

Study of $B_{(s)}^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decays in the perturbative QCD approach

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In this work, we make a detailed analysis on the penguin-dominant processes $B_{(s)}^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ in the perturbative QCD (PQCD) approach. In addition to the dominant P -wave resonance, the scalar background $f_0(980) \rightarrow K^+K^-$ is also accounted for. We improve the Gegenbauer moments in KK two-meson distribution amplitudes by fitting the PQCD factorization formulas to measured branching ratios of three-body and four-body B decays. We extract the branching ratios of two-body $B_{(s)}^0 \rightarrow \phi\phi$ decays from the corresponding four-body-decay modes and calculate the relevant polarization fractions together with two relative phases $\phi_{\parallel,\perp}$, which are consistent with the previous theoretical predictions. The PQCD predictions for the “true” triple product asymmetries (TPAs) are zero, which are expected in the standard model due to the vanishing weak phase difference, and support the current data reported by the CDF and LHCb Collaborations. A large “fake” TPA $\mathcal{A}_{\text{T-fake}}^1 = 30.4\%$ of the decay $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ is predicted for the first time, which indicates the presence of the significant final-state interactions. The TPAs of the rare decay channel $B^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ are also predicted and can be tested in the near future.

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I. INTRODUCTION

In the standard model (SM), studies of the polarization amplitudes and triple product asymmetries (TPAs) in the flavor-changing neutral current decays provide powerful tests for the presence of physics beyond the SM [1–9], especially for the decay $B_s^0 \rightarrow \phi\phi$ via a $b \rightarrow s\bar{s}s$ penguin process, where the $\phi(1020)$ is implied throughout the remainder of this paper.

The $B_s^0 \rightarrow \phi\phi$ decay is a pseudoscalar to vector-vector transition, where ϕ is reconstructed in the K^+K^- final states. According to the angular momentum conservation, there are three possible spin configurations corresponding to the polarizations of the final-state vector mesons: longitudinal polarization (A_0), and transverse polarization with spins parallel ($A_{\parallel}$) or perpendicular ($A_{\perp}$) to each other.

The first two states A_0 and $A_{\parallel}$ are CP even, while the last one $A_{\perp}$ is CP odd. Polarization amplitudes can be measured by analyzing angular distributions of final-state particles. In the factorization assumption, the longitudinal polarization should dominate based on the quark helicity analysis [10,11]. In sharp contrast to these expectations, large transverse polarization of order 50% is observed in $B \rightarrow K^*\phi$, $B \rightarrow K^*\rho$, and $B_s \rightarrow \phi\phi$ decays [12–16], which poses an interesting challenge for the theory.

Interference between the CP -even ($A_0, A_{\parallel}$) and CP -odd ($A_{\perp}$) amplitudes can generate asymmetries in angular distributions, the triple product asymmetries, which may signal unexpected CP violation due to physics beyond the SM. In recent years, TPAs have already been measured by Belle, BABAR, CDF, and LHCb [12,13,17–24]. These triple products are odd under the time reversal transformation (T), and also constitute potential signals of CP violation due to the CPT theorem. As we know, a nonvanishing direct CP violation needs the interference of at least two amplitudes with a weak phase difference $\Delta\phi$ and a strong phase difference $\Delta\delta$. The direct CP violation is proportional to $\sin\Delta\phi\sin\Delta\delta$, while TPAs go as $\sin\Delta\phi\cos\Delta\delta$. The key point is that the direct CP violation can be produced only if

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there is a nonzero strong phase difference. It has been argued that all strong phases in B decays should be rather small due to the fact that the b -quark is heavy [3]. Hence, if the strong phases are quite small, the magnitude of the direct CP violation is close to zero, but the TPA is maximal. It implies that direct CP violation and TPAs complement each other. Since no tree level operators can contribute to four-body decays $B_{(s)}^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$, there is no direct CP violation in such decay modes. However, T -odd triple products (also called “fake” TPAs), which are proportional to $\cos \Delta\phi \sin \Delta\delta$, can provide useful complementary information. Thus, it may be more promising to search for TPAs than direct CP asymmetries in $b \rightarrow s$ penguin decays.

$B_{(s)}^0 \rightarrow \phi\phi$ decays are usually treated as two-body final states on the theoretical sides and have been studied in the two-body framework using QCD factorization (QCDF) [25–27], the perturbative QCD (PQCD) approaches [28–30], the soft-collinear effective theory (SCET) [31], and the factorization-assisted topological amplitude approach (FAT) [32]. While they are at least four-body decays on the experimental side shown in Fig. 1, the vector meson ϕ decays via the strong interaction with a nonzero width. The four-body B meson decays are indeed more challenging than two-body decays, but provide a number of theoretical and phenomenological advantages. On the one hand, the four-body-decay amplitudes depend on five kinematic variables: three helicity angles and two invariant masses of final meson pairs, while the kinematics of two-body decays is fixed. On the other hand, the four-body decays not only receive the resonant and nonresonant contributions, but also involve the possible significant final-state interactions [33–35].

Four-body decays are still mostly unexplored from the theoretical point of view since the factorization formalism that describes a multibody decay in full phase space is not yet available at present. Recent studies on three-body hadronic decays of B mesons based on the symmetry principles [36–41], the QCDF [42–52], and the PQCD

approaches [53–69] look promising. It has been proposed that the factorization theorem of three-body B decays is approximately valid when two particles move collinearly and the bachelor particle recoils back [70,71]. More details can also be found in Refs. [72,73]. This situation exists particularly in the low $\pi\pi$ or $K\pi$ invariant mass region ($\lesssim 2$ GeV) of the Dalitz plot where most resonant structures are seen. The Dalitz plot is typically dominated by resonant quasi-two-body contributions along the edge. This proposal provides a theoretical framework for studies of resonant contributions based on the quasi-two-body-decay mechanism. Recently, the localized CP violation and branching fraction of the four-body decay $\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-$ have been calculated by employing a quasi-two-body QCDF approach in Refs. [74,75]. In our previous works [76–78], the PQCD factorization formalism based on the quasi-two-body-decay mechanism for four-body B meson decays has been well established. Within the framework of the PQCD approach, the branching ratios and direct CP asymmetries of four-body decays $B_{(s)}^0 \rightarrow \pi\pi\pi\pi$ have also been studied [79].

As a first step, we can only restrict ourselves to the specific kinematical configurations in which two particles each move collinearly and two pairs of final-state particles recoil back in the rest frame of the B meson; see Fig. 1. Naturally the dynamics associated with the pair of final-state mesons can be factorized into a two-meson distribution amplitude (DA) $\Phi_{h_1h_2}$ [80–86]. Thereby, the typical PQCD factorization formula for the considered four-body-decay amplitude can be described as the form of

$$\mathcal{A} = \Phi_B \otimes H \otimes \Phi_{KK} \otimes \Phi_{KK}, \quad (1)$$

where Φ_B is the universal wave function of the B meson and absorbs the nonperturbative dynamics in the process. The Φ_{KK} is the two-hadron DA, which involves the resonant and nonresonant interactions between the two mesons moving collinearly. The hard kernel H describes the dynamics of the strong and electroweak interactions in four-body hadronic decays in a similar way as the one for the corresponding two-body decays.

In this work, we study the four-body decays $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ in the PQCD approach based on k_T factorization with the relevant Feynman diagrams illustrated in Fig. 2. The invariant mass of the K^+K^- pair is restricted to be within ± 30 MeV of the known mass of the ϕ meson for comparison with the LHCb data [13]. The effect of identical particles has been considered in our work. In the considered (K^+K^-) invariant-mass range, the vector resonance ϕ is expected to contribute together with the scalar resonance $f_0(980)$. The S - and P -wave contributions are parametrized into the corresponding timelike form factors involved in the two-meson DAs. We perform a global fit of the Gegenbauer moments in two-kaon DAs associated with both longitudinal and transverse

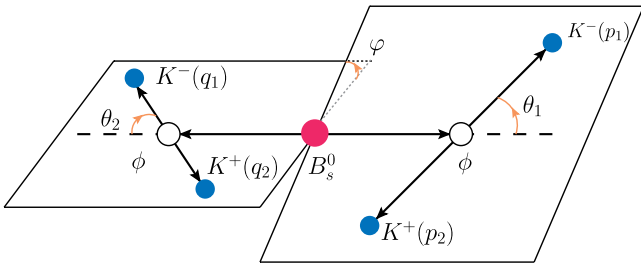


FIG. 1. Graphical definitions of the helicity angles θ_1, θ_2 , and ϕ for the $B_s^0 \rightarrow \phi\phi$ decay, with each quasi-two-body intermediate resonance decaying to two pseudoscalars ($\phi \rightarrow K^+K^-$). $\theta_{1,2}$ is denoted as the angle between the direction of motion of K^- in the ϕ rest frame and ϕ in the B_s^0 rest frame, and ϕ is the angle between the two planes defined by K^+K^- in the B_s^0 rest frame.

