Implications of the topological Chern-Simons mass in the gap equation

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In this paper, we solve the gap equation of the Yang-Mills-Gribov-Zwanziger-Chern-Simons theory by considering the first order in the Chern-Simons topological mass term, M. As a result, we find three possible solutions to the gap equation, i.e., three different Gribov parameters, two of which can be eliminated by the regime of the theory. In addition, our conclusion about the regime of the theory is different from the literature. We also show that, in our case, there is no weak-coupling regime.

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I. INTRODUCTION

It is well known that the nonperturbative non-Abelian gauge field theory is plagued by gauge copies—the famous Gribov copies [1]. Gribov noticed that the standard gauge fixing procedure by Faddeev-Popov is not sufficient to remove all equivalent gauge field inside of the Gribov region, the Faddeev-Popov operator, $\mathcal{M}^{ab} = -\partial_\mu D_\mu^{ab}$, is strictly positive, $\Omega = \{A_\mu^a; \partial_\mu A_\mu^a, \mathcal{M}^{ab} > 0\}$. In 44 years, a lot of research has been focused on the Gribov problem; see Refs. [2–8] for examples. The pioneer work was done by Zwanziger, who figured out how to implement the Gribov region in the action; hence, a new formalism called Gribov-Zwanziger formalism was created [9–11], which some authors have used to understand and explain confinement/deconfinement phase transition [12–15].

Another method to study confinement/deconfinement is through three-dimensional (3D) Euclidean Yang-Mills-Chern-Simons theory [16,17]. This method is interesting because the topological Chern-Simons (CS) mass term gives the gluon field an extra mass generating a confinement/deconfinement transition phase in 3D Euclidean Yang-Mills [12,13].

As Gribov-Zwanziger theory and Yang-Mills-Chern-Simons theory give us features about the regime of the theory, it is intriguing to assemble both theories and then to solve the gap equation for the Gribov parameter [1,9–11] and to analyze the confinement/deconfinement phase transition in the presence of the CS topological mass M at least in the first order.

In Sec. II, we introduce the Gribov-Zwanzinger formalism and the idea of confinement/deconfinement in this formalism to the reader. In Sec. III, we solve the gap equation using Yang-Mills-Gribov-Chern-Simons theory. In Sec. IV, we briefly review the gluon propagator calculus, we analyze the regime of the theory by using the Gribov parameters found in Sec. III, and we eliminate the Gribov parameters without physical meaning. Finally, we present our conclusions in Sec. V.

II. YANG-MILLS-GRIBOV-ZWANZIGER THEORY

In 1977, Gribov presented a new way to interpret quark and gluon confinement [1]. He showed that the Faddeev-Popov procedure is not enough to remove the gauge copies present in the Yang-Mills path integral in strong interaction at low temperature. He solved the problem by adding an extra restriction. This new restriction modifies the gauge sector by adding a masslike term, called Gribov mass. The presence of this term revises the gauge propagator exhibiting propagations of nonphysical excitations. This can be interpreted as signals of confinement; i.e., perturbatively physical excitations of the theory cannot be described in the infrared regime. Throughout this section, we describe the Gribov framework in more details.

A. Gribov restriction in Yang-Mills theories

It is well known from the literature that the Euclidean Yang-Mills path integral reads

$$Z[J] = \int DAe^{-S_{SYM}}, \tag{2.1}$$

where

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$$S_{\rm YM} = \frac{1}{4}\int d^d x\, F^a_{\mu\nu} F^a_{\mu\nu}, \tag{2.2}$$

with $F^a_{\mu\nu}$ being the field strength tensor, which is defined by the equation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{2.3}$$

Equation (2.1) is plagued with gauge redundancy, the famous gauge copies or Gribov copies. These gauge copies are removed by the Faddeev-Popov (FP) procedure in weak interactions and in strong interaction at high energies. However, in the nonperturbative regime, these gauge copies cannot be eliminated by the FP procedure.

By using the FP procedure, the path integral can be written as

$$Z[J] = \int DADcD\bar{c}e^{-S_{FP}}, \tag{2.4}$$

where

$$S_{FP} = S_{\rm YM} + S_{gf} \tag{2.5}$$

and

$$S_{gf} = \int d^d x (b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b). \tag{2.6}$$

In the above equations, $(\bar{c}^a, c^a)$ are the Faddeev-Popov ghosts, b^a is the Lagrange multiplier implementing the Landau gauge, $D^{ab}_\mu = (\delta^{ab}\partial_\mu + g f^{acb} A^c_\mu)$ is the covariant derivative in the adjoint representation of $SU(N)$.

