# Hadronic molecules in $B$ decays

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There are eighteen possibly existing $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}D_s^{(*)-}$ hadronic molecular states. We construct their corresponding interpolating currents and calculate their decay constants using QCD sum rules. Based on these results, we calculate their relative production rates in $B$ and $B^*$ decays through the current algebra, and calculate their relative branching ratios through the Fierz rearrangement, as summarized in Table III. Our results support the interpretations of the $X(3872)$, $Z_c(3900)^0$, $Z_c(4020)^0$, and $X_0(2900)$ as the molecular states $D\bar{D}^*$ of $J^{PC}=1^{++}$, $D\bar{D}^*$ of $J^{PC}=1^{+-}$, $D^*\bar{D}^*$ of $J^{PC}=1^{+-}$, and $D^*\bar{K}^*$ of $J^P=0^+$, respectively. Our results also suggest that the $Z_{cs}(3985)^-$, $Z_{cs}(4000)^+$, and $Z_{cs}(4220)^+$ are strange partners of the $X(3872)$, $Z_c(3900)^0$, and $Z_c(4020)^0$, respectively. In the calculations we estimate the lifetime of a weakly coupled composite particle $A=|BC\rangle$ to be $1/t_A \approx 1/t_B + 1/t_C + \Gamma_{A\to BC} + \cdots$, with $\cdots$ partial widths of other possible decay channels.

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## I. INTRODUCTION

Since the discovery of the $X(3872)$ in 2003 by Belle [1], a lot of charmoniumlike $XYZ$ states were discovered in the past twenty years [2]. These structures are good candidates of tetraquark states, which are composed by two quarks and two antiquarks. Although there is still a long way to fully understand how the strong interaction binds these quarks and antiquarks together, this subject has become one of the most intriguing research topics in particle physics. Their theoretical and experimental studies are significantly improving our understanding of the strong interaction at the low energy region [3–12].

Among all the $XYZ$ states, the $X(3872)$, $X_0(2900)$, $Z_{cs}(4000)^+$, and $Z_{cs}(4220)^+$ have been observed in $B$ decays [1,13–15]. Besides, the $Z_c(3900)^0$ and $Z_c(4020)^0$ (may) decay into the $J/\psi\pi^0$ final states [16–19], so they are also possible to be observed in the $B^-\to K^-J/\psi\pi^0$ decay. In this paper we shall investigate these six exotic structures as a whole. We shall study their mass spectra, productions in $B$ and $B^*$ decays, and decay properties. Before doing this, let us briefly review some of their information, and we refer to the reviews [3–12] for detailed discussions.

(i) The $X(3872)$ is the most puzzling state among all the charmoniumlike $XYZ$ states. It is now denoted as the $\chi_{c1}(3872)$ in PDG2020 [2], but the mass of $\chi_{c1}(2P)$ was estimated to be 3.95 GeV [20], which value is significantly larger than the mass of $X(3872)$. Consequently, various interpretations were proposed to explain it, such as a compact tetraquark state [21–28], a loosely-bound $D\bar{D}^*$ molecular state [29–37], and a hybrid charmonium state [38,39], etc. The $X(3872)$ was also studied as a conventional $c\bar{c}$ state in Refs. [40–43], and was suggested to be the mixture of a $c\bar{c}$ state with the $D\bar{D}^*$ component in Refs. [44–46].

The quantum number of $X(3872)$ was determined to be $I^GJ^{PC}=0^+1^{++}$ [47]. It has been observed in the $J/\psi\pi\pi$, $J/\psi\pi\pi\pi$, $\gamma J/\psi$, $\gamma\psi(2S)$, $\chi_{c1}\pi$, and $D^0\bar{D}^{*0}$ channels [48–55]. Especially, the $J/\psi\pi\pi$ and $J/\psi\pi\pi\pi$ channels have comparable branching ratios [56–59]:

$$\frac{\mathcal{B}(X(3872)\xrightarrow{\omega} J/\psi\pi\pi\pi)}{\mathcal{B}(X(3872)\xrightarrow{\rho} J/\psi\pi\pi)} = 1.6^{+0.4}_{-0.3}\pm 0.2, \quad (1)$$

which implies a large isospin violation.

(ii) In 2013 BESIII discovered the charged $Z_c(3900)^\pm$ in the $Y(4260)\to J/\psi\pi^+\pi^-$ decay [16], whose observation was confirmed by Belle [17] and CLEO [60]. Later BESIII observed the charged $Z_c(4020)^\pm$ in the $e^+e^-\to\pi^+\pi^-h_c$ process [61]. Because the $Z_c(3900)^\pm$ and $Z_c(4020)^\pm$ both couple strongly to charmonia and yet they are charged, these two structures are definitely not conventional $c\bar{c}$ states. There have been various theoretical models developed to explain them, such as compact tetraquark states [21,22,62–64], loosely-bound $D\bar{D}^*$ and $D^*\bar{D}^*$

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molecular states [29,65–73], hadro-quarkonia [74,75], or due to kinematical threshold effects [76–78], etc.

The $Z_c(3900)$ has been observed in the $J/\psi\pi$, $h_c\pi$, and $D\bar{D}^*$ channels [16,17,79,80], and the quantum number of its neutral one was determined to be $I^GJ^{PC} = 1^+1^{+-}$ [81]. The $Z_c(4020)$ has been observed in the $h_c\pi$ and $D^*\bar{D}^*$ channels [61,82], and the quantum number of its neutral one may also be $I^GJ^{PC} = 1^+1^{+-}$. Evidence for the $Z_c(3900)^\pm \to \eta_c\rho^\pm$ decay was reported at $\sqrt{s} = 4.226$ GeV with the relative branching ratio [83]:

$$\frac{\mathcal{B}(Z_c(3900)^\pm \to \eta_c\rho^\pm)}{\mathcal{B}(Z_c(3900)^\pm \to J/\psi\pi^\pm)} = 2.2 \pm 0.9. \quad (2)$$

This ratio was suggested in Ref. [84] to be useful to discriminate the compact tetraquark and hadronic molecule scenarios, and it has been calculated by many theoretical methods in Refs. [85–95].

(iii) In 2020 BESIII reported their observation of the $Z_{cs}(3985)^-$ in the $K^+$ recoil-mass spectra of the $e^+e^- \to K^+(D_s^-D^{*0} + D_s^{*-}D^0)$ process [96]. Later LHCb reported their observation of the $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ in the $J/\psi K^+$ invariant mass spectrum of the $B^+ \to J/\psi\phi K^+$ decay [15], and one can naturally deduce the existence of the $Z_{cs}(4000)^-$ and $Z_{cs}(4220)^-$ as their antiparticles. These structures couple strongly to charmonia and yet they are charged, so they are definitely exotic hadrons/structures. There have been some theoretical methods developed to explain the $Z_{cs}(3985)^-$, $Z_{cs}(4000)^-$, and $Z_{cs}(4220)^-$, such as compact tetraquark states [97–102], loosely-bound $DD_s^{*-}$, $D^*D_s^-$, and $D^*D_s^{*-}$ molecular states [103–113], or due to kinematical threshold effects [114–118], etc.

(iv) In 2020 LHCb reported their observation of the $X_0(2900)$ in the $D^-K^+$ invariant mass spectrum of the $B^+ \to D^+D^-K^+$ decay [13,14]. This structure has the quark content $\bar{c}\bar{s}ud$, so it is an exotic hadron with valence quarks of four different flavors. Various theoretical methods were applied to explain it, such as a compact tetraquark state [119–124], a loosely-bound $D^*\bar{K}^*$ molecular state [125–133], or due to triangle singularities [134], etc.

Summarizing the above discussions, we quickly notice that the $X(3872)$, $Z_c(3900)^0$, $Z_c(4020)^0$, and $X_0(2900)$ can be interpreted in the hadronic molecular picture as the molecular states $D\bar{D}^*$ of $J^{PC} = 1^{++}$, $D\bar{D}^*$ of $J^{PC} = 1^{+-}$, $D^*\bar{D}^*$ of $J^{PC} = 1^{+-}$, and $D^*\bar{K}^*$ of $J^P = 0^+$, respectively; the $Z_{cs}(3985)^-/Z_{cs}(4000)^-$ and $Z_{cs}(4220)^-$ can be interpreted in the hadronic molecular picture as the molecular states $DD_s^{*-}/D^*D_s^-$ of $J^P = 1^+$ and $D^*D_s^{*-}$ of $J^{PC} = 1^+$. Besides, there may exist the molecular states $D\bar{D}$ of $J^{PC} = 0^{++}$, $D^*\bar{D}^*$ of $J^{PC} = 0^{++}/2^{++}$, $D\bar{K}$ of $J^P = 0^+$, $D\bar{K}^*/D^*\bar{K}$ of $J^P = 1^+$, $D^*\bar{K}^*$ of $J^P = 1^+/2^+$, $DD_s^-$ of $J^P = 0^+$, and $D^*D_s^{*-}$ of $J^P = 0^+/2^+$.

Altogether there are eighteen possibly existing $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}D^{(*)-}$ molecular states. We use the symbol

$$|D^{(*)}\bar{D}^{(*)}/D^{(*)}\bar{K}^{(*)}/D^{(*)}\bar{D}_s^{(*)}; J^{P(C)}\rangle, \quad (3)$$

to denote them, where $\bar{D}_s^{(*)}$ is used to denote $D_s^{(*)-}$. To make this paper more understandable, we further simply our notations as follows:

(i) We shall not differentiate the charged $Z_c(3900)^\pm$ and the neutral $Z_c(3900)^0$. They are both denoted as the $Z_c(3900)$, which is interpreted as the $D\bar{D}^*$ molecular state of $J^{PC} = 1^{+-}$ in the present study. So does the $Z_c(4020)$, which is interpreted as the $D^*\bar{D}^*$ molecular state of $J^{PC} = 1^{+-}$. Note that their negative charge-conjugation parity $C = -$ is only due to the neutral ones, while the charged ones are not charge-conjugated.

(ii) We shall also denote the $Z_{cs}(3985)^-$, $Z_{cs}(4000)^\pm$, and $Z_{cs}(4220)^\pm$ as the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$, respectively. Both the $Z_{cs}(3985)$ and $Z_{cs}(4000)$ can be interpreted as the $DD_s^{*-}/D^*D_s^-$ molecular states of $J^P = 1^+$, and in the present study we shall further consider the mixing between the $DD_s^{*-}$ and $D^*D_s^-$ components, since their thresholds are quite close to each other. By doing this we can obtain strange partners of the $|D\bar{D}^*; 1^{++}\rangle$ and $|D\bar{D}^*; 1^{+-}\rangle$, which are denoted for convenience as $|D\bar{D}_s^*; 1^{++}\rangle$ and $|D\bar{D}_s^*; 1^{+-}\rangle$, respectively. We shall use them to separately explain the $Z_{cs}(3985)$ and $Z_{cs}(4000)$, but note that they are not charge-conjugated actually. See Sec. II C for detailed discussions.

The above notations can be used to well differentiate the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as well as the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$. We refer to Refs. [135–147] for some relevant theoretical studies, and to the reviews [3–12] again for their detailed discussions.

In this paper we shall systematically study the eighteen possibly existing $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states. We shall systematically construct their corresponding interpolating currents, and apply the method of QCD sum rules to calculate their decay constants. The obtained results will be used to further study their production and decay properties. This method has been applied to systematically investigate $\bar{D}^{(*)}\Sigma_c^{(*)}$ hadronic molecular states in Ref. [148].

We use the isoscalar $D\bar{D}$ molecular state of $J^{PC} = 0^{++}$, $|D\bar{D}; 0^{++}\rangle$, as an example to briefly show our procedures. First, we construct its corresponding hidden-charm tetraquark current:

$$\eta_1(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma_5 q_b(x), \tag{4}$$

which is a local meson-meson current coupling to $|D\bar{D};0^{++}\rangle$ through

$$\langle 0|\eta_1|D\bar{D};0^{++}\rangle = f_{|D\bar{D};0^{++}\rangle}. \tag{5}$$

In principle, one needs to explicitly use the nonlocal current in order to exactly describe the $D\bar{D}$ molecular state, but this cannot be done yet within the present QCD sum rule framework. In the above expressions, $a$ and $b$ are color indices; $q(x)$ is an *up* or *down* quark field, and $c(x)$ is a *charm* quark field; the decay constant $f_{|D\bar{D};0^{++}\rangle}$ can be calculated using QCD sum rules, which is an important input parameter when studying production and decay properties of $|D\bar{D};0^{++}\rangle$.

Second, we investigate the three-body $B^- \to K^- D^0 \bar{D}^0$ decay, where $|D\bar{D};0^{++}\rangle$ can be produced. The total quark content of the final states is $\bar{u}c\bar{c}s\bar{u}u$. We apply the Fierz rearrangement to carefully examine the combination of these six quarks, from which we select the current $\eta_1$, so that the relative production rate of $|D\bar{D};0^{++}\rangle$ can be estimated.

Thirdly, we apply the Fierz rearrangement [149] of the Dirac and color indices to transform the current $\eta_1$ into

$$\eta_1 \to -\frac{1}{12}\bar{q}_a\gamma_5 q_a \bar{c}_b\gamma_5 c_b + \frac{1}{12}\bar{q}_a\gamma_\mu q_a \bar{c}_b\gamma^\mu c_b + \cdots. \tag{6}$$

Accordingly, $\eta_1$ couples to the $\eta_c\eta$ and $J/\psi\omega$ channels simultaneously:

$$\begin{aligned}\langle 0|\eta_1|\eta_c\eta\rangle &\approx -\frac{1}{12}\langle 0|\bar{q}_a\gamma_5 q_a|\eta\rangle\langle 0|\bar{c}_b\gamma_5 c_b|\eta_c\rangle + \cdots,\\ \langle 0|\eta_1|J/\psi\omega\rangle &\approx \frac{1}{12}\langle 0|\bar{q}_a\gamma_\mu q_a|\omega\rangle\langle 0|\bar{c}_b\gamma^\mu c_b|J/\psi\rangle + \cdots.\end{aligned} \tag{7}$$

We can use these two equations to straightforwardly calculate the relative branching ratio of the $|D\bar{D};0^{++}\rangle$ decay into $\eta_c\eta$ to its decay into $J/\psi\omega$.

This paper is organized as follows. In Sec. II we systematically construct hidden-charm tetraquark currents corresponding to the $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states. We use them to perform QCD sum rule analyses and calculate their decay constants. The obtained results are used in Sec. III to study the productions of the $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states in $B$ and $B^*$ decays through the current algebra. In Sec. IV we use the Fierz rearrangement of the Dirac and color indices to study their decay properties, and calculate some of their relative branching ratios. The obtained results are summarized and discussed in Sec. V. This paper has a supplementary file "OPE.nb" containing all the spectral densities [150].

## II. INTERPOLATING CURRENTS

In this section we systematically construct (strange) hidden-charm and open-charm tetraquark interpolating currents corresponding to $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states. We separately construct them in the following subsections.

The isospin quantum number of these molecular states is important. In the present study,

(i) We assume that $|D\bar{D};0^{++}\rangle$, $|D^*\bar{D}^*;0^{++}\rangle$, and $|D^*\bar{D}^*;2^{++}\rangle$ have $I=0$.
(ii) The isospin of the $X(3872)$ as $|D\bar{D}^*;1^{++}\rangle$ will be separately investigated in Sec. IV A 4. Before that we simply assume it also has $I=0$.
(iii) We assume that $|D\bar{D}^*;1^{+-}\rangle$ and $|D^*\bar{D}^*;1^{+-}\rangle$ have $I=1$.
(iv) We assume that all the $D^{(*)}\bar{K}^{(*)}$ molecular states have $I=0$.
(v) All the $D^{(*)}\bar{D}_s^{(*)}$ molecular states have $I=1/2$.

Keeping this in mind, we shall use $q$ to denote either the *up* or *down* quark, so that we do not need to explicitly write the isospin contents out.

### A. $D^{(*)}\bar{D}^{(*)}$ currents

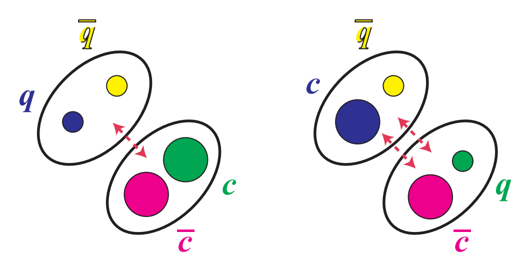
In this subsection we use the $\bar{c}$, $c$, $\bar{q}$, and $q$ ($q = u/d$) quarks to construct hidden-charm tetraquark interpolating currents. We consider two types of currents, as illustrated in Fig. 1:

$$\theta(x) = [\bar{q}_a(x)\Gamma_1^\theta q_b(x)][\bar{c}_c(x)\Gamma_2^\theta c_d(x)], \tag{8}$$

$$\eta(x) = [\bar{q}_a(x)\Gamma_1^\eta c_b(x)][\bar{c}_c(x)\Gamma_2^\eta q_d(x)], \tag{9}$$

where $a\cdots d$ are color indices and $\Gamma_{1/2}^{\theta/\eta}$ are Dirac matrices. We can relate these two configurations through the Fierz rearrangement in the Lorentz space and the color rearrangement in the color space:

$$\delta_{ab}\delta_{cd} = \frac{1}{3}\delta_{ad}\delta_{cb} + \frac{1}{2}\lambda_{ad}^n\lambda_{cb}^n. \tag{10}$$




(a) $\theta(x)$ currents (b) $\eta(x)$ currents

FIG. 1. Two types of hidden-charm tetraquark currents, $\theta(x)$ and $\eta(x)$. Quarks are shown in red/green/blue color, and antiquarks are shown in cyan/magenta/yellow color.

We shall discuss in detail this rearrangement in Sec. IV A when investigating decay properties of $D^{(*)}\bar{D}^{(*)}$ molecular states.

There can exist altogether six $D^{(*)}\bar{D}^{(*)}$ hadronic molecular states:

$$|D\bar{D};0^{++}\rangle = |D\bar{D}\rangle_{J=0}, \tag{11}$$

$$\sqrt{2}|D\bar{D}^*;1^{++}\rangle = |D\bar{D}^*\rangle_{J=1} + |D^*\bar{D}\rangle_{J=1}, \tag{12}$$

$$\sqrt{2}|D\bar{D}^*;1^{+-}\rangle = |D\bar{D}^*\rangle_{J=1} - |D^*\bar{D}\rangle_{J=1}, \tag{13}$$

$$|D^*\bar{D}^*;0^{++}\rangle = |D^*\bar{D}^*\rangle_{J=0}, \tag{14}$$

$$|D^*\bar{D}^*;1^{+-}\rangle = |D^*\bar{D}^*\rangle_{J=1}, \tag{15}$$

$$|D^*\bar{D}^*;2^{++}\rangle = |D^*\bar{D}^*\rangle_{J=2}. \tag{16}$$

Their corresponding currents are

$$\eta_1(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma_5 q_b(x), \tag{17}$$

$$\eta_2^\alpha(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma^\alpha q_b(x) - \{\gamma_5 \leftrightarrow \gamma^\alpha\}, \tag{18}$$

$$\eta_3^\alpha(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma^\alpha q_b(x) + \{\gamma_5 \leftrightarrow \gamma^\alpha\}, \tag{19}$$

$$\eta_4(x) = \bar{q}_a(x)\gamma^\mu c_a(x)\bar{c}_b(x)\gamma_\mu q_b(x), \tag{20}$$

$$\eta_5^\alpha(x) = \bar{q}_a(x)\gamma_\mu c_a(x)\bar{c}_b(x)\sigma^{\alpha\mu}\gamma_5 q_b(x) - \{\gamma_\mu \leftrightarrow \sigma^{\alpha\mu}\gamma_5\}, \tag{21}$$

$$\eta_6^{\alpha\beta}(x) = P^{\alpha\beta,\mu\nu}\bar{q}_a(x)\gamma_\mu c_a(x)\bar{c}_b(x)\gamma_\nu q_b(x), \tag{22}$$

where $P^{\alpha\beta,\mu\nu}$ is the spin-2 projection operator

$$P^{\alpha\beta,\mu\nu} = \frac{1}{2}g^{\alpha\mu}g^{\beta\nu} + \frac{1}{2}g^{\alpha\nu}g^{\beta\mu} - \frac{1}{4}g^{\alpha\beta}g^{\mu\nu}. \tag{23}$$

In the above expressions we have used the tensor field $\bar{c}_b\sigma_{\mu\nu}\gamma_5 q_b$ ($\mu,\nu = 0...3$) to construct the current $\eta_5^\alpha(x)$ of $J^{PC} = 1^{+-}$. In principle, this tensor field contains both $J^P = 1^+$ and $1^-$ components, but its negative-parity component $\bar{c}_b\sigma_{ij}\gamma_5 q_b$ ($i,\ j = 1,\ 2,\ 3$) gives the dominant contribution to $\eta_5^\alpha(x)$. Hence, the tetraquark current $\eta_5^\alpha(x)$ of $J^{PC} = 1^{+-}$ corresponds to $|D^*\bar{D}^*;1^{+-}\rangle$. Beside it, there exists another current directly corresponding to $|D^*\bar{D}^*;1^{+-}\rangle$:

$$\eta_7^{\alpha\beta}(x) = \bar{q}_a(x)\gamma^\alpha c_a(x)\bar{c}_b(x)\gamma^\beta q_b(x) - \{\alpha \leftrightarrow \beta\}, \tag{24}$$

but this current itself contains both positive- and negative-parity components, so we do not use it in the present study.