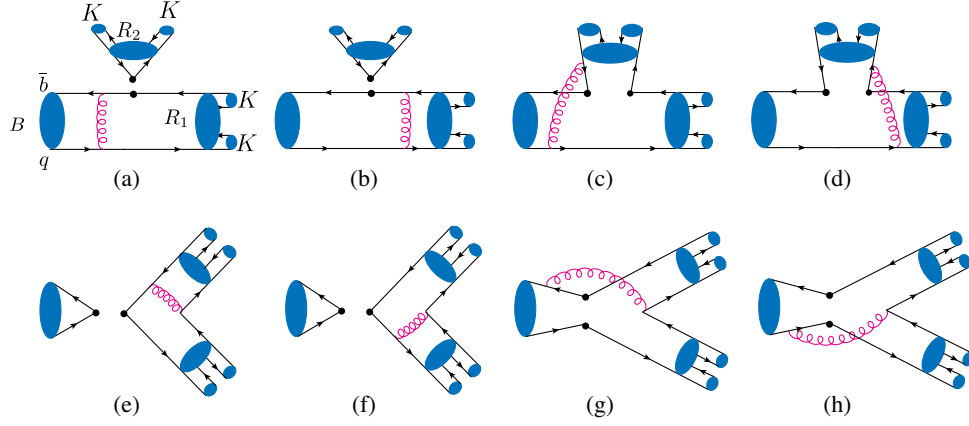


FIG. 2. Typical leading-order Feynman diagrams for the four-body decays $B \rightarrow (R_1 \rightarrow)KK(R_2 \rightarrow)KK$ with $q = (d, s)$, where the symbol $\bullet$ denotes a weak interaction vertex. The diagrams (a)–(d) represent the $B \rightarrow (R_1 \rightarrow)KK$ transition, and the diagrams (e)–(h) are for annihilation contributions.

polarizations to measured branching ratios in three-body and four-body charmless hadronic B meson decays, which will be expressed in detail in the following section. With the improved two-kaon DAs, we calculate the branching ratios and polarization fractions of each partial wave. In addition, triple-product asymmetries corresponding to the interference of the CP -odd amplitudes with the other CP -even amplitudes are predicted.

The rest of the paper is organized as follows. The kinematic variables for four-body hadronic B meson decays are defined in Sec. II. The considered S - and P -wave two-meson DAs are also parametrized, whose normalization form factors are assumed to take the Flatté and relativistic Breit-Wigner (BW) models [87,88]. We explain how to perform the global fit, and then present and discuss the

numerical results in Sec. III, which is followed by the Conclusion. Appendix A collects the explicit PQCD factorization formulas for all the decay amplitudes.

II. FRAMEWORK

A. Kinematics

Considering the four-body decay $B(p_B) \rightarrow R_1(p)R_2(q) \rightarrow P_1(p_1)P_2(p_2)Q_1(p_3)Q_2(p_4)$, as usual, we will work in the B meson rest frame. By employing the light-cone coordinates, we define the B meson momentum p_B , the total momenta of the two kaon-kaon pairs, $p = p_1 + p_2$, $q = p_3 + p_4$, and the quark momentum k_i ($i = B, p, q$) in each meson in the following form:

$$\begin{aligned} p_B &= \frac{m_B}{\sqrt{2}}(1, 1, 0_T), & p &= \frac{m_B}{\sqrt{2}}(g^+, g^-, 0_T), & q &= \frac{m_B}{\sqrt{2}}(f^-, f^+, 0_T), \\ k_B &= (0, x_B p_B^+, k_{BT}), & k_p &= (x_1 p^+, 0, k_{1T}), & k_q &= (0, x_2 q^-, k_{2T}), \end{aligned} \quad (2)$$

with the B meson mass m_B , the parton momentum fractions x_i , and the parton transverse momenta k_{iT} , $i = B, 1, 2$. The explicit expressions of $f^\pm, g^\pm$ related to the invariant masses of the meson pairs via $p^2 = \omega_1^2$ and $q^2 = \omega_2^2$ can be written as

$$\begin{aligned} g^\pm &= \frac{1}{2} \left[1 + \eta_1 - \eta_2 \pm \sqrt{(1 + \eta_1 - \eta_2)^2 - 4\eta_1} \right], \\ f^\pm &= \frac{1}{2} \left[1 - \eta_1 + \eta_2 \pm \sqrt{(1 + \eta_1 - \eta_2)^2 - 4\eta_1} \right], \end{aligned} \quad (3)$$

where $\eta_{1,2} = \frac{\omega_{1,2}^2}{m_B^2}$. For the P -wave KK pairs, the corresponding longitudinal polarization vectors are defined as

$$\epsilon_p = \frac{1}{\sqrt{2\eta_1}}(g^+, -g^-, \mathbf{0}_T), \quad \epsilon_q = \frac{1}{\sqrt{2\eta_2}}(-f^-, f^+, \mathbf{0}_T), \quad (4)$$

which satisfy the normalization $\epsilon_p^2 = \epsilon_q^2 = -1$ and the orthogonality $\epsilon_p \cdot p = \epsilon_q \cdot q = 0$.

The individual momenta p_i ($i = 1-4$) of the four final states can be derived from the relations $p = p_1 + p_2$ and $q = p_3 + p_4$, together with the on-shell conditions $p_i^2 = m_i^2$ for the final-state meson P_i or Q_i ,

$$\begin{aligned}
p_1 &= \left(\frac{m_B}{\sqrt{2}} \left(\zeta_1 + \frac{r_1 - r_2}{2\eta_1} \right) g^+, \frac{m_B}{\sqrt{2}} \left(1 - \zeta_1 + \frac{r_1 - r_2}{2\eta_1} \right) g^-, \mathbf{p}_T \right), \\
p_2 &= \left(\frac{m_B}{\sqrt{2}} \left(1 - \zeta_1 - \frac{r_1 - r_2}{2\eta_1} \right) g^+, \frac{m_B}{\sqrt{2}} \left(\zeta_1 - \frac{r_1 - r_2}{2\eta_1} \right) g^-, -\mathbf{p}_T \right), \\
p_3 &= \left(\frac{m_B}{\sqrt{2}} \left(1 - \zeta_2 + \frac{r_3 - r_4}{2\eta_2} \right) f^-, \frac{m_B}{\sqrt{2}} \left(\zeta_2 + \frac{r_3 - r_4}{2\eta_2} \right) f^+, \mathbf{q}_T \right), \\
p_4 &= \left(\frac{m_B}{\sqrt{2}} \left(\zeta_2 - \frac{r_3 - r_4}{2\eta_2} \right) f^-, \frac{m_B}{\sqrt{2}} \left(1 - \zeta_2 - \frac{r_3 - r_4}{2\eta_2} \right) f^+, -\mathbf{q}_T \right), \\
\mathbf{p}_T^2 &= \zeta_1(1 - \zeta_1)\omega_1^2 + \alpha_1, \quad \mathbf{q}_T^2 = \zeta_2(1 - \zeta_2)\omega_2^2 + \alpha_2,
\end{aligned} \tag{5}$$

with the factors

$$\alpha_1 = -\frac{r_1 + r_2}{2\eta_1} + \frac{(r_1 - r_2)^2}{4\eta_1^2}, \quad \alpha_2 = -\frac{r_3 + r_4}{2\eta_2} + \frac{(r_3 - r_4)^2}{4\eta_2^2}, \tag{6}$$

and the mass ratios $r_i = m_i^2/m_B^2$, m_i being the masses of the final-state mesons.