As we said previously, in Ref. [1], Gribov proved that the FP procedure is not enough to remove the gauge redundancy in (2.1); i.e., in (2.4), we are still overcounting gauge field configurations. To solve this problem, Gribov demonstrated that in the Landau gauge, $\partial_\mu A^a_\mu = 0$, we have to restrict (2.4) to a region where the FP operator is positive ($\mathcal{M}^{ab} > 0$). This region is called the Gribov region and is defined as

$$\Omega = \{A^a_\mu; \partial_\mu A^a_\mu = 0; \mathcal{M}^{ab} = -(\partial^2 \delta^{ab} - g f^{abc} A^c_\mu \partial_\mu) > 0\}. \tag{2.7}$$

The restricted path integral (2.4) reads

$$\begin{aligned} Z[J] &= \int_\Omega DADcD\bar{c}e^{-S_{\rm YM}-S_{gf}} \\ &= \int DADcD\bar{c}\mathcal{V}(\Omega)e^{-S_{\rm YM}-S_{gf}}. \end{aligned} \tag{2.8}$$

After imposing the Gribov region on the path integral by assuming a condition over the FP operator, we need to analyze the ghost propagator in the presence of an external gauge field,

$$\mathcal{G}(k, A) = \frac{1}{N^2 - 1}\delta^{ab}(\mathcal{M}^{-1})^{ab}(k, A). \tag{2.9}$$

As Gribov pointed out in Ref. [1], due to the presence of the external gauge field, the above ghost propagator can be written as

$$\mathcal{G}(k, A) = \frac{1}{k^2}(1 + \sigma(k, A)), \tag{2.10}$$

where $\sigma(k, A)$ is the so-called ghost form factor. In the absence of the gauge field, this factor goes to zero. As a result, we obtain the free ghost propagator again. From here, we can see that there is a relation between $\sigma(k, A)$ and the restriction $\mathcal{V}(\Omega)$ in (2.8). At first order in the gauge fields, the connected ghost two-point function reads

$$\begin{aligned} \mathcal{G}(k,A) &= \frac{1}{k^2}\left(1 + \frac{k_\mu k_\nu}{k^2}\frac{Ng^2}{Vd(N^2-1)}\int \frac{d^d p}{(2\pi)^4}\frac{A^a_\mu(k)A^a_\nu(-k)}{(k-p)^2}\right) \\ &\approx \frac{1}{k^2}\frac{1}{(1-\sigma(k,A))}. \end{aligned} \tag{2.11}$$

A condition called the no-pole condition is implemented,

$$\sigma(k, A) < 1. \tag{2.12}$$

In the limit $k \to 0$, the ghost form factor reads

$$\sigma(0, A) = \frac{1}{V}\frac{1}{d}\frac{Ng^2}{N^2 - 1}\int \frac{d^d p}{(2\pi)^4}\frac{A^a_\mu(p)A^a_\mu(-p)}{p^2}. \tag{2.13}$$

The no-pole condition keeps the path integral (2.8) inside the Gribov region: $\mathcal{V}(\Omega) \neq 0$. Thus, $\mathcal{V}(\Omega) = \theta(1 - \sigma(0, A))$ implements the condition in the path integral by means of a step function. The integral representation of it reads

$$\mathcal{V}(\Omega) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon}\frac{d\beta}{2\pi i\beta}e^{\beta(1-\sigma(0,A))}. \tag{2.14}$$

The Gribov region is obtained by inserting (2.14) into (2.8). The final Gribov-Zwanziger action is obtained by assuming the saddle-point approximation as it is shown in the next section.

B. Gluon propagator

As a consequence of the restriction, the ghost form factor introduces a gauge bilinear term in the action. Hence, the gauge sector is modified. This can be observed in the gluon propagator. By taking the quadratic part in the gauge field and integrating it, we have that

$$\langle A^a_\mu(k)A^b_\nu(p)\rangle = \delta(p+k)\mathcal{N}\int \frac{d\beta}{2i\pi\beta}e^{f(\beta)}(K^{ab}_{\mu\nu})^{-1}$$
$$f(\beta) = \beta - \ln\beta - \frac{d-1}{d}(N^2-1)V\int\frac{d^dp}{(2\pi)^d}\ln\left(p^2+\frac{\beta N g^2}{N^2-1}\frac{2}{dV}\frac{1}{p^2}\right), \tag{2.15}$$

where

$$K^{ab}_{\mu\nu} = \delta^{ab}\left(\left(\beta\frac{1}{V}\frac{1}{d}\frac{Ng^2}{N^2-1}\frac{1}{k^2}+k^2\right)\delta_{\mu\nu}+\left(\frac{1}{\alpha}-1\right)k_\mu k_\nu\right). \tag{2.16}$$

The saddle-point approximation is implemented in order to solve the integral over β, (2.14),

$$Z_{quad} \approx e^{f(\beta_0)}, \tag{2.17}$$

where β_0 is the minimum of the $f(\beta)$ and Z_{quad} is the partition function (2.8) in the quadratic approximation. From the minimum condition $f'(\beta_0)=0$, we obtain the following result:

$$1 = \frac{d-1}{d}Ng^2\int\frac{d^dp}{(2\pi)^d)}\frac{1}{p^4+\gamma^4}. \tag{2.18}$$