We use $X$ to generally denote the $D^{(*)}\bar{D}^{(*)}$ molecular states, and assume that the currents $\eta_{1\cdots6}^{\alpha_1\cdots\alpha_J}$ of spin-$J$ couple to them through

$$\langle 0|\eta^{\alpha_1\cdots\alpha_J}|X\rangle = f_X\epsilon^{\alpha_1\cdots\alpha_J}, \tag{25}$$

where $f_X$ is the decay constant, and $\epsilon^{\alpha_1\cdots\alpha_J}$ is the traceless and symmetric polarization tensor.

We apply the method of QCD sum rules [151,152] to study the $D^{(*)}\bar{D}^{(*)}$ hadronic molecular states through the currents $\eta_{1\ldots6}$. We calculate their spectral densities using the method of operator product expansion (OPE) at the leading order of $\alpha_s$ and up to the $D(\text{imension}) = 8$ terms. In the calculations we calculate the perturbative term, the quark condensates $\langle\bar{q}q\rangle/\langle\bar{s}s\rangle$, the gluon condensate $\langle g_s^2GG\rangle$, the quark-gluon mixed condensates $\langle g_s\bar{q}\sigma Gq\rangle/\langle g_s\bar{s}\sigma Gs\rangle$, and their combinations $\langle\bar{q}q\rangle^2$, $\langle\bar{q}q\rangle\langle\bar{s}s\rangle$, $\langle\bar{q}q\rangle\langle g_s\bar{q}\sigma Gq\rangle$, $\langle\bar{q}q\rangle\langle g_s\bar{s}\sigma Gs\rangle$, and $\langle\bar{s}s\rangle\langle g_s\bar{q}\sigma Gq\rangle$. We take into account the *charm* and *strange* quark masses, but ignore the *up* and *down* quark masses. We adopt the factorization assumption of vacuum saturation for higher dimensional condensates. The $D = 3$ quark condensates $\langle\bar{q}q\rangle/\langle\bar{s}s\rangle$ and the $D = 5$ mixed condensates $\langle g_s\bar{q}\sigma Gq\rangle/\langle g_s\bar{s}\sigma Gs\rangle$ are both multiplied by the charm quark mass $m_c$, which are thus important power corrections.

The obtained spectral densities are too length, so we list them in the supplementary file "OPE.nb" [150]. In the calculations we use the following values for various QCD sum rule parameters [2,153–161]:

$$\begin{aligned} m_s &= 96^{+8}_{-4}\ \text{MeV}, \\ m_c &= 1.275^{+0.025}_{-0.035}\ \text{GeV}, \\ \langle\bar{q}q\rangle &= -(0.240 \pm 0.010)^3\ \text{GeV}^3, \\ \langle\bar{s}s\rangle &= (0.8 \pm 0.1) \times \langle\bar{q}q\rangle, \\ \langle g_s^2GG\rangle &= 0.48 \pm 0.14\ \text{GeV}^4, \\ \langle g_s\bar{q}\sigma Gq\rangle &= -M_0^2 \times \langle\bar{q}q\rangle, \\ \langle g_s\bar{s}\sigma Gs\rangle &= -M_0^2 \times \langle\bar{s}s\rangle, \\ M_0^2 &= 0.8 \pm 0.2\ \text{GeV}^2, \end{aligned} \tag{26}$$

where the running mass in the $\overline{MS}$ scheme is used for the *charm* quark.

Based on these spectral densities, we calculate masses and decay constants of the $D^{(*)}\bar{D}^{(*)}$ hadronic molecular states. The results are summarized in Table I, supporting the interpretations of the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as the $|D\bar{D}^*;1^{++}\rangle$, $|D\bar{D}^*;1^{+-}\rangle$, and $|D^*\bar{D}^*;1^{+-}\rangle$ molecular states, respectively. The accuracy of our QCD sum rule results is moderate but not enough to extract their binding energies. Therefore, our results can only suggest but not determine: (a) whether these $D^{(*)}\bar{D}^{(*)}$ molecular states exist or not, and (b) whether they are bound states or resonance states. Note that we have not taken into account the

TABLE I. Masses and decay constants of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states, extracted from the currents $\eta_{1\cdots6}$, $\xi_{1\cdots6}$, and $\zeta_{1\cdots6}$, respectively.

| | | | Working Regions | | | | | |
|---|---|---|---|---|---|---|---|---|
| Currents | Configuration | $s_0^{\min}$[GeV$^2$] | $s_0$[GeV$^2$] | $M_B^2$[GeV$^2$] | Pole [%] | Mass [GeV] | $f_X$[GeV$^5$] | Candidate |
| $\eta_1$ | $\lvert D\bar{D};0^{++}\rangle$ | 16.6 | $18.0\pm1.0$ | 2.77–3.08 | 40–50 | $3.73^{+0.08}_{-0.08}$ | $(1.42^{+0.28}_{-0.25})\times10^{-2}$ | |
| $\eta_2^\alpha$ | $\lvert D\bar{D}^*;1^{++}\rangle$ | 17.3 | $19.0\pm1.0$ | 2.77–3.15 | 40–51 | $3.87^{+0.08}_{-0.09}$ | $(2.16^{+0.40}_{-0.36})\times10^{-2}$ | $X(3872)$ |
| $\eta_3^\alpha$ | $\lvert D\bar{D}^*;1^{+-}\rangle$ | 17.3 | $19.0\pm1.0$ | 2.77–3.15 | 40–51 | $3.87^{+0.08}_{-0.09}$ | $(2.16^{+0.40}_{-0.36})\times10^{-2}$ | $Z_c(3900)$ |
| $\eta_4$ | $\lvert D^*\bar{D}^*;0^{++}\rangle$ | 18.7 | $20.0\pm1.0$ | 2.72–3.00 | 40–49 | $4.01^{+0.11}_{-0.11}$ | $(3.07^{+0.58}_{-0.52})\times10^{-2}$ | |
| $\eta_5^\alpha$ | $\lvert D^*\bar{D}^*;1^{+-}\rangle$ | 19.6 | $20.0\pm1.0$ | 3.05–3.14 | 40–42 | $4.06^{+0.14}_{-0.13}$ | $(4.96^{+0.89}_{-0.80})\times10^{-2}$ | $Z_c(4020)$ |
| $\eta_6^{\alpha\beta}$ | $\lvert D^*\bar{D}^*;2^{++}\rangle$ | 17.9 | $20.0\pm1.0$ | 2.68–3.20 | 40–55 | $4.04^{+0.13}_{-0.12}$ | $(1.71^{+0.30}_{-0.27})\times10^{-2}$ | |
| $\xi_1$ | $\lvert D\bar{K};0^{+}\rangle$ | 9.2 | $10.0\pm1.0$ | 2.08–2.26 | 40–47 | $2.72^{+0.10}_{-0.10}$ | $(0.77^{+0.18}_{-0.17})\times10^{-2}$ | |
| $\xi_2^\alpha$ | $\lvert D\bar{K}^*;1^{+}\rangle$ | 10.4 | $11.0\pm1.0$ | 2.24–2.37 | 40–45 | $2.89^{+0.10}_{-0.11}$ | $(0.86^{+0.19}_{-0.17})\times10^{-2}$ | |
| $\xi_3^\alpha$ | $\lvert D^*\bar{K};1^{+}\rangle$ | 10.0 | $11.0\pm1.0$ | 2.13–2.35 | 40–49 | $2.85^{+0.10}_{-0.11}$ | $(0.82^{+0.18}_{-0.17})\times10^{-2}$ | |
| $\xi_4$ | $\lvert D^*\bar{K}^*;0^{+}\rangle$ | 10.6 | $11.5\pm1.0$ | 2.05–2.23 | 40–48 | $2.91^{+0.10}_{-0.11}$ | $(1.29^{+0.31}_{-0.28})\times10^{-2}$ | $X_0(2900)$ |
| $\xi_5^\alpha$ | $\lvert D^*\bar{K}^*;1^{+}\rangle$ | 12.0 | $12.5\pm1.0$ | 2.46–2.58 | 40–44 | $3.13^{+0.15}_{-0.12}$ | $(2.96^{+0.62}_{-0.55})\times10^{-2}$ | |
| $\xi_6^{\alpha\beta}$ | $\lvert D^*\bar{K}^*;2^{+}\rangle$ | 11.8 | $12.5\pm1.0$ | 2.40–2.56 | 40–45 | $3.09^{+0.12}_{-0.11}$ | $(1.10^{+0.23}_{-0.21})\times10^{-2}$ | |
| $\zeta_1$ | $\lvert D\bar{D}_s;0^{+}\rangle$ | 18.4 | $19.0\pm1.0$ | 3.08–3.22 | 40–44 | $3.86^{+0.07}_{-0.08}$ | $(1.74^{+0.32}_{-0.29})\times10^{-2}$ | |
| $\zeta_2^\alpha$ | $\lvert D\bar{D}_s^*;1^{++}\rangle$ | 19.3 | $20.0\pm1.0$ | 3.14–3.31 | 40–45 | $3.99^{+0.08}_{-0.08}$ | $(2.65^{+0.46}_{-0.42})\times10^{-2}$ | $Z_{cs}(3985)$ |
| $\zeta_3^\alpha$ | $\lvert D\bar{D}_s^*;1^{+-}\rangle$ | 19.3 | $20.0\pm1.0$ | 3.14–3.31 | 40–45 | $3.99^{+0.08}_{-0.08}$ | $(2.65^{+0.46}_{-0.42})\times10^{-2}$ | $Z_{cs}(4000)$ |
| $\zeta_4$ | $\lvert D^*\bar{D}_s^*;0^{+}\rangle$ | 21.1 | $22.0\pm1.0$ | 3.20–3.38 | 40–45 | $4.20^{+0.08}_{-0.08}$ | $(4.44^{+0.77}_{-0.69})\times10^{-2}$ | |
| $\zeta_5^\alpha$ | $\lvert D^*\bar{D}_s^*;1^{+}\rangle$ | 22.6 | $22.0\pm1.0$ | ~3.69 | ~37 | $4.22^{+0.09}_{-0.09}$ | $(6.96^{+1.15}_{-1.03})\times10^{-2}$ | $Z_{cs}(4220)$ |
| $\zeta_6^{\alpha\beta}$ | $\lvert D^*\bar{D}_s^*;2^{+}\rangle$ | 20.0 | $22.0\pm1.0$ | 3.16–3.62 | 40–52 | $4.20^{+0.09}_{-0.10}$ | $(2.39^{+0.36}_{-0.34})\times10^{-2}$ | |

radiative corrections in our QCD sum rule calculations, which lead to even more theoretical uncertainties. However, in the present study we are more concerned about the ratios, i.e., the relative production rates and the relative branching ratios, whose uncertainties are significantly reduced. We refer to Refs. [162–184] for more QCD sum rule studies.

### B. $D^{(*)}\bar{K}^{(*)}$ currents

In this subsection we use the $c$, $s$, $\bar{q}$, and $\bar{q}$ ($q = u/d$) quarks to construct open-charm tetraquark interpolating currents. We consider the following type of currents, as illustrated in Fig. 2:

$$\xi(x) = [\bar{q}_a(x)\Gamma_1^\xi c_b(x)][\bar{q}_c(x)\Gamma_2^\xi s_d(x)], \tag{27}$$

where $\Gamma_{1/2}^\xi$ are Dirac matrices. We shall use the Fierz rearrangement to study this configuration in Sec. IV B.

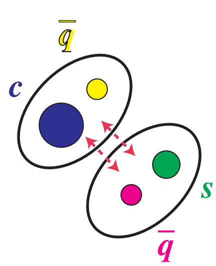



FIG. 2. Open-charm tetraquark currents $\xi(x)$.

There can exist altogether six $D^{(*)}\bar{K}^{(*)}$ hadronic molecular states:

$$|D\bar{K};0^+\rangle = |D\bar{K}\rangle_{J=0}, \tag{28}$$

$$|D\bar{K}^*;1^+\rangle = |D\bar{K}^*\rangle_{J=1}, \tag{29}$$

$$|D^*\bar{K};1^+\rangle = |D^*\bar{K}\rangle_{J=1}, \tag{30}$$

$$|D^*\bar{K}^*;0^+\rangle = |D^*\bar{K}^*\rangle_{J=0}, \tag{31}$$

$$|D^*\bar{K}^*;1^+\rangle = |D^*\bar{K}^*\rangle_{J=1}, \tag{32}$$

$$|D^*\bar{K}^*;2^+\rangle = |D^*\bar{K}^*\rangle_{J=2}. \tag{33}$$

Their corresponding currents are

$$\xi_1(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{q}_b(x)\gamma_5 s_b(x), \tag{34}$$

$$\xi_2^\alpha(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{q}_b(x)\gamma^\alpha s_b(x), \tag{35}$$

$$\xi_3^\alpha(x) = \bar{q}_a(x)\gamma^\alpha c_a(x)\bar{q}_b(x)\gamma_5 s_b(x), \tag{36}$$

$$\xi_4(x) = \bar{q}_a(x)\gamma^\mu c_a(x)\bar{q}_b(x)\gamma_\mu s_b(x), \tag{37}$$

$$\xi_5^\alpha(x) = \bar{q}_a(x)\gamma_\mu c_a(x)\bar{q}_b(x)\sigma^{\alpha\mu}\gamma_5 s_b(x) - \{\gamma_\mu \leftrightarrow \sigma^{\alpha\mu}\gamma_5\}, \tag{38}$$

$$\xi_6^{\alpha\beta}(x) = P^{\alpha\beta,\mu\nu}\bar{q}_a(x)\gamma_\mu c_a(x)\bar{q}_b(x)\gamma_\nu s_b(x). \tag{39}$$

We use the currents $\xi_{1\cdots6}$ to perform numerical analyses, and calculate masses and decay constants of $D^{(*)}\bar{K}^{(*)}$ molecular states. The obtained results are summarized in Table I. Our results support the interpretation of the $X_0(2900)$ as the $|D^*\bar{K}^*;0^+\rangle$ molecular state. However, masses of $|D\bar{K};0^+\rangle$ and $|D^*\bar{K};1^+\rangle$ are significantly larger than the $D\bar{K}$ and $D^*\bar{K}$ thresholds at about 2360 MeV and 2500 MeV, respectively. This diversity may be due to the nature of $K$ mesons as Nambu-Goldstone bosons. Again, the accuracy of our QCD sum rule results is moderate but not enough to determine: (a) whether these $D^{(*)}\bar{K}^{(*)}$ molecular states exist or not, and (b) whether they are bound states or resonance states.

### C. $D^{(*)}\bar{D}_s^{(*)}$ currents

In this subsection we use the $c$, $\bar{c}$, $s$, and $\bar{q}$ ($q = u/d$) quarks to construct strange hidden-charm tetraquark interpolating currents. We consider the following type of currents:

$$\zeta(x) = [\bar{q}_a(x)\Gamma_1^\zeta c_b(x)][\bar{c}_c(x)\Gamma_2^\zeta s_d(x)], \tag{40}$$

where $\Gamma_{1/2}^\zeta$ are Dirac matrices. Their Fierz rearrangements are quite similar to those for $\eta(x)$, just with one light *up*/*down* quark replaced by another *strange* quark.

There can exist altogether six $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states:

$$|D\bar{D}_s;0^+\rangle = |DD_s^-\rangle_{J=0}, \tag{41}$$

$$\sqrt{2}|D\bar{D}_s^*;1^{++}\rangle = |DD_s^{*-}\rangle_{J=1} + |D^*D_s^-\rangle_{J=1}, \tag{42}$$

$$\sqrt{2}|D\bar{D}_s^*;1^{+-}\rangle = |DD_s^{*-}\rangle_{J=1} - |D^*D_s^-\rangle_{J=1}, \tag{43}$$

$$|D^*\bar{D}_s^*;0^+\rangle = |D^*D_s^{*-}\rangle_{J=0}, \tag{44}$$

$$|D^*\bar{D}_s^*;1^+\rangle = |D^*D_s^{*-}\rangle_{J=1}, \tag{45}$$

$$|D^*\bar{D}_s^*;2^+\rangle = |D^*D_s^{*-}\rangle_{J=2}. \tag{46}$$

Their corresponding currents are

$$\zeta_1(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma_5 s_b(x), \tag{47}$$

$$\zeta_2^\alpha(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma^\alpha s_b(x) - \{\gamma_5 \leftrightarrow \gamma^\alpha\}, \tag{48}$$

$$\zeta_3^\alpha(x) = \bar{q}_a(x)\gamma_5 c_a(x)\bar{c}_b(x)\gamma^\alpha s_b(x) + \{\gamma_5 \leftrightarrow \gamma^\alpha\}, \tag{49}$$

$$\zeta_4(x) = \bar{q}_a(x)\gamma^\mu c_a(x)\bar{c}_b(x)\gamma_\mu s_b(x), \tag{50}$$

$$\zeta_5^\alpha(x) = \bar{q}_a(x)\gamma_\mu c_a(x)\bar{c}_b(x)\sigma^{\alpha\mu}\gamma_5 s_b(x) - \{\gamma_\mu \leftrightarrow \sigma^{\alpha\mu}\gamma_5\}, \tag{51}$$

$$\zeta_6^{\alpha\beta}(x) = P^{\alpha\beta,\mu\nu}\bar{q}_a(x)\gamma_\mu c_a(x)\bar{c}_b(x)\gamma_\nu s_b(x). \tag{52}$$

In the above expressions we have considered the mixing between the $DD_s^{*-}$ and $D^*D_s^-$ components, because their thresholds are very close to each other. After doing this, $|D\bar{D}_s^*;1^{++}\rangle$ is the strange partner of $|D\bar{D}^*;1^{++}\rangle$, so we denote its quantum number as $J^{PC} = 1^{++}$ for convenience; $|D\bar{D}_s^*;1^{+-}\rangle$ is the strange partner of $|D\bar{D}^*;1^{+-}\rangle$, so we also denote its quantum number as $J^{PC} = 1^{+-}$.

We use the currents $\zeta_{1\cdots6}$ to perform numerical analyses, and calculate masses and decay constants of $D^{(*)}\bar{D}_s^{(*)}$ molecular states. The obtained results are summarized in Table I. Our results support the interpretations of the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$ as the $|D\bar{D}_s^*;1^{++}\rangle$, $|D\bar{D}_s^*;1^{+-}\rangle$, and $|D^*\bar{D}_s^*;1^+\rangle$ molecular states, respectively. However, the accuracy of our QCD sum rule results is not enough to differentiate the $Z_{cs}(3985)$ and $Z_{cs}(4000)$.

To better understand $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states, we shall further study their production and decay properties in the following sections. Especially, the decay constants $f_X$ calculated in this section are important input parameters.

## III. PRODUCTIONS THROUGH THE CURRENT ALGEBRA

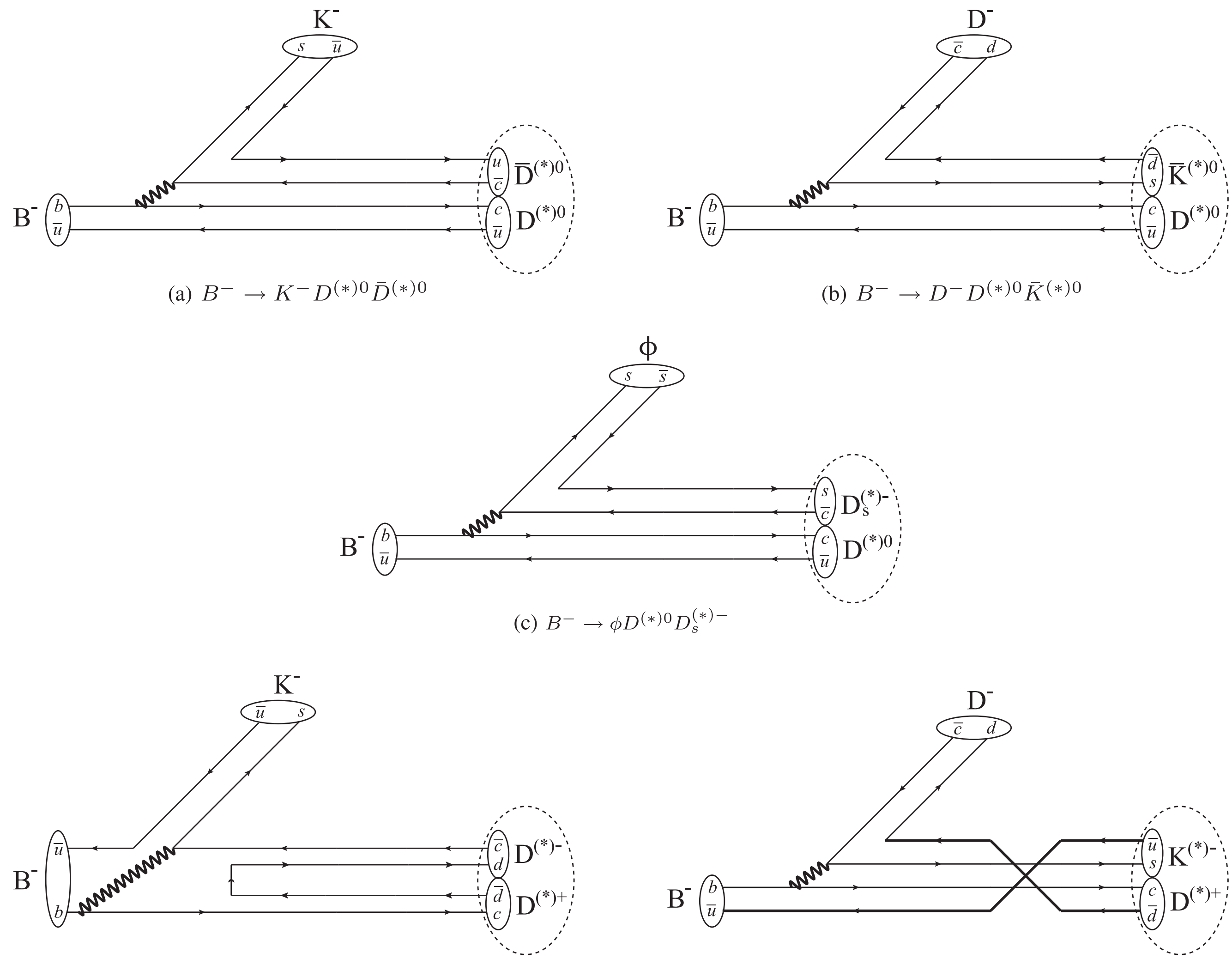
In this section we study productions of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states in $B$ and $B^*$ decays through the current algebra. We shall calculate their relative production rates, such as $\mathcal{B}(B^- \to K^- X)$: $\mathcal{B}(B^- \to K^- X')$, with $X$ and $X'$ different molecular states. This method has been applied in Ref. [148] to systematically study productions of $\bar{D}^{(*)}\Sigma_c^{(*)}$ hadronic molecular states in $\Lambda_b^0$ decays.

