1}{z}\right)\right\}S^{\mathrm{L}abc}_{\alpha\beta\gamma,\rho\sigma}\left(\frac{1}{z'},\frac{1}{z}\right)\right] \\ &+\mathrm{Re}\left[i\Omega^\alpha_\mu\iint d\left(\frac{1}{z}\right)d\left(\frac{1}{z'}\right)z\mathrm{Tr}_s\left\{\tilde{\Delta}^\mu\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\tilde{S}^{\mathrm{L}}_{\alpha,\rho\sigma}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\right\}\right], \end{aligned} \quad (43)$$

where $\Omega_\mu^\alpha = g_\mu^\alpha - P_h^\alpha w_\mu$ and $\hat{\Gamma}^{\mu\nu}(z)$, $\hat{\Gamma}_\partial^{\mu\nu\lambda}(z)$, $\hat{\Gamma}_{FA}^{\mu\nu\lambda}(\frac{1}{z'},\frac{1}{z})$, $\hat{\Gamma}_{FS}^{\mu\nu\lambda}(\frac{1}{z'},\frac{1}{z})$, and $\tilde{\Delta}^\mu(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z})$ are given by Eqs. (1), (2), (4), (5), and (7), respectively. For the hard part, we have used the notation $S_{\alpha\beta,\rho\sigma}(\frac{1}{z})$ for $S_{\alpha\beta,\rho\sigma}(\frac{P_h}{z})$ and $S^{\mathrm{L}abc}_{\alpha\beta\gamma,\rho\sigma}(\frac{1}{z'},\frac{1}{z})$ for $S^{\mathrm{L}abc}_{\alpha\beta\gamma,\rho\sigma}(\frac{P_h}{z'},\frac{P_h}{z})$, etc., suppressing $P_h$ for short. In the last term of Eq. (43), $\tilde{S}^{\mathrm{L}}_{\alpha,\rho\sigma}$ is defined from $\tilde{S}^{\mathrm{L}a}_{\alpha,\rho\sigma}$ in Eq. (30) by $(\tilde{S}^{\mathrm{L}a}_{\alpha,\rho\sigma})_{rs} = \frac{1}{2N} t^a_{rs} \tilde{S}^{\mathrm{L}}_{\alpha,\rho\sigma}$, where $r$, $s$ indicates the color indices for the quark and $\mathrm{Tr}_s$ denotes the trace in the spinor space. The LO diagrams for the hard parts of Figs. 2(a), 2(b), and 2(d) are, respectively, shown in Figs. 3, 4, and 5. It is

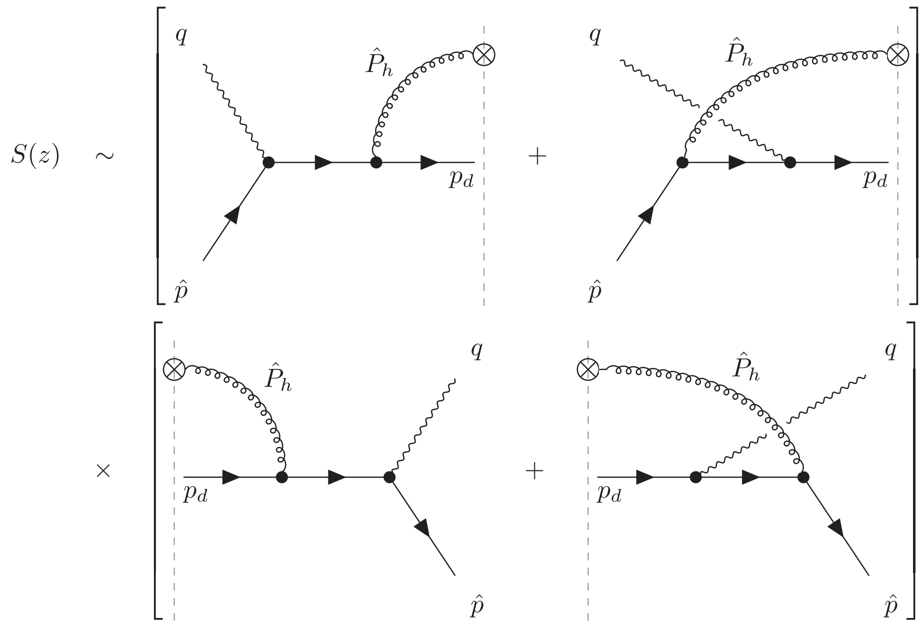


FIG. 3. The lowest-order Feynman diagrams for $S_{\alpha\beta,\rho\sigma}(\frac{1}{z})$ in Eq. (43). We set $\hat{p} \equiv xp$ and $\hat{P}_h \equiv P_h/z$. The symbol ⊗ indicates the fragmentation to the final hadron, and $p_d$ is the momentum of an unobserved parton in the final state.

