Data-based analysis of the forward-backward asymmetry in $B^{\pm} \to K^{\pm}K^{\mp}K^{\pm}$

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An analysis of the forward-backward asymmetry (FBA) in the decay $B^{\pm} \to K^{\pm}K^{\mp}K^{\pm}$ is carried out based on the LHCb data. It is found that the large FBA observed for the invariant mass of the K^+K^- pair around 1.5 GeV can be explained by the interference of the amplitudes between the resonances with even and odd spins, where the former can be the spin-0 $f_0(1500)$ resonance plus a nonresonance s wave, while the latter is a spin-1 resonance which is probably $\rho^0(1450)$. This is in contradiction with the conclusion of former experimental analysis [e.g., *BABAR*, Phys. Rev. D **85**, 112010 (2012)], according to which the analysis showed no signal of spin-odd resonances at all when the invariant mass of the K^+K^- pair was around 1.5 GeV. According to the analysis of the current paper the existence of the spin-odd resonances such as $\rho^0(1450)$ is inevitable for the explanation of the large FBA in this region. The analysis also shows that the CP asymmetry of the decay channel $B^{\pm} \to \rho^0(1450)K^{\pm}$ is about $(-3.4 \pm 3.0)\%$. We suggest our experimental colleagues perform a closer analysis of this channel. We also suggest to perform the measurements of the FBAs (as well as the forward-backward CP asymmetry) in other three-body decay channels of beauty and charmed mesons, as this is helpful for resonance analysis.

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I. INTRODUCTION

The observables forward-backward asymmetries (FBAs) have played an important role in the history of particle physics. Examples include the discovery of the parity violation of weak interaction [1–3], the precision measurement of the Z boson [4], and the study of the lepton universality [5–7]. The introduction of the FBA and the FBA-induced CP asymmetry (FB-CPA) to the hadronic multibody decays of beauty and charmed mesons provides a good approach for isolating the interfering effects between nearby resonances [8], which is helpful for the understanding of the behavior of the CP violation, the resonance spectroscopy, as well as the low-energy quantum chromodynamics (QCD).

The decays of B meson are excellent probes for new physics indirectly via the study of CP violation (CPV) and rare decays, as well as good places for improving our understanding of QCD at low energy via spectroscopy study of resonances, among which the hadronic multibody decays of B mesons becomes increasingly important. For the former case, the lepton universality in decays $B \to K^{(*)}l^+l^-$ has gained a great deal of attention from both the theoretical and experimental sides [9–16]. For the latter, QCD exotic states such as the pentaquark states were also first observed in hadronic multibody decays of B meson [17–20].

This three-body B meson decay channel $B^{\pm} \to K^{\pm}K^{\mp}K^{\pm}$ has been studied experimentally by *BABAR* [21,22], Belle [23], and LHCb [24], in which a structure referred to as $f_X(1500)$ when the invariant mass of one K^+K^- pair is around 1.5 GeV was reported by *BABAR* and Belle. Although it can be explained by $f_0(1500)$ or a combination of some even-spin resonances such as $f_0(1500)$ and $f'_2(1525)$ for the *BABAR* and the Belle cases, the nature of $f_X(1500)$ is still unclear. Recent theoretical investigations via perturbative QCD approach indicates that the $f_X(1500)$ structure is probably the spin-1 resonance $\rho^0(1450)$ [25–27].

The LHCb data in Ref. [24] provide us the opportunity to investigate the nature of $f_X(1500)$ via the FBA and the FB-CPA in the decay channel $B^{\pm} \to K^{\pm}K^{\mp}K^{\pm}$. The evident large FBA when the invariant mass of K^+K^- pair is around 1.5 GeV, which can even be clearly seen from the events distributions in the Dalitz plots of this channel, implies strongly the presence of a spin-odd resonance, which could probably be $\rho^0(1450)$, for reasons which will be explained in this paper. This is clearly in contradiction with the former analysis performed by *BABAR* and Belle.

The remainder of this paper is structured as follows. In Sec. II, we present the definition of the FBA and the FB-CPA for the three-body decays of B meson, followed with a brief discussion. In Sec. III, we perform the analysis of the FBA and the FB-CPA of $B^{\pm} \to K^{\pm}K^{\mp}K^{\pm}$ based on

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the data of LHCb in Ref. [24], according to which it is found that the large FBA strongly indicates the presence of the p-wave resonances $\rho^0(1450)$. In the last section, we make our conclusion.