This is the so-called gap equation, where we have defined $\gamma^4 = \frac{\beta_0 N}{N^2-1}\frac{2}{dV}g^2$ as the Gribov mass. This equation determines the value of γ^4; i.e., the Gribov parameter is not a loose parameter in our framework. It is a self-consistent parameter determined by the gap equation (2.18). After computing the inverse of (2.16), the gloun propagator (2.15) becomes

$$\langle A^a_\mu(k)A^b_\nu(p)\rangle = \delta(p+k)\delta^{ab}\frac{k^2}{k^4+\gamma^4}P_{\mu\nu}(k). \tag{2.19}$$

A direct consequence of the presence of the Girbov parameter in the gluon propagator is the fact that

$$\frac{1}{k^4+\gamma^4} = \frac{1}{2i\gamma^2}\left(\frac{1}{k^2+i\gamma^2}-\frac{1}{k^2-i\gamma^2}\right). \tag{2.20}$$

This means that we have propagation of excitations with complex masses; i.e., these are not physical excitations. This is one way to interpret confinement where in the strong coupled regime of the theory cannot describe perturbatively physical excitations.

By restricting the path integral using the Gribov region, we add a nonlocal mass term for the gauge field into the action, accounting for nonperturbative effects. However, such a nonlocal term can be rewritten as a local form [10]. This action is known as the Gribov-Zwanziger action; see Ref. [4] for more detail about the calculus of this action. In an arbitrary dimension and in general linear covariant gauge, it is given by the equation

$$S = \int d^dx\frac{1}{4}(F^a_{\mu\nu})^2 + \int d^dx\left(\frac{\alpha}{2}b^ab^a + \mathrm{i}b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\right)$$
$$+\int d^dx[\bar{\varphi}^{ac}_\mu\partial_\nu D^{ab}_\nu\varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu\partial_\nu(D^{ab}_\nu\omega^{bc}_\mu) - g(\partial_\nu\bar{\omega}^{an}_\mu)f^{abc}D^{bm}_\nu c^m\varphi^{cn}_\mu]$$
$$-\gamma^2 g\int d^dx\left[f^{abc}A^a_\mu\varphi^{bc}_\mu + f^{abc}A^a_\mu\bar{\varphi}^{bc}_\mu + \frac{d}{g}(N_c^2-1)\gamma^2\right], \tag{2.21}$$

with γ is the Gribov parameter; $F^a_{\mu\nu}$ is the field strength tensor, which is defined by (2.3); $(\phi,\bar{\phi})$ is a pair of complex-conjugate bosonic fields; $(\omega,\bar{\omega})$ are anticommuting complex-conjugate fields; the fields $(\bar{c}^a, c^a)$ are the Faddeev-Popov ghosts; α is the gauge parameter, which is zero for the Landau gauge, $\partial_\mu A_\mu = 0$; b^a accounts for the Lagrange multiplier implementing the gauge condition; and $D^{ab}_\mu = (\delta^{ab}\partial_\mu + gf^{acb}A^c_\mu)$ is the covariant derivative in the adjoint representation of $SU(N)$.

III. YANG-MILLS-GRIBOV-ZWANZIGER-CHERN-SIMONS GLUON PROPAGATOR

The starting point of our investigation is the local Gribov-Zwanziger action in linear covariant gauge in 3D dimensions Euclidean space. Therefore, Eq. (2.21) became

$$S = \int d^3x \frac{1}{4}(F^a_{\mu\nu})^2 + \int d^3x \left(\frac{\alpha}{2} b^a b^a + \mathrm{i} b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b\right) + \int d^3x [\bar{\varphi}^{ac}_\mu \partial_\nu D^{ab}_\nu \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \partial_\nu (D^{ab}_\nu \omega^{bc}_\mu) - g(\partial_\nu \bar{\omega}^{an}_\mu) f^{abc} D^{bm}_\nu c^m \varphi^{cn}_\mu] - \gamma^2 g \int d^3x \left[f^{abc} A^a_\mu \varphi^{bc}_\mu + f^{abc} A^a_\mu \bar{\varphi}^{bc}_\mu + \frac{3}{g}(N_c^2 - 1)\gamma^2\right], \quad (3.1)$$

with, now, $\gamma^4 = \frac{2\beta N g^2}{3V(N^2-1)}$.