Comparing Eqs. (5) and (2), one can see that the meson momentum fractions are modified by the final-state meson masses,

$$\frac{p_1^+}{p^+} = \zeta_1 + \frac{r_1 - r_2}{2\eta_1}, \quad \frac{q_1^-}{q^-} = \zeta_2 + \frac{r_3 - r_4}{2\eta_2}. \tag{7}$$

The relation between $\zeta_{1,2}$ and the polar angle $\theta_{1,2}$ in the dimeson rest frame in Fig. 1 can be obtained easily,

$$2\zeta_1 - 1 = \sqrt{1 + 4\alpha_1} \cos\theta_1, \quad 2\zeta_2 - 1 = \sqrt{1 + 4\alpha_2} \cos\theta_2, \tag{8}$$

with the upper and lower limits of $\zeta_{1,2}$,

$$\zeta_{1\max,\min} = \frac{1}{2}[1 \pm \sqrt{1 + 4\alpha_1}], \quad \zeta_{2\max,\min} = \frac{1}{2}[1 \pm \sqrt{1 + 4\alpha_2}]. \tag{9}$$

B. Distribution amplitudes

Without the end point singularities in the evaluations, the distribution amplitudes are one of the most significant nonperturbative inputs in the PQCD approach. In this section, we will briefly introduce the B meson DAs, the S -, P -wave two-kaon DAs, as well as the timelike form factors used in our calculations. In what follows the subscripts S , P are always related to the corresponding partial waves.

The light-cone hadronic matrix element for a B meson is parametrized as [89–94]

$$\begin{aligned}
&\int d^4z e^{i\mathbf{k}_1 \cdot \mathbf{z}} \langle 0 | q_\beta(z) \bar{b}_\alpha(0) | B(p_B) \rangle \\
&= \frac{i}{\sqrt{2N_c}} \left\{ (\not{p}_B + m_B) \gamma_5 \left[\phi_B(\mathbf{k}_1) - \frac{\not{p} - \not{p}}{\sqrt{2}} \bar{\phi}_B(\mathbf{k}_1) \right] \right\}_{\beta\alpha}, \tag{10}
\end{aligned}$$

where q represents a d or s quark. The two wave functions ϕ_B and $\bar{\phi}_B$ in the above decomposition, related to ϕ_B^+ and ϕ_B^- defined in the literature [95] via $\phi_B = (\phi_B^+ + \phi_B^-)/2$ and $\bar{\phi}_B = (\phi_B^+ - \phi_B^-)/2$, obey the normalization conditions

$$\int \frac{d^4\mathbf{k}_1}{(2\pi)^4} \phi_B(\mathbf{k}_1) = \frac{f_B}{2\sqrt{2N_c}}, \quad \int \frac{d^4\mathbf{k}_1}{(2\pi)^4} \bar{\phi}_B(\mathbf{k}_1) = 0. \tag{11}$$

It has been shown that the contribution from $\bar{\phi}_B$ is of next-to-leading power and numerically suppressed [90,91,96], compared to the leading-power contribution from ϕ_B . Taking the PQCD evaluation of the $B \rightarrow \pi$ transition form factor $F_0^{B \rightarrow \pi}$ in Ref. [96] as an example, we find that the $\bar{\phi}_B$ contribution to $F_0^{B \rightarrow \pi}$ is about 20% of the ϕ_B one. The higher-twist B meson DAs have been systematically investigated in the heavy quark effective theory [97], and are decomposed according to definite twist and conformal spin assignments up to twist 6. In principle, all the next-to-leading-power sources should be included for a consistent and complete analysis, which, however, goes beyond the scope of the present formalism. Therefore, we focus only on the leading-power component

$$\Phi_B = \frac{i}{\sqrt{2N_c}} (\not{p}_B + m_B) \gamma_5 \phi_B(x, b), \tag{12}$$

with the impact parameter b being conjugate to the parton transverse momentum k_{BT} . The B meson DA $\phi_B(x, b)$ is chosen as the model form widely adopted in the PQCD approach [89–93,98],

$$\phi_B(x, b) = N_B x^2 (1-x)^2 \exp \left[-\frac{m_B^2 x^2}{2\omega_B^2} - \frac{1}{2} (\omega_B b)^2 \right], \tag{13}$$

where the constant N_B is related to the B meson decay constant f_B through the normalization condition

$\int_0^1 dx \phi_B(x, b=0) = f_B/(2\sqrt{2N_c})$. The shape parameter takes the values $\omega_B = 0.40$ GeV for the B^0 meson and $\omega_{B_s} = 0.48$ GeV [89,99–101] for the B_s^0 meson with 10% variation in the numerical study below.

The S -wave two-kaon DA can be written in the following form [102]:

$$\Phi_S(z, \omega) = \frac{1}{\sqrt{2N_c}} [\not{p} \phi_S^0(z, \omega^2) + \omega \phi_S^s(z, \omega^2) + \omega(\not{p} - 1) \phi_S^t(z, \omega^2)], \quad (14)$$

in which the asymptotic forms of the individual twist-2 and twist-3 components ϕ_S^0 and $\phi_S^{s,t}$ are parametrized as [80–83]

$$\phi_S^0(z, \omega^2) = \frac{9F_S(\omega^2)}{\sqrt{2N_c}} a_S z(1-z)(1-2z), \quad (15)$$

$$\phi_S^s(z, \omega^2) = \frac{F_S(\omega^2)}{2\sqrt{2N_c}}, \quad (16)$$

$$\phi_S^t(z, \omega^2) = \frac{F_S(\omega^2)}{2\sqrt{2N_c}} (1-2z), \quad (17)$$

with the timelike scalar form factor $F_S(\omega^2)$. The Gegenbauer moment a_S in Eq. (15) has adopted the same value as that determined in Ref. [103]: $a_S = 0.80 \pm 0.16$.

The corresponding P -wave two-kaon DAs related to both longitudinal and transverse polarizations are decomposed, up to twist 3, into [57]

$$\Phi_P^L(z, \zeta, \omega) = \frac{1}{\sqrt{2N_c}} \left[\omega \not{p} \phi_P^0(z, \omega^2) + \omega \phi_P^s(z, \omega^2) + \frac{\not{p}_1 \not{p}_2 - \not{p}_2 \not{p}_1}{\omega(2\zeta - 1)} \phi_P^t(z, \omega^2) \right] (2\zeta - 1), \quad (18)$$

$$\Phi_P^T(z, \zeta, \omega) = \frac{1}{\sqrt{2N_c}} \left[\gamma_5 \not{p}_T \not{p} \phi_P^T(z, \omega^2) + \omega \gamma_5 \not{p}_T \phi_P^a(z, \omega^2) + i\omega \frac{\epsilon^{\mu\nu\rho\sigma} \gamma_\mu \epsilon_{T\nu} p_\rho n_{-\sigma}}{p \cdot n_-} \phi_P^v(z, \omega^2) \right] \cdot \sqrt{\zeta(1-\zeta) + \alpha_1}. \quad (19)$$