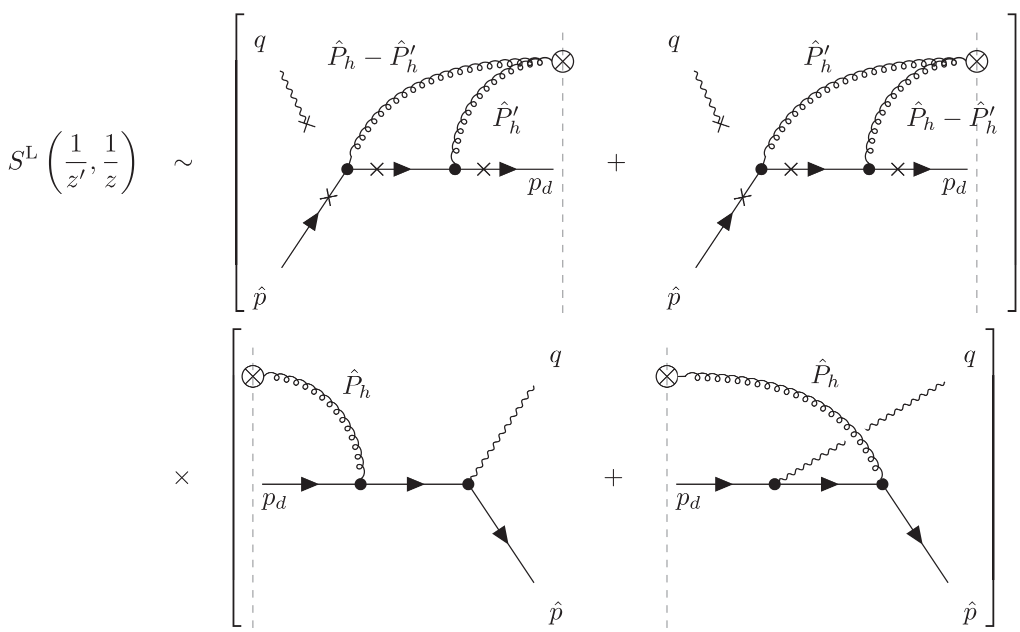


FIG. 4. The lowest-order Feynman diagrams for $S^{\mathrm{L}abc}_{\alpha\beta\gamma,\rho\sigma}(\frac{1}{z'},\frac{1}{z})$ in Eq. (43). We set $\hat{P}'_h \equiv P_h/z'$. Three crosses (×) on the quark line in the upper diagrams indicate that the virtual photon line with a cross at one end needs to be attached to one of these crosses, and all three diagrams have to be included. Thus, the number of diagrams for $S^{\mathrm{L}abc}_{\alpha\beta\gamma,\rho\sigma}$ is $(3+3)\times 2 = 12$. The meaning of the other symbols are the same as in Fig. 3.

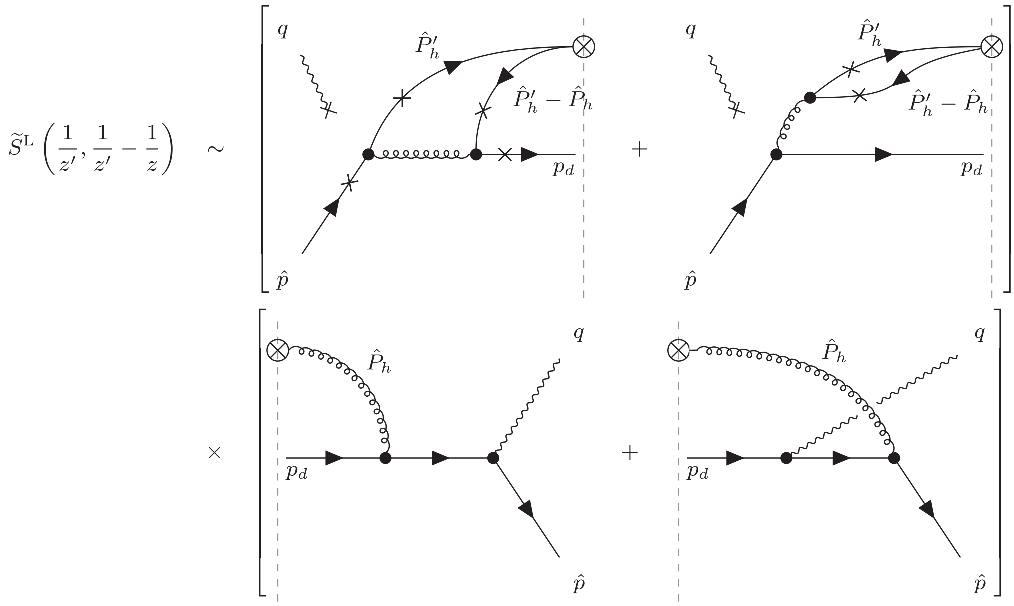


FIG. 5. The lowest-order Feynman diagrams for $\tilde{S}^{\rm L}_{\alpha,\rho\sigma}(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z})$. The meaning of the symbols is the same as in Fig. 4. The total number of diagrams for $\tilde{S}^{\rm L}_{\alpha,\rho\sigma}$ is $(4+2)\times 2 = 12$.

easy to show that the hadronic tensor $w_{\rho\sigma}$ satisfies the electromagnetic gauge invariance $q^\rho w_{\rho\sigma} = q^\sigma w_{\rho\sigma} = 0$, owing to the WT identity in QED.

### C. Spin-dependent cross section

The calculation of $L^{\rho\sigma}W_{\rho\sigma}$ in Eq. (24) can be done in the same way as Ref. [1]: $W^{\rho\sigma}$ can be expanded in terms of the six tensors [20] $\mathscr{V}_k^{\rho\sigma}$ $(k = 1, \ldots, 4, 8, 9)$ defined by

$$\mathscr{V}_1^{\mu\nu} = X^\mu X^\nu + Y^\mu Y^\nu, \qquad \mathscr{V}_2^{\mu\nu} = g^{\mu\nu} + Z^\mu Z^\nu,$$
$$\mathscr{V}_3^{\mu\nu} = T^\mu X^\nu + X^\mu T^\nu, \qquad \mathscr{V}_4^{\mu\nu} = X^\mu X^\nu - Y^\mu Y^\nu,$$
$$\mathscr{V}_8^{\mu\nu} = T^\mu Y^\nu + Y^\mu T^\nu, \qquad \mathscr{V}_9^{\mu\nu} = X^\mu Y^\nu + Y^\mu X^\nu. \qquad (44)$$

By introducing the inverses of $\mathscr{V}_k^{\rho\sigma}$ and $\tilde{\mathscr{V}}_k^{\rho\sigma}$ satisfying $\mathscr{V}_k^{\rho\sigma}\tilde{\mathscr{V}}_{k'\rho\sigma} = \delta_{kk'}$, as

$$\tilde{\mathscr{V}}_1^{\mu\nu} = \frac{1}{2}(2T^\mu T^\nu + X^\mu X^\nu + Y^\mu Y^\nu), \quad \tilde{\mathscr{V}}_2^{\mu\nu} = T^\mu T^\nu,$$
$$\tilde{\mathscr{V}}_3^{\mu\nu} = -\frac{1}{2}(T^\mu X^\nu + X^\mu T^\nu), \quad \tilde{\mathscr{V}}_4^{\mu\nu} = \frac{1}{2}(X^\mu X^\nu - Y^\mu Y^\nu),$$
$$\tilde{\mathscr{V}}_8^{\mu\nu} = -\frac{1}{2}(T^\mu Y^\nu + Y^\mu T^\nu), \quad \tilde{\mathscr{V}}_9^{\mu\nu} = \frac{1}{2}(X^\mu Y^\nu + Y^\mu X^\nu), \quad (45)$$

$W^{\mu\nu}$ can be expanded as

$$W^{\mu\nu} = \sum_{k=1,\ldots,4,8,9} \mathscr{V}_k^{\mu\nu}[W_{\rho\sigma}\tilde{\mathscr{V}}_k^{\rho\sigma}]. \qquad (46)$$