Now, by coupling the Chern-Simons action,

$$S_{\mathrm{CS}} = -\frac{\mathrm{i}M}{2} \epsilon^{\mu\nu\lambda} \int d^3x \left(A^a_\mu \partial_\lambda A^a_\nu - \frac{2}{3!} g f^{abc} A^a_\mu A^b_\nu A^c_\lambda\right), \quad (3.2)$$

to (3.1), we obtain the Yang-Mills-Gribov-Zwanziger-Chern-Simons (YMGZCS) action:

$$S_{\mathrm{YMGZCS}} = \int d^3x \left[\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{iM}{2} \epsilon_{\mu\nu\lambda} \left(A^a_\nu \partial_\lambda A^a_\mu - \frac{2}{3!} g f^{abc} A^a_\mu A^b_\nu A^c_\lambda\right)\right] + \int d^3x \left(\frac{\alpha}{2} b^a b^a + \mathrm{i} b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b\right) + \int d^3x [\bar{\varphi}^{ac}_\mu \partial_\nu D^{ab}_\nu \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \partial_\nu (D^{ab}_\nu \omega^{bc}_\mu) - g(\partial_\nu \bar{\omega}^{an}_\mu) f^{abc} D^{bm}_\nu c^m \varphi^{cn}_\mu] - \gamma^2 g \int d^3x \left[f^{abc} A^a_\mu \varphi^{bc}_\mu + f^{abc} A^a_\mu \bar{\varphi}^{bc}_\mu + \frac{3}{g}(N_c^2 - 1)\gamma^2\right]. \quad (3.3)$$

The gluon propagator poles in Yang-Milss-Chern-Simons theory in the presence of Gribov ambiguity have already been analyzed in the literature [12]. However, in Ref. [12], the authors considered only the zero order of the gap equation expansion in the CS mass term. Despite the zero order being the most dominant term, we will see that the first order in the CS mass has notable physical implications.

A. Three solutions for the gap equation

In this section, we analyze the contribution of the Chern-Simons mass term to the gap equation. The gap equation is a self-consistent condition obtained through the saddle-point approximation, see Sec. II B, which becomes exact in the thermodynamic limit [1,9–11]. In other words, the gap equation can be obtained by taking the first derivative of the vacuum energy density, $\mathcal{E}_v$, with respect to β, computed at the specific value β^* that minimizes $\mathcal{E}_v$. The vacuum energy density, $\mathcal{E}_v$, is given by [12]

$$-V\mathcal{E}_v = \beta - \ln\beta - \frac{1}{2}\mathrm{Tr}\ln Q^{ab}_{\mu\nu}. \quad (3.4)$$

Then, taking into account the saddle point in the thermodynamic limit by holding $\gamma^4 = \frac{2\beta N g^2}{3V(N^2-1)}$ finite, we find

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^4 + \gamma_*^4}{(k^4 + \gamma_*^4)^2 + k^6 M^2} = \frac{3}{2Ng^2}. \quad (3.5)$$

The calculus of (3.5) is similar to what has been done in Ref. [13]. By solving this integral (step-by-step calculation is found in Appendix), we find the equation

$$\gamma^3 - \frac{Ng^2}{6\sqrt{2}\pi}\gamma^2 + \frac{5Ng^2}{192\sqrt{2}\pi}M^2 = 0, \quad (3.6)$$

which is a cubic equation. As a result, there are three solutions for this equations; i.e., there are three local minimums for the vacuum energy density. By construction of the Gribov-Zwanziger theory, in the perturbative regime, the gap equation solution ensures that the functional integral of the gauge fields is taken in the region where the gauge field configurations are associated with the smallest eigenvalues of the Fadeev-Popov operator [4,9]. Therefore, although there are three possible solutions for the gap equation, we are guaranteeing that the gauge field configurations belonging to the integration domain correspond to those associated with the smallest eigenvalues of the Fadeev-Popov operator at the leading order in g^2 [18]. Also, remember that inside the Gribov region we still have gauge copies, which means the Fadeev-Popov operator still has zero modes. The only region free of copies is called the fundamental modular region [4], but nobody knows how to handle it. Consequently, we will not find a global minimum for the vacuum energy density inside of the Gribov region. Hence, it sounds reasonable to find more than one solution for the gap equation; it means multiple local minima. In Sec. IV, we find the physical Gribov parameter and eliminate the others that are spurious.

The discriminant of (3.6) determines if these roots are complex or real:

(i) If the discriminant of (3.6) is positive,

$$\Delta = -(1215\pi^2 M^2 - 16g^4 N^2) > 0, \quad (3.7)$$

there are three real roots, which means three real values for the Gibov parameter, γ. On that account, M has limited values to satisfy (3.7),

$$0 < M < \frac{4g^2N}{9\sqrt{15}\pi}. \tag{3.8}$$

Also, here, we have assumed positive values of Chern-Simons mass $M > 0$.

By using the François Viète's formula, we write down all three real solutions for (3.6),

$$\gamma_t = \frac{g^2N}{18\sqrt{2}\pi} + \frac{g^2N}{9\sqrt{2}\pi}\cos\left[\frac{2\pi t}{3} - \frac{1}{3}\arccos\left(1 - \frac{1215\pi^2M^2}{8g^4N^2}\right)\right], \tag{3.9}$$

where $t = 0, 1, 2$. Notice, if we take $M = 0$, we recover the result from Ref. [12]. From these three solutions, only one of them has physical meaning.