As depicted in Fig. 3, the quark content of the initial state $B^-$ is $\bar{u}b$. When it decays, first the $b$ quark decays into a $c$ quark by emitting a $W^-$ boson, and this $W^-$ boson translates into a pair of $\bar{c}$ and $s$ quarks, both of which are Cabibbo-favored; then they pick up an isoscalar quark-antiquark pair $\bar{u}u + \bar{d}d + \bar{s}s$ from the vacuum; finally they hadronize into the $D^{(*)}\bar{D}^{(*)}\bar{K}^{(*)}$ and $D^{(*)}\bar{D}_s^{(*)}\phi$ final states:

$$B^- = \bar{u}b \to \bar{u}c\bar{c}s \to \bar{u}c\bar{c}s(\bar{u}u + \bar{d}d + \bar{s}s) \to D^{(*)}\bar{D}^{(*)}\bar{K}^{(*)} + D^{(*)}\bar{D}_s^{(*)}\phi + \cdots. \tag{53}$$

Among all the possible final states, $K^-D^{(*)0}\bar{D}^{(*)0}$, $K^-D^{(*)+}D^{(*)-}$, $D^-D^{(*)0}\bar{K}^{(*)0}$, and $D^-D^{(*)+}K^{(*)-}$ can further produce the neutral $D^{(*)}\bar{D}^{(*)}$ and $D^{(*)}\bar{K}^{(*)}$ molecular states, and $\phi D^{(*)0}D_s^{(*)-}$ can further produce the charged $D^{(*)}\bar{D}_s^{(*)}$ molecular states.

We shall develop three Fierz rearrangements in Sec. III A to describe the production mechanisms depicted in Figs. 3(a)–3(c), and use them to perform numerical analyses separately in Secs. III B–III D. Productions of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states in $B^*$ decays will be similarly investigated in Sec. III E.




(a) $B^- \to K^- D^{(*)0}\bar{D}^{(*)0}$ (b) $B^- \to D^- D^{(*)0}\bar{K}^{(*)0}$

(c) $B^- \to \phi D^{(*)0} D_s^{(*)-}$

(d) $B^- \to K^- D^{(*)+} D^{(*)-}$ (e) $B^- \to D^- D^{(*)+} K^{(*)-}$

FIG. 3. Possible production mechanisms of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states in $B^-$ decays.

### A. Fierz rearrangement

To describe the production mechanism depicted in Fig. 3(a), we use the color rearrangement given in Eq. (10), and apply the Fierz transformation to interchange the $s_d$ and $u_f$ quarks:

$$B^- \longrightarrow J_{B^-} = [\delta^{ab}\bar{u}_a\gamma_5 b_b] \tag{54}$$

$$\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \tag{55}$$

$$\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \times [\delta^{ef}\bar{u}_e u_f] \tag{56}$$

$$\stackrel{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3} \times \bar{u}_a\gamma_\rho(1-\gamma_5)c_b \times \bar{c}_c\gamma^\rho(1-\gamma_5)s_d \times \bar{u}_e u_f + \cdots \tag{57}$$

$$\begin{aligned} \stackrel{\text{Fierz: } s_d\leftrightarrow u_f}{=} & -\frac{1}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma_5 u_f] \times [\delta^{ed}\bar{u}_e\gamma^\rho(1-\gamma_5)s_d] \\ & + \frac{1}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma^\rho(1-\gamma_5)u_f] \times [\delta^{ed}\bar{u}_e\gamma_5 s_d] \\ & - \frac{i}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\sigma^{\mu\rho}\gamma_5 u_f] \times [\delta^{ed}\bar{u}_e\gamma_\mu(1-\gamma_5)s_d] + \cdots \end{aligned} \tag{58}$$

$$= +\frac{1}{24} \times (\eta_3^\alpha - \eta_2^\alpha) \times [\bar{u}_a\gamma_\alpha\gamma_5 s_a] + \frac{1}{12} \times \eta_4 \times [\bar{u}_a\gamma_5 s_a] + \frac{i}{24} \times \eta_5^\alpha \times [\bar{u}_a\gamma_\alpha\gamma_5 s_a] + \cdots. \tag{59}$$

Similarly, we describe the production mechanism depicted in Fig. 3(b) as:

$$B^- \longrightarrow J_{B^-} = [\delta^{ab}\bar{u}_a\gamma_5 b_b] \tag{60}$$

$$\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \tag{61}$$

$$\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \times [\delta^{ef}\bar{d}_e d_f] \tag{62}$$

$$\stackrel{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3} \times \bar{u}_a\gamma_\rho(1-\gamma_5)c_b \times \bar{c}_c\gamma^\rho(1-\gamma_5)s_d \times \bar{d}_e d_f + \cdots \tag{63}$$

$$\begin{aligned} \stackrel{\text{Fierz}:\ s_d\leftrightarrow d_f}{=} & -\frac{1}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma_5 d_f] \times [\delta^{ed}\bar{d}_e\gamma^\rho(1-\gamma_5)s_d] \\ & + \frac{1}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma^\rho(1-\gamma_5)d_f] \times [\delta^{ed}\bar{d}_e\gamma_5 s_d] \\ & - \frac{i}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma_\mu(1-\gamma_5)d_f] \times [\delta^{ed}\bar{d}_e\sigma^{\mu\rho}\gamma_5 s_d] + \cdots \end{aligned} \tag{64}$$

$$= -\frac{1}{12} \times \xi_3^\alpha \times [\bar{c}_a\gamma_\alpha\gamma_5 d_a] - \frac{1}{12} \times \xi_4 \times [\bar{c}_a\gamma_5 d_a] + \frac{i}{24} \times \xi_5^\alpha \times [\bar{c}_a\gamma_\alpha\gamma_5 d_a] + \cdots, \tag{65}$$

and the production mechanism depicted in Fig. 3(c) as:

$$B^- \longrightarrow J_{B^-} = [\delta^{ab}\bar{u}_a\gamma_5 b_b] \tag{66}$$

$$\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \tag{67}$$

$$\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \times [\delta^{ef}\bar{s}_e s_f] \tag{68}$$

$$\stackrel{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3} \times \bar{u}_a\gamma_\rho(1-\gamma_5)c_b \times \bar{c}_c\gamma^\rho(1-\gamma_5)s_d \times \bar{s}_e s_f + \cdots \tag{69}$$

$$\begin{aligned} \stackrel{\text{Fierz}:\ s_d\leftrightarrow s_f}{=} & -\frac{1}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\gamma_5 s_f] \times [\delta^{ed}\bar{s}_e\gamma^\rho(1-\gamma_5)s_d] \\ & - \frac{i}{12} \times [\delta^{ab}\bar{u}_a\gamma_\rho(1-\gamma_5)c_b] \times [\delta^{cf}\bar{c}_c\sigma^{\mu\rho}\gamma_5 s_f] \times [\delta^{ed}\bar{s}_e\gamma_\mu(1-\gamma_5)s_d] + \cdots \end{aligned} \tag{70}$$

$$= -\frac{1}{24} \times (\zeta_3^\alpha - \zeta_2^\alpha) \times [\bar{s}_a\gamma_\alpha s_a] - \frac{i}{24} \times \zeta_5^\alpha \times [\bar{s}_a\gamma_\alpha s_a] + \cdots. \tag{71}$$

The brief explanations to the Fierz rearrangement given in Eq. (54) are as follows:

(i) Eq. (55) describes the Cabibbo-favored weak decay process, $b \to c + \bar{c}s$, via the $V$-$A$ current.

(ii) Eq. (56) describes the production of the $\bar{u}u$ pair from the vacuum via the $^3P_0$ quark pair creation mechanism.

(iii) In Eq. (57) we apply the color rearrangement given in Eq. (10).

(iv) In Eq. (58) we apply the Fierz transformation to interchange the $s_d$ and $u_f$ quarks.

(v) In Eq. (59) we combine the four quarks $\bar{u}_a c_b \bar{c}_c u_f$ together so that $D^{(*)0}\bar{D}^{(*)0}$ molecular states can be produced.

The other two Fierz rearrangements given in Eqs. (60) and (66) can be similarly investigated.

In the above expression, we only consider $\eta_{1\cdots6}$, $\xi_{1\cdots6}$, and $\zeta_{1\cdots6}$ defined in Eqs. (17)–(22), (34)–(39), and (47)–(52). These currents couple to $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states through $S$-wave. Besides, there may exist some other currents coupling to these states through $P$-wave, but we do not take them into account in the present study. Hence, some other molecular states such as $|D\bar{D};0^{++}\rangle$ may still be produced in $B^-$ decays, and omissions of these currents cause some theoretical uncertainties.

Beside the three production mechanisms depicted in Figs. 3(a)–(c), there exist some other possible production mechanisms, such as those depicted in Figs. 3(d) and 3(e). However, the mechanism depicted in Fig. 3(d) contains the color factor $\frac{\delta^{ad}\delta^{cf}\delta^{eb}}{9}\times\bar{u}_a c_b \bar{c}_c s_d \bar{d}_e d_f$, and the one depicted in Fig. 3(e) contains the color factor $\frac{\delta^{ad}\delta^{cf}\delta^{eb}}{9}\times\bar{u}_a c_b \bar{c}_c s_d \bar{d}_e d_f$, so both of them are suppressed.

Very recently, the LHCb Collaboration reported their observations of $X(4630)$ and $X(4685)$ in the $B^+\to K^+X\to K^+J/\psi\phi$ decay process [15]. Although these two structures were also observed in $B$ decays, their production mechanisms may not be (completely) the same as those depicted in Fig. 3(a)–(c), so we do not investigate them in the present study.

### B. Productions of $D^{(*)}\bar{D}^{(*)}$ molecules

In this subsection we use the Fierz rearrangement given in Eq. (54) to perform numerical analyses, and calculate relative production rates of $D^{(*)}\bar{D}^{(*)}$ molecular states in $B^-\to K^-X$ decays. To do this we need the following couplings to the $K^-$ meson:

$$\langle 0|\bar{u}_a i\gamma_5 s_a|K^-(q)\rangle = \lambda_K,$$
$$\langle 0|\bar{u}_a \gamma_\mu\gamma_5 s_a|K^-(q)\rangle = iq_\mu f_K, \tag{72}$$

where $f_K = 155.6$ MeV [2] and $\lambda_K = \frac{f_K^2 m_K}{m_q+m_s}$ with the isospin-averaged current quark mass $m_q=\frac{m_u+m_d}{2}$.

We extract from Eq. (54) the following decay channels:

(1) The decay of $B^-$ into $|D\bar{D}^*;1^{++}\rangle K^-$ is contributed by $\eta_2^\alpha\times[\bar{u}_a\gamma_\alpha\gamma_5 s_a]$:

$$\langle B^-(q)|D\bar{D}^*;1^{++}(\epsilon_1,q_1)K^-(q_2)\rangle$$
$$= -\frac{id_1}{24} f_K f_{|D\bar{D}^*;1^{++}\rangle}\epsilon_1\cdot q_2. \tag{73}$$

The decay constant $f_{|D\bar{D}^*;1^{++}\rangle}$ has been calculated in the previous section and given in Table I. The overall factor $d_1$ is related to: (a) the coupling of $J_{B^-}$ to $B^-$, (b) the weak and $^3P_0$ decay processes described by Eqs. (55) and (56), (c) the isospin factor between $|D^0\bar{D}^{*0};1^{++}\rangle$ and $|D\bar{D}^*;1^{++}\rangle$, and (d) the dynamical process depicted in Fig. 3(a).

(2) The decay of $B^-$ into $|D\bar{D}^*;1^{+-}\rangle K^-$ is contributed by $\eta_3^\alpha\times[\bar{u}_a\gamma_\alpha\gamma_5 s_a]$:

$$\langle B^-(q)|D\bar{D}^*;1^{+-}(\epsilon_1,q_1)K^-(q_2)\rangle$$
$$= \frac{i\,d_1}{24} f_K f_{|D\bar{D}^*;1^{+-}\rangle}\epsilon_1\cdot q_2. \tag{74}$$

(3) The decay of $B^-$ into $|D^*\bar{D}^*;0^{++}\rangle K^-$ is contributed by $\eta_4\times[\bar{u}_a\gamma_5 s_a]$:

$$\langle B^-(q)|D^*\bar{D}^*;0^{++}(q_1)K^-(q_2)\rangle$$
$$= \frac{d_1}{12}\lambda_K f_{|D^*\bar{D}^*;0^{++}\rangle}. \tag{75}$$

(4) The decay of $B^-$ into $|D^*\bar{D}^*;1^{+-}\rangle K^-$ is contributed by $\eta_5^\alpha\times[\bar{u}_a\gamma_\alpha\gamma_5 s_a]$:

$$\langle B^-(q)|D^*\bar{D}^*;1^{+-}(\epsilon_1,q_1)K^-(q_2)\rangle$$
$$= -\frac{d_1}{24} f_K f_{|D^*\bar{D}^*;1^{+-}\rangle}\epsilon_1\cdot q_2. \tag{76}$$

In the present study we adopt the possible interpretations of the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as the $|D\bar{D}^*;1^{++}\rangle$, $|D\bar{D}^*;1^{+-}\rangle$, and $|D^*\bar{D}^*;1^{+-}\rangle$ molecular states, respectively. As summarized in Table I, the masses of the $D^{(*)}\bar{D}^{(*)}$ molecular states calculated through the QCD sum rule method have considerable uncertainties, based on which we are not able to study their production and decay properties, e.g., we are not able to calculate the partial decay width of the $|D\bar{D}^*;1^{+-}\rangle$ molecular state into the $D\bar{D}^*$ channel due to the ambiguous phase space, as will be studied in Sec. IV A 5. Hence, we assume the masses of the $D^{(*)}\bar{D}^{(*)}$ molecular states to be either the realistic masses or at the lowest $D^{(*)}\bar{D}^{(*)}$ thresholds [2]:

$$\begin{aligned}
M_{|D\bar{D};0^{++}\rangle} &\approx M_{D^0}+M_{\bar{D}^0} = 3730\ \text{MeV},\\
M_{|D\bar{D}^*;1^{++}\rangle} &= M_{X(3872)} = 3871.69\ \text{MeV},\\
M_{|D\bar{D}^*;1^{+-}\rangle} &= M_{Z_c(3900)} = 3888.4\ \text{MeV},\\
M_{|D^*\bar{D}^*;0^{++}\rangle} &\approx M_{D^{*0}}+M_{\bar{D}^{*0}} = 4014\ \text{MeV},\\
M_{|D^*\bar{D}^*;1^{+-}\rangle} &\approx M_{Z_c(4020)} = 4024.1\ \text{MeV},\\
M_{|D^*\bar{D}^*;2^{++}\rangle} &\approx M_{D^{*0}}+M_{\bar{D}^{*0}} = 4014\ \text{MeV}.
\end{aligned} \tag{77}$$

Then we can evaluate the above production amplitudes to obtain the following partial decay widths:

$$\begin{aligned}
\Gamma(B^-\to K^-|D\bar{D};0^{++}\rangle) &= 0,\\
\Gamma(B^-\to K^-|D\bar{D}^*;1^{++}\rangle) &= d_1^2 0.40\times10^{-10}\ \text{GeV}^{13},\\
\Gamma(B^-\to K^-|D\bar{D}^*;1^{+-}\rangle) &= d_1^2 0.38\times10^{-10}\ \text{GeV}^{13},\\
\Gamma(B^-\to K^-|D^*\bar{D}^*;0^{++}\rangle) &= d_1^2 4.21\times10^{-10}\ \text{GeV}^{13},\\
\Gamma(B^-\to K^-|D^*\bar{D}^*;1^{+-}\rangle) &= d_1^2 1.35\times10^{-10}\ \text{GeV}^{13},\\
\Gamma(B^-\to K^-|D^*\bar{D}^*;2^{++}\rangle) &= 0.
\end{aligned} \tag{78}$$

### C. Productions of $D^{(*)}\bar{K}^{(*)}$ molecules

Similarly, we use the Fierz rearrangement given in Eq. (60) to perform numerical analyses, and calculate relative production rates of $D^{(*)}\bar{K}^{(*)}$ molecular states in $B^- \to D^- X$ decays:

(5) The decay of $B^-$ into $|D^*\bar{K};1^+\rangle D^-$ is contributed by $\xi_3^\alpha \times [\bar{c}_a\gamma_\alpha\gamma_5 d_a]$:

$$\langle B^-(q)|D^*\bar{K};1^+(\epsilon_1,q_1)D^-(q_2)\rangle = -\frac{id_2}{12} f_D f_{|D^*\bar{K};1^+\rangle}\epsilon_1\cdot q_2. \quad (79)$$

The overall factor $d_2$ is related to: (a) the coupling of $J_{B^-}$ to $B^-$, (b) the weak and $^3P_0$ decay processes described by Eqs. (61) and (62), (c) the isospin factor between $|D^{*0}\bar{K}^0;1^+\rangle$ and $|D^*\bar{K};1^+\rangle$, and (d) the dynamical process depicted in Fig. 3(b).

(6) The decay of $B^-$ into $|D^*\bar{K}^*;0^+\rangle D^-$ is contributed by $\xi_4 \times [\bar{c}_a\gamma_5 d_a]$:

$$\langle B^-(q)|D^*\bar{K}^*;0^+(q_1)D^-(q_2)\rangle = -\frac{d_2}{12}\lambda_D f_{|D^*\bar{K}^*;0^+\rangle}. \quad (80)$$

(7) The decay of $B^-$ into $|D^*\bar{K}^*;1^+\rangle D^-$ is contributed by $\xi_5^\alpha \times [\bar{c}_a\gamma_\alpha\gamma_5 d_a]$:

$$\langle B^-(q)|D^*\bar{K}^*;1^+(\epsilon_1,q_1)D^-(q_2)\rangle = -\frac{d_2}{24} f_D f_{|D^*\bar{K}^*;1^+\rangle}\epsilon_1\cdot q_2. \quad (81)$$

In the above expressions $f_D$ and $\lambda_D$ are decay constants of the $D^-$ meson, whose definitions and values can be found in Table II.

In the present study we adopt the possible interpretation of the $X_0(2900)$ as the $|D^*\bar{K}^*;0^+\rangle$ molecular state. Accordingly, we assume masses of $D^{(*)}\bar{K}^{(*)}$ molecular states to be either the realistic mass or at the lowest $D^{(*)}\bar{K}^{(*)}$ thresholds [2]:

$$\begin{aligned} M_{|D\bar{K};0^+\rangle} &\approx M_{D^0}+M_{K^+} = 2359\text{ MeV},\\ M_{|D\bar{K}^*;1^+\rangle} &\approx M_{D^0}+M_{K^{*+}} = 2756\text{ MeV},\\ M_{|D^*\bar{K};1^+\rangle} &\approx M_{D^{*0}}+M_{K^+} = 2501\text{ MeV},\\ M_{|D^*\bar{K}^*;0^+\rangle} &= M_{X_0(2900)} = 2866\text{ MeV},\\ M_{|D^*\bar{K}^*;1^+\rangle} &\approx M_{D^{*0}}+M_{K^{*+}} = 2899\text{ MeV},\\ M_{|D^*\bar{K}^*;2^+\rangle} &\approx M_{D^{*0}}+M_{K^{*+}} = 2899\text{ MeV}. \end{aligned} \quad (82)$$

Then we can evaluate the above production amplitudes to obtain the following partial decay widths:

$$\begin{aligned} \Gamma(B^-\to D^-|D\bar{K};0^+\rangle) &= 0,\\ \Gamma(B^-\to D^-|D\bar{K}^*;1^+\rangle) &= 0,\\ \Gamma(B^-\to D^-|D^*\bar{K};1^+\rangle) &= d_2^2 2.13\times10^{-10}\text{ GeV}^{13},\\ \Gamma(B^-\to D^-|D^*\bar{K}^*;0^+\rangle) &= d_2^2 3.18\times10^{-10}\text{ GeV}^{13},\\ \Gamma(B^-\to D^-|D^*\bar{K}^*;1^+\rangle) &= d_2^2 2.23\times10^{-10}\text{ GeV}^{13},\\ \Gamma(B^-\to D^-|D^*\bar{K}^*;2^+\rangle) &= 0. \end{aligned} \quad (83)$$

### D. Productions of $D^{(*)}\bar{D}_s^{(*)}$ molecules

Similarly, we use the Fierz rearrangement given in Eq. (66) to perform numerical analyses, and calculate relative production rates of $D^{(*)}\bar{D}_s^{(*)}$ molecular states in $B^- \to \phi X$ decays:

(8) The decay of $B^-$ into $|D\bar{D}_s^*;1^{++}\rangle\phi$ is contributed by $\zeta_2^\alpha \times [\bar{s}_a\gamma_\alpha s_a]$:

$$\langle B^-(q)|D\bar{D}_s^*;1^{++}(\epsilon_1,q_1)\phi(\epsilon_2,q_2)\rangle = \frac{d_3}{24} m_\phi f_\phi f_{|D\bar{D}_s^*;1^{++}\rangle}\epsilon_1\cdot\epsilon_2. \quad (84)$$

The overall factor $d_3$ is related to: (a) the coupling of $J_{B^-}$ to $B^-$, (b) the weak and $^3P_0$ decay processes described by Eqs. (67) and (68), and (c) the dynamical process depicted in Fig. 3(c). In this case there does not exist the isospin factor.