Then, one obtains

$$L^{\mu\nu}W_{\mu\nu} = \sum_{k=1,\ldots,4,8,9}[L_{\mu\nu}\mathscr{V}_k^{\mu\nu}][W_{\rho\sigma}\tilde{\mathscr{V}}_k^{\rho\sigma}]$$
$$= Q^2 \sum_{k=1,\ldots,4,8,9} \mathscr{A}_k(\phi-\chi)[W_{\rho\sigma}\tilde{\mathscr{V}}_k^{\rho\sigma}], \qquad (47)$$

where $\mathscr{A}_k(\varphi) \equiv L_{\mu\nu}\mathscr{V}_k^{\mu\nu}/Q^2$ are given by

$$\mathscr{A}_1(\varphi) = 1 + \cosh^2\psi, \qquad \mathscr{A}_2(\varphi) = -2,$$
$$\mathscr{A}_3(\varphi) = -\cos\varphi \sinh 2\psi, \qquad \mathscr{A}_4(\varphi) = \cos 2\varphi \sinh^2\psi,$$
$$\mathscr{A}_8(\varphi) = -\sin\varphi \sinh 2\psi, \qquad \mathscr{A}_9(\varphi) = \sin 2\varphi \sinh^2\psi, \qquad (48)$$

with $\psi$ defined in Eq. (19). From Eqs. (47) and (48), one sees the cross section can be decomposed into the five structure functions with different azimuthal dependences which are carried by the $\mathscr{A}_k(\varphi)$'s. Substituting Eqs. (1), (2), (4), (5), and (7) into Eq, (43), we find that the cross section (24) takes the following structure:

$$\frac{d^6\sigma}{dx_{bj}dQ^2dz_f dq_T^2 d\phi d\chi} = \frac{\alpha_{em}^2\alpha_s M_h}{16\pi^2 x_{bj}^2 S_{ep}^2 Q^2}\sum_k \mathscr{A}_k(\varphi)\mathcal{S}_k \iint dx d\left(\frac{1}{z}\right)\frac{z^3}{x} f_1(x)\delta\left(\frac{q_T^2}{Q^2} - \left(1-\frac{1}{\hat{x}}\right)\left(1-\frac{1}{\hat{z}}\right)\right)$$
$$\times\left\{\frac{1}{z}\Delta\hat{G}_{3\bar{T}}(z)\hat{\sigma}^k_{\rm int} + \hat{G}_T^{(1)}(z)\hat{\sigma}^k_{NDG} + \frac{1}{z}\frac{\partial\hat{G}_T^{(1)}(z)}{\partial(1/z)}\hat{\sigma}^k_{DG} + \Delta\hat{H}_T^{(1)}(z)\hat{\sigma}^k_{NDH} + \frac{1}{z}\frac{\partial\Delta\hat{H}_T^{(1)}(z)}{\partial(1/z)}\hat{\sigma}^k_{DH}\right.$$
$$+\frac{1}{2}\int d\left(\frac{1}{z'}\right)\left[\sum_{i=1}^{3}{\rm Im}\hat{N}_i\left(\frac{1}{z'},\frac{1}{z}\right)\left(\frac{1}{1/z-1/z'}\hat{\sigma}^{-(1)}_{i,k} + \frac{1}{z}\left(\frac{1}{1/z-1/z'}\right)^2\hat{\sigma}^{-(2)}_{i,k} + z'\hat{\sigma}^{-(3)}_{i,k} + \frac{z'^2}{z}\hat{\sigma}^{-(4)}_{i,k}\right)\right.$$
$$\left.+\sum_{i=1}^{3}\hat{O}_i\left(\frac{1}{z'},\frac{1}{z}\right)\left(\frac{1}{1/z-1/z'}\hat{\sigma}^{+(1)}_{i,k} + \frac{1}{z}\left(\frac{1}{1/z-1/z'}\right)^2\hat{\sigma}^{+(2)}_{i,k} + z'\hat{\sigma}^{+(3)}_{i,k} + \frac{z'^2}{z}\hat{\sigma}^{+(4)}_{i,k}\right)\right]$$
$$+\int d\left(\frac{1}{z'}\right)\frac{2}{C_F}\left[{\rm Im}\tilde{D}_{FT}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\left(\hat{\sigma}^k_{DF} + \frac{1}{z}\frac{1}{1/z-1/z'}\hat{\sigma}^k_{DF2} + \frac{z'}{z}\hat{\sigma}^k_{DF3}\right.\right.$$
$$\left.+\frac{1}{1-(1-q_T^2/Q^2)z_f/z'}\hat{\sigma}^k_{DF4} + \frac{1}{1-(1-q_T^2/Q^2)z_f(1/z-1/z')}\hat{\sigma}^k_{DF5}\right)$$
$$+{\rm Im}\tilde{G}_{FT}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\left(\hat{\sigma}^k_{GF} + \frac{1}{z}\frac{1}{1/z-1/z'}\hat{\sigma}^k_{GF2} + \frac{z'}{z}\hat{\sigma}^k_{GF3}\right.$$
$$\left.\left.\left.+\frac{1}{1-(1-q_T^2/Q^2)z_f/z'}\hat{\sigma}^k_{GF4} + \frac{1}{1-(1-q_T^2/Q^2)z_f(1/z-1/z')}\hat{\sigma}^k_{GF5}\right)\right]\right\}, \tag{49}$$

where

$$\mathcal{S}_{1,2,3,4} \equiv \sin\Phi_S, \qquad \mathcal{S}_{8,9} \equiv \cos\Phi_S,$$
$$\hat{x} = x_{bj}/x, \qquad \hat{z} = z_f/z \tag{50}$$

and we have set

$$\hat{N}_3\left(\frac{1}{z'},\frac{1}{z}\right) \equiv -\hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right),$$
$$\hat{O}_3\left(\frac{1}{z'},\frac{1}{z}\right) \equiv \hat{O}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right) \tag{51}$$

for convenience. Partonic hard parts for each FF can be computed from the corresponding diagrams, Figs. 3–5. We have reached the form (49) based on the observation that the $z'$ dependence of the hard parts for the dynamical FFs appears in the cross section only through the factors explicitly shown in Eq. (49) (see Appendix C in Ref. [15]), and, hence, we can define all the partonic hard cross section $\hat{\sigma}$'s in Eq. (49) as the functions of $\hat{x}$, $\hat{z}$, $Q$, and $q_T$. In addition, we found by explicit calculation of the LO diagrams that

$$\hat{\sigma}^{\pm(3)}_{i,k} = \hat{\sigma}^{\pm(1)}_{i,k}, \tag{52}$$
$$\hat{\sigma}^k_{DF} = \hat{\sigma}^k_{GF} = 0. \tag{53}$$