(ii) If $\Delta = 0$, we obtain

$$M = \frac{4g^2N}{9\sqrt{15}\pi}, \tag{3.10}$$

and the solutions of the gap equation are

$$\gamma_t = \frac{g^2N}{18\sqrt{2}\pi} + \frac{g^2N}{9\sqrt{2}\pi}\sin\left(\frac{2\pi t}{3} + \frac{\pi}{6}\right); \tag{3.11}$$

there are three roots, but two of them are similar $\gamma_0 = \gamma_1$. We see that the value of the topological mass given by (3.10) is the maximum allowed value of M to obtain real roots for the gap equation.

(iii) If $\Delta < 0$, i.e.,

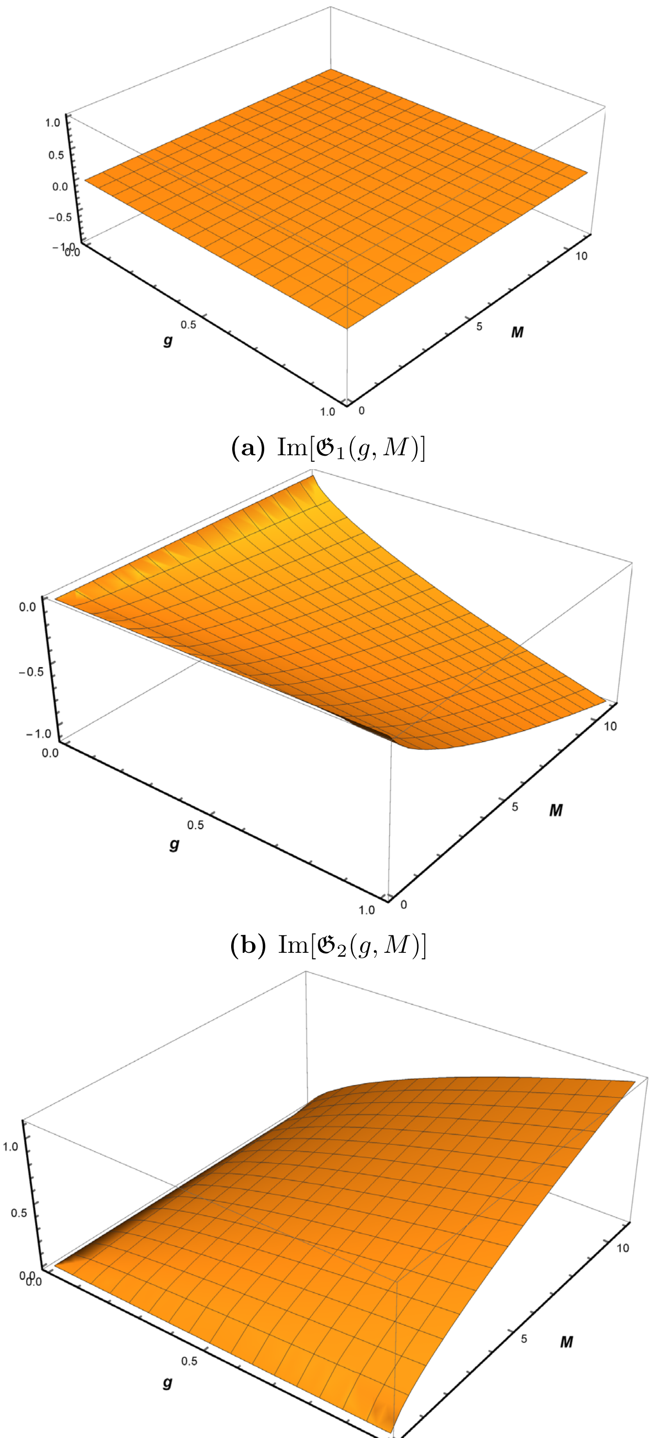
$$M > \frac{4g^2N}{9\sqrt{15}\pi},$$

there are one real root $\mathfrak{G}_1$, which is exactly equal the real root γ_2 from the case $\Delta > 0$, and two complex conjugate roots, $\mathfrak{G}_2$ and $\mathfrak{G}_3$. We do not show their expression in this paper because they are complicated expressions in function of g and M. Instead, we only plot their imaginary part; see Fig. 1. Of course, the imaginary part of $\mathfrak{G}_1$ is null, since it is the real root.

Here, it is interesting to notice that the weak-coupling constant regime is given by the condition without restriction to Gribov horizon, that is

$$M > \frac{Ng^2}{6\pi},$$

please for more details about this equation see Ref. [12]. This weak-coupling condition is bigger than the maximum value of M given by (3.10), $\frac{Ng^2}{6\pi} > \frac{4g^2N}{9\sqrt{15}\pi}$. Therefore, in the case of real roots for the gap equation, the regime is always in the strong-coupling regime, since M is always smaller than $Ng^2/6\pi$. This result is different from the one in Ref. [12].