II. THE FBA AND THE FB-*CP*A IN THREE-BODY DECAYS OF HEAVY MESON

For a three-body decay process of a beauty or a charmed meson, $H \to M_1 M_2 M_3$, with M_j ($j = 1, 2, 3$) being three pseudoscalar mesons which can only decay via electroweak interactions, we denote the angle between the momentum of M_1 and that of the initial particle H in the c.m. frame of the $M_1 M_2$ system as θ (see Fig. 1 for illustration), and the invariant mass squared of $M_i M_j$ system as m_{ij}^2. One has the relation

$$\cos\theta \equiv \frac{\vec{p}_1^* \cdot \vec{p}_3^*}{|\vec{p}_1^*||\vec{p}_3^*|} = \frac{m_{23}^2 - (m_{23,\max}^2 + m_{23,\min}^2)/2}{(m_{23,\max}^2 - m_{23,\min}^2)/2}, \quad (1)$$

where $\vec{p}_j^*$ is the momentum of M_j in the c.m. frame of the $M_1 M_2$ system; $m_{23,\max(\min)}^2$ is the maximum (minimum) value of m_{23}^2 constrained by the phase space. The FBA, which describes the preference of the flying direction of M_1 with respect to that of H in the c.m. frame of the $M_1 M_2$ system, is defined as [8]

$$A^{FB} = \frac{\int_0^1 \langle|\mathcal{A}|^2\rangle d\cos\theta - \int_{-1}^0 \langle|\mathcal{A}|^2\rangle d\cos\theta}{\int_{-1}^1 \langle|\mathcal{A}|^2\rangle d\cos\theta}, \quad (2)$$

where the notion "$\langle\cdots\rangle$" represents integration over the invariant mass squared m_{12}^2, $\langle|\mathcal{A}|^2\rangle \equiv \int_a^b \frac{(m_{23,\max}^2 - m_{23,\min}^2)}{2} \times |\mathcal{A}|^2 dm_{12}^2$, with $[a,b]$ the interval that the integration was performed on. By expressing the decay amplitudes in terms of partial waves,

$$\mathcal{A} = \sum_l a_l P_l(\cos\theta), \quad (3)$$

one find that the FBA depends on the interferences of even and odd partial waves:

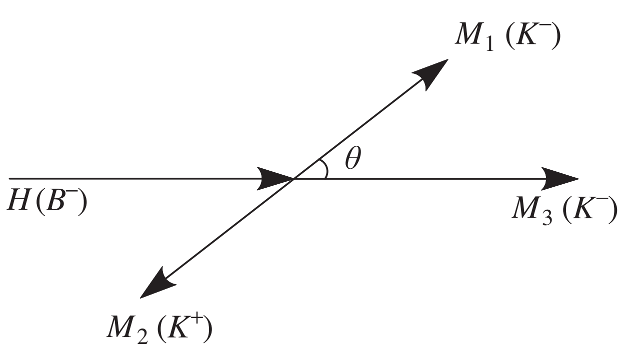

FIG. 1. The definition of θ in the c.m. frame of the $M_1 M_2$ system.

$$A^{FB} = \frac{2}{\sum_j [\langle|a_j|^2\rangle/(2j+1)]} \sum_{\substack{\text{even } l \\ \text{odd } k}} f_{lk} \Re(\langle a_l a_k^* \rangle), \quad (4)$$

where $f_{lk} \equiv \int_0^1 P_l P_k d\cos\theta = \frac{(-)^{(l+k+1)/2} l! k!}{2^{l+k-1}(l-k)(l+k+1)[(l/2)!]^2 \{[(k-1)/2]!\}^2}$ [28]. From this equation, one can see that the numerator contains *only* the interference term between even- and odd waves. This implies that large FBA around even- (odd)-wave resonances usually indicates the interference with nearby odd- (even)-wave contributions.[1] It is impossible to generate a large FBA with only the presence of even or odd waves.