(9) The decay of $B^-$ into $|D\bar{D}_s^*;1^{+-}\rangle\phi$ is contributed by $\zeta_3^\alpha \times [\bar{s}_a\gamma_\alpha s_a]$:

$$\langle B^-(q)|D\bar{D}_s^*;1^{+-}(\epsilon_1,q_1)\phi(\epsilon_2,q_2)\rangle = -\frac{d_3}{24} m_\phi f_\phi f_{|D\bar{D}_s^*;1^{+-}\rangle}\epsilon_1\cdot\epsilon_2. \quad (85)$$

(10) The decay of $B^-$ into $|D^*\bar{D}_s^*;1^+\rangle\phi$ is contributed by $\zeta_5^\alpha \times [\bar{s}_a\gamma_\alpha s_a]$:

$$\langle B^-(q)|D^*\bar{D}_s^*;1^+(\epsilon_1,q_1)\phi(\epsilon_2,q_2)\rangle = -\frac{i\,d_3}{24} m_\phi f_\phi f_{|D^*\bar{D}_s^*;1^+\rangle}\epsilon_1\cdot\epsilon_2. \quad (86)$$

In the above expressions $f_\phi$ is decay constant of the $\phi$ meson. Besides, we also need $f_\phi^T$, and their definitions are

$$\begin{aligned} \langle 0|\bar{s}_a\gamma_\mu s_a|\phi(\epsilon,q)\rangle &= m_\phi f_\phi \epsilon_\mu,\\ \langle 0|\bar{s}_a\sigma_{\mu\nu} s_a|\phi(\epsilon,q)\rangle &= i f_\phi^T(q_\mu\epsilon_\nu - q_\nu\epsilon_\mu), \end{aligned} \quad (87)$$

where $f_\phi = 233$ MeV [2] and $f_\phi^T = 177$ MeV [196].

In the present study we adopt the possible interpretations of the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$ as the $|D\bar{D}_s^*;1^{++}\rangle$, $|D\bar{D}_s^*;1^{+-}\rangle$, and $|D^*\bar{D}_s^*;1^+\rangle$ molecular states, respectively. Accordingly, we assume masses of $D^{(*)}\bar{D}_s^{(*)}$ molecular states to be either the realistic masses or at the lowest $D^{(*)}\bar{D}_s^{(*)}$ thresholds [2]:

TABLE II. Couplings of meson operators to meson states, where color indices are omitted for simplicity. All the isovector meson operators have the quark content $\bar{q}\Gamma q = (\bar{u}\Gamma u - \bar{d}\Gamma d)/\sqrt{2}$, and all the isoscalar light meson operators have the quark content $\bar{q}\Gamma q = (\bar{u}\Gamma u + \bar{d}\Gamma d)/\sqrt{2}$.

| Operators–$J_{(\mu\nu)}$ | $I^G J^{PC}$ | Mesons | $I^G J^{PC}$ | Couplings | Decay Constants |
|---|---|---|---|---|---|
| $\bar{c}c$ | $0^+0^{++}$ | $\chi_{c0}(1P)$ | $0^+0^{++}$ | $\langle 0\|J\|\chi_{c0}\rangle = m_{\chi_{c0}} f_{\chi_{c0}}$ | $f_{\chi_{c0}} = 343$ MeV [185] |
| $\bar{c}i\gamma_5 c$ | $0^+0^{-+}$ | $\eta_c$ | $0^+0^{-+}$ | $\langle 0\|J\|\eta_c\rangle = \lambda_{\eta_c}$ | $\lambda_{\eta_c} = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}$ |
| $\bar{c}\gamma_\mu c$ | $0^-1^{--}$ | $J/\psi$ | $0^-1^{--}$ | $\langle 0\|J_\mu\|J/\psi\rangle = m_{J/\psi} f_{J/\psi}\epsilon_\mu$ | $f_{J/\psi} = 418$ MeV [186] |
| $\bar{c}\gamma_\mu\gamma_5 c$ | $0^+1^{++}$ | $\eta_c$<br>$\chi_{c1}(1P)$ | $0^+0^{-+}$<br>$0^+1^{++}$ | $\langle 0\|J_\mu\|\eta_c\rangle = ip_\mu f_{\eta_c}$<br>$\langle 0\|J_\mu\|\chi_{c1}\rangle = m_{\chi_{c1}} f_{\chi_{c1}}\epsilon_\mu$ | $f_{\eta_c} = 387$ MeV [186]<br>$f_{\chi_{c1}} = 335$ MeV [187] |
| $\bar{c}\sigma_{\mu\nu} c$ | $0^-1^{\pm -}$ | $J/\psi$<br>$h_c(1P)$ | $0^-1^{--}$<br>$0^-1^{+-}$ | $\langle 0\|J_{\mu\nu}\|J/\psi\rangle = if^T_{J/\psi}(p_\mu\epsilon_\nu - p_\nu\epsilon_\mu)$<br>$\langle 0\|J_{\mu\nu}\|h_c\rangle = if^T_{h_c}\epsilon_{\mu\nu\alpha\beta}\epsilon^\alpha p^\beta$ | $f^T_{J/\psi} = 410$ MeV [186]<br>$f^T_{h_c} = 235$ MeV [186] |
| $\bar{c}q$ | $0^+$ | $\bar{D}_0^*$ | $0^+$ | $\langle 0\|J\|\bar{D}_0^*\rangle = m_{D_0^*} f_{D_0^*}$ | $f_{D_0^*} = 410$ MeV [188] |
| $\bar{c}i\gamma_5 q$ | $0^-$ | $\bar{D}$ | $0^-$ | $\langle 0\|J\|\bar{D}\rangle = \lambda_D$ | $\lambda_D = \frac{f_D m_D^2}{m_c+m_d}$ |
| $\bar{c}\gamma_\mu q$ | $1^-$ | $\bar{D}^*$ | $1^-$ | $\langle 0\|J_\mu\|\bar{D}^*\rangle = m_{D^*} f_{D^*}\epsilon_\mu$ | $f_{D^*} = 253$ MeV [189] |
| $\bar{c}\gamma_\mu\gamma_5 q$ | $1^+$ | $\bar{D}$<br>$\bar{D}_1$ | $0^-$<br>$1^+$ | $\langle 0\|J_\mu\|\bar{D}\rangle = ip_\mu f_D$<br>$\langle 0\|J_\mu\|\bar{D}_1\rangle = m_{D_1} f_{D_1}\epsilon_\mu$ | $f_D = 211.9$ MeV [2]<br>$f_{D_1} = 356$ MeV [188] |
| $\bar{c}\sigma_{\mu\nu} q$ | $1^\pm$ | $\bar{D}^*$<br>… | $1^-$<br>$1^+$ | $\langle 0\|J_{\mu\nu}\|\bar{D}^*\rangle = if^T_{D^*}(p_\mu\epsilon_\nu - p_\nu\epsilon_\mu)$<br>… | $f^T_{D^*} \approx 220$ MeV<br>… |
| $\bar{c}s$ | $0^+$ | … | $0^+$ | … | … |
| $\bar{c}i\gamma_5 s$ | $0^-$ | $\bar{D}_s$ | $0^-$ | $\langle 0\|J\|\bar{D}_s\rangle = \lambda_{D_s}$ | $\lambda_{D_s} = \frac{f_{D_s} m_{D_s}^2}{m_c+m_s}$ |
| $\bar{c}\gamma_\mu s$ | $1^-$ | $\bar{D}_s^*$ | $1^-$ | $\langle 0\|J_\mu\|\bar{D}_s^*\rangle = m_{D_s^*} f_{D_s^*}\epsilon_\mu$ | $f_{D_s^*} = 301$ MeV [2] |
| $\bar{c}\gamma_\mu\gamma_5 s$ | $1^+$ | $\bar{D}_s$<br>… | $0^-$<br>$1^+$ | $\langle 0\|J_\mu\|\bar{D}\rangle = ip_\mu f_{D_s}$<br>… | $f_{D_s} = 257$ MeV [2] |
| $\bar{c}\sigma_{\mu\nu} s$ | $1^\pm$ | $\bar{D}_s^*$<br>… | $1^-$<br>$1^+$ | $\langle 0\|J_{\mu\nu}\|\bar{D}_s^*\rangle = if^T_{D_s^*}(p_\mu\epsilon_\nu - p_\nu\epsilon_\mu)$<br>… | $f^T_{D_s^*} \approx f_{D_s^*} \times \frac{f^T_{D^*}}{f_{D^*}}$<br>… |
| $\bar{q}q$ | $0^+0^{++}$ | $f_0(500)$ | $0^+0^{++}$ | $\langle 0\|J\|f_0\rangle = m_{f_0} f_{f_0}$ | $f_{f_0} \sim 380$ MeV [190] |
| $\bar{q}i\gamma_5 q$ | $0^+0^{-+}$ | $\eta$ | $0^+0^{-+}$ | … | … |
| $\bar{q}\gamma_\mu q$ | $0^-1^{--}$ | $\omega$ | $0^-1^{--}$ | $\langle 0\|J_\mu\|\omega\rangle = m_\omega f_\omega\epsilon_\mu$ | $f_\omega \approx f_\rho = 216$ MeV [191] |
| $\bar{q}\gamma_\mu\gamma_5 q$ | $0^+1^{++}$ | $\eta$<br>$f_1(1285)$ | $0^+0^{-+}$<br>$0^+1^{++}$ | $\langle 0\|J_\mu\|\eta\rangle = ip_\mu f_\eta$<br>… | $f_\eta = 97$ MeV [192,193]<br>… |
| $\bar{q}\sigma_{\mu\nu} q$ | $0^-1^{\pm -}$ | $\omega$<br>$h_1(1170)$ | $0^-1^{--}$<br>$0^-1^{+-}$ | $\langle 0\|J_{\mu\nu}\|\omega\rangle = if^T_\omega(p_\mu\epsilon_\nu - p_\nu\epsilon_\mu)$<br>$\langle 0\|J_{\mu\nu}\|h_1\rangle = if^T_{h_1}\epsilon_{\mu\nu\alpha\beta}\epsilon^\alpha p^\beta$ | $f^T_\omega \approx f^T_\rho = 159$ MeV [191]<br>$f^T_{h_1} \approx f^T_{b_1} = 180$ MeV [194] |
| $\bar{q}q$ | $1^-0^{++}$ | … | $1^-0^{++}$ | … | … |
| $\bar{q}i\gamma_5 q$ | $1^-0^{-+}$ | $\pi^0$ | $1^-0^{-+}$ | $\langle 0\|J\|\pi^0\rangle = \lambda_\pi$ | $\lambda_\pi = \frac{f_\pi m_\pi^2}{m_u+m_d}$ |
| $\bar{q}\gamma_\mu q$ | $1^+1^{--}$ | $\rho^0$ | $1^+1^{--}$ | $\langle 0\|J_\mu\|\rho^0\rangle = m_\rho f_\rho\epsilon_\mu$ | $f_\rho = 216$ MeV [191] |
| $\bar{q}\gamma_\mu\gamma_5 q$ | $1^-1^{++}$ | $\pi^0$<br>$a_1(1260)$ | $1^-0^{-+}$<br>$1^-1^{++}$ | $\langle 0\|J_\mu\|\pi^0\rangle = ip_\mu f_\pi$<br>$\langle 0\|J_\mu\|a_1\rangle = m_{a_1} f_{a_1}\epsilon_\mu$ | $f_\pi = 130.2$ MeV [2]<br>$f_{a_1} = 254$ MeV [195] |
| $\bar{q}\sigma_{\mu\nu} q$ | $1^+1^{\pm -}$ | $\rho^0$<br>$b_1(1235)$ | $1^+1^{--}$<br>$1^+1^{+-}$ | $\langle 0\|J_{\mu\nu}\|\rho^0\rangle = if^T_\rho(p_\mu\epsilon_\nu - p_\nu\epsilon_\mu)$<br>$\langle 0\|J_{\mu\nu}\|b_1\rangle = if^T_{b_1}\epsilon_{\mu\nu\alpha\beta}\epsilon^\alpha p^\beta$ | $f^T_\rho = 159$ MeV [191]<br>$f^T_{b_1} = 180$ MeV [194] |

$$
\begin{aligned}
M_{|D\bar{D}_s;0^+\rangle} &\approx M_{D^0} + M_{D_s^-} = 3833\ \text{MeV},\\
M_{|D\bar{D}_s^*;1^{++}\rangle} &\approx M_{Z_{cs}(3985)} = 3982.5\ \text{MeV},\\
M_{|D\bar{D}_s^*;1^{+-}\rangle} &\approx M_{Z_{cs}(4000)} = 4003\ \text{MeV},\\
M_{|D^*\bar{D}_s^*;0^+\rangle} &= M_{D^{*0}} + M_{D_s^{*-}} = 4119\ \text{MeV},\\
M_{|D^*\bar{D}_s^*;1^+\rangle} &\approx M_{Z_{cs}(4220)} = 4216\ \text{MeV},\\
M_{|D^*\bar{D}_s^*;2^+\rangle} &\approx M_{D^{*0}} + M_{D_s^{*-}} = 4119\ \text{MeV}.
\end{aligned} \tag{88}
$$

Then we can evaluate the above production amplitudes to obtain the following partial decay widths:

$$
\begin{aligned}
&\Gamma(B^- \to \phi|D\bar{D}_s;0^+\rangle) = 0,\\
&\Gamma(B^- \to \phi|D\bar{D}_s^*;1^{++}\rangle) = d_3^2 2.53\times 10^{-10}\ \text{GeV}^{13},\\
&\Gamma(B^- \to \phi|D\bar{D}_s^*;1^{+-}\rangle) = d_3^2 2.38\times 10^{-10}\ \text{GeV}^{13},\\
&\Gamma(B^- \to \phi|D^*\bar{D}_s^*;0^+\rangle) = 0,\\
&\Gamma(B^- \to \phi|D^*\bar{D}_s^*;1^+\rangle) = d_3^2 5.77\times 10^{-10}\ \text{GeV}^{13},\\
&\Gamma(B^- \to \phi|D^*\bar{D}_s^*;2^+\rangle) = 0.
\end{aligned} \tag{89}
$$

From Eqs. (78), (83), and (89), we can derive the relative production rate $\mathcal{R}_1 \equiv \frac{\mathcal{B}(B^-\to K^-X/D^-X/\phi X)}{\mathcal{B}(B^-\to D^-|D^*\bar{K}^*;0^+\rangle)/d_2}$. We summarize the obtained results in Table III, which will be further discussed in Sec. V. The difference among the three overall factors $d_{1,2,3}$ is partly caused by the different dynamical processes described by Eqs. (59), (65), and (71), i.e., the attraction between $D^{(*)}$ and $\bar{D}^{(*)}$, the attraction between $D^{(*)}$ and $\bar{K}^{(*)}$, and the attraction between $D^{(*)}$ and $\bar{D}_s^{(*)}$ can be different. However, we are not able to estimate this in the present study.

### E. Productions in $B^*$ decays

In this subsection we follow the same procedures to study productions of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states in $B^*$ decays.

Similar to Eq. (54), we apply the Fierz transformation to obtain:

$$
\begin{aligned}
B^{*-} \longrightarrow J^\alpha_{B^{*-}} &= [\delta^{ab}\bar{u}_a\gamma^\alpha b_b]\\
&\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d]\\
&\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d]\times[\delta^{ef}\bar{u}_e u_f]\\
&\overset{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3}\times\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b\times\bar{c}_c\gamma^\rho(1-\gamma_5)s_d\times\bar{u}_e u_f+\cdots\\
&\overset{\text{Fierz: } s_d\leftrightarrow u_f}{=} +\frac{1}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma_5 u_f]\times[\delta^{ed}\bar{u}_e\gamma^\alpha(1-\gamma_5)s_d]\\
&\quad -\frac{1}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma^\alpha(1-\gamma_5)u_f]\times[\delta^{ed}\bar{u}_e\gamma_5 s_d]\\
&\quad +\frac{1}{12}\times[\delta^{ab}\bar{u}_a i\sigma^{\alpha\rho}\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma^\rho(1-\gamma_5)u_f]\times[\delta^{ed}\bar{u}_e\gamma_5 s_d]+\cdots\\
&= -\frac{1}{12}\times\eta_1\times[\bar{u}_a\gamma^\alpha\gamma_5 s_a]-\frac{1}{24}\times(\eta_2^\alpha+\eta_3^\alpha)\times[\bar{u}_a\gamma_5 s_a]-\frac{i}{24}\times\eta_5^\alpha\times[\bar{u}_a\gamma_5 s_a]+\cdots.
\end{aligned} \tag{90}
$$

Similar to Eq. (60), we apply the Fierz transformation to obtain:

$$
\begin{aligned}
B^{*-} \longrightarrow J^\alpha_{B^{*-}} &= [\delta^{ab}\bar{u}_a\gamma^\alpha b_b]\\
&\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d]\\
&\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d]\times[\delta^{ef}\bar{d}_e d_f]\\
&\overset{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3}\times\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b\times\bar{c}_c\gamma^\rho(1-\gamma_5)s_d\times\bar{d}_e d_f+\cdots\\
&\overset{\text{Fierz: } s_d\leftrightarrow u_f}{=} +\frac{1}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma_5 d_f]\times[\delta^{ed}\bar{d}_e\gamma^\alpha(1-\gamma_5)s_d]\\
&\quad -\frac{1}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma^\alpha(1-\gamma_5)d_f]\times[\delta^{ed}\bar{d}_e\gamma_5 s_d]\\
&\quad -\frac{1}{12}\times[\delta^{ab}\bar{u}_a i\sigma^{\alpha\rho}\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma_5 d_f]\times[\delta^{ed}\bar{d}_e\gamma^\rho(1-\gamma_5)s_d]+\cdots\\
&= +\frac{1}{12}\times\xi_1\times[\bar{c}_a\gamma^\alpha\gamma_5 d_a]+\frac{1}{12}\times\xi_2^\alpha\times[\bar{c}_a\gamma_5 d_a]+\frac{i}{24}\times\xi_5^\alpha\times[\bar{c}_a\gamma_5 d_a]+\cdots.
\end{aligned} \tag{91}
$$

Similar to Eq. (66), we apply the Fierz transformation to obtain:

$$\begin{aligned}
B^{*-} &\longrightarrow J^{\alpha}_{B^{*-}} = [\delta^{ab}\bar{u}_a\gamma^\alpha b_b] \\
&\xrightarrow{\text{weak}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d] \\
&\xrightarrow{\text{QPC}} [\delta^{ab}\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b]\times[\delta^{cd}\bar{c}_c\gamma^\rho(1-\gamma_5)s_d]\times[\delta^{ef}\bar{s}_e s_f] \\
&\overset{\text{color}}{=} \frac{\delta^{ab}\delta^{cf}\delta^{ed}}{3}\times\bar{u}_a\gamma^\alpha\gamma_\rho(1-\gamma_5)c_b\times\bar{c}_c\gamma^\rho(1-\gamma_5)s_d\times\bar{s}_e s_f+\cdots \\
&\overset{\text{Fierz: } s_d\leftrightarrow s_f}{=} +\frac{1}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma_5 s_f]\times[\delta^{ed}\bar{s}_e\gamma^\alpha(1-\gamma_5)s_d] \\
&\quad -\frac{i}{12}\times[\delta^{ab}\bar{u}_a\gamma_5 c_b]\times[\delta^{cf}\bar{c}_c\gamma_\mu(1-\gamma_5)s_f]\times[\delta^{ed}\bar{s}_e\sigma^{\alpha\mu}\gamma_5 s_d]+\cdots \\
&= +\frac{1}{12}\times\zeta_1\times[\bar{s}_a\gamma^\alpha s_a]-\frac{i}{24}\times(\zeta_{2\mu}+\zeta_{3\mu})\times[\bar{s}_a\sigma^{\alpha\mu}\gamma_5 s_a]+\cdots. \qquad (92)
\end{aligned}$$

From Eq. (90), we obtain the following partial decay widths

$$\begin{aligned}
\Gamma(B^{*-}\to K^-|D\bar{D};0^{++}\rangle) &= d_4^2 0.16\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to K^-|D\bar{D}^*;1^{++}\rangle) &= d_4^2 0.62\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to K^-|D\bar{D}^*;1^{+-}\rangle) &= d_4^2 0.61\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to K^-|D^*\bar{D}^*;0^{++}\rangle) &= 0, \\
\Gamma(B^{*-}\to K^-|D^*\bar{D}^*;1^{+-}\rangle) &= d_4^2 2.84\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to K^-|D^*\bar{D}^*;2^{++}\rangle) &= 0. \qquad (93)
\end{aligned}$$

From Eq. (91), we obtain the following partial decay widths

$$\begin{aligned}
\Gamma(B^{*-}\to D^-|D\bar{K};0^{+}\rangle) &= d_5^2 0.18\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to D^-|D\bar{K}^*;1^{+}\rangle) &= d_5^2 1.69\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to D^-|D^*\bar{K};1^{+}\rangle) &= 0, \\
\Gamma(B^{*-}\to D^-|D^*\bar{K}^*;0^{+}\rangle) &= 0, \\
\Gamma(B^{*-}\to D^-|D^*\bar{K}^*;1^{+}\rangle) &= d_5^2 4.40\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to D^-|D^*\bar{K}^*;2^{+}\rangle) &= 0. \qquad (94)
\end{aligned}$$

From Eq. (92), we obtain the following partial decay widths

$$\begin{aligned}
\Gamma(B^{*-}\to \phi|D\bar{D}_s;0^{+}\rangle) &= d_6^2 1.78\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to \phi|D\bar{D}_s^*;1^{++}\rangle) &= d_6^2 1.09\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to \phi|D\bar{D}_s^*;1^{+-}\rangle) &= d_6^2 1.02\times10^{-10}\ \text{GeV}^{13}, \\
\Gamma(B^{*-}\to \phi|D^*\bar{D}_s^*;0^{+}\rangle) &= 0, \\
\Gamma(B^{*-}\to \phi|D^*\bar{D}_s^*;1^{+}\rangle) &= 0, \\
\Gamma(B^{*-}\to \phi|D^*\bar{D}_s^*;2^{+}\rangle) &= 0. \qquad (95)
\end{aligned}$$

In the above expressions $d_{4,5,6}$ are three overall factors.