(a) $\mathrm{Im}[\mathfrak{G}_1(g, M)]$

(b) $\mathrm{Im}[\mathfrak{G}_2(g, M)]$

(c) $\mathrm{Im}[\mathfrak{G}_3(g, M)]$

FIG. 1. Plot of imaginary part of γ when $\Delta < 0$, $N = 3$.

After we have found the solutions for the gap equation, we can investigate the gluon propagator poles, analyze the regime of theory for each Gribov parameter in the next section, and find the physical Gribov parameter, i.e., the one that describes nature.

IV. REGIME OF THE YANG-MILLS-GRIBOV-ZWANZIGER-CHERN-SIMONS THEORY

To calculate the gluon propagator from this theory, it is necessary to take only the quadratic part in the gauge field of action (3.3) and integrate it out. Following these steps, one should end up with

$$S = \int \frac{d^3k}{(2\pi)^3}\left(-\frac{1}{2}\tilde{A}^a_\mu(k)Q^{ab}_{\mu\nu}\tilde{A}^b_\nu(-k)\right), \qquad (4.1)$$

where

$$Q^{ab}_{\mu\nu} = \delta^{ab}\left(\frac{k^4+\gamma^4}{k^2}\delta_{\mu\nu} + \left(\frac{1}{\alpha}-1\right)k_\mu k_\nu + M\epsilon_{\mu\nu\lambda}k_\lambda\right) \qquad (4.2)$$

and γ^4 is the Gribov parameter. To obtain the propagator, we have to compute the inverse of (4.2), which can be obtained through the following expression:

$$Q^{ab}_{\mu\nu}(Q^{bc}_{\nu\delta})^{-1} = \delta^{ac}\delta_{\mu\delta}. \qquad (4.3)$$

The ansatz for the inverse of (4.2) reads

$$(Q^{bc}_{\nu\delta})^{-1} = \delta^{bc}\left(F(k)\delta_{\nu\delta} + B(k)\frac{k_\nu k_\delta}{k^2} + C(k)M\epsilon_{\delta\nu\alpha}\frac{k_\alpha}{k^2}\right), \qquad (4.4)$$

where the coefficients are dimensionless.

The Landau gauge is recovered in the limit $\alpha \to 0$, and the propagator reads

$$\langle A^a_\mu(k)A^b_\nu(-k)\rangle = \delta^{ab}F(k)\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{k^4}{(k^4+\gamma^4)}M\epsilon_{\mu\nu\alpha}\frac{k_\alpha}{k^2}\right]. \qquad (4.5)$$

The overall factor $F(k)$ is given by

$$F(k) = \frac{(k^4+\gamma^4)k^2}{(k^4+\gamma^4)^2 + k^6M^2}. \qquad (4.6)$$

As it is pointed out in Ref. [12], the poles of the propagator (4.5) are found by determining the roots of the polynomial

$$\begin{aligned} P(k^2) &= (k^4+\gamma^4)^2 + k^6M^2 \\ &= (k^2+m_1^2)(k^2+m_2^2)(k^2+m_3^2)(k^2+m_4^2), \end{aligned} \qquad (4.7)$$

where m_i stands for the solutions of the polynomial $P(k^2)$. The discriminant of $P(k^2)$ is

$$\Delta_p = 256M^4\gamma^{20} - 27M^8\gamma^{16}. \qquad (4.8)$$

As a result, there are four complex roots for $P(k^2)$, if $\Delta_p > 0$ or $\gamma > \sqrt[4]{27}M/4$. It means that there is no physical excitation. Then, we are in the confinement phase. There are two complex and two real roots for $P(k^2)$, if $\Delta_p < 0$ or $\gamma < \sqrt[4]{27}M/4$. This time, there is physical excitation, since there are two real roots. Hence, we found the deconfinement phase.

In Ref. [12], the authors investigated the regime of the theory comparing M and g, since, there, $\gamma \propto g^2$ and it is not a function of M. Nonetheless, in Sec. III A, it has shown that γ also depends on M; then, it must be taken into account, too. To do so, we separately investigate all solutions from (3.6) using Wolfram *Mathematica* software. Therefore, let us analyze the regime of the theory via the discriminant of the polynomial $P(k^2)$ (4.8) and study all cases for γ from Sec. III A, i.e., $\gamma = \{\gamma_0, \gamma_1, \gamma_2, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3\}$. By replacing each value of γ in (4.8), we get the Tables I and II.