The various twists ϕ_P^i in the above equations can be expanded in terms of the Gegenbauer polynomials:

$$\phi_P^0(z, \omega^2) = \frac{3F_P^{\parallel}(\omega^2)}{\sqrt{2N_c}} z(1-z) \left[1 + a_{2\phi}^0 \frac{3}{2} (5(1-2z)^2 - 1) \right], \quad (20)$$

$$\phi_P^s(z, \omega^2) = \frac{3F_P^{\perp}(\omega^2)}{2\sqrt{2N_c}} (1-2z), \quad (21)$$

$$\phi_P^t(z, \omega^2) = \frac{3F_P^{\perp}(\omega^2)}{2\sqrt{2N_c}} (1-2z)^2, \quad (22)$$

$$\phi_P^T(z, \omega^2) = \frac{3F_P^{\perp}(\omega^2)}{\sqrt{2N_c}} z(1-z) \left[1 + a_{2\phi}^T \frac{3}{2} (5(1-2z)^2 - 1) \right], \quad (23)$$

$$\phi_P^a(z, \omega^2) = \frac{3F_P^{\parallel}(\omega^2)}{4\sqrt{2N_c}} (1-2z), \quad (24)$$

$$\phi_P^v(z, \omega^2) = \frac{3F_P^{\parallel}(\omega^2)}{8\sqrt{2N_c}} [1 + (1-2z)^2], \quad (25)$$

with the Gegenbauer coefficients $a_{2\phi}^{0,T}$ and the two P -wave form factors $F_P^{\parallel}(\omega^2)$ and $F_P^{\perp}(\omega^2)$. The moment $a_{2\phi}^0$ in the longitudinal twist-2 component ϕ_P^0 has already been determined in a recent global analysis from the three-body B decays in the PQCD approach [67]. We will update the fitting result in the following section by taking the additional four-body decay $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ into account, while the moment $a_{2\phi}^T$ associated with the transverse twist-2 component ϕ_P^T is determined in a global analysis for the first time. Since the amounts of the current experimental data are not yet enough for fixing the Gegenbauer moments in the twist-3 DAs $\phi_P^{s,t}$ and $\phi_P^{v,a}$, they have been set to the asymptotic forms in the present work.

The elastic rescattering effects in a final-state meson pair can be absorbed into the timelike form factors $F_P^{\parallel,\perp}(\omega^2)$ in the two-meson DAs according to the Watson theorem [104]. For the narrow resonance ϕ , we usually employ the relativistic BW line shape for the form factors $F_P^{\parallel}(\omega^2)$ [105]. The explicit formula is expressed as [88]

$$F_P^{\parallel}(\omega^2) = \frac{m_\phi^2}{m_\phi^2 - \omega^2 - im_\phi \Gamma_\phi(\omega^2)}, \quad (26)$$

where $m_\phi = 1.0195$ GeV is the ϕ meson mass. The mass-dependent width $\Gamma_\phi(\omega)$ is defined as

$$\Gamma_\phi(\omega^2) = \Gamma_\phi \left(\frac{m_\phi}{\omega} \right) \left(\frac{k(\omega)}{k(m_\phi)} \right)^{(2L_R+1)}, \quad (27)$$

with the natural width of the ϕ meson $\Gamma_\phi = 4.25$ MeV [106]. The $k(\omega)$ is the momentum vector of the resonance decay product measured in the resonance rest frame, while $k(m_\phi)$ is the value of $k(\omega)$ when $\omega = m_\phi$. The explicit expression of kinematic variables $k(\omega)$ is defined in the $h_1 h_2$ center-of-mass frame

$$k(\omega) = \frac{\sqrt{\lambda(\omega^2, m_{h_1}^2, m_{h_2}^2)}}{2\omega}, \quad (28)$$

with the Källén function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$ and m_{h_1, h_2} as the final-state mass. The orbital angular momentum L_R in the two-meson system is set to $L_R = 1$ for a P -wave state. Because of the limited studies on the form factor $F_P^\perp(\omega^2)$, we use the two decay constants $f_\phi^{(T)}$ of the intermediate particle to determine the ratio $F_P^\perp(\omega^2)/F_P^\parallel(\omega^2) \approx (f_\phi^T/f_\phi)$.

For scalar resonance $f_0(980)$, we adopt the Flatté parametrization where the resulting line shape is above and below the threshold of the intermediate particle [87]. If the coupling of a resonance to the channel opening nearby is very strong, the Flatté parametrization shows a scaling invariance and does not allow for an extraction of individual partial decay widths. Thus, we employ the modified Flatté model suggested by Bugg [107] following the LHCb Collaboration [108,109],

$$F_S(\omega^2) = \frac{m_{f_0(980)}^2}{m_{f_0(980)}^2 - \omega^2 - im_{f_0(980)}(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}\rho_{KK}F_{KK}^2)}. \quad (29)$$

The coupling constants $g_{\pi\pi} = 0.167 \text{ GeV}$ and $g_{KK} = 3.47g_{\pi\pi}$ [108,109] describe the $f_0(980)$ decay into the final states $\pi^+\pi^-$ and K^+K^- , respectively. The exponential factor $F_{KK} = e^{-\alpha q_k^2}$ is introduced above the $K\bar{K}$ threshold to reduce the ρ_{KK} factor as invariant mass increases, where q_k is the momentum of the kaon in the $K\bar{K}$ rest frame and $\alpha = 2.0 \pm 0.25 \text{ GeV}^{-2}$ [107,108]. The phase space factors $\rho_{\pi\pi}$ and ρ_{KK} read as [87,108,110]

$$\begin{aligned} \rho_{\pi\pi} &= \frac{2}{3} \sqrt{1 - \frac{4m_{\pi^\pm}^2}{\omega^2}} + \frac{1}{3} \sqrt{1 - \frac{4m_{\pi^0}^2}{\omega^2}}, \\ \rho_{KK} &= \frac{1}{2} \sqrt{1 - \frac{4m_{K^\pm}^2}{\omega^2}} + \frac{1}{2} \sqrt{1 - \frac{4m_{K^0}^2}{\omega^2}}. \end{aligned} \quad (30)$$

C. Helicity amplitudes

The differential branching fraction for $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ in the $B_{(s)}^0$ meson rest frame is expressed as

$$\frac{d^5\mathcal{B}}{d\Omega} = \frac{\tau_B k(\omega_1)k(\omega_2)k(\omega_1, \omega_2)}{16(2\pi)^6 m_B^2} |A|^2, \quad (31)$$

where $d\Omega$ with $\Omega \equiv \{\theta_1, \theta_2, \varphi, \omega_1, \omega_2\}$ stands for the five-dimensional measure spanned by the three helicity angles and the two invariant masses, and

$$k(\omega_1, \omega_2) = \frac{\sqrt{[m_B^2 - (\omega_1 + \omega_2)^2][m_B^2 - (\omega_1 - \omega_2)^2]}}{2m_B} \quad (32)$$

is the momentum of the K^+K^- pair in the $B_{(s)}$ meson rest frame.

The four-body phase space has been derived in the analysis of the $K \rightarrow \pi\pi l\nu$ decay [111], the semileptonic $\bar{B} \rightarrow D(D^*)\pi l\nu$ decays [112], the semileptonic baryonic decays [113,114], and the four-body baryonic decays [115]. One can confirm that Eq. (31) is equivalent to those in Refs. [113,115] by appropriate variable changes. Replacing the helicity angle θ by the meson momentum fraction ζ via Eq. (8), Eq. (31) is turned into

$$\frac{d^5\mathcal{B}}{d\zeta_1 d\zeta_2 d\omega_1 d\omega_2 d\varphi} = \frac{\tau_B k(\omega_1)k(\omega_2)k(\omega_1, \omega_2)}{4(2\pi)^6 m_B^2 \sqrt{1+4\alpha_1} \sqrt{1+4\alpha_2}} |A|^2. \quad (33)$$

The $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decay comprises a mixture of CP eigenstates and can be disentangled by means of an angular analysis in the helicity basis. In this basis, the decay is described by three angles θ_1 , θ_2 , and φ , depicted in Fig. 1, where $\theta_{1,2}$ is the angle between the K^- direction in the ϕ rest frame and the ϕ direction in the B_s^0 rest frame, and φ is the angle between the two ϕ meson decay planes.