From Eqs. (93)–(95), we can derive the relative production rate $\mathcal{R}_2 \equiv \frac{\mathcal{B}(B^{*-}\to K^-X/D^-X/\phi X)}{\mathcal{B}(B^{*-}\to K^-|D\bar{D};0^{++}\rangle)/d_4}$. We summarize the obtained results in Table III, which will be further discussed in Sec. V. Again, the difference among the three overall factors $d_{4,5,6}$ is partly due to that the attraction between $D^{(*)}$ and $\bar{D}^{(*)}$, the attraction between $D^{(*)}$ and $\bar{K}^{(*)}$, and the attraction between $D^{(*)}$ and $\bar{D}_s^{(*)}$ can be different.

## IV. DECAY PROPERTIES THROUGH THE FIERZ REARRANGEMENT

The Fierz rearrangement [149] of the Dirac and color indices has been applied in Refs. [19,190] to study strong decay properties of the $X(3872)$ and $Z_c(3900)$ as $D\bar{D}^*$ molecular states. In this section we follow the same procedures to study decay properties of other $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states, separately in Secs. IV A–IV C. We shall study their two-body and three-body hadronic decays as well as the four-body decay process $X\to J/\psi\omega\to J/\psi\pi\pi\pi$.

This method has been systematically applied to study light baryon and tetraquark currents in Refs. [197–205], and applied to study strong decay properties of the $P_c$ and $X(6900)$ states in Refs. [206,207]. A similar arrangement of the spin and color indices in the nonrelativistic case can be found in Refs. [18,208–213], which was applied to study decay properties of the $P_c$ and $XYZ$ states.

Before doing this, we note that the Fierz rearrangement in the Lorentz space is a matrix identity, valid if each quark/antiquark field in the initial and final currents is at the same location. For example, we can apply the Fierz rearrangement to transform a nonlocal current $\eta = [\bar{q}(x)c(x)][\bar{c}(y)q(y)]$ into the combination of many nonlocal currents $\theta = [\bar{q}(x)q(y)][\bar{c}(y)c(x)]$, with all the quark fields remaining at the same locations. We shall keep this in mind, and omit the coordinates in this section.

In the calculations we need couplings of charmonium operators to charmonium states, which are listed in Table II. We also need couplings of charmed/light meson operators to charmed/light meson states, which are also listed in Table II. We refer to Refs. [19,190] for detailed discussions of these couplings.

### A. Decay properties of $D^{(*)}\bar{D}^{(*)}$ molecules

In this subsection we perform the Fierz rearrangement for the currents $\eta_{1,4,5,6}$, and use the obtained results to study strong decay properties of $D^{(*)}\bar{D}^{(*)}$ molecular states. The results obtained in Refs. [19,190] using $\eta^{\alpha}_{2,3}$ are also summarized here for completeness.

We adopt the possible interpretations of the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as the $|D\bar{D}^*;1^{++}\rangle$, $|D\bar{D}^*;1^{+-}\rangle$, and $|D^*\bar{D}^*;1^{+-}\rangle$ molecular states, respectively. Masses of $D^{(*)}\bar{D}^{(*)}$ molecular states are accordingly chosen, as given in Eqs. (77). Besides, we assume that $|D\bar{D}^*;1^{+-}\rangle$ and $|D^*\bar{D}^*;1^{+-}\rangle$ have $I=1$; $|D\bar{D};0^{++}\rangle$, $|D^*\bar{D}^*;0^{++}\rangle$, and $|D^*\bar{D}^*;2^{++}\rangle$ have $I=0$; the isospin of the $X(3872)$ as $|D\bar{D}^*;1^{++}\rangle$ will be separately investigated in Sec. IVA 4.

#### 1. $|D\bar{D};0^{++}\rangle$

The current $\eta_1$ corresponds to the $|D\bar{D};0^{++}\rangle$ molecular state. We can apply the Fierz rearrangement to interchange the $c_a$ and $q_b$ quarks, and transform it into:

$$\begin{aligned}\eta_1 &= \bar{q}_a\gamma_5 c_a \bar{c}_b\gamma_5 q_b\\ &\to -\frac{1}{12}\bar{q}_a q_a \bar{c}_b c_b - \frac{1}{12}\bar{q}_a\gamma_5 q_a\bar{c}_b\gamma_5 c_b\\ &\quad+\frac{1}{12}\bar{q}_a\gamma_\mu q_a\bar{c}_b\gamma^\mu c_b - \frac{1}{12}\bar{q}_a\gamma_\mu\gamma_5 q_a\bar{c}_b\gamma^\mu\gamma_5 c_b\\ &\quad-\frac{1}{24}\bar{q}_a\sigma_{\mu\nu}q_a\bar{c}_b\sigma^{\mu\nu}c_b+\cdots,\end{aligned}\tag{96}$$

where we have only kept the leading-order fall-apart decays described by color-singlet-color-singlet meson-meson currents, but neglect the $\mathcal{O}(\alpha_s)$ corrections described by color-octet-color-octet meson-meson currents.

As an example, we investigate $|D\bar{D};0^{++}\rangle$ through $\eta_1(x)$ and the above Fierz rearrangement. As depicted in Fig. 4(a), when the $\bar{q}_a$ and $q_d$ quarks meet each other and the $c_b$ and $\bar{c}_c$ quarks meet each other at the same time, $|D\bar{D};0^{++}\rangle$ can decay into one charmonium meson and one light meson:

$$\begin{aligned}[\delta^{ab}\bar{q}_a c_b][\delta^{cd}\bar{c}_c q_d] &\overset{\text{color}}{=} \frac{1}{3}\delta^{ad}\delta^{cb}\bar{q}_a c_b\bar{c}_c q_d+\cdots\\ &\xrightarrow{\text{Fierz}}\frac{1}{3}[\delta^{ad}\bar{q}_a q_d][\delta^{cb}c_b c_b]+\cdots.\end{aligned}\tag{97}$$

This decay process can be described by the Fierz rearrangement given in Eq. (96).

Assuming the isospin of $|D\bar{D};0^{++}\rangle$ to be $I=0$, we extract the following three decay channels:

(1) The decay of $|D\bar{D};0^{++}\rangle$ into $\eta_c\eta$ is contributed by both $\bar{q}_a\gamma_5 q_a\times\bar{c}_b\gamma_5 c_b$ and $\bar{q}_a\gamma_\mu\gamma_5 q_a\times\bar{c}_b\gamma^\mu\gamma_5 c_b$:

$$\begin{aligned}&\langle D\bar{D};0^{++}(p)|\eta_c(p_1)\eta(p_2)\rangle\\ &\quad=\frac{a_1}{12}\lambda_{\eta_c}\lambda_\eta+\frac{a_1}{12}f_{\eta_c}f_\eta p_1\cdot p_2,\end{aligned}\tag{98}$$

where $a_1$ is an overall factor, related to the coupling of $\eta_1$ to $|D\bar{D};0^{++}\rangle$ as well as the dynamical process depicted in Fig. 4(a).

(2) The decay of $|D\bar{D};0^{++}\rangle$ into $J/\psi\omega$ is contributed by both $\bar{q}_a\gamma_\mu q_a\times\bar{c}_b\gamma^\mu c_b$ and $\bar{q}_a\sigma_{\mu\nu}q_a\times\bar{c}_b\sigma^{\mu\nu}c_b$:

$$\begin{aligned}&\langle D\bar{D};0^{++}(p)|J/\psi(\epsilon_1,p_1)\omega(\epsilon_2,p_2)\rangle\\ &=\frac{a_1}{12}m_{J/\psi}f_{J/\psi}m_\omega f_\omega\epsilon_1\cdot\epsilon_2\\ &\quad+\frac{a_1}{12}f^T_{J/\psi}f^T_\omega(\epsilon_1\cdot\epsilon_2 p_1\cdot p_2-\epsilon_1\cdot p_2\epsilon_2\cdot p_1).\end{aligned}\tag{99}$$

(3) The decay of $|D\bar{D};0^{++}\rangle$ into $\chi_{c0}f_0(500)$ is contributed by $\bar{q}_a q_a\times\bar{c}_b c_b$:

$$\begin{aligned}&\langle D\bar{D};0^{++}(p)|\chi_{c0}(p_1)f_0(p_2)\rangle\\ &\quad=-\frac{a_1}{12}m_{\chi_{c0}}f_{\chi_{c0}}m_{f_0}f_{f_0}.\end{aligned}\tag{100}$$

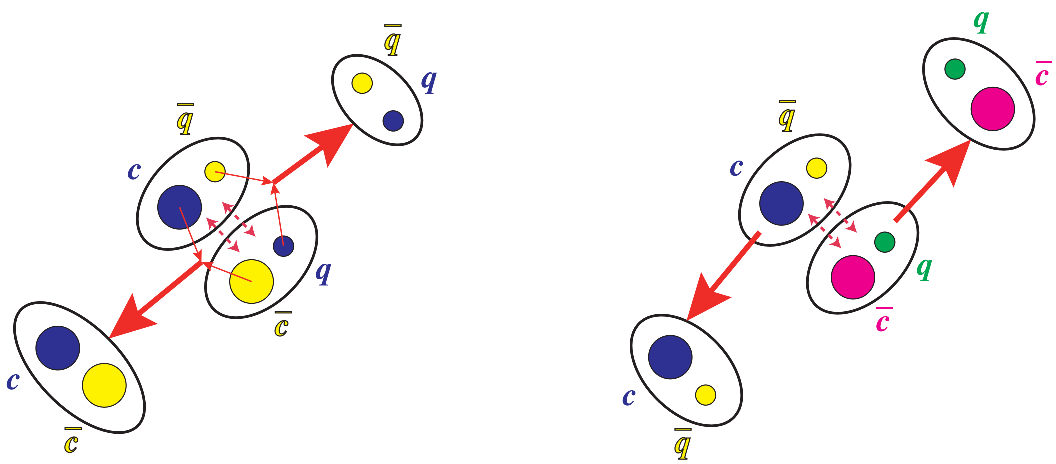
Assuming the mass of $|D\bar{D};0^{++}\rangle$ to be about $M_{D^0}+M_{\bar{D}^0}=3730$ MeV, we summarize the above amplitudes to obtain the following partial decay widths:

$$\begin{aligned}&\Gamma(|D\bar{D};0^{++}\rangle\to\eta_c\eta)=a_1^2 1.7\times10^{-4}\ \text{GeV}^7,\\ &\Gamma(|D\bar{D};0^{++}\rangle\to J/\psi\omega\to J/\psi\pi\pi\pi)\\ &\qquad=a_1^2 1.1\times10^{-10}\ \text{GeV}^7,\\ &\Gamma(|D\bar{D};0^{++}\rangle\to\chi_{c0}f_0(500)\to\chi_{c0}\pi\pi)\\ &\qquad=a_1^2 9.7\times10^{-10}\ \text{GeV}^7.\end{aligned}\tag{101}$$

There are two different terms, $A\equiv\bar{q}_a\gamma_5 q_a\times\bar{c}_b\gamma_5 c_b$ and $B\equiv\bar{q}_a\gamma_\mu\gamma_5 q_a\times\bar{c}_b\gamma^\mu\gamma_5 c_b$, both of which can contribute to the decay of $|D\bar{D};0^{++}\rangle$ into the $\eta_c\eta$ final states. There are also two different terms, $C\equiv\bar{q}_a\gamma_\mu q_a\times\bar{c}_b\gamma^\mu c_b$ and $D\equiv\bar{q}_a\sigma_{\mu\nu}q_a\times\bar{c}_b\sigma^{\mu\nu}c_b$, both of which can contribute to the decay of $|D\bar{D};0^{++}\rangle$ into the $J/\psi\omega$ final states. Phase angles among them, such as the phase angle between the two coupling constants $f_{J/\psi}$ and $f^T_{J/\psi}$, cannot be well determined in the present study. This causes some uncertainties, which will be investigated in Appendix B.

#### 2. $|D^*\bar{D}^*;0^{++}\rangle$

The current $\eta_4$ corresponds to the $|D^*\bar{D}^*;0^{++}\rangle$ molecular state. We can apply the Fierz rearrangement and transform it into:




(a) Fierz rearrangement: $\eta \to \theta$ (b) direct fall-apart process

FIG. 4. Fall-apart decay processes of $D^{(*)}\bar{D}^{(*)}$ molecular states, investigated through the $\eta(x)$ currents.

$$
\begin{aligned}
\eta_4 &= \bar{q}_a\gamma^\mu c_a\bar{c}_b\gamma_\mu q_b \\
&\to -\frac{1}{3}\bar{q}_a q_a\bar{c}_b c_b + \frac{1}{3}\bar{q}_a\gamma_5 q_a\bar{c}_b\gamma_5 c_b + \frac{1}{6}\bar{q}_a\gamma_\mu q_a\bar{c}_b\gamma^\mu c_b \\
&\quad + \frac{1}{6}\bar{q}_a\gamma_\mu\gamma_5 q_a\bar{c}_b\gamma^\mu\gamma_5 c_b + \cdots. \qquad (102)
\end{aligned}
$$

Assuming the isospin of $|D^*\bar{D}^*;0^{++}\rangle$ to be $I=0$ and its mass to be about $M_{D^{*0}} + M_{\bar{D}^{*0}} = 4014$ MeV, we obtain the following partial decay widths:

$$
\begin{aligned}
\Gamma(|D^*\bar{D}^*;0^{++}\rangle \to \eta_c\eta) &= a_4^2 4.0\times 10^{-3}\ \mathrm{GeV}^7, \\
\Gamma(|D^*\bar{D}^*;0^{++}\rangle \to J/\psi\omega) &= a_4^2 4.8\times 10^{-6}\ \mathrm{GeV}^7, \\
\Gamma(|D^*\bar{D}^*;0^{++}\rangle \to \chi_{c0}f_0(500)) &= a_4^2 4.1\times 10^{-6}\ \mathrm{GeV}^7, \qquad (103)
\end{aligned}
$$

where $a_4$ is an overall factor.

Besides, $|D^*\bar{D}^*;0^{++}\rangle$ can directly fall apart into the $D^*$ and $\bar{D}^*$ mesons, and further decay into the $D^*\bar{D}\pi$ and $D\bar{D}^*\pi$ final states. This process is depicted in Fig. 4(b). We estimate its partial width to be

$$
\begin{aligned}
&\Gamma(|D^*\bar{D}^*;0^{++}\rangle \to D^*\bar{D}^* \to D^*\bar{D}\pi + D\bar{D}^*\pi) \\
&\quad = a_4'^2 1.7\times 10^{-6}\ \mathrm{GeV}^7, \qquad (104)
\end{aligned}
$$

where $a_4'$ is another overall factor. It is probably larger than $a_4$, and we define their ratio to be $t \equiv a_4'/a_4$. This parameter measures the dynamical difference between the process depicted in Fig. 4(a) and the one depicted Fig. 4(b).

The uncertainty of Eq. (104) is quite large, because we choose the mass of $|D^*\bar{D}^*;0^{++}\rangle$ to be just at the $D^{*0}\bar{D}^{*0}$ threshold, so that only the $D^{*0}\bar{D}^0\pi^0$ and $D^0\bar{D}^{*0}\pi^0$ final states are kinematically allowed. However, if its realistic mass turns out to be slightly larger, some other final states such as $D^{*0}\bar{D}^{*0}$ and $D^{*0}D^+\pi^-$ may also become possible. Moreover, phase spaces of the $D^{*0}\bar{D}^0\pi^0$ and $D^0\bar{D}^{*0}\pi^0$ final states depend significantly on the mass of $|D^*\bar{D}^*;0^{++}\rangle$.

### 3. $|D^*\bar{D}^*;2^{++}\rangle$

The current $\eta_6^{\alpha\beta}$ corresponds to the $|D^*\bar{D}^*;2^{++}\rangle$ molecular state. We can apply the Fierz rearrangement and transform it into:

$$
\begin{aligned}
\eta_6^{\alpha\beta} &= P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu c_a\bar{c}_b\gamma_\nu q_b \\
&\to -\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu q_a\bar{c}_b\gamma_\nu c_b \\
&\quad -\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu\gamma_5 q_a\bar{c}_b\gamma_\nu\gamma_5 c_b \\
&\quad +\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\sigma_{\mu\rho} q_a\bar{c}_b\sigma_{\nu\rho} c_b + \cdots. \qquad (105)
\end{aligned}
$$

Assuming the isospin of $|D^*\bar{D}^*;2^{++}\rangle$ to be $I=0$ and its mass to be about $M_{D^{*0}} + M_{\bar{D}^{*0}} = 4014$ MeV, we obtain the following partial decay widths:

$$
\begin{aligned}
\Gamma(|D^*\bar{D}^*;2^{++}\rangle \to \eta_c\eta) &= a_6^2 5.1\times 10^{-7}\ \mathrm{GeV}^7, \\
\Gamma(|D^*\bar{D}^*;2^{++}\rangle \to J/\psi\omega) &= a_6^2 3.7\times 10^{-5}\ \mathrm{GeV}^7, \qquad (106)
\end{aligned}
$$

where $a_6$ is an overall factor.

Besides, $|D^*\bar{D}^*;2^{++}\rangle$ can directly fall apart into the $D^*$ and $\bar{D}^*$ mesons, and further decay into the $D^*\bar{D}\pi$ and $D\bar{D}^*\pi$ final states:

$$
\begin{aligned}
&\Gamma(|D^*\bar{D}^*;2^{++}\rangle \to D^*\bar{D}^* \to D^*\bar{D}\pi + D\bar{D}^*\pi) \\
&\quad = a_6'^2 4.9\times 10^{-6}\ \mathrm{GeV}^7, \qquad (107)
\end{aligned}
$$

where $a_6' \approx t\times a_6$ is another overall factor.

#### 4. $|D\bar{D}^*; 1^{++}\rangle$

The current $\eta_2^\alpha$ corresponds to the $|D\bar{D}^*; 1^{++}\rangle$ molecular state. Based on this current, we have systematically studied decay properties of the $X(3872)$ as $|D\bar{D}^*; 1^{++}\rangle$ in Ref. [190]. We summarize some of those results here.

The isospin breaking effect of the $X(3872)$ is significant and important to understand its nature. In Ref. [190] we assumed it to be the combination of both $I = 0$ and $I = 1$ $|D\bar{D}^*; 1^{++}\rangle$ states:

$$X(3872) = \cos\theta |D\bar{D}^*; 0^+1^{++}\rangle + \sin\theta |D\bar{D}^*; 1^-1^{++}\rangle. \tag{108}$$

After fine-tuning the isospin-breaking angle to be $\theta = \pm 15°$, the BESIII measurement [59],

$$\frac{\mathcal{B}(X(3872) \to J/\psi\omega \to J/\psi\pi\pi\pi)}{\mathcal{B}(X(3872) \to J/\psi\rho \to J/\psi\pi\pi)} = 1.6^{+0.4}_{-0.3} \pm 0.2, \tag{109}$$

can be explained.

In the present study we further select the positive angle $\theta = +15°$, so that the $D^0\bar{D}^{*0}$ component is more than the $D^+D^{*-}$ one:

$$\begin{aligned} X(3872) &= \cos 15° |D\bar{D}^*; 0^+1^{++}\rangle + \sin 15° |D\bar{D}^*; 1^-1^{++}\rangle \\ &= 0.8684 |D^0\bar{D}^{*0}; 1^{++}\rangle + 0.4959 |D^+D^{*-}; 1^{++}\rangle. \end{aligned} \tag{110}$$

Using this angle, decay properties of the $X(3872)$ as $|D\bar{D}^*; 1^{++}\rangle$ were systematically studied in Ref. [190], and the results are summarized in Eq. (133) of Sec. IV C.

Besides, productions of the $X(3872)$ as $|D\bar{D}^*; 1^{++}\rangle$ in $B$ and $B^*$ decays are increased from Eqs. (78) and (93) to be

$$\begin{aligned} \Gamma(B^- \to K^- |D\bar{D}^*; 1^{++}\rangle) &= d_1^2 0.60 \times 10^{-10}\ \mathrm{GeV}^{13}, \\ \Gamma(B^{*-} \to K^- |D\bar{D}^*; 1^{++}\rangle) &= d_3^2 0.93 \times 10^{-10}\ \mathrm{GeV}^{13}. \end{aligned} \tag{111}$$

The two ratios $\mathcal{R}_1$ and $\mathcal{R}_2$ given in Table III need to be accordingly modified.