The FB-CPA is defined as the difference between FBAs of the pair of CP-conjugate processes, which reads

$$A_{CP}^{FB} \equiv \frac{1}{2}(A^{FB} - \bar{A}^{FB}), \quad (5)$$

where $\bar{A}^{FB}$ is the FBA of the CP-conjugate process $\bar{H} \to \bar{M}_1 \bar{M}_2 \bar{M}_3$; the factor $1/2$ is introduced so as to make sure the value of the A_{CP}^{FB} lies between -1 and 1. One immediately see from Eqs. (4) and (5) that the FB-CPA has the ability of isolating CPVs originated from the interference of even and odd waves.

III. DATA-BASED ANALYSIS OF FBA AND FB-*CP*A IN $B^\pm \to K^\pm K^\mp K^\pm$

Thanks to the high statistics, the LHCb is able to investigate the B-meson decays—including the branching fractions and the CPAs of the three-body decays of B meson—in an unprecedented precision [29,30]. A very detailed analysis has been carried out by LHCb for the decay process $B^\pm \to K^\pm K^\mp K^\pm$ mentioned in Sec. I, from which rich resonance structures and regional CPAs can be clearly seen throughout the Dalitz plots [24]. Besides, signal yields projected in bins of the invariant mass of one of the K^+K^- pair were also investigated. For each bin, the signal yield was divided into two parts according to whether $\cos\theta > 0$ or $\cos\theta < 0$, where θ was defined as the angle between the momenta of the two Kaons with the same-sign charge in the c.m. frame of the K^+K^- pair with lower invariant mass, m_{low} (also see Fig. 1 for illustration). Based on the data of Fig. 6 of the LHCb's paper [24], we obtain the measured FBAs, FB-CPA, and regional CPA for each bin of the decay process $B^\pm \to K^\pm K^\mp K^\pm$, which are presented in Fig. 2, respectively. One interesting behavior of the FB-CPA is that its value almost does not change for m_{low} ranges from 1 to 1.8 GeV, which deserves investigations from both the experimental and theoretical sides. However, since this is not what we focus on in this paper, we will simply skip this point from now on.

[1] Since M_1 and M_2 are spin-0 particles, the spin of the resonance decaying into them equals the angular momentum between them.

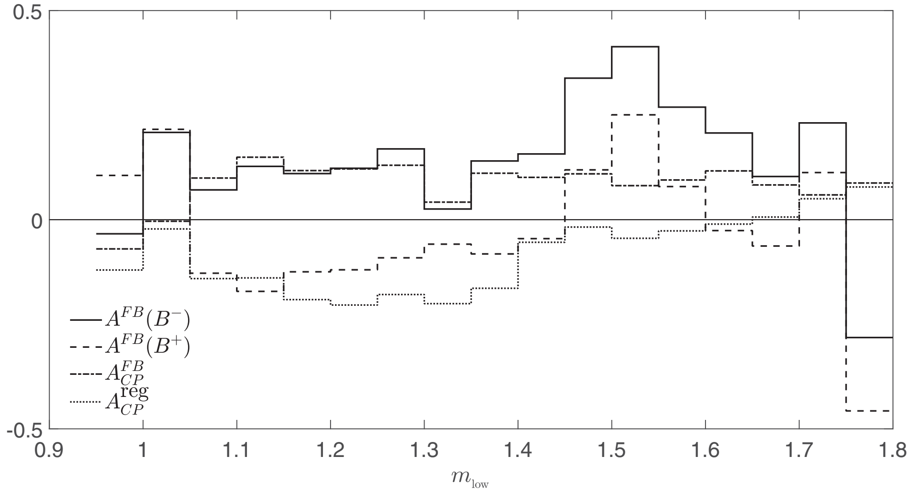

FIG. 2. Various observables extracted from the LHCb data in Ref. [24]. The solid and the dashed lines are the FBA for $B^- \to K^-K^+K^-$ and $B^+ \to K^+K^-K^+$, respectively; the dashed-dotted and the dotted lines are the FB-CPA A_{CP}^{FB} and regional CPA $A_{CP,}^{\rm reg}$ respectively.