From Table I, we see that γ_2 and $\mathfrak{G}_2$ do not describe the confinement phase. Therefore, gluons always behave as free particles. At this point, γ_2 can be discarded, since, clearly, it does not describe the reality; it is well known that gluons have a confinement phase. From the same table, we also realize there is no physical excitation when the Gribov parameter is given by γ_0. It means gluons are always confined. From here, it seems γ_0 also does not describe nature either. Then, it should not be considered as a physical parameter of theory. From Table II, we notice that the complex conjugate roots from the gap equation do not contribute to establishing the regime of the theory.

TABLE I. The regime of the theory defined by the real roots from the gap equation. Δ_p is given by (4.8).

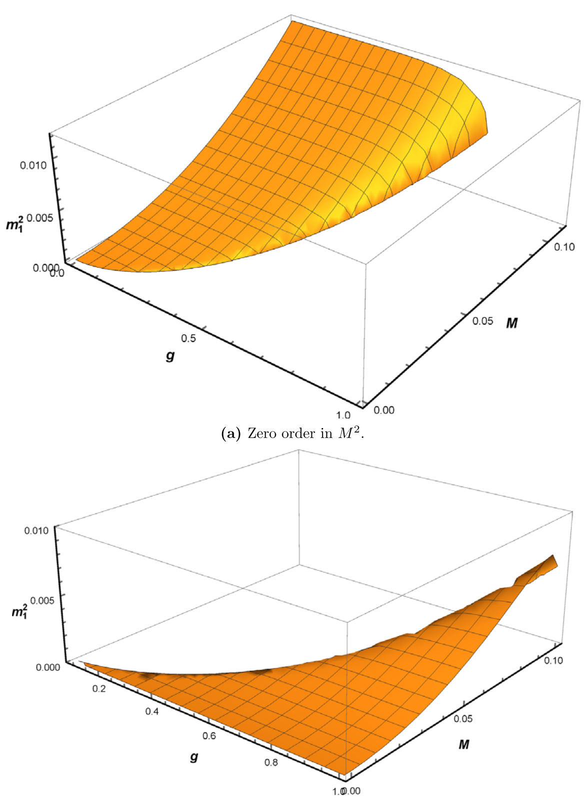
	γ_0	γ_1	γ_2	$\mathfrak{G}_1$
Confinement $\Delta_p > 0$	$0 < M \leq \frac{4g^2N}{9\sqrt{15}\pi}$	$\frac{(6\sqrt{6}-5\sqrt{2})}{54\sqrt[4]{3}\pi}Ng^2 < M \leq \frac{4g^2N}{9\sqrt{15}\pi}$	False	False
Deconfinement $\Delta_p < 0$	False	$0 < M < \frac{(6\sqrt{6}-5\sqrt{2})}{54\sqrt[4]{3}\pi}Ng^2$	$0 < M \leq \frac{4g^2N}{9\sqrt{15}\pi}$	$M > \frac{4g^2N}{9\sqrt{15}\pi}$

TABLE II. The regime of the theory defined by the complex conjugate roots from the gap equation. Δ_p is given by (4.8).

	$\mathfrak{G}_2$	$\mathfrak{G}_3$
Confinement $\Delta_p > 0$	False	False
Deconfinement $\Delta_p < 0$	False	False

The only Gribov parameter that gives us both phases (confinement and deconfinement) is γ_1; see Table I. However, by choosing the solution γ_1, the CS mass term assumes smaller values in the deconfinement phase than in the confinement phase, leading to an alternative conclusion than the one in Ref. [12]. Furthermore, in the confinement defined by γ_1, see Table I, we see that when g assumes large values M also can assume large values. Therefore, all excitations in the theory are confined for large values of g and also for large values of M.

To visualize how the masses of the gluon propagator change by considering the first order in M, we plot m_1^2, the first pole from the polynomial polynomial $P(k^2)$, at zero order in M and at first order in M using γ_1; see Fig. 2. Clearly, they are very different from each other. Also, from this figure, we show that the poles of the gluon propagator are affected if the CS mass term is considered in the gap equation.



(b) First order in M^2 by using γ_1.

FIG. 2. m_1^2 in function of M and g, $N = 3$.

V. CONCLUSION

In this paper, three solutions for the gap equation dependent on (g, M) are presented, meaning that there are three local minima for the vacuum energy density. However, only one has physical meaning. In Ref. [12], M can assume all possible positive values. In our cases, this is not always true; M assumes all possible positive values if and only if we are working with γ_2 as shown in Table I. However, if γ_2 is chosen, the confinement phase is not present, which means gluons are always free; in other words, only the deconfinement phase is observed in this case. Hence, γ_2 does not describe nature. From Table I, we can also conclude that gluons are always confined when the Gribov parameter is given by γ_0. And, yes, for g strong and M small, there is confinement in agreement with the study of Ref. [12], but there is no deconfinement phase in the case of γ_0. Again, it does not seem to represent nature. Therefore, γ_0 is not a physical parameter, and it may be ignored. The only Gribov parameter that gives us both phases (confinement and deconfinement) is γ_1. In spite of that, with γ_1, the CS mass term assumes smaller values than in the confinement phase, in contrast to the conclusion in Ref. [12].