Because of the proximity of the ϕ resonance to the scalar $f_0(980)$ resonance, there are irreducible scalar resonant contributions to four-body $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays. Thereby, a K^+K^- pair can be produced in the S - or P -wave configuration in the selected invariant mass regions. One decomposes the decay amplitudes into six helicity components: $h = VV$ (3), VS (2), and SS , each with a corresponding amplitude A_h , where V denotes a vector meson and S denotes a scalar meson. The first three, commonly referred to as the P -wave amplitudes, are associated with the final states, where both K^+K^- pairs come from intermediate vector mesons. In the transversity basis, a P -wave decay amplitude can be decomposed into three components: A_0 , for which the polarizations of the final-state vector mesons are longitudinal to their momenta; and $A_\parallel$ ($A_\perp$), for which the polarizations are transverse to the momenta and parallel (perpendicular) to each other. As the S -wave K^+K^- pair can arise from R_1 or R_2 labeled in Fig. 2(a), the corresponding single S -wave amplitude is denoted by A_{VS} . The double S -wave amplitude A_{SS} is associated with the final state, where both two-meson pairs are generated in the S wave. A randomized choice is made for which the ϕ meson is used to determine θ_1 and which is used to determine θ_2 . Thus, the total decay amplitude A is a coherent sum of the P -, S -, and double S -wave components. Specifically, these helicity amplitudes for the $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays denote

$$\begin{aligned}
A_{VV}: B_{(s)}^0 &\rightarrow \phi(\rightarrow K^+K^-)\phi(\rightarrow K^+K^-), \\
A_{VS}: B_{(s)}^0 &\rightarrow \phi(\rightarrow K^+K^-)f_0(980)(\rightarrow K^+K^-), \\
A_{SS}: B_{(s)}^0 &\rightarrow f_0(980)(\rightarrow K^+K^-)f_0(980)(\rightarrow K^+K^-). \quad (34)
\end{aligned}$$

By including the $\zeta_{1,2}$ dependencies instead of $\theta_{1,2}$ and azimuth-angle dependencies relying on Eq. (8), the total decay amplitude in Eq. (33) can be written as

$$\begin{aligned}
A = A_0 \frac{2\zeta_1 - 1}{\sqrt{1 + 4\alpha_1}} \frac{2\zeta_2 - 1}{\sqrt{1 + 4\alpha_2}} + A_{\parallel} 2\sqrt{2} \sqrt{\frac{\zeta_1(1 - \zeta_1) + \alpha_1}{1 + 4\alpha_1}} \sqrt{\frac{\zeta_2(1 - \zeta_2) + \alpha_2}{1 + 4\alpha_2}} \cos \varphi \\
+ iA_{\perp} 2\sqrt{2} \sqrt{\frac{\zeta_1(1 - \zeta_1) + \alpha_1}{1 + 4\alpha_1}} \sqrt{\frac{\zeta_2(1 - \zeta_2) + \alpha_2}{1 + 4\alpha_2}} \sin \varphi + A_{VS} \left(\frac{2\zeta_1 - 1}{\sqrt{1 + 4\alpha_1}} + \frac{2\zeta_2 - 1}{\sqrt{1 + 4\alpha_2}} \right) + A_{SS}. \quad (35)
\end{aligned}$$

On the basis of Eq. (33), we can obtain the branching ratio form,

$$\mathcal{B}_h = \frac{\tau_B}{4(2\pi)^6 m_B^2} \frac{2\pi}{9} C_h \int d\omega_1 d\omega_2 k(\omega_1) k(\omega_2) k(\omega_1, \omega_2) |A_h|^2, \quad (36)$$

where the invariant masses $\omega_{1,2}$ are integrated over the chosen K^+K^- mass window. The coefficients C_h are the results of the integrations over $\zeta_1, \zeta_2, \varphi$ in terms of Eq. (36) and listed as follows:

$$C_h = \begin{cases} (1 + 4\alpha_1)(1 + 4\alpha_2), & h = 0, \parallel, \perp, \\ 3(1 + 4\alpha_1), & h = VS, \\ 9, & h = SS. \end{cases} \quad (37)$$

The CP -averaged branching ratio and the direct CP asymmetry in each component are defined as

$$\mathcal{B}_h^{\text{avg}} = \frac{1}{2}(\mathcal{B}_h + \bar{\mathcal{B}}_h), \quad \mathcal{A}_h^{\text{dir}} = \frac{\bar{\mathcal{B}}_h - \mathcal{B}_h}{\bar{\mathcal{B}}_h + \mathcal{B}_h}, \quad (38)$$

respectively, where $\bar{\mathcal{B}}_h$ is the branching ratio of the corresponding CP -conjugate channel. The sum of the six components yields the total branching ratio and the overall direct- CP asymmetry,

$$\mathcal{B}_{\text{total}} = \sum_h \mathcal{B}_h, \quad \mathcal{A}_{CP}^{\text{dir}} = \frac{\sum_h \bar{\mathcal{B}}_h - \sum_h \mathcal{B}_h}{\sum_h \bar{\mathcal{B}}_h + \sum_h \mathcal{B}_h}, \quad (39)$$

respectively.

For the VV decays, the polarization fractions f_{λ} with $\lambda = 0, \parallel$, and $\perp$ and two relative phases $\phi_{\parallel}, \phi_{\perp}$ are described as

$$f_{\lambda} = \frac{|A_{\lambda}|^2}{|A_0|^2 + |A_{\parallel}|^2 + |A_{\perp}|^2}, \quad \phi_{\parallel, \perp} = \arg \frac{A_{\parallel, \perp}}{A_0}, \quad (40)$$

with the normalization relation $f_0 + f_{\parallel} + f_{\perp} = 1$.

D. Triple product asymmetries

Consider a four-body decay $B \rightarrow R_1(\rightarrow P_1 P_2) R_2(\rightarrow Q_1 Q_2)$, in which one measures the four particles' momenta in the B rest frame. We define $\hat{n}_{R_i}$ ($i = 1, 2$) as a unit vector perpendicular to the R_i decay plane and $\hat{z}_{R_1}$ as a unit vector in the direction of R_1 in the B rest frame. Thus we have

$$\hat{n}_{R_1} \cdot \hat{n}_{R_2} = \cos \varphi, \quad \hat{n}_{R_1} \times \hat{n}_{R_2} = \sin \varphi \hat{z}, \quad (41)$$

implying a T -odd scalar triple product

$$(\hat{n}_{R_1} \times \hat{n}_{R_2}) \cdot \hat{z} = \sin \varphi, \quad 2(\hat{n}_{R_1} \cdot \hat{n}_{R_2})(\hat{n}_{R_1} \times \hat{n}_{R_2}) \cdot \hat{z} = \sin 2\varphi. \quad (42)$$

One can define a TPA as an asymmetry between the number of decays involving positive and negative values of $\sin \varphi$ or $\sin 2\varphi$,

$$\mathcal{A}_T^1 = \frac{\Gamma(\cos \theta_1 \cos \theta_2 \sin \varphi > 0) - \Gamma(\cos \theta_1 \cos \theta_2 \sin \varphi < 0)}{\Gamma(\cos \theta_1 \cos \theta_2 \sin \varphi > 0) + \Gamma(\cos \theta_1 \cos \theta_2 \sin \varphi < 0)}, \quad (43)$$

$$\mathcal{A}_T^2 = \frac{\Gamma(\sin 2\varphi > 0) - \Gamma(\sin 2\varphi < 0)}{\Gamma(\sin 2\varphi > 0) + \Gamma(\sin 2\varphi < 0)}. \quad (44)$$

It has been found that TPAs originate from the interference of the CP -odd amplitudes $A_{\perp}$ with the other CP -even amplitudes A_0 and $A_{\parallel}$. According to Eq. (8), the TPAs associated with $A_{\perp}$ for the considered four-body decays are derived from the partially integrated differential decay rates as [4,23]

$$\begin{aligned}\mathcal{A}_T^1 &= \frac{\Gamma((2\zeta_1 - 1)(2\zeta_2 - 1) \sin \varphi > 0) - \Gamma((2\zeta_1 - 1)(2\zeta_2 - 1) \sin \varphi < 0)}{\Gamma((2\zeta_1 - 1)(2\zeta_2 - 1) \sin \varphi > 0) + \Gamma((2\zeta_1 - 1)(2\zeta_2 - 1) \sin \varphi < 0)} \\ &= -\frac{2\sqrt{2}}{\pi\mathcal{D}} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_{0\parallel}^*],\end{aligned}\quad (45)$$

$$\begin{aligned}\mathcal{A}_T^2 &= \frac{\Gamma(\sin 2\varphi > 0) - \Gamma(\sin 2\varphi < 0)}{\Gamma(\sin 2\varphi > 0) + \Gamma(\sin 2\varphi < 0)} \\ &= -\frac{4}{\pi\mathcal{D}} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_{0\parallel}^*],\end{aligned}\quad (46)$$

with the denominator

$$\mathcal{D} = \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) (|A_0|^2 + |A_\parallel|^2 + |A_\perp|^2). \quad (47)$$