Another characterized feature of Fig. 2 lies in the obvious large FBAs associated with some resonance's sturcture when the invariant mass of the K^+K^- pair is around 1.5 GeV for both the CP conjugate processes $B^\pm \to K^\pm K^\mp K^\pm$, which indicates strongly the interference of odd- and even-partial waves according to the analysis in the last section. The FBA of this region is so large that it can even be clearly seen from the events distributions in the Dalitz plots. In what follows, we will focus on this phase-space region. To be more specific, our analysis in this paper is performed only for $m_{\rm low}$ ranges between 1.30 and 1.65 GeV in order to exclude the potential pollution of resonances such as $\phi(1020)$ and $f_0(1710)$.[2] The corresponding event yields for $\cos\theta > 0$ and $\cos\theta < 0$, as well as the FBAs and FB-CPAs, are presented in Table I for all bins of $m_{\rm low}$ ranging between 1.30 and 1.65 GeV, where the uncertainties are statistical only.

There are several resonances that could contribute to $B^\pm \to K^\pm K^\mp K^\pm$ in this region, including $f_0(1500)$, $\rho^0(1450)$, $X(1575)$, $f_2'(1525)$, etc., among which the presence of $f_0(1500)$ and $f_2'(1525)$ has been reported by *BABAR* [22]. After trying various fitting scenarios, we found that the best fit to the LHCb data of Ref. [24] for $m_{\rm low}$ ranges between 1.30 and 1.65 GeV constitutes the resonances $f_0(1500)$ and $\rho^0(1450)$, plus a nonresonance s wave. The decay amplitude of $B^- \to K^-K^+K^-$ can then be parametrized as

$$\mathcal{A}_{B^-\to K^-K^+K^-} = \frac{c_1 e^{i\delta_1}\cos\theta}{m_{\rm low}^2 - m_\rho^2 + im_\rho\Gamma_\rho} + \frac{c_0 e^{i\delta_0}}{m_{\rm low}^2 - m_f^2 + im_f\Gamma_f} + \frac{c_{NS}e^{i\delta_{NS}}}{m_f\Gamma_f}, \tag{6}$$

where c_l and δ_l ($l = 0, 1, NS$) are the corresponding amplitudes (excluding the corresponding propagators and the Legendre polynomials) and the relative phases, respectively, $m_{\rho/f}$ and $\Gamma_{\rho/f}$ are, respectively, the mass and the decay width of the resonance $\rho^0(1450)/f_0(1500)$. The factor $1/m_f\Gamma_f$ in the last term is introduced to make sure that c_{NS} has the same dimension as c_0 and c_1. The amplitude of $B^+ \to K^+K^-K^+$ can be obtained by replacing c_l and δ_l by $\bar{c}_l$ and $\bar{\delta}_l$, respectively. The event yield of each bin in Fig. 3 is fitted according to

$$\mathcal{N}_{B\pm,i}(\cos\theta \gtrless 0)/0.05\ \text{GeV} = \int_{\cos\theta\gtrless 0} [R|\mathcal{A}_{B^\pm\to K^-K^+K^\pm}|^2]_{m_{\rm low}=\bar{m}_{{\rm low},i}} d\cos\theta, \tag{7}$$

where $R = \sqrt{(m_{\rm low}^2 - 4m_K^2)[m_B^2 - (m_{\rm low} - m_K)^2][m_B^2 - (m_{\rm low} + m_K)^2]}$ is the phase-space factor, and $\bar{m}_{{\rm low},i}$ is the mean value of $m_{\rm low}$ of bin i. Note that we have absorbed all the factors which are irrelevant to the discussions of FBAs and FB-CPAs into the amplitudes c_l. Once this has been done, the amplitudes c_l's become dimensionless.

[2]Both $\phi(1020)$ and $f_0(1710)$ have little influence on the observed large FBA. For $\phi(1020)$, although it is one of the dominant resonances, it is far away from the region of the observed large FBA and its width is narrow enough. Hence, its effects on the observed large FBA is negligible even if it is a vector resonance. For $f_0(1710)$, on the other hand, although it is not far away from the region of the observed large FBA and has a relatively large decay width, it cannot be the reason for the observed large FBA as it is a spin-even scalar resonance.

TABLE I. The event yields, FBAs, and FB-CPAs of each bin for $m_{\rm low}$ ranging between 1.30 and 1.65 GeV, where the uncertainties are statistical only. The FBAs and FB-CPAs are obtained according to the definitions $A_i^{FB} \equiv [N_i(\cos\theta > 0) - N_i(\cos\theta < 0)]/[N_i(\cos\theta > 0) + N_i(\cos\theta < 0)]$, $\bar{A}_i^{FB} \equiv [\bar{N}_i(\cos\theta > 0) - \bar{N}_i(\cos\theta < 0)]/[\bar{N}_i(\cos\theta > 0) + \bar{N}_i(\cos\theta < 0)]$, and $A_{CP,i}^{FB} \equiv \frac{1}{2}(A_i^{FB} - \bar{A}_i^{FB})$, respectively.