In order to transform the cross section (49) into a more concise form, we note the following points: (I) Owing to the symmetry property under $1/z' \leftrightarrow 1/z - 1/z'$ of $\hat{N}_1$ and $\hat{O}_1$ (6) and the relations (51), the terms of $\hat{\sigma}^{\pm(3)}_{i,k}$ and $\hat{\sigma}^{\pm(4)}_{i,k}$ can be combined, respectively, with those of $\hat{\sigma}^{\pm(1)}_{i,k}$ and $\hat{\sigma}^{\pm(2)}_{i,k}$, taking into account the relation (52). (II) Using Eqs. (8), (10), and (11), one can eliminate the intrinsic FF and the derivative of the kinematical FFs in favor of the kinematical and the dynamical FFs. This way, we finally obtain the twist-3 gluon FF contribution to $ep \to e\Lambda^\uparrow X$ as[2]

$$\frac{d^6\sigma}{dx_{bj}dQ^2dz_f dq_T^2 d\phi d\chi} = \frac{\alpha_{em}^2\alpha_s M_h}{16\pi^2 x_{bj}^2 S_{ep}^2 Q^2}\sum_k \mathscr{A}_k(\phi-\chi)\mathcal{S}_k \int_{x_{\rm min}}^1 \frac{dx}{x}\int_{z_{\rm min}}^1 \frac{dz}{z} z^2 f_1(x)\delta\left(\frac{q_T^2}{Q^2} - \left(1-\frac{1}{\hat{x}}\right)\left(1-\frac{1}{\hat{z}}\right)\right)$$
$$\times\left\{\hat{G}_T^{(1)}(z)\hat{\sigma}^k_G + \Delta\hat{H}_T^{(1)}(z)\hat{\sigma}^k_H\right.$$
$$+\int d\left(\frac{1}{z'}\right)\left[\frac{1}{1/z-1/z'}{\rm Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{N1} + \hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{N2} + \hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{N3}\right)\right.$$

[2] As was noted at the end of Sec. II, the kinematical FFs $\hat{G}_T^{(1)}(z)$ and $\Delta\hat{H}_T^{(1)}(z)$ can, in principle, be eliminated in terms of the dynamical FFs.

$$+\frac{1}{z}\left(\frac{1}{1/z-1/z'}\right)^2\mathrm{Im}\left(\hat{N}_1\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DN1}+\hat{N}_2\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DN2}+\hat{N}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DN3}\right)$$
$$+\frac{1}{1/z-1/z'}\mathrm{Im}\left(\hat{O}_1\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{O1}+\hat{O}_2\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{O2}+\hat{O}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{O3}\right)$$
$$+\frac{1}{z}\left(\frac{1}{1/z-1/z'}\right)^2\mathrm{Im}\left(\hat{O}_1\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DO1}+\hat{O}_2\left(\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DO2}+\hat{O}_2\left(\frac{1}{z}-\frac{1}{z'},\frac{1}{z}\right)\hat{\sigma}^k_{DO3}\right)\Bigg]$$
$$+\int d\left(\frac{1}{z'}\right)\frac{2}{C_F}\Bigg[\mathrm{Im}\tilde{D}_{FT}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\Bigg(\hat{\sigma}^k_{DF1}+\frac{1}{z}\frac{1}{1/z-1/z'}\hat{\sigma}^k_{DF2}+\frac{z'}{z}\hat{\sigma}^k_{DF3}$$
$$+\frac{1}{1-(1-q_T^2/Q^2)z_f/z'}\hat{\sigma}^k_{DF4}+\frac{1}{1-(1-q_T^2/Q^2)z_f(1/z-1/z')}\hat{\sigma}^k_{DF5}\Bigg)$$
$$+\mathrm{Im}\tilde{G}_{FT}\left(\frac{1}{z'},\frac{1}{z'}-\frac{1}{z}\right)\Bigg(\frac{1}{z}\frac{1}{1/z-1/z'}\hat{\sigma}^k_{GF2}+\frac{z'}{z}\hat{\sigma}^k_{GF3}$$
$$+\frac{1}{1-(1-q_T^2/Q^2)z_f/z'}\hat{\sigma}^k_{GF4}+\frac{1}{1-(1-q_T^2/Q^2)z_f(1/z-1/z')}\hat{\sigma}^k_{GF5}\Bigg)\Bigg]\Bigg\}, \tag{54}$$

where the lower limits of $x$ and $z$ are, respectively, given by $x_{\min}=x_{bj}(1+\frac{z_f}{1-z_f}\frac{q_T^2}{Q^2})$ and $z_{\min}=z_f(1+\frac{x_{bj}}{1-x_{bj}}\frac{q_T^2}{Q^2})$. The partonic hard cross sections which appear newly in Eq. (54) are defined from those in Eq. (49) as

$$\hat{\sigma}^k_G=\frac{1}{2}\hat{\sigma}^k_{\mathrm{int}}+\hat{\sigma}^k_{NDG}+2\hat{\sigma}^k_{DG}, \tag{55}$$

$$\hat{\sigma}^k_H=\frac{1}{2}\hat{\sigma}^k_{\mathrm{int}}+\hat{\sigma}^k_{NDH}+4\hat{\sigma}^k_{DH}, \tag{56}$$

$$\hat{\sigma}^k_{N1}=2\hat{\sigma}^k_{\mathrm{int}}+4\hat{\sigma}^k_{DG}+8\hat{\sigma}^k_{DH}, \tag{57}$$

$$\hat{\sigma}^k_{N2}=\hat{\sigma}^k_{\mathrm{int}}+8\hat{\sigma}^k_{DH}+\frac{1}{2}(\hat{\sigma}^{-(1)}_{2,k}-\hat{\sigma}^{-(1)}_{3,k}), \tag{58}$$

$$\hat{\sigma}^k_{N3}=-\hat{\sigma}^k_{\mathrm{int}}-4\hat{\sigma}^k_{DG}+\frac{1}{2}(\hat{\sigma}^{-(1)}_{2,k}-\hat{\sigma}^{-(1)}_{3,k}), \tag{59}$$