In addition, in our case, the coupling constant is not small enough to get the weak-coupling regime, since M is never bigger than $\frac{Ng^2}{6\pi}$. It means that there are always Gribov ambiguities and we are never in the perturbative regime. The weak-coupling regime is only reached with the solution given by $\mathfrak{G}_1$—a solution for the gap equation that only defines deconfinement phase, see Table I, which is physically meaningless. Therefore, in this paper, we have shown that the gap equation solutions and the regime of the theory are evidently affected by considering the first order in M in the gap equation.

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APPENDIX: SOLVING THE GAP EQUATION

The integral on the left side of (3.5) can be solved by the residues theorem. However, the solutions for the gap equation (3.5) are only given by graphic analysis. The analysis of the equation will be done by comparing its left side in Fig. 3 with its right side. In the nonperturbative regime, i.e., in the infrared, g^2 is strong, and the right side of (3.5) is small. In Fig. 3, we can see that the left side from (3.5) is also very small for γ^4 large and M small. In the perturbative regime, i.e., when g^2 is small, the right side of (3.5) is large. In the left side of the same equation, large values are reached when M is big or γ^4 is small, which is consistent with theory, since $\gamma \propto \exp(-1/g^2)$. In Fig. 3, one

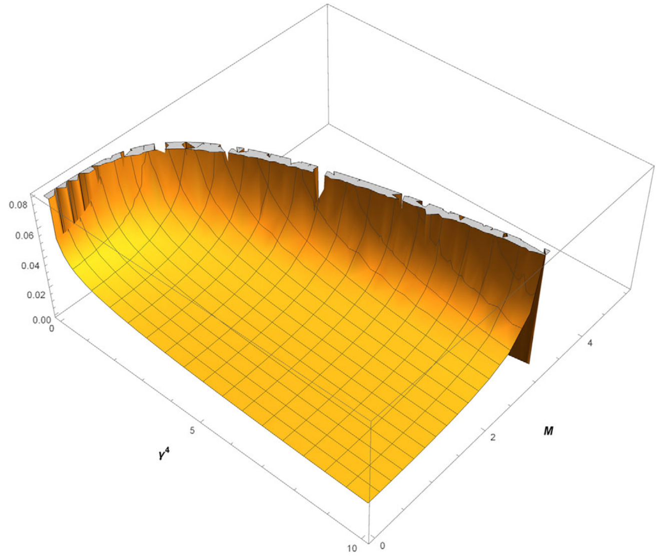

FIG. 3. The left side from (3.5).

can observe that the left side of the gap equation (3.5) is not defined by all values of the Chern-Simons mass M, for $M > 3$ in the plot, the function is not defined.

As we are working with Gribov formalism, we are only interested in the IR regime, where Gribov copies show up. By the graphic analysis above, we see that the IR regime is given by a small M. Then, we can expand the left side of (3.5) in M^2 using Taylor series. Hence, the gap equation can be written as follows:

$$\int \frac{d^3k}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{(-i)^{2n}(k^{6n}M^{2n})}{(\gamma^4+k^4)^{2n+1}} = \frac{3}{2Ng^2} \tag{A1}$$

$$\sum_{n=0}^{\infty} (-i)^{2n} \frac{4\pi}{(2\pi)^3} \int dk \frac{k^{6n}M^{2n}}{(\gamma^4+k^4)^{2n+1}} k^2 = \frac{3}{2Ng^2}. \tag{A2}$$

In this paper, we only consider the two first terms of (A2). This equation has been solved in zero order in Ref. [12], as was already said before.

To solve (A2), we used the residue theorem

$$\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx = \pi \mathrm{i}\mathrm{Res}[f(z)]_{z=z_0}$$
$$= -\pi \mathrm{i}\mathrm{Res}[f(z)]_{z=-z_0}, \tag{A3}$$

where z_0 is the pole of the function $f(z)$. In spherical coordinates, the zero-order term from (A2) is

$$\frac{1}{2\pi^2}\int_0^\infty \frac{k^2}{k^4+\gamma^4} dk = \frac{1}{4\pi\sqrt{2}\gamma}, \tag{A4}$$

which is the result from Ref. [12]. The first-order term in M^2 of (A2) is

$$-\frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{k^8 M^2}{(k^2-i\gamma^2)^3(k^2+i\gamma^2)^3} dk = -\frac{5M^2}{128\sqrt{2}\pi\gamma^3}. \tag{A5}$$

By replacing (A4) and (A5) in (A2), we obtain

$$\gamma^3 - \frac{Ng^2}{6\sqrt{2}\pi}\gamma^2 + \frac{5Ng^2}{192\sqrt{2}\pi} M^2 = 0. \tag{A6}$$

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