	B^-			B^+			
Bin (GeV)	$N_i(\cos\theta > 0)$	$N_i(\cos\theta < 0)$	$A_i^{FB}(\%)$	$\bar{N}_i(\cos\theta > 0)$	$\bar{N}_i(\cos\theta < 0)$	$\bar{A}_i^{FB}(\%)$	$A_{CP,i}^{FB}(\%)$
1.30–1.35	683 ± 26	649 ± 25	2.6 ± 2.7	942 ± 31	1059 ± 33	-5.9 ± 2.2	4.2 ± 1.9
1.35–1.40	926 ± 30	698 ± 26	14.0 ± 2.5	1038 ± 32	1223 ± 35	-8.2 ± 2.1	11.1 ± 1.7
1.40–1.45	1399 ± 37	1019 ± 32	15.7 ± 2.0	1286 ± 36	1408 ± 38	-4.5 ± 1.9	10.1 ± 1.4
1.45–1.50	1995 ± 45	986 ± 31	33.9 ± 1.7	1728 ± 42	1360 ± 37	11.9 ± 1.8	11.0 ± 1.2
1.50–1.55	1702 ± 41	706 ± 27	41.4 ± 1.9	1646 ± 41	986 ± 31	25.1 ± 1.9	8.1 ± 1.3
1.55–1.60	1351 ± 37	778 ± 28	26.9 ± 2.1	1212 ± 35	1034 ± 32	7.9 ± 2.1	9.5 ± 1.4
1.60–1.65	1022 ± 32	671 ± 26	20.7 ± 2.4	842 ± 29	887 ± 30	-2.6 ± 2.4	11.7 ± 1.7

The fitted curves are also presented in Fig. 3 with inputs all taken from Ref. [31], while the numerical values of the fitted parameters are presented in Table II. The goodness of the corresponding fits are 0.92 and 0.86 for $B^- \to K^\mp K^\pm K^\mp$ and $B^+ \to K^\pm K^\mp K^\pm$, indicating that the data around 1.5 GeV can be reasonably described by the resonances $\rho^0(1450)$ and $f_0(1500)$, with $\rho^0(1450)$ the dominant one. This is in contradiction with the conclusion of *BABAR* in Ref. [22], according to which the analysis showed no signal of the spin-1 resonance $\rho^0(1450)$. With those fitted parameters, one can also obtain the CP asymmetries of $B^\pm \to \rho^0(1450)K^\pm$, which is $A_{CP}(B^\pm \to \rho^0(1450)K^\pm) \equiv \frac{|c_1|^2 - |\bar{c}_1|^2}{|c_1|^2 + |\bar{c}_1|^2} = (-3.4 \pm 3.0)\%$.

With the central values of the fitted parameters, the FBAs of $B^\pm \to K^\pm K^\mp K^\pm$ are depicted in Fig. 4, along with comparisons with those obtained from the data. One can see from this figure that the fitted FBAs fit with the data quite well, which is not a surprising result since this fit is in essence optimized according to the FBAs. On the other hand, the fitted FB-CPAs and regional CPAs, which are presented in Fig. 5, show less accordance with those from the data. This is also understandable since both of the FB-CPAs and the regional CPAs represent "fine structures" of the decay $B^\pm \to K^\pm K^\mp K^\pm$ comparing with the FBAs. Our analysis is too simple to describe such fine structures as the FB-CPAs and the regional CPAs well. But, be that as it may, these fitted curves in Fig. 5, especially that of the FB-CPA, still show the tendency that are in accordance with those from the data.

We also try other fitting scenarios, which are presented in Table III, along with the goodness of each scenario. For example, we have tried to fit the data by replacing $\rho^0(1450)$ by $X(1575)$, which has been observed by the BES Collaboration in the channel $J/\psi \to K^+K^-\pi^0$ a long time ago [32], but the goodness of this fit is bad. We have also tried to fit the data by replacing $\rho^0(1450)$ by nonresonance p wave. However, the fit is bad also, indicating that the large FBA for $m_{\rm low}^2$ around 1.5 GeV cannot be explained by nonresonance p-wave contributions either. From Table III one can see that the fitting scenario which was presented in detail above represents the best among all those in this table.