The above TPAs contain the integrands $\text{Im}(A_\perp A_{0\parallel}^*) = |A_\perp||A_{0\parallel}^*| \sin(\Delta\phi + \Delta\delta)$, where $\Delta\phi$ and $\Delta\delta$ denote the

weak and strong phase differences between the amplitudes $A_\perp$ and $A_{0\parallel}$, respectively. As already noted, $\text{Im}(A_\perp A_{0\parallel}^*)$ can be nonzero even if the weak phases vanish. Thus, it is not quite accurate to identify a nonzero TPA as a signal of CP violation. To obtain a true CP violation signal, one has to compare the TPAs in the B and $\bar{B}$ meson decays. The helicity amplitude for the CP -conjugated process can be inferred from Eq. (35) through $A_0 \rightarrow \bar{A}_0$, $A_\parallel \rightarrow \bar{A}_\parallel$, and $A_\perp \rightarrow -\bar{A}_\perp$, in which the $\bar{A}_\lambda$ are obtained from the A_λ by changing the sign of the weak phases. Thus, the TPAs $\bar{\mathcal{A}}_T^i$ for the charge-conjugate process are defined similarly, but with a multiplicative minus sign.

One therefore constructs the “true” and “fake” asymmetries by combining $\mathcal{A}_T^i$ and $\bar{\mathcal{A}}_T^i$ [4],

$$\begin{aligned}\mathcal{A}_{T\text{-true}}^1 &\equiv \frac{[\Gamma(T > 0) + \bar{\Gamma}(\bar{T} > 0)] - [\Gamma(T < 0) + \bar{\Gamma}(\bar{T} < 0)]}{[\Gamma(T > 0) + \bar{\Gamma}(\bar{T} > 0)] + [\Gamma(T < 0) + \bar{\Gamma}(\bar{T} < 0)]} \\ &= -\frac{2\sqrt{2}}{\pi(\mathcal{D} + \bar{\mathcal{D}})} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_0^* - \bar{A}_\perp \bar{A}_0^*],\end{aligned}\quad (48)$$

$$\begin{aligned}\mathcal{A}_{T\text{-true}}^2 &\equiv \frac{[\Gamma(\sin 2\varphi > 0) + \bar{\Gamma}(\sin 2\bar{\varphi} > 0)] - [\Gamma(\sin 2\varphi < 0) + \bar{\Gamma}(\sin 2\bar{\varphi} < 0)]}{[\Gamma(\sin 2\varphi > 0) + \bar{\Gamma}(\sin 2\bar{\varphi} > 0)] + [\Gamma(\sin 2\varphi < 0) + \bar{\Gamma}(\sin 2\bar{\varphi} < 0)]} \\ &= -\frac{4}{\pi(\mathcal{D} + \bar{\mathcal{D}})} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_{0\parallel}^* - \bar{A}_\perp \bar{A}_{0\parallel}^*],\end{aligned}\quad (49)$$

$$\begin{aligned}\mathcal{A}_{T\text{-fake}}^1 &\equiv \frac{[\Gamma(T > 0) - \bar{\Gamma}(\bar{T} > 0)] - [\Gamma(T < 0) - \bar{\Gamma}(\bar{T} < 0)]}{[\Gamma(T > 0) + \bar{\Gamma}(\bar{T} > 0)] + [\Gamma(T < 0) + \bar{\Gamma}(\bar{T} < 0)]} \\ &= -\frac{2\sqrt{2}}{\pi(\mathcal{D} + \bar{\mathcal{D}})} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_0^* + \bar{A}_\perp \bar{A}_0^*],\end{aligned}\quad (50)$$

$$\begin{aligned}\mathcal{A}_{T\text{-fake}}^2 &\equiv \frac{[\Gamma(\sin 2\varphi > 0) - \bar{\Gamma}(\sin 2\bar{\varphi} > 0)] - [\Gamma(\sin 2\varphi < 0) - \bar{\Gamma}(\sin 2\bar{\varphi} < 0)]}{[\Gamma(\sin 2\varphi > 0) + \bar{\Gamma}(\sin 2\bar{\varphi} > 0)] + [\Gamma(\sin 2\varphi < 0) + \bar{\Gamma}(\sin 2\bar{\varphi} < 0)]} \\ &= -\frac{4}{\pi(\mathcal{D} + \bar{\mathcal{D}})} \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) \text{Im}[A_\perp A_{0\parallel}^* + \bar{A}_\perp \bar{A}_{0\parallel}^*],\end{aligned}\quad (51)$$

with $\bar{\Gamma}$ being the decay rate of the CP -conjugate process, $T = (2\zeta_1 - 1)(2\zeta_2 - 1) \sin \varphi$ and $\bar{T} = (2\zeta_1 - 1)(2\zeta_2 - 1) \sin \bar{\varphi}$, and the denominator is

$$\bar{\mathcal{D}} = \int d\omega_1 d\omega_2 k(\omega_1)k(\omega_2)k(\omega_1, \omega_2) (|\bar{A}_0|^2 + |\bar{A}_\parallel|^2 + |\bar{A}_\perp|^2) \quad (52)$$

for the CP -conjugate decay.

It is shown that the terms $\text{Im}[A_{\perp}A_{0,\parallel}^* - \bar{A}_{\perp}\bar{A}_{0,\parallel}^*]$ in Eqs. (48) and (49) are proportional to $\sin \Delta\phi \cos \Delta\delta$, which are nonzero only in the presence of the weak phase difference. Then TPAs provide an alternative measure of CP violation. Furthermore, compared with direct CP asymmetries, $\mathcal{A}_{T\text{-true}}^i$ does not suffer the suppression from the strong phase difference and is maximal when the strong phase difference vanishes. For the special case of the involved neutral intermediate states $B_{(s)}^0 \rightarrow \phi\phi$ modes, in which each helicity amplitude involves the same single weak phase in the SM. This results in $\mathcal{A}_{T\text{-true}}^i = 0$ due to the vanishing weak phase difference. The true TPAs for the neutral modes are thus expected to be zero in the SM. If such asymmetries are observed experimentally, it is probably a signal of new physics. While for the term $\text{Im}[A_{\perp}A_{0,\parallel}^* + \bar{A}_{\perp}\bar{A}_{0,\parallel}^*] \propto \cos \Delta\phi \sin \Delta\delta$, the $\mathcal{A}_{T\text{-fake}}^i$ can be nonzero when the weak phase difference vanishes. Such a quantity is referred to as a fake asymmetry (CP conserving), which reflects the effect of strong phases [4,5], instead of CP violation.

III. NUMERICAL ANALYSIS

In this section, we calculate the branching ratios ($\mathcal{B}$), the polarization fractions f_{λ} , and relative phases $\phi_{\parallel,\perp}$ (rad), together with TPAs, respectively. The related input parameters for the numerical calculations are collected in Table I. The decay constants used are the values from Refs. [28,67], while the meson masses, Wolfenstein parameters, and the lifetimes are taken from the PDG review [106]. We neglect uncertainties on the constants since they are negligible with respect to other sources of uncertainties.