$$\hat{\sigma}^k_{DN1}=2\hat{\sigma}^k_{DG}+4\hat{\sigma}^k_{DH}+\frac{1}{2}(\hat{\sigma}^{-(2)}_{1,k}-\hat{\sigma}^{-(4)}_{1,k}), \tag{60}$$

$$\hat{\sigma}^k_{DN2}=2\hat{\sigma}^k_{DG}+4\hat{\sigma}^k_{DH}+\frac{1}{2}(\hat{\sigma}^{-(2)}_{2,k}-\hat{\sigma}^{-(4)}_{3,k}), \tag{61}$$

$$\hat{\sigma}^k_{DN3}=-4\hat{\sigma}^k_{DG}+\frac{1}{2}(\hat{\sigma}^{-(4)}_{2,k}-\hat{\sigma}^{-(2)}_{3,k}), \tag{62}$$

$$\hat{\sigma}^k_{O1}=\hat{\sigma}^{+(1)}_{1,k}, \tag{63}$$

$$\hat{\sigma}^k_{O2}=\frac{1}{2}(\hat{\sigma}^{+(1)}_{2,k}+\hat{\sigma}^{+(1)}_{3,k}), \tag{64}$$

$$\hat{\sigma}^k_{O3}=\frac{1}{2}(\hat{\sigma}^{+(1)}_{2,k}+\hat{\sigma}^{+(1)}_{3,k}), \tag{65}$$

$$\hat{\sigma}^k_{DO1}=\frac{1}{2}(\hat{\sigma}^{+(2)}_{1,k}+\hat{\sigma}^{+(4)}_{1,k}), \tag{66}$$

$$\hat{\sigma}^k_{DO2}=\frac{1}{2}(\hat{\sigma}^{+(2)}_{2,k}+\hat{\sigma}^{+(4)}_{3,k}), \tag{67}$$

$$\hat{\sigma}^k_{DO3}=\frac{1}{2}(\hat{\sigma}^{+(4)}_{2,k}+\hat{\sigma}^{+(2)}_{3,k}), \tag{68}$$

$$\hat{\sigma}^k_{DF1}=-\hat{\sigma}^k_{\mathrm{int}}-2\hat{\sigma}^k_{DG}-4\hat{\sigma}^k_{DH}, \tag{69}$$

and others are the same as those appearing in Eq. (49). Although $\hat{\sigma}^k_{DF}=0$ as shown in Eq. (53), $\hat{\sigma}^k_{DF1}$ term appears in Eq. (54) due to the relations (8), (10), and (11).

To write down the partonic hard cross sections in Eq. (54), we further take into account the following relations:

$$\hat{\sigma}^k_{O2}=\hat{\sigma}^k_{O3}, \tag{70}$$

$$\hat{\sigma}^k_{DF1}=-\frac{1}{2}\hat{\sigma}^k_{N1}, \tag{71}$$

$$\hat{\sigma}^k_{DN1}=\hat{\sigma}^k_{DN2}=\hat{\sigma}^k_{DO1}=\hat{\sigma}^k_{DO2}, \tag{72}$$

$$\hat{\sigma}^k_{DN3}=-\hat{\sigma}^k_{DO3}. \tag{73}$$

The relations (70) and (71) are obvious from Eqs. (64), (65), (57), and (69), and Eqs. (72) and (73) are obtained by explicit calculation of the LO diagrams.

Then, the independent hard cross sections are given as follows:

$$\hat{\sigma}_G^1 = C_F \frac{2Q^2}{q_T^3} \frac{(-1+\hat{z})^2(-1-\hat{x}^2+\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}_G^2 = C_F \frac{8}{q_T} \hat{x}(-1+\hat{z}),$$
$$\hat{\sigma}_G^3 = C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(\hat{z}-\hat{x}-2\hat{z}\,\hat{x}+\hat{z}^2(-2+4\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_G^4 = \frac{1}{2}\hat{\sigma}_G^2,$$
$$\hat{\sigma}_G^8 = C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_G^9 = C_F \frac{4}{q_T} \frac{\hat{x}(-1+\hat{z})}{\hat{z}}, \tag{74}$$

$$\hat{\sigma}_H^1 = -C_F \frac{4Q^2}{q_T^3} \frac{(-1+\hat{z})^2}{\hat{z}^2}, \qquad \hat{\sigma}_H^2 = 0, \qquad \hat{\sigma}_H^3 = -C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(-1+2\hat{z})}{\hat{z}^2},$$
$$\hat{\sigma}_H^4 = -C_F \frac{4}{q_T} \frac{(-1+\hat{z})}{\hat{z}}, \qquad \hat{\sigma}_H^8 = \hat{\sigma}_H^3, \qquad \hat{\sigma}_H^9 = \hat{\sigma}_H^4, \tag{75}$$

$$\hat{\sigma}_{N1}^1 = C_F \frac{8Q^2}{q_T^3} \frac{(-1+\hat{z})^2(3\hat{z}(1-2\hat{x})\hat{x}+\hat{x}(1+\hat{x})+\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}_{N1}^2 = C_F \frac{32}{q_T} \frac{\hat{x}(-1+\hat{z})^2}{\hat{z}},$$
$$\hat{\sigma}_{N1}^3 = C_F \frac{8Q}{q_T^2} \frac{(-1+\hat{z})^2(-1-2\hat{x}+\hat{z}(-2+4\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{N1}^4 = -C_F \frac{8}{q_T} \frac{(-1+\hat{z})(1-2(-1+\hat{z})x)}{\hat{z}},$$
$$\hat{\sigma}_{N1}^8 = -C_F \frac{8Q}{q_T^2} \frac{(-1+\hat{z})^2}{\hat{z}^2}, \qquad \hat{\sigma}_{N1}^9 = -C_F \frac{8}{q_T} \frac{(-1+\hat{z})}{\hat{z}}, \tag{76}$$

$$\hat{\sigma}_{N2}^1 = -C_F \frac{4Q^2}{q_T^3} \frac{(-1+\hat{z})^2(-1+(-1+3\hat{z})\hat{x})}{\hat{z}^3}, \qquad \hat{\sigma}_{N2}^2 = -C_F \frac{8}{q_T} \frac{\hat{x}(-1+\hat{z})}{\hat{z}},$$
$$\hat{\sigma}_{N2}^3 = -C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(-3(1+\hat{x})+\hat{z}(3+4\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{N2}^4 = -C_F \frac{4}{q_T} \frac{(-1+\hat{z})(2+\hat{x})}{\hat{z}},$$
$$\hat{\sigma}_{N2}^8 = -C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(-3-\hat{x}+\hat{z}(3+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{N2}^9 = \hat{\sigma}_{N2}^4, \tag{77}$$