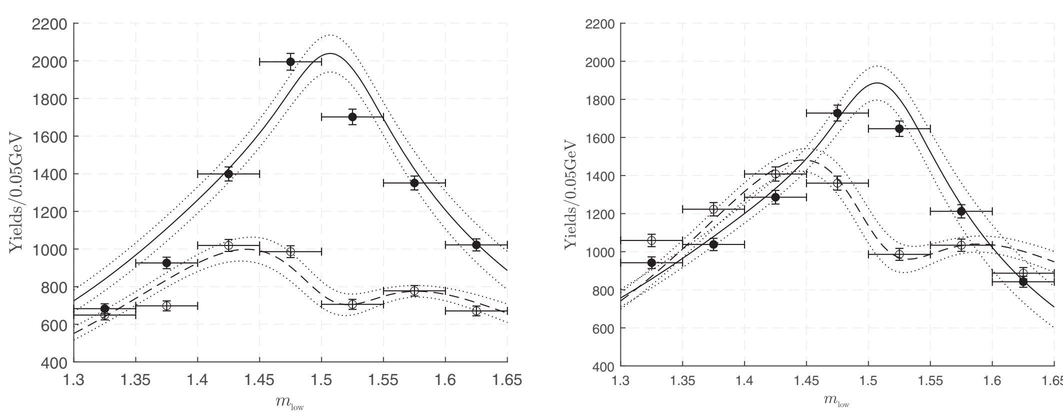

FIG. 3. The event yields from the data of LHCb (only the statistical error depicted) and its corresponding fitted curves of both the CP-conjugate decay channels $B^- \to K^-K^+K^-$ (left) and $B^+ \to K^+K^-K^+$ (right) projected in bins of the invariant mass of the K^+K^- ranging between 1.3 and 1.6 GeV, for $\cos\theta > 0$ (solid circle for the data, solid line for the fitted curve) and $\cos\theta < 0$ (hollow circle for the data, dashed line for the fitted curve). The dotted lines represent the range of the 1σ confidence-level fits.

TABLE II. The fitted values of the parameters c_l, δ_l, $\bar{c}_l$, and $\bar{\delta}_l$, respectively. The phases δ_1 and $\bar{\delta}_1$ are fixed to 0.

Resonance	c_l	δ_l	$\bar{c}_l$	$\bar{\delta}_l$
$\rho^0(1450)$	30.7 ± 0.5	0	31.7 ± 0.8	0
$f_0(1500)$	1.78 ± 0.38	-0.03 ± 0.15	2.12 ± 0.33	-0.27 ± 0.13
Nonresonance s-wave	0.26 ± 0.29	2.20 ± 0.80	1.06 ± 0.28	2.15 ± 0.19

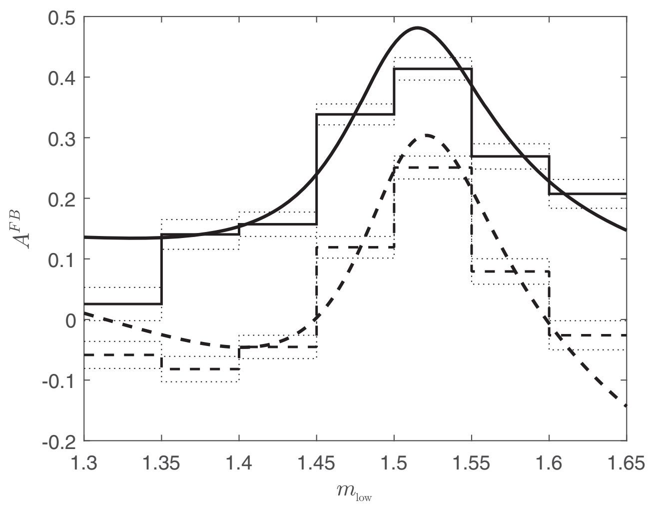

FIG. 4. The best-fit FBAs of $B^- \to K^-K^+K^-$ (solid curve) and $B^+ \to K^+K^-K^+$ (dashed curve) comparing with those extracted from the data of LHCb. The corresponding step lines are the FBAs extracted from the data. The dotted step lines represent the statistical uncertainties.