#### 5. $|D\bar{D}^*; 1^{+-}\rangle$ and $|D^*\bar{D}^*; 1^{+-}\rangle$

The currents $\eta_3^\alpha$ and $\eta_5^\alpha$ correspond to the $|D\bar{D}^*; 1^{+-}\rangle$ and $|D^*\bar{D}^*; 1^{+-}\rangle$ molecular states, respectively. Based on these two currents, we have systematically studied decay properties of the $Z_c(3900)$ as $|D\bar{D}^*; 1^{+-}\rangle$ in Ref. [19]. We summarize some of those results here. Especially, in Ref. [19] we have considered the mixing between $|D\bar{D}^*; 1^{+-}\rangle$ and $|D^*\bar{D}^*; 1^{+-}\rangle$:

$$\begin{aligned} |D\bar{D}^*; 1^{+-}\rangle' &= +\cos\phi |D\bar{D}^*\rangle_{1^{+-}} + \sin\phi |D^*\bar{D}^*\rangle_{1^{+-}}, \\ |D^*\bar{D}^*; 1^{+-}\rangle' &= -\sin\phi |D\bar{D}^*\rangle_{1^{+-}} + \cos\phi |D^*\bar{D}^*\rangle_{1^{+-}}, \end{aligned} \tag{112}$$

in order to explain the BESIII measurement [83]:

$$\frac{\mathcal{B}(Z_c(3900)^\pm \to \eta_c\rho^\pm)}{\mathcal{B}(Z_c(3900)^\pm \to J/\psi\pi^\pm)} = 2.2 \pm 0.9. \tag{113}$$

In the present study we do not consider this mixing any more. We just use the single current $\eta_3^\alpha$ to study decay properties of the $Z_c(3900)$ as $|D\bar{D}^*; 1^{+-}\rangle$, and use the single current $\eta_5^\alpha$ to study the $Z_c(4020)$ as $|D^*\bar{D}^*; 1^{+-}\rangle$. The results are summarized in Eqs. (134) and (135) of Sec. IV C. Actually, as shown in Table IV of Appendix B, the above BESIII measurement given in Eq. (113) can also be explained if we take into account those unknown phase angles among different coupling constants.

### B. Decay properties of $D^{(*)}\bar{K}^{(*)}$ molecules

In this subsection we perform the Fierz rearrangement for $\xi_{1\cdots6}$, and use the obtained results to study strong decay properties of $D^{(*)}\bar{K}^{(*)}$ molecular states. Again, we adopt the possible interpretation of the $X_0(2900)$ as the $|D^*\bar{K}^*; 0^+\rangle$ molecular state. Masses of $D^{(*)}\bar{K}^{(*)}$ molecular states are accordingly chosen, as given in Eqs. (82).

#### 1. $|D\bar{K}; 0^+\rangle$ and $|D^*\bar{K}^*; 0^+\rangle$

The currents $\xi_1$ and $\xi_4$ correspond to the $|D\bar{K}; 0^+\rangle$ and $|D^*\bar{K}^*; 0^+\rangle$ molecular states, respectively. We can apply the Fierz rearrangement and transform them into:

$$\begin{aligned} \xi_1 &= \bar{q}_a\gamma_5 c_a \bar{q}_b\gamma_5 s_b \\ &\to -\frac{1}{12}\bar{q}_a s_a \bar{q}_b c_b - \frac{1}{12}\bar{q}_a\gamma_5 s_a \bar{q}_b\gamma_5 c_b \\ &\quad + \frac{1}{12}\bar{q}_a\gamma_\mu s_a \bar{q}_b\gamma^\mu c_b - \frac{1}{12}\bar{q}_a\gamma_\mu\gamma_5 s_a \bar{q}_b\gamma^\mu\gamma_5 c_b \\ &\quad - \frac{1}{24}\bar{q}_a\sigma_{\mu\nu} s_a \bar{q}_b\sigma^{\mu\nu} c_b + \cdots, \end{aligned} \tag{114}$$

$$\begin{aligned} \xi_4 &= \bar{q}_a\gamma^\mu c_a \bar{q}_b\gamma_\mu s_b \\ &\to -\frac{1}{3}\bar{q}_a s_a \bar{q}_b c_b + \frac{1}{3}\bar{q}_a\gamma_5 s_a \bar{q}_b\gamma_5 c_b \\ &\quad + \frac{1}{6}\bar{q}_a\gamma_\mu s_a \bar{q}_b\gamma^\mu c_b + \frac{1}{6}\bar{q}_a\gamma_\mu\gamma_5 s_a \bar{q}_b\gamma^\mu\gamma_5 c_b + \cdots. \end{aligned} \tag{115}$$

Assuming the mass of $|D\bar{K}; 0^+\rangle$ to be about $M_{D^0} + M_{K^+} = 2359$ MeV, all its strong decay channels turn out to be kinematically forbidden.

Assuming the mass of $|D^*\bar{K}^*; 0^+\rangle$ to be $M_{X_0(2900)} = 2866$ MeV, we obtain the following partial decay widths:

$$\Gamma(|D^*\bar{K}^*;0^+\rangle \to D\bar{K}) = b_4^2 1.6\times 10^{-5}\ \mathrm{GeV}^7,$$
$$\Gamma(|D^*\bar{K}^*;0^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) = b_4^2 5.4\times 10^{-8}\ \mathrm{GeV}^7, \quad (116)$$

where $b_4$ is an overall factor.

Besides, $|D^*\bar{K}^*;0^+\rangle$ can directly fall apart into the $D^*$ and $\bar{K}^*$ mesons, and further decay into the $D^*\bar{K}\pi$ final states:

$$\Gamma(|D^*\bar{K}^*;0^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) = b_4'^2 1.9\times 10^{-6}\ \mathrm{GeV}^7, \quad (117)$$

where $b_4' \approx t\times b_4$ is another overall factor.

### 2. $|D\bar{K}^*;1^+\rangle$, $|D^*\bar{K};1^+\rangle$, and $|D^*\bar{K}^*;1^+\rangle$

The currents $\xi_2^\alpha$, $\xi_3^\alpha$, and $\xi_5^\alpha$ correspond to the $|D\bar{K}^*;1^+\rangle$, $|D^*\bar{K};1^+\rangle$, and $|D^*\bar{K}^*;1^+\rangle$ molecular states, respectively. We can apply the Fierz rearrangement and transform them into:

$$\begin{aligned}\xi_2^\alpha &= \bar{q}_a\gamma_5 c_a \bar{q}_b\gamma^\alpha s_b \to -\frac{1}{12}\bar{q}_a\gamma_5 s_a\bar{q}_b\gamma^\alpha c_b - \frac{1}{12}\bar{q}_a\gamma^\alpha s_a\bar{q}_b\gamma_5 c_b \\ &+\frac{1}{12}\bar{q}_a\gamma^\alpha\gamma_5 s_a\bar{q}_b c_b - \frac{1}{12}\bar{q}_a s_a\bar{q}_b\gamma^\alpha\gamma_5 c_b \\ &-\frac{i}{12}\bar{q}_a\gamma_\mu\gamma_5 s_a\bar{q}_b\sigma^{\alpha\mu}c_b - \frac{i}{12}\bar{q}_a\sigma^{\alpha\mu}s_a\bar{q}_b\gamma_\mu\gamma_5 c_b \\ &+\frac{i}{12}\bar{q}_a\gamma_\mu s_a\bar{q}_b\sigma^{\alpha\mu}\gamma_5 c_b - \frac{i}{12}\bar{q}_a\sigma^{\alpha\mu}\gamma_5 s_a\bar{q}_b\gamma_\mu c_b + \cdots, \end{aligned} \quad (118)$$

$$\begin{aligned}\xi_3^\alpha &= \bar{q}_a\gamma^\alpha c_a \bar{q}_b\gamma_5 s_b \to -\frac{1}{12}\bar{q}_a\gamma_5 s_a\bar{q}_b\gamma^\alpha c_b - \frac{1}{12}\bar{q}_a\gamma^\alpha s_a\bar{q}_b\gamma_5 c_b \\ &-\frac{1}{12}\bar{q}_a\gamma^\alpha\gamma_5 s_a\bar{q}_b c_b + \frac{1}{12}\bar{q}_a s_a\bar{q}_b\gamma^\alpha\gamma_5 c_b \\ &-\frac{i}{12}\bar{q}_a\gamma_\mu\gamma_5 s_a\bar{q}_b\sigma^{\alpha\mu}c_b - \frac{i}{12}\bar{q}_a\sigma^{\alpha\mu}s_a\bar{q}_b\gamma_\mu\gamma_5 c_b \\ &-\frac{i}{12}\bar{q}_a\gamma_\mu s_a\bar{q}_b\sigma^{\alpha\mu}\gamma_5 c_b + \frac{i}{12}\bar{q}_a\sigma^{\alpha\mu}\gamma_5 s_a\bar{q}_b\gamma_\mu c_b + \cdots, \end{aligned} \quad (119)$$

$$\begin{aligned}\xi_5^\alpha &= \bar{q}_a\gamma_\mu c_a\bar{q}_b\sigma^{\alpha\mu}\gamma_5 s_b - \{\gamma_\mu \leftrightarrow \sigma^{\alpha\mu}\gamma_5\} \\ &\to -\frac{i}{2}\bar{q}_a\gamma_5 s_a\bar{q}_b\gamma^\alpha c_b + \frac{i}{2}\bar{q}_a\gamma^\alpha s_a\bar{q}_b\gamma_5 c_b \\ &+\frac{1}{6}\bar{q}_a\gamma_\mu\gamma_5 s_a\bar{q}_b\sigma^{\alpha\mu}c_b - \frac{1}{6}\bar{q}_a\sigma^{\alpha\mu}s_a\bar{q}_b\gamma_\mu\gamma_5 c_b \\ &+\cdots. \end{aligned} \quad (120)$$

Assuming the mass of $|D\bar{K}^*;1^+\rangle$ to be about $M_{D^0}+M_{K^{*+}} = 2756$ MeV, we obtain the following partial decay widths:

$$\Gamma(|D\bar{K}^*;1^+\rangle \to D^*\bar{K}) = b_2^2 1.7\times 10^{-7}\ \mathrm{GeV}^7,$$
$$\Gamma(|D\bar{K}^*;1^+\rangle \to D\bar{K}^* \to D\bar{K}\pi) = b_2^2 4.7\times 10^{-9}\ \mathrm{GeV}^7,$$
$$\Gamma(|D\bar{K}^*;1^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) = b_2^2 7.0\times 10^{-10}\ \mathrm{GeV}^7, \quad (121)$$

where $b_2$ is an overall factor.

Besides, $|D\bar{K}^*;1^+\rangle$ can directly fall apart into the $D$ and $\bar{K}^*$ mesons, and further decay into the $D\bar{K}\pi$ final states:

$$\Gamma(|D\bar{K}^*;1^+\rangle \to D\bar{K}^* \to D\bar{K}\pi) = b_2'^2 2.9\times 10^{-6}\ \mathrm{GeV}^7, \quad (122)$$

where $b_2' \approx t\times b_2$ is another overall factor.

Assuming the mass of $|D^*\bar{K};1^+\rangle$ to be about $M_{D^{*0}}+M_{K^+} = 2501$ MeV, we obtain the following partial decay widths:

$$\Gamma(|D^*\bar{K};1^+\rangle \to D\bar{K}^* \to D\bar{K}\pi) = b_3^2 5.4\times 10^{-16}\ \mathrm{GeV}^7,$$
$$\Gamma(|D^*\bar{K};1^+\rangle \to D^*\bar{K} \to D\pi\bar{K}) = b_3^2 1.3\times 10^{-12}\ \mathrm{GeV}^7, \quad (123)$$

where $b_3$ is an overall factor.

Besides, $|D^*\bar{K};1^+\rangle$ can directly fall apart into the $D^*$ and $\bar{K}$ mesons, and further decay into the $D\pi\bar{K}$ final states:

$$\Gamma(|D^*\bar{K};1^+\rangle \to D^*\bar{K} \to D\pi\bar{K}) = b_3'^2 3.2\times 10^{-10}\ \mathrm{GeV}^7, \quad (124)$$

where $b_3' \approx t\times b_3$ is another overall factor.

Assuming the mass of $|D^*\bar{K}^*;1^+\rangle$ to be about $M_{D^{*0}}+M_{K^{*+}} = 2899$ MeV, we obtain the following partial decay widths:

$$\Gamma(|D^*\bar{K}^*;1^+\rangle \to D\bar{K}^*) = b_5^2 3.8\times 10^{-6}\ \mathrm{GeV}^7,$$
$$\Gamma(|D^*\bar{K}^*;1^+\rangle \to D^*\bar{K}) = b_5^2 1.2\times 10^{-5}\ \mathrm{GeV}^7, \quad (125)$$

where $b_5$ is an overall factor.

Besides, $|D^*\bar{K}^*;1^+\rangle$ can directly fall apart into the $D^*$ and $\bar{K}^*$ mesons, and further decay into the $D^*\bar{K}\pi$ and $D\pi\bar{K}^*$ final states:

$$\Gamma(|D^*\bar{K}^*;1^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) = b_5'^2 1.1\times 10^{-5}\ \mathrm{GeV}^7,$$
$$\Gamma(|D^*\bar{K}^*;1^+\rangle \to D^*\bar{K}^* \to D\pi\bar{K}^*) = b_5'^2 1.1\times 10^{-9}\ \mathrm{GeV}^7, \quad (126)$$

where $b_5' \approx t\times b_5$ is another overall factor.

### 3. $|D^*\bar{K}^*;2^+\rangle$

The current $\xi_6^{\alpha\beta}$ corresponds to the $|D^*\bar{K}^*;2^+\rangle$ molecular state. We can apply the Fierz rearrangement and transform it into:

$$\begin{aligned}\xi_6^{\alpha\beta} &= P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu c_a\bar{q}_b\gamma_\nu s_b\\ &\to -\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu s_a\bar{q}_b\gamma_\nu c_b\\ &\quad -\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\gamma_\mu\gamma_5 s_a\bar{q}_b\gamma_\nu\gamma_5 c_b\\ &\quad +\frac{1}{6}P^{\alpha\beta,\mu\nu}\bar{q}_a\sigma_{\mu\rho} s_a\bar{q}_b\sigma_{\nu\rho} c_b + \cdots. \end{aligned} \tag{127}$$

Assuming the mass of $|D^*\bar{K}^*;2^+\rangle$ to be about $M_{D^{*0}} + M_{K^{*+}} = 2899$ MeV, we obtain the following partial decay widths:

$$\begin{aligned}\Gamma(|D^*\bar{K}^*;2^+\rangle \to D\bar{K}) &= b_6^2 3.0\times 10^{-7}\ \mathrm{GeV}^7,\\ \Gamma(|D^*\bar{K}^*;2^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) &= b_6^2 1.3\times 10^{-6}\ \mathrm{GeV}^7,\\ \Gamma(|D^*\bar{K}^*;2^+\rangle \to D^*\bar{K}^* \to D\pi\bar{K}^*) &= b_6^2 7.4\times 10^{-11}\ \mathrm{GeV}^7,\end{aligned} \tag{128}$$

where $b_6$ is an overall factor.

Besides, $|D^*\bar{K}^*;2^+\rangle$ can directly fall apart into the $D^*$ and $\bar{K}^*$ mesons, and further decay into the $D^*\bar{K}\pi$ and $D\pi\bar{K}^*$ final states:

$$\begin{aligned}\Gamma(|D^*\bar{K}^*;2^+\rangle \to D^*\bar{K}^* \to D^*\bar{K}\pi) &= b_6'^2 1.7\times 10^{-5}\ \mathrm{GeV}^7,\\ \Gamma(|D^*\bar{K}^*;2^+\rangle \to D^*\bar{K}^* \to D\pi\bar{K}^*) &= b_6'^2 9.9\times 10^{-10}\ \mathrm{GeV}^7,\end{aligned} \tag{129}$$

where $b_6' \approx t\times b_6$ is another overall factor.

### C. Summary of decay properties

We summarize the relative branching ratios obtained in Secs. IV A and IV B here. The Fierz rearrangements for $\zeta_{1\cdots6}$ are quite similar to those for $\eta_{1\cdots6}$, just with one *up*/*down* quark replaced by another *strange* quark. Hence, decay properties of $D^{(*)}\bar{D}_s^{(*)}$ molecular states can be similarly investigated, and we also summarize their results here. We use the parameter $t \approx a_i'/a_i \approx b_i'/b_i$ $(i = 1\ldots6)$ to measure the dynamical difference between the process depicted in Fig. 4(a) and the one depicted in Fig. 4(b).

We obtain the following relative branching ratios for $|D\bar{D};0^{++}\rangle$, $|D^*\bar{D}^*;0^{++}\rangle$, and $|D^*\bar{D}^*;2^{++}\rangle$, all of which are assumed to have $I = 0$:

$$\begin{aligned}&\mathcal{B}(|D\bar{D};0^{++}\rangle \to \eta_c\eta : J/\psi\omega(\to\pi\pi\pi) : \chi_{c0}f_0(500)(\to\pi\pi))\\ \approx\ &1 : 10^{-6} : 10^{-5},\end{aligned} \tag{130}$$

$$\begin{aligned}&\mathcal{B}(|D^*\bar{D}^*;0^{++}\rangle \to \eta_c\eta : J/\psi\omega : \chi_{c0}f_0(500) : D^*\bar{D}^*(\to\bar{D}\pi))\\ \approx\ &1 : 0.001 : 0.001 : 10^{-4}t,\end{aligned} \tag{131}$$

$$\begin{aligned}&\mathcal{B}(|D^*\bar{D}^*;2^{++}\rangle \to \eta_c\eta : J/\psi\omega : D^*\bar{D}^*(\to\bar{D}\pi))\\ \approx\ &1 : 73 : 9.5t.\end{aligned} \tag{132}$$

Decay properties of the $X(3872)$ as $|D\bar{D}^*;1^{++}\rangle$ have been systematically studied in Ref. [190], where we obtain

$$\begin{aligned}&\mathcal{B}(|D\bar{D}^*;1^{++}\rangle \to J/\psi\omega(\to\pi\pi\pi) : J/\psi\rho(\to\pi\pi) : \chi_{c0}\pi : \eta_c f_0(\to\pi\pi) : \chi_{c1}f_0(\to\pi\pi) : D\bar{D}^*(\to\bar{D}\pi))\\ \approx\ &1 : 0.63(\text{input}) : 0.015 : 0.091 : 0.086 : 7.4t.\end{aligned} \tag{133}$$

In the calculations we have assumed the $X(3872)$ to be the combination of both $I = 0$ and $I = 1$ $|D\bar{D}^*;1^{++}\rangle$ molecular states.

Decay properties of the $Z_c(3900)$ as $|D\bar{D}^*;1^{+-}\rangle$ have been systematically studied in Ref. [19]. In the present study we further study the $Z_c(4020)$ as $|D^*\bar{D}^*;1^{+-}\rangle$. Altogether, we obtain

$$\begin{aligned}&\mathcal{B}(|D\bar{D}^*;1^{+-}\rangle \to J/\psi\pi : \eta_c\rho : h_c\pi : \chi_{c1}\rho(\to\pi\pi) : D\bar{D}^*(\to\bar{D}\pi))\\ \approx\ &1 : 0.092 : 0.011 : 10^{-6} : 74t,\end{aligned} \tag{134}$$

$$\begin{aligned}&\mathcal{B}(|D^*\bar{D}^*;1^{+-}\rangle \to J/\psi\pi : \eta_c\rho : h_c\pi : \chi_{c1}\rho(\to\pi\pi) : D^*\bar{D}^*(\to\bar{D}\pi))\\ \approx\ &1 : 0.33 : 0.002 : 10^{-5} : 11t.\end{aligned} \tag{135}$$

In the calculations we have assumed that the $Z_c(3900)$ and $Z_c(4020)$ both have $I = 1$.

Assuming the mass of $|D\bar{K};0^+\rangle$ to be about $M_{D^0}+M_{K^+}=2359$ MeV, all its strong decay channels turn out to be kinematically forbidden.