$$\hat{\sigma}_{N3}^1 = -C_F \frac{4Q^2}{q_T^3} \frac{(-1+\hat{z})^2(3\hat{z}(2-3\hat{x})\hat{x}+\hat{x}(1+\hat{x})+2\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}_{N3}^2 = -C_F \frac{8}{q_T} \frac{\hat{x}(-1+\hat{z})(-3+4\hat{z})}{\hat{z}},$$
$$\hat{\sigma}_{N3}^3 = -C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(1+\hat{z}(7-20\hat{x})+5\hat{x}+8\hat{z}^2(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{N3}^4 = \frac{1}{2}\hat{\sigma}_{N3}^2,$$
$$\hat{\sigma}_{N3}^8 = -C_F \frac{2Q}{q_T^2} \frac{(-1+\hat{z})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{N3}^9 = -C_F \frac{4}{q_T} \frac{\hat{x}(-1+\hat{z})}{\hat{z}}, \tag{78}$$

$$\hat{\sigma}_{DN1}^1 = C_F \frac{2Q^2}{q_T^3} \frac{(-1+\hat{z})^2(2\hat{z}(1-3\hat{x})\hat{x}+(1+\hat{x})^2+\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}_{DN1}^2 = \frac{1}{4}\hat{\sigma}_{N1}^2,$$
$$\hat{\sigma}_{DN1}^3 = C_F \frac{4Q}{q_T^2} \frac{(-1+\hat{z})^2(-1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{DN1}^4 = -C_F \frac{4}{q_T} \frac{(-1+\hat{z})(1+\hat{x}-\hat{x}\,\hat{z})}{\hat{z}},$$
$$\hat{\sigma}_{DN1}^8 = \frac{1}{2}\hat{\sigma}_{N1}^8, \qquad \hat{\sigma}_{DN1}^9 = \frac{1}{2}\hat{\sigma}_{N1}^9, \tag{79}$$

$$\hat{\sigma}_{DN3}^1 = -C_F \frac{4Q^2}{q_T^3} \frac{(-1+\hat{z})^2(1+2\hat{z}(2-3\hat{x})\hat{x}+\hat{x}^2+\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}_{DN3}^2 = -2\hat{\sigma}_{DN1}^2,$$
$$\hat{\sigma}_{DN3}^3 = -C_F \frac{8Q}{q_T^2} \frac{(-1+\hat{z})^2(-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}_{DN3}^4 = -\hat{\sigma}_{DN1}^2, \qquad \hat{\sigma}_{DN3}^8 = 0, \qquad \hat{\sigma}_{DN3}^9 = 0, \tag{80}$$

$$\hat{\sigma}^1_{O1} = -C_F \frac{8Q^2}{q_T^3}\frac{(-1+\hat{z})^2(-1-\hat{x}+\hat{z}(-2+3\hat{x}))}{\hat{z}^3}, \qquad \hat{\sigma}^2_{O1} = -C_F\frac{16}{q_T}\frac{\hat{x}(-1+\hat{z})}{\hat{z}},$$
$$\hat{\sigma}^3_{O1} = -C_F \frac{4Q}{q_T^2}\frac{(-1+\hat{z})(-1-3\hat{x}+\hat{z}(-1+4\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}^4_{O1} = \frac{1}{2}\hat{\sigma}^2_{O1},$$
$$\hat{\sigma}^8_{O1} = -C_F \frac{4Q}{q_T^2}\frac{(-1+\hat{z})(-1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}^9_{O1} = \frac{1}{2}\hat{\sigma}^2_{O1}, \tag{81}$$

$$\hat{\sigma}^1_{O2} = C_F \frac{4Q^2}{q_T^3}\frac{(-1+\hat{z})^2(1+\hat{x}+3\hat{z}(2-3\hat{x})\hat{x}+2\hat{x}^2+\hat{z}^2(1-6\hat{x}+6\hat{x}^2))}{\hat{x}\hat{z}^3}, \qquad \hat{\sigma}^2_{O2} = -C_F\frac{8}{q_T}\frac{\hat{x}(-1+\hat{z})(3-2\hat{z})}{\hat{z}},$$
$$\hat{\sigma}^3_{O2} = C_F \frac{2Q}{q_T^2}\frac{(-1+\hat{z})(1+\hat{z}(5-16\hat{x})+7\hat{x}+\hat{z}^2(-4+8\hat{x}))}{\hat{z}^2}, \qquad \hat{\sigma}^4_{O2} = \frac{1}{2}\hat{\sigma}^2_{O2}, \qquad \hat{\sigma}^8_{O2} = \frac{1}{2}\hat{\sigma}^8_{O1}, \qquad \hat{\sigma}^9_{O2} = \frac{1}{2}\hat{\sigma}^9_{O1}, \tag{82}$$

$$\hat{\sigma}^1_{DF2} = \frac{C_F}{N}\frac{Q^2}{q_T^3}\frac{(-1+\hat{z})(1+\hat{x}-\hat{z}\,\hat{x})}{\hat{x}\hat{z}^2}, \qquad \hat{\sigma}^2_{DF2} = 0,$$
$$\hat{\sigma}^3_{DF2} = -\frac{C_F}{N}\frac{Q}{q_T^2}\frac{(-1+\hat{z})}{\hat{z}}, \qquad \hat{\sigma}^4_{DF2} = -\frac{C_F}{N}\frac{1}{q_T},$$
$$\hat{\sigma}^8_{DF2} = \hat{\sigma}^3_{DF2}, \qquad \hat{\sigma}^9_{DF2} = \hat{\sigma}^4_{DF2}, \tag{83}$$

$$\hat{\sigma}^1_{DF3} = -\frac{C_F}{N}\frac{1}{q_T}\frac{3\hat{z}(1-2\hat{x})\hat{x}+\hat{x}(1+\hat{x})+\hat{z}^2(1-6\hat{x}+6\hat{x}^2)}{\hat{z}^2}, \qquad \hat{\sigma}^2_{DF3} = -\frac{C_F}{N}\frac{4q_T}{Q^2}\hat{x}^2,$$
$$\hat{\sigma}^3_{DF3} = -\frac{C_F}{N}\frac{1}{Q}\frac{\hat{x}(-1-2\hat{x}+\hat{z}(-2+4\hat{x}))}{\hat{z}}, \qquad \hat{\sigma}^4_{DF3} = -\frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(-1+2(-1+\hat{z})\hat{x})}{\hat{z}},$$
$$\hat{\sigma}^8_{DF3} = \frac{C_F}{N}\frac{1}{Q}\frac{\hat{x}}{\hat{z}}, \qquad \hat{\sigma}^9_{DF3} = \frac{C_F}{N}\frac{1}{q_T}\frac{-1+\hat{x}}{\hat{z}}, \tag{84}$$