A. Global fit

According to Eqs. (20)–(25), the total amplitudes A related to both longitudinal (L) and transverse (N, T) components for the four-body decays $B_{(s)}^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ can be expanded in terms of the Gegenbauer moments from the two-meson DAs. As a result, we can decompose the squared amplitudes into the linear combinations of the Gegenbauer moments $a_{2\phi}^{0,T}$ and their products

$$|A^L|^2 = M_0^L + a_{2\phi}^0 M_1^L + (a_{2\phi}^0)^2 M_2^L + (a_{2\phi}^0)^3 M_3^L + (a_{2\phi}^0)^4 M_4^L, \quad (53)$$

$$|A^{(i)}|^2 = M_0^{(i)} + a_{2\phi}^T M_1^{(i)} + (a_{2\phi}^T)^2 M_2^{(i)} + (a_{2\phi}^T)^3 M_3^{(i)} + (a_{2\phi}^T)^4 M_4^{(i)}, i = N, T. \quad (54)$$

While for three-body decays $B_{(s)} \rightarrow (\pi, K)\phi \rightarrow (\pi, K)KK$, analogously, the squared amplitudes $|A|^2$ can be parametrized as follow:

$$|A|^2 = M_0 + a_{2\phi}^0 M_1 + (a_{2\phi}^0)^2 M_2. \quad (55)$$

We then compute the coefficients M , which involve only the Gegenbauer polynomials, to establish the database for our global fit.

Similar to the proposal in Refs. [67,99], we adopt the standard nonlinear least- χ^2 (lsq) method [116], in which the χ^2 function is defined for n pieces of experimental data $v_i \pm \delta v_i$ with the errors δv_i and the fitted corresponding theoretical values v_i^{th} as

$$\chi^2 = \sum_{i=1}^n \left(\frac{v_i - v_i^{\text{th}}}{\delta v_i} \right)^2. \quad (56)$$

In general, we should include the maximal amount of data in the fit in order to minimize statistical uncertainties. However, those measurements with significance lower than 3σ do not impose stringent constraints and need not be taken into account in principle. Therefore, the Gegenbauer moments $a_{2\phi}^{0(T)}$ for the twist-2 KK DAs $\phi_P^{0(T)}$ can be obtained by fitting the formulas in Eqs. (53)–(55) with the Gegenbauer-moment-independent database to the five pieces of $B^{0(+)} \rightarrow K^{0(+)}\phi \rightarrow K^{0(+)}KK$ and $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ data, including three branching ratios and two polarization fractions as summarized in Table II,

$$a_{2\phi}^0 = 0.40 \pm 0.06, \quad a_{2\phi}^T = 1.48 \pm 0.07, \quad (57)$$

whose errors mainly arise from experimental uncertainties. For comparison, the updated fitting results are also listed in Table II and match well with the data within errors.

Note that our $a_{2\phi}^0$, determined with $\chi^2/\text{d.o.f.} = 1.3$, is distinct from the value $a_{2\phi}^0 = -0.31 \pm 0.19$ in Ref. [67], which can be understood from the following clarification. The additional new four-body decay $B_s^0 \rightarrow \phi\phi \rightarrow (KK)(KK)$ included in the present work is dominated by $B_s^0 \rightarrow (\phi \rightarrow)KK$ transition form factors, the $\mathcal{B}$ of which

TABLE I. The decay constants are taken from Refs. [28,67]. Other parameters are from PDG 2020 [106].

Mass (GeV)	$m_{B_s} = 5.37$	$m_B = 5.28$	$m_{K^\pm} = 0.494$	
Wolfenstein parameters	$\lambda = 0.22650$	$A = 0.790$	$\bar{\rho} = 0.141$	$\bar{\eta} = 0.357$
Decay constants (GeV)	$f_{B_s} = 0.23$	$f_B = 0.21$	$f_{\phi(1020)} = 0.215$	$f_{\phi(1020)}^T = 0.186$
Lifetime (ps)	$\tau_{B_s} = 1.51$	$\tau_{B^0} = 1.52$		

TABLE II. Experimental data for branching ratios and polarization fractions [106], and the theoretical results derived from the fitted Gegenbauer moments in Eq. (57). For simplicity, only the theoretical errors from the Gegenbauer moments are presented.

Channel	Data			Fit		
	$\mathcal{B}(10^{-6})$			$\mathcal{B}(10^{-6})$		
$B^+ \rightarrow K^+ \phi \rightarrow K^+ KK$	8.8 ± 0.7			$8.8^{+0.3}_{-0.2}$		
$B^0 \rightarrow K^0 \phi \rightarrow K^0 KK$	7.3 ± 0.7			$8.5^{+0.2}_{-0.3}$		
Channel	Data			Fit		
	$\mathcal{B}(10^{-6})$	$f_0(\%)$	$f_\perp(\%)$	$\mathcal{B}(10^{-6})$	$f_0(\%)$	$f_\perp(\%)$
$B_s^0 \rightarrow \phi\phi \rightarrow (KK)(KK)$	18.7 ± 1.5	37.8 ± 1.3	29.2 ± 0.9	$18.1^{+0.7}_{-0.8}$	$38.2^{+2.4}_{-2.5}$	$30.9^{+1.2}_{-0.9}$

could be more sensitive to the Gegenbauer moment $a_{2\phi}^0$. Hence, the measured $B_s^0 \rightarrow \phi\phi \rightarrow (KK)(KK)$ branching ratio can give an effective constraint on the global fit of the KK two-meson DAs, and the corresponding fitting result of the $a_{2\phi}^0$ could be changed a lot: from -0.31 ± 0.19 to 0.40 ± 0.06 .

One can also observe that the $a_{2\phi}^T$ fitted in this work is slightly larger than unity as shown in Eq. (57), which is not favored in view of the convergence of the Gegenbauer expansion. We then added one more Gegenbauer moment $a_{4\phi}^T$ in the twist-2 transverse component ϕ_P^T . Naturally, Eq. (54) should be replaced with the following form:

$$\begin{aligned}
|A^{(i)}|^2 = & M_0^{(i)} + a_{2\phi}^T M_1^{(i)} + (a_{2\phi}^T)^2 M_2^{(i)} + (a_{2\phi}^T)^3 M_3^{(i)} + (a_{2\phi}^T)^4 M_4^{(i)} + (a_{4\phi}^T) M_5^{(i)} + (a_{4\phi}^T)^2 M_6^{(i)} \\
& + (a_{4\phi}^T)^3 M_7^{(i)} + (a_{4\phi}^T)^4 M_8^{(i)} + (a_{2\phi}^T a_{4\phi}^T) M_9^{(i)} + (a_{2\phi}^T)^2 (a_{4\phi}^T) M_{10}^{(i)} + (a_{2\phi}^T)^3 (a_{4\phi}^T) M_{11}^{(i)} \\
& + (a_{2\phi}^T) (a_{4\phi}^T)^2 M_{12}^{(i)} + (a_{2\phi}^T)^2 (a_{4\phi}^T)^2 M_{13}^{(i)} + (a_{2\phi}^T) (a_{4\phi}^T)^3 M_{14}^{(i)}, \quad i = N, T,
\end{aligned} \tag{58}$$

and a fit with $\chi^2/\text{d.o.f.} = 1.3$ is attained,

$$a_{2\phi}^0 = 0.40 \pm 0.06, \quad a_{2\phi}^T = 0.85 \pm 0.32, \quad a_{4\phi}^T = 0.77 \pm 0.39. \tag{59}$$

The outcomes of both $a_{2\phi}^T$ and $a_{4\phi}^T$ in Eq. (59) are all smaller than unit, implying that the contributions from the higher-order Gegenbauer moment is significant. In principle, we should introduce the same number of Gegenbauer moments for the two twist-2 DAs ϕ_P^0 and ϕ_P^T . However, it is not practical to include many parameters in the fit because of the limited amount of experimental data at present. Consequently, we will adopt the $a_{2\phi}^{0,T}$ as presented in Eq. (57) in this work. Anyway, the above results show that $a_{2\phi}^T$ can be reduced efficiently by including the higher moment $a_{4\phi}^T$ in the fit when more experimental data with improved precision are available in the future.

B. S-wave contributions

The PQCD predictions for the branching ratios of various components and their sum in the $B_{(s)}^0 \rightarrow (K^+ K^-)(K^+ K^-)$ decays are summarized in Table III, in which the theoretical uncertainties are estimated from three different sources. The first error is due to the shape

parameters ω_B in the $B_{(s)}$ meson DAs with 10% variation. The second one comes from the Gegenbauer moments in various twist DAs of the KK pair with different intermediate resonances. The last one is caused by the variation of the hard scale t from $0.75t$ to $1.25t$ (without changing $1/b_i$) and the QCD scale $\Lambda_{\text{QCD}} = 0.25 \pm 0.05$ GeV, which characterizes the effect of the next-to-leading-order QCD contributions. The three uncertainties are comparable, and their combined impacts could exceed 50%, implying that the nonperturbative parameters in the DAs of the initial and final states need to be constrained more precisely, and the higher-order correction to four-body B meson decays is critical. It should be stressed that these considered modes are induced only by penguin operators in the PQCD approach at leading order as can be seen easily from Appendix A; their direct CP violations are naturally zero without the interference between the tree and penguin amplitudes.