We obtain the following relative branching ratios for $|D^*\bar{K};1^+\rangle$:

$$\begin{aligned} &\mathcal{B}(|D^*\bar{K};1^+\rangle \to D\bar{K}^*(\to\bar{K}\pi) \quad : \quad \bar{K}D^*(\to D\pi)) \\ \approx\quad & \qquad\qquad\qquad\qquad 10^{-6} \quad : \quad t+0.004. \end{aligned} \tag{136}$$

The isospin-averaged width of the $\bar{K}^*$ meson is about $\Gamma_{K^*}\approx 49.1$ MeV [2], which value is quite large and cannot be neglected. We refer to Appendix A for more discussions. Taking this into account, we estimate widths of $|D\bar{K}^*;1^+\rangle$, $|D^*\bar{K}^*;0^+\rangle$, $|D^*\bar{K}^*;1^+\rangle$, and $|D^*\bar{K}^*;2^+\rangle$ to be

$$\Gamma_{|D^{(*)}\bar{K}^*;J^+\rangle}\approx\Gamma_{\bar{K}^*}+\Gamma_{|D^{(*)}\bar{K}^*;J^+\rangle\to D\bar{K}/D^*\bar{K}/D\bar{K}^*/D^*\bar{K}^*/\cdots}, \tag{137}$$

with the following relative branching ratios

$$\begin{aligned} &\mathcal{B}(|D\bar{K}^*;1^+\rangle \to D\bar{K}^*(\to\bar{K}\pi) \quad : \quad D^*\bar{K} \quad : \quad D^*\bar{K}^*(\to\bar{K}\pi)) \\ \approx\quad & \qquad\qquad t+0.002 \quad : \quad 0.057 \quad : \quad 10^{-4}, \end{aligned} \tag{138}$$

$$\begin{aligned} &\mathcal{B}(|D^*\bar{K}^*;0^+\rangle \to D\bar{K} \quad : \quad D^*\bar{K}^*(\to\bar{K}\pi)) \\ \approx\quad & \qquad\qquad 8.5 \quad : \quad t+0.028, \end{aligned} \tag{139}$$

$$\begin{aligned} &\mathcal{B}(|D^*\bar{K}^*;1^+\rangle \to D\bar{K}^* \quad : \quad D^*\bar{K} \quad : \quad D^*\bar{K}^*(\to\bar{K}\pi) \quad : \quad \bar{K}^*D^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 0.34 \quad : \quad 1.1 \quad : \quad t \quad : \quad 10^{-4}t, \end{aligned} \tag{140}$$

$$\begin{aligned} &\mathcal{B}(|D^*\bar{K}^*;2^+\rangle \to D\bar{K} \quad : \quad D^*\bar{K}^*(\to\bar{K}\pi) \quad : \quad \bar{K}^*D^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 0.018 \quad : \quad t+0.074 \quad : \quad 10^{-4}t. \end{aligned} \tag{141}$$

We obtain the following relative branching ratios for $D^{(*)}\bar{D}_s^{(*)}$ molecular states:

$$\begin{aligned} &\mathcal{B}(|D\bar{D}_s;0^+\rangle \to \eta_c\bar{K} \quad : \quad J/\psi\bar{K}^*(\to\bar{K}\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 0.001, \end{aligned} \tag{142}$$

$$\begin{aligned} &\mathcal{B}(|D\bar{D}_s^*;1^{++}\rangle \to J/\psi\bar{K}^*(\to\bar{K}\pi) \quad : \quad \chi_{c0}\bar{K} \quad : \quad D\bar{D}_s^* \quad : \quad \bar{D}_sD^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 0.012 \quad : \quad 25 \quad : \quad 68, \end{aligned} \tag{143}$$

$$\begin{aligned} &\mathcal{B}(|D\bar{D}_s^*;1^{+-}\rangle \to J/\psi\bar{K} \quad : \quad \eta_c\bar{K}^* \quad : \quad D\bar{D}_s^* \quad : \quad \bar{D}_sD^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 0.093 \quad : \quad 41 \quad : \quad 87, \end{aligned} \tag{144}$$

$$\begin{aligned} &\mathcal{B}(|D^*\bar{D}_s^*;0^+\rangle \to \eta_c\bar{K} \quad : \quad J/\psi\bar{K}^* \quad : \quad \bar{D}_s^*D^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 0.088 \quad : \quad 0.001, \end{aligned} \tag{145}$$

$$\begin{aligned} &\mathcal{B}(|D_s\bar{D}_s^*;1^+\rangle \to J/\psi\bar{K} \quad : \quad \eta_c\bar{K}^* \quad : \quad h_c\bar{K} \quad : \quad \chi_{c1}\bar{K}^*(\to\bar{K}\pi) \quad : \quad D_s^*\bar{D}^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 0.36 \quad : \quad 0.002 \quad : \quad 10^{-7} \quad : \quad 31, \end{aligned} \tag{146}$$

$$\begin{aligned} &\mathcal{B}(|D^*\bar{D}_s^*;2^+\rangle \to \eta_c\bar{K} \quad : \quad J/\psi\bar{K}^* \quad : \quad \bar{D}_s^*D^*(\to D\pi)) \\ \approx\quad & \qquad\qquad 1 \quad : \quad 27 \quad : \quad 0.055. \end{aligned} \tag{147}$$

In the calculations we have adopted the possible interpretations of the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$ as the $|D\bar{D}_s^*;1^{++}\rangle$, $|D\bar{D}_s^*;1^{+-}\rangle$, and $|D^*\bar{D}_s^*;1^+\rangle$ molecular states, respectively. Masses of $D^{(*)}\bar{D}_s^{(*)}$ molecular states are accordingly chosen, as given in Eqs. (88).

## V. SUMMARY AND DISCUSSIONS

In this paper we systematically investigate the eighteen possibly existing $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ ($\bar{D}_s^{(*)} \equiv D_s^{(*)-}$) hadronic molecular states, including $|D\bar{D};0^{++}\rangle$, $|D\bar{D}^*;1^{++}/1^{+-}\rangle$, $|D^*\bar{D}^*;0^{++}/1^{+-}/2^{++}\rangle$, $|D\bar{K};0^{+}\rangle$, $|D\bar{K}^*;1^{+}\rangle$, $|D^*\bar{K}^*;0^{+}/1^{+}/2^{+}\rangle$, $|D\bar{D}_s;0^{+}\rangle$, $|D\bar{D}_s^*;1^{++}/1^{+-}\rangle$, and $|D^*\bar{D}_s^*;0^{+}/1^{+}/2^{+}\rangle$. Note that the $D\bar{D}_s^*$ molecules are not charge-conjugated, but for convenience we use $|D\bar{D}_s^*;1^{++}\rangle$ and $|D\bar{D}_s^*;1^{+-}\rangle$ to denote strange partners of $|D\bar{D}^*;1^{++}\rangle$ and $|D\bar{D}^*;1^{+-}\rangle$, respectively. By doing this, we can differentiate the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as well as the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$.

We systematically construct their corresponding interpolating currents, through which we study their mass spectra as well as their production and decay properties. The isospin of these molecular states is important, and in the present study we assume: (a) $|D\bar{D}^*;1^{+-}\rangle$ and $|D^*\bar{D}^*;1^{+-}\rangle$ have $I=1$, (b) $|D\bar{D};0^{++}\rangle$, $|D^*\bar{D}^*;0^{++}\rangle$, and $|D^*\bar{D}^*;2^{++}\rangle$ have $I=0$, (c) the isospin of the $X(3872)$ as $|D\bar{D}^*;1^{++}\rangle$ is separately investigated in Sec. IV A 4, (d) all the $D^{(*)}\bar{K}^{(*)}$ molecular states have $I=0$, and (e) all the $D^{(*)}\bar{D}_s^{(*)}$ molecular states have $I=1/2$.

Firstly, we use the method of QCD sum rules to calculate masses and decay constants of the $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states, and the obtained results are summarized in Table I. Our results support: (a) the interpretations of the $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ as the $|D\bar{D}^*;1^{++}\rangle$, $|D\bar{D}^*;1^{+-}\rangle$, and $|D^*\bar{D}^*;1^{+-}\rangle$ molecular states, respectively; (b) the interpretation of the $X_0(2900)$ as the $|D^*\bar{K}^*;0^{+}\rangle$ molecular state; (c) the interpretations of the $Z_{cs}(3985)$, $Z_{cs}(4000)$, and $Z_{cs}(4220)$ as the $|D\bar{D}_s^*;1^{++}\rangle$, $|D\bar{D}_s^*;1^{+-}\rangle$, and $|D^*\bar{D}_s^*;1^{+}\rangle$ molecular states, respectively. The uncertainty/accuracy is moderate but not enough to extract the binding energy as well as to differentiate the $Z_{cs}(3985)$ and $Z_{cs}(4000)$. Hence, our QCD sum rule results can only suggest but not determine: (a) whether these molecular states exist or not, and (b) whether they are bound states or resonance states. To better understand them, we further study their production and decay properties, and the decay constants $f_X$ extracted from QCD sum rules are important input parameters.

Secondly, we use the current algebra to study productions of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states in $B$ and $B^*$ decays. We derive the relative production rates

$$\mathcal{R}_1 \equiv \frac{\mathcal{B}(B^- \to K^-X/D^-X/\phi X)}{\mathcal{B}(B^- \to D^-|D^*\bar{K}^*;0^+\rangle)/d_2}, \tag{148}$$

$$\mathcal{R}_2 \equiv \frac{\mathcal{B}(B^{*-} \to K^-X/D^-X/\phi X)}{\mathcal{B}(B^{*-} \to K^-|D\bar{D};0^{++}\rangle)/d_4}, \tag{149}$$

and the obtained results are summarized in Table III. In the calculations we only consider $\eta_{1\cdots6}$, $\xi_{1\cdots6}$, and $\zeta_{1\cdots6}$ defined in Eqs. (17)–(22), (34)–(39), and (47)–(52), which couple to these molecular states through $S$-wave. Besides, there may exist some other currents coupling to them through $P$-wave, which are not taken into account in the present study. Consequently, some other molecular states such as $|D\bar{D};0^{++}\rangle$ may still be produced in $B^-$ decays, and omissions of these currents cause some theoretical uncertainties.

Third, we use the Fierz rearrangement of the Dirac and color indices to study strong decay properties of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ molecular states. We calculate some of their relative branching ratios, and the obtained results are summarized in Table III. In the calculations we only consider the leading-order fall-apart decays described by color-singlet-color-singlet meson-meson currents, but neglect the $\mathcal{O}(\alpha_s)$ corrections described by color-octet-color-octet meson-meson currents. Hence, there may be some other possible decay channels, and omissions of the $\mathcal{O}(\alpha_s)$ corrections cause some theoretical uncertainties.

Generally speaking, the uncertainty of our QCD sum rule results is moderate, as given in Table I, while uncertainties of relative branching ratios and relative production rates are much larger. In the present study we use local currents and work under the naive factorization scheme, so our uncertainties are significantly larger than the well-developed QCD factorization scheme [214–216], whose uncertainty is at the 5% level [217]. On the other hand, we only calculate the ratios, which significantly reduces our uncertainties [218]. Hence, we roughly estimate uncertainties of relative branching ratios to be at the $X^{+100\%}_{-50\%}$ level, and uncertainties of relative production rates to be at the $X^{+200\%}_{-67\%}$ level.

There are three unfixed parameters in Table III: the parameters $t' \approx d_2/d_1 \approx d_5/d_4$ and $t'' \approx d_3/d_1 \approx d_6/d_4$ (partly) measure the dynamical difference among Fig. 3(a)–(c), and the parameter $t \approx a_i'/a_i \approx b_i'/b_i$ ($i = 1\ldots6$) measures the dynamical difference between Figs. 4(a) and 4(b). Simply assuming them to be $t \approx t' \approx t'' \approx 1$ (local currents), we use the results given in Table III to draw conclusions as follows:

### A. $X_0(2900)$ as $|D^*\bar{K}^*;0^+\rangle$

Our QCD sum rule analyses support the interpretation of the $X_0(2900)$ as the $|D^*\bar{K}^*;0^+\rangle$ molecular state. Given its width to be about $\Gamma_{X_0(2900)} = 57.2$ MeV [219], we use Eqs. (137) and (139) to estimate:

$$\Gamma_{X_0(2900)} \approx \Gamma_{\bar{K}^*} + \Gamma_{X\to D\bar{K}} + \Gamma_{X\to D^*\bar{K}^*\to D^*\bar{K}\pi}, \tag{150}$$

where

TABLE III. Relative branching ratios of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states and their relative production rates in $B$ and $B^*$ decays. In the 2nd and 3rd columns we show their relative production rates $\mathcal{R}_1 \equiv \frac{\mathcal{B}(B^- \to K^- X/D^- X/\phi X)}{\mathcal{B}(B^- \to D^- |D^* \bar{K}^*; 0^+\rangle)/d_2}$ and $\mathcal{R}_2 \equiv \frac{\mathcal{B}(B^{*-} \to K^- X/D^- X/\phi X)}{\mathcal{B}(B^{*-} \to K^- |D\bar{D}; 0^{++}\rangle)/d_4}$. In the 4th-16th/9th/13th columns we show their relative branching ratios, where we use the symbol "$AB$" to generally denote the two-body decay $X \to AB$, the three-body decay process $X \to AB \to Ab_1 b_2$, or the four-body decay process $X \to AB \to Ab_1 b_2 b_3$, depending on which channel is kinematically allowed. The number 0.63 with † is fixed by the BESIII measurement $\frac{\mathcal{B}(X(3872) \to J/\psi\omega \to J/\psi\pi\pi\pi)}{\mathcal{B}(X(3872) \to J/\psi\rho \to J/\psi\pi\pi)} = 1.6^{+0.4}_{-0.3} \pm 0.2$ [59], which is related to the isospin breaking effect of the $X(3872)$ interpreted as the $|D\bar{D}^*; 1^{++}\rangle$ molecular state. The parameters $t' \approx d_2/d_1 \approx d_5/d_4$ and $t'' \approx d_3/d_1 \approx d_6/d_4$ (partly) measure the dynamical difference among Fig. 3(a)–(c), and the parameter $t \approx a_i'/a_i \approx b_i'/b_i$ ($i = 1 \ldots 6$) measures the dynamical difference between Figs. 4(a) and 4(b).

| | Productions | | Decay Channels | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\eta_c\eta$ | $J/\psi\omega$ | $\eta_c f_0$ | $\chi_{c0} f_0$ | $\chi_{c1} f_0$ | $J/\psi\pi$ | $\chi_{c0}\pi$ | $h_c\pi$ | $\eta_c\rho$ | $J/\psi\rho$ | $\chi_{c1}\rho$ | $D\bar{D}^*$ | $D^*\bar{D}^*$ | Candidate |
| $\vert D\bar{D}; 0^{++}\rangle$ | ⋯ | $d_4$ | 1 | $10^{-6}$ | ⋯ | $10^{-5}$ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |
| $\vert D\bar{D}^*; 1^{++}\rangle$ | $0.19 d_1$ | $5.7 d_4$ | ⋯ | 1 | 0.091 | ⋯ | 0.086 | ⋯ | 0.015 | ⋯ | ⋯ | $0.63^\dagger$ | ⋯ | $7.4t$ | ⋯ | $X(3872)$ |
| $\vert D\bar{D}^*; 1^{+-}\rangle$ | $0.12 d_1$ | $3.7 d_4$ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | 1 | ⋯ | 0.011 | 0.092 | ⋯ | $10^{-6}$ | $74t$ | ⋯ | $Z_c(3900)$ |
| $\vert D^*\bar{D}^*; 0^{++}\rangle$ | $1.3 d_1$ | ⋯ | 1 | 0.001 | ⋯ | 0.001 | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | $10^{-4} t$ | ⋯ |
| $\vert D^*\bar{D}^*; 1^{+-}\rangle$ | $0.42 d_1$ | $17 d_4$ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | 1 | ⋯ | 0.002 | 0.33 | ⋯ | $10^{-5}$ | ⋯ | $11t$ | $Z_c(4020)$ |
| $\vert D^*\bar{D}^*; 2^{++}\rangle$ | ⋯ | ⋯ | 1 | 73 | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | $9.5t$ | ⋯ |

| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\Gamma_{\bar{K}^*}$ [MeV] | $D\bar{K}$ | $D\bar{K}^*$ | $D^*\bar{K}$ | $D^*\bar{K}^*$ | $D^*\bar{K}^*$ | Candidate |
|---|---|---|---|---|---|---|---|---|---|
| $\vert D\bar{K}; 0^+\rangle$ | ⋯ | $1.1 d_5$ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |
| $\vert D\bar{K}^*; 1^+\rangle$ | ⋯ | $10 d_5$ | 49.1 MeV | ⋯ | $t + 0.002$ | 0.057 | $10^{-4}$ | ⋯ | ⋯ |
| $\vert D^*\bar{K}; 1^+\rangle$ | $0.67 d_2$ | ⋯ | ⋯ | ⋯ | $10^{-6}$ | $t + 0.004$ | ⋯ | ⋯ | ⋯ |
| $\vert D^*\bar{K}^*; 0^+\rangle$ | $d_2$ | ⋯ | 49.1 MeV | 8.5 | ⋯ | ⋯ | $t + 0.028$ | ⋯ | $X_0(2900)$ |
| $\vert D^*\bar{K}^*; 1^+\rangle$ | $0.70 d_2$ | $27 d_5$ | 49.1 MeV | ⋯ | 0.34 | 1.1 | $t$ | $10^{-4} t$ | ⋯ |
| $\vert D^*\bar{K}^*; 2^+\rangle$ | ⋯ | ⋯ | 49.1 MeV | 0.018 | ⋯ | ⋯ | $t + 0.074$ | $10^{-4} t$ | ⋯ |

| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\eta_c\bar{K}$ | $J/\psi\bar{K}$ | $\chi_{c0}\bar{K}$ | $h_c\bar{K}$ | $\eta_c\bar{K}^*$ | $J/\psi\bar{K}^*$ | $\chi_{c1}\bar{K}^*$ | $D\bar{D}_s^*$ | $D^*\bar{D}_s$ | $D^*\bar{D}_s^*$ | Candidate |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vert D\bar{D}_s; 0^+\rangle$ | ⋯ | $11 d_6$ | 1 | ⋯ | ⋯ | ⋯ | ⋯ | 0.001 | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |
| $\vert D\bar{D}_s^*; 1^{++}\rangle$ | $0.79 d_3$ | $6.6 d_6$ | ⋯ | ⋯ | 0.012 | ⋯ | ⋯ | 1 | ⋯ | 25 | 68 | ⋯ | $Z_{cs}(3985)$ |
| $\vert D\bar{D}_s^*; 1^{+-}\rangle$ | $0.75 d_3$ | $6.2 d_6$ | ⋯ | 1 | ⋯ | ⋯ | 0.093 | ⋯ | ⋯ | 41 | 87 | ⋯ | $Z_{cs}(4000)$ |
| $\vert D^*\bar{D}_s^*; 0^+\rangle$ | ⋯ | ⋯ | 1 | ⋯ | ⋯ | ⋯ | ⋯ | 0.088 | ⋯ | ⋯ | ⋯ | 0.001 | ⋯ |
| $\vert D^*\bar{D}_s^*; 1^+\rangle$ | $1.8 d_3$ | ⋯ | ⋯ | 1 | ⋯ | 0.002 | 0.36 | ⋯ | $10^{-7}$ | ⋯ | ⋯ | 31 | $Z_{cs}(4220)$ |
| $\vert D^*\bar{D}_s^*; 2^+\rangle$ | ⋯ | ⋯ | 1 | ⋯ | ⋯ | ⋯ | ⋯ | 27 | ⋯ | ⋯ | ⋯ | 0.055 | ⋯ |

$$
\begin{aligned}
\Gamma_{\bar{K}^*} &\approx 49.1 \text{ MeV}, \\
\Gamma_{X \to D^0 \bar{K}^0} &\approx 3.6 \text{ MeV}, \\
\Gamma_{X \to D^+ K^-} &\approx 3.6 \text{ MeV}, \\
\Gamma_{X \to D^{*0} \bar{K}^{*0} \to D^{*0} (\bar{K}\pi)^0} &\approx 0.4 \text{ MeV}, \\
\Gamma_{X \to D^{*+} K^{*-} \to D^{*+} (\bar{K}\pi)^-} &\approx 0.4 \text{ MeV}.
\end{aligned} \tag{151}
$$

According to the first term $\Gamma_{\bar{K}^*}$, we propose to confirm the $X_0(2900)$ in its dominant $D^*\bar{K}\pi$ decay channel.

It is interesting to notice that in the present study the neutral $X_0(2900)^0$ is produced in the $B^- \to D^-[D^{*0}\bar{K}^{*0}]$ channel (Fig. 3(b)) and not in the suppressed $B^- \to D^-[D^{*+}K^{*-}]$ channel (Fig. 3(e)). However, it equally decays into the $D^0\bar{K}^0$ and $D^+K^-$ final states. Recalling that the branching ratio of the $B^- \to D^- D^0 \bar{K}^0$ decay is significantly larger than that of the $B^- \to D^- D^+ K^-$ decay [2]:

$$
\begin{aligned}
\mathcal{B}(B^- \to D^- D^0 \bar{K}^0) &= (1.55 \pm 0.21) \times 10^{-3}, \\
\mathcal{B}(B^- \to D^- D^+ K^-) &= (2.2 \pm 0.7) \times 10^{-4},
\end{aligned} \tag{152}
$$

the nonsuppressed signal of the neutral $X_0(2900)^0$ in the suppressed $B^- \to D^- D^+ K^-$ decay channel is probably much more significant than in the $B^- \to D^- D^0 \bar{K}^0$ decay channel. This behavior can be used to test its isospin breaking effect.

### B. $X_0(2900)$ versus $X(3872)$

The $X(3872)$ and $X_0(2900)$ have both been observed in $B$ decays. Some of their experimental measurements are [2,13,14]:

$$
\begin{aligned}
&\mathcal{B}(B^- \to K^- X(3872) \xrightarrow{\omega} K^- J/\psi\pi\pi\pi) \\
&\qquad = (6.0 \pm 2.2) \times 10^{-6},
\end{aligned} \tag{153}
$$

$$\mathcal{B}(B^- \to D^-D^+K^-) = (2.2 \pm 0.7) \times 10^{-4}, \quad (154)$$

$$\frac{\mathcal{B}(B^- \to D^-X_0(2900) \to D^-D^+K^-)}{\mathcal{B}(B^- \to D^-D^+K^-)} = (5.6 \pm 1.4 \pm 0.5) \times 10^{-2}, \quad (155)$$

from which we can further derive

$$\frac{\mathcal{B}(B^- \to D^-X_0(2900) \to D^-D^+K^-)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} = 2.1 \pm 1.1. \quad (156)$$

From Table III we can derive:

$$\frac{\mathcal{B}(B^- \to D^-|D^*\bar{K}^*\rangle_{0^+} \to D^-D^+K^-)}{\mathcal{B}(B^- \to K^-|D\bar{D}^*\rangle_{1^{++}} \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 3.1, \quad (157)$$

with the uncertainty roughly at the $X^{+200\%}_{-66\%}$ level.

The two values given in Eqs. (156) and (157) are well consistent with each other, supporting the interpretations of the $X(3872)$ and $X_0(2900)$ as the $|D\bar{D}^*; 1^{++}\rangle$ and $|D^*\bar{K}^*; 0^+\rangle$ molecular states, respectively.