$$\hat{\sigma}^1_{DF4} = \frac{C_F}{N}\frac{Q^2}{q_T^3}\frac{(-1+\hat{z})(-1+\hat{z}+5\hat{x}-6\hat{z}\,\hat{x}-6\hat{x}^2+6\hat{z}\hat{x}^2)}{\hat{x}^2}$$
$$+\frac{C_F}{q_T}\frac{(\hat{x}-1)^2(1+\hat{x})-\hat{z}(\hat{x}-1)^2(1+6\hat{x})-\hat{z}^3(1-6\hat{x}+6\hat{x}^2)+\hat{z}^2(1+\hat{x}-6\hat{x}^2+6\hat{x}^3)}{\hat{z}\,\hat{x}(-1+\hat{z}+\hat{x})},$$
$$\hat{\sigma}^2_{DF4} = \frac{C_F}{N}\frac{4}{q_T}\hat{z}(-1+\hat{z}) - C_F\frac{4q_T}{Q^2}\frac{\hat{z}(1+\hat{z}-\hat{x})\hat{x}}{(-1+\hat{z}+\hat{x})},$$
$$\hat{\sigma}^3_{DF4} = \frac{C_F}{N}\frac{2Q}{q_T^2}\frac{(-1+\hat{z})(-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}} + \frac{2C_F}{Q}\frac{(\hat{z}+\hat{z}^2+\hat{x}-2\hat{z}\,\hat{x}-2\hat{z}^2\hat{x}-\hat{x}^2+2\hat{z}\hat{x}^2)}{(-1+\hat{z}+\hat{x})},$$
$$\hat{\sigma}^4_{DF4} = \frac{C_F}{N}\frac{1}{q_T}\frac{(1+\hat{x}+2\hat{z}^2\hat{x}-\hat{z}(1+2\hat{x}))}{\hat{x}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1+\hat{z}^2\hat{x}+\hat{x}^2-\hat{z}(1+\hat{x}^2))}{\hat{x}(-1+\hat{z}+\hat{x})},$$
$$\hat{\sigma}^8_{DF4} = \frac{C_F}{N}\frac{2}{Q}\hat{z} - C_F\frac{4}{Q}\frac{\hat{z}(-1+\hat{x})}{(-1+\hat{z}+\hat{x})},$$
$$\hat{\sigma}^9_{DF4} = \frac{C_F}{N}\frac{1}{q_T}\frac{(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}(-1+\hat{z}+\hat{x})}, \tag{85}$$

$$\hat{\sigma}^1_{DF5} = \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})^2(1+\hat{x}+6\hat{z}^2\hat{x}-\hat{z}(1+6\hat{x}))}{\hat{z}^2\hat{x}} - \frac{C_F}{q_T}\frac{(\hat{x}-1)^2(1+\hat{x})-\hat{z}(\hat{x}-1)^2(1+6\hat{x})-\hat{z}^3(1-6\hat{x}+6\hat{x}^2)+\hat{z}^2(1+\hat{x}-6\hat{x}^2+6\hat{x}^3)}{\hat{z}\,\hat{x}(-1+\hat{z}+\hat{x})},$$

$$\hat{\sigma}^2_{DF5} = \frac{C_F}{N}\frac{4q_T}{Q^2}(-1+\hat{x})\hat{x} + C_F\frac{4q_T}{Q^2}\frac{\hat{z}(1+\hat{z}-x)\hat{x}}{(-1+\hat{z}+\hat{x})},$$

$$\hat{\sigma}^3_{DF5} = \frac{C_F}{N}\frac{2}{Q}\frac{(-1+\hat{x})(-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}} - \frac{2C_F}{Q}\frac{(\hat{z}+\hat{z}^2+\hat{x}-2\hat{z}\,\hat{x}-2\hat{z}^2\hat{x}-\hat{x}^2+2\hat{z}\hat{x}^2)}{(-1+\hat{z}+\hat{x})},$$

$$\hat{\sigma}^4_{DF5} = \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(-1+\hat{z}+\hat{x}-2\hat{z}\,\hat{x}-2\hat{x}^2+2\hat{z}\hat{x}^2)}{\hat{x}\,\hat{z}} + C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1+\hat{z}^2\hat{x}+\hat{x}^2-\hat{z}(1+\hat{x}^2))}{\hat{x}(-1+\hat{z}+\hat{x})},$$

$$\hat{\sigma}^8_{DF5} = -\frac{C_F}{N}\frac{2}{Q}(-1+\hat{x}) + C_F\frac{4}{Q}\frac{\hat{z}(-1+\hat{x})}{(-1+\hat{z}+\hat{x})},$$

$$\hat{\sigma}^9_{DF5} = -\frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}\,\hat{z}} + C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}(-1+\hat{z}+\hat{x})}, \tag{86}$$

$$\hat{\sigma}^1_{GF2} = -\frac{C_F}{N},\frac{Q^2}{q_T^3}\frac{(-1+\hat{z})(1-\hat{x}+\hat{z}\,\hat{x})}{\hat{x}\hat{z}^2}, \qquad \hat{\sigma}^2_{GF2} = 0, \qquad \hat{\sigma}^3_{GF2} = \hat{\sigma}^3_{DF2},$$

$$\hat{\sigma}^4_{GF2} = \hat{\sigma}^4_{DF2}, \qquad \hat{\sigma}^8_{GF2} = \hat{\sigma}^8_{DF2}, \qquad \hat{\sigma}^9_{GF2} = \hat{\sigma}^9_{DF2}, \tag{87}$$

$$\hat{\sigma}^1_{GF3} = -\frac{C_F}{N}\frac{1}{q_T}\frac{\hat{z}(5-6\hat{x})\hat{x}+\hat{x}(-1+\hat{x})+\hat{z}^2(1-6\hat{x}+6\hat{x}^2)}{\hat{z}^2}, \qquad \hat{\sigma}^2_{GF3} = \hat{\sigma}^2_{DF3},$$