TABLE III. Various fitting scenarios that were performed. The goodness χ^2/d.o.f. for each fit is obtained with the inclusion of uncertainties from $m_{\rm low}$, which were simply estimated as $0.05\ \mathrm{GeV}/2 = 0.025\ \mathrm{GeV}$ for each bin. The table is ordered according to the goodness of each fit.

Fitting scenario			χ^2/d.o.f.	
s wave	p wave	d wave	B^-	B^+
$f_0(1500)$ Nonresonance s wave	$\rho^0(1450)$		0.92	0.86
$f_0(1500)$	$\rho^0(1450)$	Nonresonance d wave	0.94	1.92
$f_0(1500)$	$\rho^0(1450)$		1.07	2.10
$f_0(1500)$	$\rho^0(1450)$	$f_2'(1525)$	1.10	2.57
	$\rho^0(1450)$	$f_2'(1525)$	1.76	2.70
Nonresonance s wave	$\rho^0(1450)$	$f_2'(1525)$	2.61	2.75
Nonresonance s wave	$\rho^0(1450)$		3.95	3.40
$f_0(1500)$	Nonresonance p wave	$f_2'(1525)$	10.1	41.1
$f_0(1500)$	$X(1575)$	$f_2'(1525)$	9.59	45.8

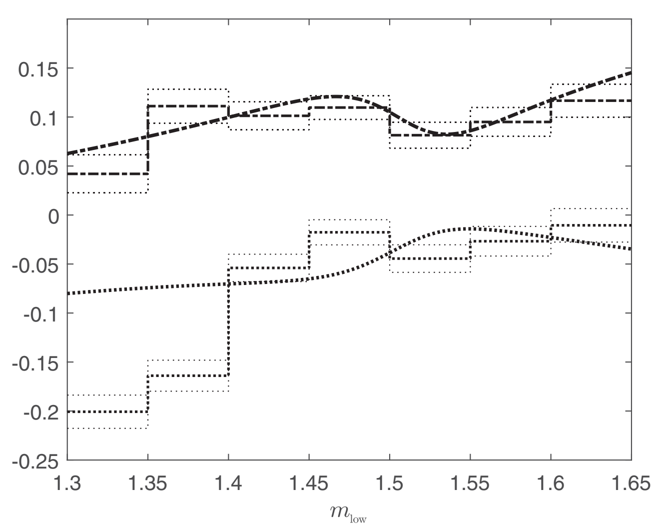

FIG. 5. The best-fit FB-CPA (dashed-dotted) and regional CPA (thick dotted) of $B^\pm \to K^\pm K^\mp K^\pm$. The corresponding step lines are the FBAs extracted from the data, while the curves are plotted with the input of the parameters taking values of the best fit. The thin dotted lines represent the statistical uncertainties.

IV. SUMMARY AND CONCLUSION

A general analysis of the FBA and the FB-CPA were presented in this paper. According to the analysis of this paper, the FBA as well as the FB-CPA are sensitive to the interfering effects of even- and odd waves in three-body decays of beauty and charmed mesons. This makes them serve as good tools for the resonance structure analysis in the aforementioned decay processes. We suggest our experimental colleagues perform the measurements of the FBAs (as well as the FB-CPAs) in three-body decay channels of beauty and charmed mesons.

Enlightened by the notably large FBAs embedded in the LHCb data in Ref. [24], we performed a data-based analysis of the FBAs and CPAs of the decay $B^\pm \to K^\pm K^\mp K^\pm$. We found that the large FBA observed when the invariant mass of the K^+K^- pair lies around 1.5 GeV can be interpreted as the interference of the amplitudes between the resonances $f_0(1500)$ with $\rho^0(1450)$. The analysis shows the existence of the decay channel $B^\pm \to \rho^0(1450)K^\pm$, with CP asymmetry of $A_{CP}(B^\pm \to \rho^0(1450)K^\pm) = (-3.4 \pm 3.0)\%$. This is in contradiction with the conclusion of *BABAR* in Ref. [22], according to which the analysis showed no signal of the spin-1 resonance $\rho^0(1450)$. We suggest our experimental colleagues make a closer analysis of the decay channel $B^\pm \to K^\pm K^\mp K^\pm$.

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