### C. $Z_{cs}(3985)$ as $|D\bar{D}_s^*; 1^{++}\rangle$ and $Z_{cs}(4000)$ as $|D\bar{D}_s^*; 1^{+-}\rangle$

Since the $DD_s^{*-}$ and $D^*D_s^-$ thresholds are very close to each other, in the present study we use them to combine two mixed states, as defined in Eqs. (42)–(43):

$$\sqrt{2}|D\bar{D}_s^*; 1^{++}\rangle = |DD_s^{*-}\rangle_{J=1} + |D^*D_s^-\rangle_{J=1},$$
$$\sqrt{2}|D\bar{D}_s^*; 1^{+-}\rangle = |DD_s^{*-}\rangle_{J=1} - |D^*D_s^-\rangle_{J=1}.$$

They are strange partner states of $|D\bar{D}^*; 1^{++}\rangle$ and $|D\bar{D}^*; 1^{+-}\rangle$, so we simply denote their quantum numbers as $J^{PC} = 1^{++}$ and $1^{+-}$. Our QCD sum rule analyses support the interpretations of the $Z_{cs}(3985)$ and $Z_{cs}(4000)$ as the $|D\bar{D}_s^*; 1^{++}\rangle$ and $|D\bar{D}_s^*; 1^{+-}\rangle$ molecular states, respectively, i.e., they are strange partners of the $X(3872)$ and $Z_c(3900)$, respectively.

Our results given in Table III suggest that the $Z_{cs}(4000)$ as $|D\bar{D}_s^*; 1^{+-}\rangle$ can be observed in the $B^- \to \phi K^- J/\psi$ decay, but the $Z_{cs}(3985)$ as $|D\bar{D}_s^*; 1^{++}\rangle$ cannot. This is just consistent with the BESIII and LHCb observations [15,96]. To verify the above interpretations, we propose to search for the $Z_{cs}(3985)$ in the $B^- \to \phi J/\psi K^{*-} \to \phi J/\psi(\bar{K}\pi)^-$ decay process. From Table III we can derive:

$$\frac{\mathcal{B}(B^- \to \phi Z_{cs}(3985) \xrightarrow{K^*} \phi J/\psi(\bar{K}\pi)^-)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 0.41, \quad (158)$$

Together with Eq. (153) we can further derive:

$$\mathcal{B}(B^- \to \phi Z_{cs}(3985) \xrightarrow{K^*} \phi J/\psi(\bar{K}\pi)^-) \approx 2.4 \times 10^{-6}. \quad (159)$$

### D. $Z_{cs}(4000)$ and $Z_{cs}(4220)$ versus $X(3872)$

The $Z_{cs}(4000)$ and $Z_{cs}(4220)$ have both been observed in the $B^- \to \phi J/\psi K^-$ decay. Some of their experimental measurements are [2,15]:

$$\mathcal{B}(B^- \to \phi J/\psi K^-) = (5.0 \pm 0.4) \times 10^{-5}, \quad (160)$$

$$\frac{\mathcal{B}(B^- \to \phi Z_{cs}(4000) \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to \phi J/\psi K^-)} = (9.4 \pm 2.1 \pm 3.4) \times 10^{-2}, \quad (161)$$

$$\frac{\mathcal{B}(B^- \to \phi Z_{cs}(4220) \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to \phi J/\psi K^-)} = (10 \pm 4^{+10}_{-7}) \times 10^{-2}. \quad (162)$$

Together with Eq. (153), we can further derive

$$\frac{\mathcal{B}(B^- \to \phi Z_{cs}(4000) \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} = 0.78 \pm 0.44,$$
$$\frac{\mathcal{B}(B^- \to \phi Z_{cs}(4220) \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} = 0.83^{+0.95}_{-0.74}. \quad (163)$$

From Table III we can derive:

$$\frac{\mathcal{B}(B^- \to \phi|D\bar{D}_s^*\rangle_{1^{+-}} \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to K^-|D\bar{D}^*\rangle_{1^{++}} \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 0.28,$$
$$\frac{\mathcal{B}(B^- \to \phi|D^*\bar{D}_s^*\rangle_{1^{+-}} \to \phi J/\psi K^-)}{\mathcal{B}(B^- \to K^-|D\bar{D}^*\rangle_{1^{++}} \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 2.7, \quad (164)$$

with the uncertainty roughly at the $X^{+200\%}_{-66\%}$ level.

The two sets of values given in Eqs. (163) and (164) are consistent with each other, supporting the interpretations of the $Z_{cs}(4000)$ and $Z_{cs}(4220)$ as the $|D\bar{D}_s^*; 1^{+-}\rangle$ and $|D^*\bar{D}_s^*; 1^{+-}\rangle$ molecular states, respectively. To verify these interpretations, we propose to confirm the $Z_{cs}(4000)$ in the $B^- \to \phi D^0 D_s^{*-}$ and $B^- \to \phi D^{*0} D_s^-$ decays, and the $Z_{cs}(4220)$ in the $B^- \to \phi D^{*0} D_s^{*-}$ decay.

### E. $Z_c(3900)$ and $Z_c(4020)$

Our QCD sum rule analyses support the interpretations of the $Z_c(3900)$ and $Z_c(4020)$ as the $|D\bar{D}^*; 1^{+-}\rangle$ and $|D^*\bar{D}^*; 1^{+-}\rangle$ molecular states, respectively. From Table III we can derive:

$$\frac{\mathcal{B}(B^- \to K^-Z_c(3900) \to K^-J/\psi\pi^0)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 0.078,$$
$$\frac{\mathcal{B}(B^- \to K^-Z_c(4020) \to K^-J/\psi\pi^0)}{\mathcal{B}(B^- \to K^-X(3872) \xrightarrow{\omega} K^-J/\psi\pi\pi\pi)} \approx 1.7. \quad (165)$$

Together with Eq. (153) and the experiment measurement [2],

$$\mathcal{B}(B^- \to J/\psi \bar{K}^0 \pi^-) = (1.14 \pm 0.11) \times 10^{-3}, \tag{166}$$

we can further derive:

$$\frac{\mathcal{B}(B^- \to K^- Z_c(3900) \to K^- J/\psi \pi^0)}{\mathcal{B}(B^- \to K^- J/\psi \pi^0)} \approx 0.1\%,$$
$$\frac{\mathcal{B}(B^- \to K^- Z_c(4020) \to K^- J/\psi \pi^0)}{\mathcal{B}(B^- \to K^- J/\psi \pi^0)} \approx 1.7\%. \tag{167}$$

Accordingly, we propose to search for the $Z_c(4020)$ in the $B^- \to K^- Z_c \to K^- J/\psi \pi^0$ decay process, but note that uncertainties of the above estimations are roughly at the $X^{+200\%}_{-66\%}$ level.

### F. $B^*$ decays

We propose to confirm the $X(3872)$, $Z_c(3900)$, $Z_c(4020)$, $Z_{cs}(3985)$, and $Z_{cs}(4000)$ in $B^*$ decays, although the identification of $B^*$ in its hadronic decays is still not easy. Based on the interpretations of the present study, we can derive from Table III:

$$\frac{\mathcal{B}(B^{*-} \to K^- Z_c(3900) \to K^- J/\psi \pi^0)}{\mathcal{B}(B^{*-} \to K^- X(3872) \xrightarrow{\omega} K^- J/\psi \pi\pi\pi)} \approx 0.08,$$
$$\frac{\mathcal{B}(B^{*-} \to K^- Z_c(4020) \to K^- J/\psi \pi^0)}{\mathcal{B}(B^{*-} \to K^- X(3872) \xrightarrow{\omega} K^- J/\psi \pi\pi\pi)} \approx 2.2,$$
$$\frac{\mathcal{B}(B^{*-} \to \phi Z_{cs}(3985) \xrightarrow{K^*} \phi J/\psi (\bar{K}\pi)^-)}{\mathcal{B}(B^{*-} \to K^- X(3872) \xrightarrow{\omega} K^- J/\psi \pi\pi\pi)} \approx 0.11,$$
$$\frac{\mathcal{B}(B^{*-} \to \phi Z_{cs}(4000) \to \phi J/\psi K^-)}{\mathcal{B}(B^{*-} \to K^- X(3872) \xrightarrow{\omega} K^- J/\psi \pi\pi\pi)} \approx 0.08. \tag{168}$$

### G. Other possibly existing states

To end this paper, we propose to search for the possibly existing $|D^*\bar{D}^*; 0^{++}\rangle$, $|D^*\bar{K}; 1^+\rangle$, and $|D^*\bar{K}^*; 1^+\rangle$ molecular states in $B$ decays:

(i) We propose to search for the isoscalar $|D^*\bar{D}^*; 0^{++}\rangle$ molecular state in the $B^- \to K^- X \to K^- \eta_c \eta$ decay, and the isovector one in the $B^- \to K^- X \to K^- \eta_c \pi$ decay.

(ii) We propose to search for the $|D^*\bar{K}; 1^+\rangle$ molecular state in the $B^- \to D^- X \to D^- D \bar{K} \pi$ decay.

(iii) We propose to search for the $|D^*\bar{K}^*; 1^+\rangle$ molecular state in the $B^- \to D^- X \to D^- D \bar{K} \pi$ and $B^- \to D^- X \to D^- D^* \bar{K} \pi$ decays.

Estimations on their relative production rates and relative branching ratios can be found in Table III, although there are large uncertainties coming from their unknown (hadron) masses. Before doing this, we propose to search for the relevant decay processes first, i.e., the $B^- \to K^- \eta_c \eta$, $B^- \to K^- \eta_c \pi$, $B^- \to D^- D \bar{K} \pi$, and $B^- \to D^- D^* \bar{K} \pi$ decays. Besides, we propose to search for the possibly existing $|D\bar{D}; 0^{++}\rangle$, $|D\bar{K}; 0^+\rangle$, $|D\bar{K}^*; 1^+\rangle$, $|D^*\bar{K}^*; 1^+\rangle$, and $|D\bar{D}_s; 0^+\rangle$ molecular states in $B^*$ decays.

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## APPENDIX A: WIDTH OF COMPOSITE PARTICLE

In this Appendix we generally study the width of a composite particle $A$, which is composed by two subparticles $B$ and $C$. We discuss several cases:

Case A: The two subparticles $B$ and $C$ are both stable. If the mass of $A$ is below the $BC$ threshold, that is $M_A \le M_B + M_C$, the composite particle $A$ is also stable. If $M_A > M_B + M_C$, the width of $A$ is $\Gamma_A = \Gamma_{A \to B+C}$.

Case B: The subparticle $C$ can decay into $c_1$ and $c_2$, but its width is not very large. If $M_A \le M_B + M_{c_1} + M_{c_2}$, the composite particle $A$ is stable. If $M_A > M_B + M_C$, the width of $A$ can be estimated as $\Gamma_A \approx \Gamma_{A \to B+C}$. If $M_B + M_{c_1} + M_{c_2} < M_A \le M_B + M_C$, the width of $A$ can be estimated as the width of the decay process $\Gamma_A \approx \Gamma_{A \to B+C \to B+c_1+c_2}$.

Case C: The subparticle $C$ can decay into $c_1$ and $c_2$, but its width is not so small. We further assume that $M_A > M_B + M_C$, so that $A$ can decay into $B$ and $C$. The width of this process is $\Gamma_{A \to B+C}$, proportional to $g^2_{A \to BC}$, with $g_{A \to BC}$ the relevant coupling constant.

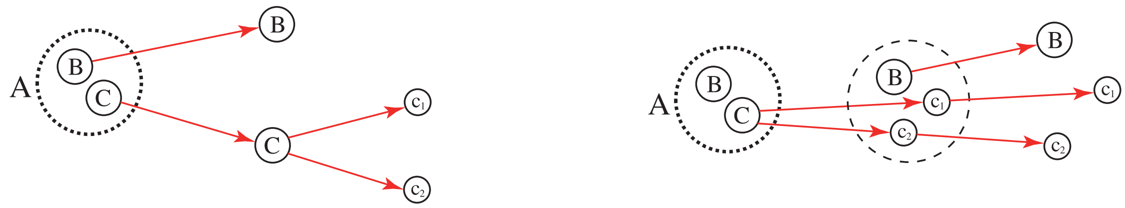
Besides, the subparticle $C$ can decay inside the composite particle $A$, so that $A$ also decays. The width of this process is probably smaller than $\Gamma_C$. Altogether, we arrive at $\Gamma_A \lesssim \Gamma_C + \Gamma_{A \to B+C}$.

Case D: The subparticle $C$ can decay into $c_1$ and $c_2$, but its width is not so small. We further assume that $M_B + M_{c_1} + M_{c_2} < M_A \le M_B + M_C$, so that $A$ can decay into $BC$ and further into $Bc_1c_2$. The width of this process is $\Gamma_{A \to B+C \to B+c_1+c_2}$, again proportional to $g^2_{A \to BC}$.

Besides, the subparticle $C$ can decay inside the composite particle $A$, so that $A$ also decays. Altogether, we arrive at $\Gamma_A \lesssim \Gamma_C + \Gamma_{A \to B+C \to B+c_1+c_2}$.

In the above discussions we just want to argue that the decay process depicted in Fig. 5(a) and the one depicted in Fig. 5(b) are not the same. The former depends on the coupling constant $g_{A \to BC}$, while the latter depends on widths of subparticles but not depends on $g_{A \to BC}$. It might not be reasonable to take these two decay processes as the same one, although they might not be fully independent.

Generally assuming that the two subparticles $B$ and $C$ are both unstable as well as $M_A > M_B + M_C$ or $M_B + M_{c_1} + M_{c_2} < M_A \le M_B + M_C$, we arrive at




(a) $A \to B + C \to B + c_1 + c_2$. (b) C decays, so that $A$ also decays.

FIG. 5. Two possible decay processes of the composite particle $A$ composed by two subparticles $B$ and $C$.

$$\Gamma_A \lesssim \Gamma_B + \Gamma_C + \Gamma_{A\to BC(\to c_1 c_2)} + \cdots, \qquad \text{(A1)}$$

with $\cdots$ partial widths of other possible decay channels. Consequently, we can estimate the lifetime of the composite particle $A$ through

$$1/t_A \lesssim 1/t_B + 1/t_C + \Gamma_{A\to BC(\to c_1 c_2)} + \cdots. \qquad \text{(A2)}$$

In the present study we simply use

$$\Gamma_A \approx \Gamma_B + \Gamma_C + \Gamma_{A\to BC(\to c_1 c_2)} + \cdots, \qquad \text{(A3)}$$

for a weakly-coupled composite system. This formula may be useful to examine the nature of the particle $A$, *i.e.*, to discriminate whether it is a compact multiquark state or a composite hadronic molecular state.

## APPENDIX B: UNCERTAINTIES FROM PHASE ANGLES

As discussed in Sec. IV A 1, there are two different terms, $A \equiv \bar{q}_a\gamma_5 q_a \times \bar{c}_b\gamma_5 c_b$ and $B \equiv \bar{q}_a\gamma_\mu\gamma_5 q_a \times \bar{c}_b\gamma^\mu\gamma_5 c_b$, both of which can contribute to the decay of $|D\bar{D};0^{++}\rangle$ into the

TABLE IV. Relative branching ratios of $D^{(*)}\bar{D}^{(*)}$, $D^{(*)}\bar{K}^{(*)}$, and $D^{(*)}\bar{D}_s^{(*)}$ hadronic molecular states and their relative production rates in $B$ and $B^*$ decays. See the caption of Table III for detailed explanations. In this table we take into account those unknown phase angles among different coupling constants, so that the BESIII measurement $\frac{\mathcal{B}(Z_c(3900)^\pm\to\eta_c\rho^\pm)}{\mathcal{B}(Z_c(3900)^\pm\to J/\psi\pi^\pm)} = 2.2 \pm 0.9$ [83] can be explained with the interpretation of $Z_c(3900)$ as the $|D\bar{D}^*;1^{+-}\rangle$ molecular state.

| | Productions | | Decay Channels | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\eta_c\eta$ | $J/\psi\omega$ | $\eta_c f_0$ | $\chi_{c0}f_0$ | $\chi_{c1}f_0$ | $J/\psi\pi$ | $\chi_{c0}\pi$ | $h_c\pi$ | $\eta_c\rho$ | $J/\psi\rho$ | $\chi_{c1}\rho$ | $D\bar{D}^*$ | $D^*\bar{D}^*$ |
| $\lvert D\bar{D};0^{++}\rangle$ | $\cdots$ | $d_4$ | 0.9–1.0 | $10^{-6}$ | $\cdots$ | $10^{-5}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\lvert D\bar{D}^*;1^{++}\rangle$ | $0.19d_1$ | $5.7d_4$ | $\cdots$ | 1 | 0.09 | $\cdots$ | 0.09 | $\cdots$ | 0.02 | $\cdots$ | $\cdots$ | $0.63^\dagger$ | $\cdots$ | $7.4t$ | $\cdots$ |
| $\lvert D\bar{D}^*;1^{+-}\rangle$ | $0.12d_1$ | $3.7d_4$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | 1.0–5.6 | $\cdots$ | 0.01 | 0.1–2.4 | $\cdots$ | $10^{-6}$ | $74t$ | $\cdots$ |
| $\lvert D^*\bar{D}^*;0^{++}\rangle$ | $1.3d_1$ | $\cdots$ | 0.9–1.0 | 0.001 | $\cdots$ | 0.001 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $10^{-4}t$ |
| $\lvert D^*\bar{D}^*;1^{+-}\rangle$ | $0.42d_1$ | $17d_4$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | 1.0–1.9 | $\cdots$ | 0.002 | 0.33–0.89 | $\cdots$ | $10^{-5}$ | $\cdots$ | $11t$ |
| $\lvert D^*\bar{D}^*;2^{++}\rangle$ | $\cdots$ | $\cdots$ | 1 | 5–73 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $9.5t$ |

| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\Gamma_{\bar{K}^*}$ [MeV] | $D\bar{K}$ | $D\bar{K}^*$ | $D^*\bar{K}$ | $D^*\bar{K}^*$ | $D^*\bar{K}^*$ |
|---|---|---|---|---|---|---|---|---|
| $\lvert D\bar{K};0^+\rangle$ | $\cdots$ | $1.1d_5$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\lvert D\bar{K}^*;1^+\rangle$ | $\cdots$ | $10d_5$ | 49.1 MeV | $\cdots$ | $t$ | 0.06–0.25 | $10^{-4}$ | $\cdots$ |
| $\lvert D^*\bar{K};1^+\rangle$ | $0.67d_2$ | $\cdots$ | $\cdots$ | $\cdots$ | $10^{-6}$ | $t$ | $\cdots$ | $\cdots$ |
| $\lvert D^*\bar{K}^*;0^+\rangle$ | $d_2$ | $\cdots$ | 49.1 MeV | 3.5–8.5 | $\cdots$ | $\cdots$ | $t$ | $\cdots$ |
| $\lvert D^*\bar{K}^*;1^+\rangle$ | $0.70d_2$ | $27d_5$ | 49.1 MeV | $\cdots$ | 0.34–0.75 | 1.1–2.0 | $t$ | $10^{-4}t$ |
| $\lvert D^*\bar{K}^*;2^+\rangle$ | $\cdots$ | $\cdots$ | 49.1 MeV | 0.02 | $\cdots$ | $\cdots$ | $t$ | $10^{-4}t$ |

| Configuration | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\eta_c\bar{K}$ | $J/\psi\bar{K}$ | $\chi_{c0}\bar{K}$ | $h_c\bar{K}$ | $\eta_c\bar{K}^*$ | $J/\psi\bar{K}^*$ | $\chi_{c1}\bar{K}^*$ | $D\bar{D}_s^*$ | $D^*\bar{D}_s$ | $D^*\bar{D}_s^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert D\bar{D}_s;0^+\rangle$ | $\cdots$ | $11d_6$ | 0.2–1.0 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | 0.001 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\lvert D\bar{D}_s^*;1^{++}\rangle$ | $0.79d_3$ | $6.6d_6$ | $\cdots$ | $\cdots$ | 0.01 | $\cdots$ | $\cdots$ | 1 | $\cdots$ | 25 | 68 | $\cdots$ |
| $\lvert D\bar{D}_s^*;1^{+-}\rangle$ | $0.75d_3$ | $6.2d_6$ | $\cdots$ | 1.0–8.3 | $\cdots$ | $\cdots$ | 0.1–3.1 | $\cdots$ | $\cdots$ | 41 | 87 | $\cdots$ |
| $\lvert D^*\bar{D}_s^*;0^+\rangle$ | $\cdots$ | $\cdots$ | 0.3–1.0 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | 0.09 | $\cdots$ | $\cdots$ | $\cdots$ | 0.001 |
| $\lvert D^*\bar{D}_s^*;1^+\rangle$ | $1.8d_3$ | $\cdots$ | $\cdots$ | 1.0–2.3 | $\cdots$ | 0.002 | 0.4–1.0 | $\cdots$ | $10^{-7}$ | $\cdots$ | $\cdots$ | 31 |
| $\lvert D^*\bar{D}_s^*;2^+\rangle$ | $\cdots$ | $\cdots$ | 1 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | 2–27 | $\cdots$ | $\cdots$ | $\cdots$ | 0.06 |

$\eta_c\eta$ final states; there are also two different terms, $C \equiv \bar{q}_a\gamma_\mu q_a \times \bar{c}_b\gamma^\mu c_b$ and $D \equiv \bar{q}_a\sigma_{\mu\nu}q_a \times \bar{c}_b\sigma^{\mu\nu}c_b$, both of which can contribute to the decay of $|D\bar{D};0^{++}\rangle$ into the $J/\psi\omega$ final states.

Besides, there can be more than one terms contributing to some other decay channels. Phase angles among them, such as the phase angle between the two coupling constants $f_{J/\psi}$ and $f^T_{J/\psi}$, cannot be well determined in the present study. In this Appendix we rotate these phase angles and redo all the calculations. Their relevant uncertainties are summarized in Table IV.

Especially, if we take into account these unknown phase angles, the BESIII measurement [83],

$$\frac{\mathcal{B}(Z_c(3900)^\pm \to \eta_c\rho^\pm)}{\mathcal{B}(Z_c(3900)^\pm \to J/\psi\pi^\pm)} = 2.2 \pm 0.9, \tag{B1}$$

can be explained with the interpretation of $Z_c(3900)$ as the $|D\bar{D}^*;1^{+-}\rangle$ molecular state.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