$$\hat{\sigma}^3_{GF3} = -\frac{C_F}{N}\frac{1}{Q}\frac{\hat{x}(-1+2\hat{z})(-1+2\hat{x})}{\hat{z}}, \qquad \hat{\sigma}^4_{GF3} = -\frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(1+2(-1+\hat{z})\hat{x})}{\hat{z}},$$

$$\hat{\sigma}^8_{GF3} = -\hat{\sigma}^8_{DF3}, \qquad \hat{\sigma}^9_{GF3} = -\hat{\sigma}^9_{DF3}, \tag{88}$$

$$\hat{\sigma}^1_{GF4} = \frac{C_F}{N}\frac{Q^2}{q_T^3}\frac{(-1+\hat{z})(-1+\hat{z}+7\hat{x}-6\hat{z}\,\hat{x}-6\hat{x}^2+6\hat{z}\hat{x}^2)}{\hat{x}^2} - C_F\frac{1}{q_T}\frac{(-1+\hat{x})^2-6\hat{z}(-1+\hat{x})\hat{x}+\hat{z}^2(1-6\hat{x}+6\hat{x}^2)}{\hat{z}\,\hat{x}},$$

$$\hat{\sigma}^2_{GF4} = \frac{C_F}{N}\frac{4}{q_T}\hat{z}(-1+\hat{z}) - C_F\frac{4q_T}{Q^2}\hat{z}\,\hat{x},$$

$$\hat{\sigma}^3_{GF4} = \frac{C_F}{N}\frac{2Q}{q_T^2}\frac{(-1+\hat{z})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}} - C_F\frac{2}{Q}(1-\hat{x}+\hat{z}(-1+2\hat{x})),$$

$$\hat{\sigma}^4_{GF4} = \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{z}+\hat{x}-2\hat{z}\,\hat{x}+2\hat{z}^2\hat{x})}{\hat{x}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1+(-1+\hat{z})\hat{x})}{\hat{x}},$$

$$\hat{\sigma}^8_{GF4} = \frac{C_F}{N}\frac{2Q}{q_T^2}(-1+\hat{z}) - C_F\frac{2}{Q},$$

$$\hat{\sigma}^9_{GF4} = \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{z}-\hat{x}+2\hat{z}\,\hat{x})}{\hat{x}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})}{\hat{x}}, \tag{89}$$

$$
\begin{aligned}
\hat{\sigma}^1_{GF5} &= \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})^2(-1+\hat{z}+\hat{x}-6\hat{z}\,\hat{x}+6\hat{z}^2\hat{x})}{\hat{z}^2\hat{x}} - C_F\frac{1}{q_T}\frac{(-1+\hat{x})^2-6\hat{z}(-1+\hat{x})\hat{x}+\hat{z}^2(1-6\hat{x}+6\hat{x}^2)}{\hat{z}\,\hat{x}},\\
\hat{\sigma}^2_{GF5} &= \frac{C_F}{N}\frac{4q_T}{Q^2}(-1+\hat{x})\hat{x} - C_F\frac{4q_T}{Q^2}\hat{z}\,\hat{x},\\
\hat{\sigma}^3_{GF5} &= \frac{C_F}{N}\frac{2}{Q}\frac{(-1+\hat{x})(1-\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{z}} - C_F\frac{2}{Q}(1-\hat{x}+\hat{z}(-1+2\hat{x})),\\
\hat{\sigma}^4_{GF5} &= \frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(-1+\hat{z}+3\hat{x}-2\hat{z}\,\hat{x}-2\hat{x}^2+2\hat{z}\hat{x}^2)}{\hat{x}\,\hat{z}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})(1+(-1+\hat{z})\hat{x})}{\hat{x}},\\
\hat{\sigma}^8_{GF5} &= -\frac{C_F}{N}\frac{2}{Q}\frac{(-1+\hat{z})(-1+\hat{x})}{\hat{z}} - C_F\frac{2}{Q},\\
\hat{\sigma}^9_{GF5} &= -\frac{C_F}{N}\frac{1}{q_T}\frac{(-1+\hat{x})(1-3\hat{x}+\hat{z}(-1+2\hat{x}))}{\hat{x}\,\hat{z}} - C_F\frac{2}{q_T}\frac{(-1+\hat{x})}{\hat{x}}.
\end{aligned}
\tag{90}
$$

Equations (74)–(90) and the relations (70)–(73) specify all the partonic cross sections in the final formula (54).

## IV. SUMMARY

In this paper, we have studied the transversely polarized spin-1/2 hyperon production in semi-inclusive deep inelastic scattering (SIDIS), $ep \to e\Lambda^{\uparrow}X$. Specifically, we have derived the LO twist-3 gluon FF contribution to the polarized cross section. Since the twist-3 gluon FFs are related to the $q\bar{q}g$ FFs through the EOM relations and the LIRs, we consistently took into account the latter contribution together. This has completed the twist-3 LO cross section for this process together with the results for the contribution from the twist-3 DF and the twist-3 quark FFs derived in Ref. [1]. The final result for the cross section is given in Eq. (54). It consists of five components with different azimuthal structures as

$$
\begin{aligned}
\frac{d^6\sigma}{dx_{bj}dQ^2dz_f dq_T^2 d\phi d\chi} &= \mathcal{F}_1\sin\Phi_S + \mathcal{F}_2\sin\Phi_S\cos\varphi\\
&\quad + \mathcal{F}_3\sin\Phi_S\cos 2\varphi\\
&\quad + \mathcal{F}_4\cos\Phi_S\sin\varphi\\
&\quad + \mathcal{F}_5\cos\Phi_S\sin 2\varphi,
\end{aligned}
\tag{91}
$$

where $\varphi = \phi - \chi$ is the relative azimuthal angle between the lepton ($\phi$) and the hadron ($\chi$) planes and $\Phi_S$ is the azimuthal angle of the transverse spin vector of $\Lambda^{\uparrow}$ measured from the hadron plane with the structure functions $\mathcal{F}_{1,2,3,4,5}$ written as the convolution of the twist-3 FFs and the quark DF in the proton and the partonic hard cross sections. The LO cross section given in Ref. [1] and the present study contains several unknown nonperturbative functions, and their determination requires global analyses of many twist-3 processes in which the same twist-3 functions appear. Information from analyses of small-$P_T$ data in terms of the TMD factorization is also of great help to constrain some of the twist-3 functions. In any case, our twist-3 cross section formula is the starting point of analyzing the large-$P_T$ hyperon polarization in SIDIS, which we hope to be measured in the future EIC experiment.

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