In contrast to the vector mesons, the identification of scalar mesons is a long-standing puzzle, and the underlying structure of scalar mesons is not theoretically well established (for a review, see Ref. [106]). Based on the assumption that $f_0(980)$ is a pure $s\bar{s}$ state, different kinds of theoretical approaches have been applied to study the $B_{(s)}$ meson decays involving $f_0(980)$ in the final states, for instance: (a) the charmonium decay $B_s \rightarrow J/\psi f_0(980)$ was

TABLE III. PQCD predictions for the branching ratios of various components and their sum in the $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays. The theoretical uncertainties are attributed to the variations of the shape parameter $\omega_{B_{(s)}}$ in the $B_{(s)}$ meson DA, of the Gegenbauer moments in various twist DAs of the KK pair, and of the hard scale t and the QCD scale Λ_{QCD} .

Components	$B_s^0 \rightarrow (K^+K^-)(K^+K^-)$	$B^0 \rightarrow (K^+K^-)(K^+K^-)$
B_0	$(1.73^{+0.62+0.13+0.77}_{-0.43-0.13-0.65}) \times 10^{-6}$	$(3.98^{+0.06+0.07+0.06}_{-0.05-0.06-0.07}) \times 10^{-9}$
$B_{\parallel}$	$(1.40^{+0.10+0.11+0.58}_{-0.09-0.10-0.59}) \times 10^{-6}$	$(4.48^{+0.08+0.05+0.08}_{-0.12-0.04-0.05}) \times 10^{-11}$
$B_{\perp}$	$(1.40^{+0.13+0.10+0.58}_{-0.11-0.10-0.62}) \times 10^{-6}$	$(1.01^{+0.08+0.13+0.37}_{-0.02-0.24-0.47}) \times 10^{-12}$
B_{VS}	$(3.66^{+1.80+0.56+1.36}_{-1.02-0.52-0.98}) \times 10^{-8}$	$(5.50^{+0.60+1.70+1.30}_{-0.20-1.46-1.16}) \times 10^{-11}$
B_{SS}	$(4.38^{+2.18+2.05+2.60}_{-1.35-1.40-2.00}) \times 10^{-9}$	$(1.17^{+0.22+0.96+0.14}_{-0.20-0.59-0.20}) \times 10^{-11}$
B_{total}	$(4.57^{+0.86+0.35+1.95}_{-0.64-0.34-1.87}) \times 10^{-6}$	$(4.09^{+0.07+0.10+0.08}_{-0.06-0.08-0.08}) \times 10^{-9}$

analyzed in the light-cone QCD sum rule and factorization assumption [117], as well as the generalized factorization and $SU(3)$ flavor symmetry [118]; (b) in Ref. [119], the $\bar{B}_s \rightarrow f_0(980)$ transition form factor was calculated from the light-cone sum rules with B -meson DAs. As a first approximation, the scalar meson $f_0(980)$ is taken into account in the $s\bar{s}$ density operator in the present work. The S -wave timelike form factor $F_S(\omega^2)$ adopted to parametrize the S -wave two-kaon DAs has been determined in Refs. [66,78,103].

By using the numerical results given in Table III, the S -wave fractions defined as

$$f_{\sigma} = \frac{\mathcal{B}_{\sigma}}{\mathcal{B}_{\text{total}}}, \quad f_{S\text{-wave}} = \sum_{\sigma} f_{\sigma}, \quad \sigma = VS, SS \quad (60)$$

are also calculated in our work, and the corresponding results are presented in Table IV. The total S -wave fraction of the two considered modes is estimated to be less than 2%, which is in good agreement with the experimental analysis that the contributions from the S -wave components are negligible [13].

In general, the branching ratios of two-body $B_{(s)}^0 \rightarrow R_1 R_2$ decays can be extracted from the corresponding four-body-decay modes in Table III under the narrow width approximation

$$\begin{aligned} \mathcal{B}(B_{(s)}^0 \rightarrow R_1 R_2 \rightarrow (K^+K^-)(K^+K^-)) \\ \approx \mathcal{B}(B_{(s)}^0 \rightarrow R_1 R_2) \times \mathcal{B}(R_1 \rightarrow K^+K^-) \times \mathcal{B}(R_2 \rightarrow K^+K^-). \end{aligned} \quad (61)$$

Before evaluating $\mathcal{B}(f_0(980) \rightarrow K^+K^-)$, we first define the ratio between $f_0(980) \rightarrow K^+K^-$ and $f_0(980) \rightarrow \pi^+\pi^-$:

$$R_{K/\pi} = \frac{\mathcal{B}(f_0(980) \rightarrow K^+K^-)}{\mathcal{B}(f_0(980) \rightarrow \pi^+\pi^-)}. \quad (62)$$

In recent years, *BABAR* Collaboration has measured the ratio of the partial decay width of $f_0(980) \rightarrow K^+K^-$ to $f_0(980) \rightarrow \pi^+\pi^-$ of $R_{K/\pi}^{\text{exp}} = 0.69 \pm 0.32$ using the $B \rightarrow KK^+K^-$ and $B \rightarrow K\pi^+\pi^-$ decays [120]. Meanwhile, BES also performed a partial wave analysis of $\chi_{c0} \rightarrow f_0(980)f_0(980) \rightarrow \pi^+\pi^-\pi^+\pi^-$ and $\chi_{c0} \rightarrow f_0(980)f_0(980) \rightarrow \pi^+\pi^-K^+K^-$ in the $\psi(2S) \rightarrow \gamma\chi_{c0}$ decay and extracted the ratio as $R_{K/\pi}^{\text{exp}} = 0.25^{+0.17}_{-0.11}$ [121,122]. Their average yielded $R_{K/\pi}^{\text{exp}} = 0.35^{+0.15}_{-0.14}$ [123]. Utilizing $\mathcal{B}(f_0(980) \rightarrow \pi^+\pi^-) = 0.50$, which is taken from [124] and in agreement with the value of $\mathcal{B}(f_0(980) \rightarrow \pi^+\pi^-) = 0.46$ reported by LHCb [110], one can obtain the branching fraction $\mathcal{B}(f_0(980) \rightarrow K^+K^-) = 0.175$.

Relying on the ratio $R_{K/\pi}$ and $\mathcal{B}(\phi \rightarrow K^+K^-) = 49.2\%$, the branching ratio of three-body decay $B_s^0 \rightarrow \phi(f_0(980) \rightarrow \pi^+\pi^-)$ is calculated as follows:

$$\begin{aligned} \mathcal{B}(B_s^0 \rightarrow \phi(f_0(980) \rightarrow \pi^+\pi^-)) \\ = \frac{\mathcal{B}(B_s^0 \rightarrow \phi f_0(980) \rightarrow (K^+K^-)(K^+K^-))}{\mathcal{B}(\phi \rightarrow K^+K^-)} \\ \cdot R_{\pi/K} = (1.06^{+0.67}_{-0.44}) \times 10^{-7}. \end{aligned} \quad (63)$$

On the experimental side, the LHCb Collaboration has reported the measurement $\mathcal{B}(B_s^0 \rightarrow \phi(f_0(980) \rightarrow \pi^+\pi^-)) =$

TABLE IV. PQCD predictions of the S -wave fractions in the $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays. The sources of the theoretical errors are the same as in Table III.

Modes	$f_{VS}(\%)$	$f_{SS}(\%)$	$f_{S\text{-wave}}(\%)$
$B_s^0 \rightarrow (K^+K^-)(K^+K^-)$	$0.801^{+0.205+0.057+0.192}_{-0.129-0.059-0.031}$	$0.096^{+0.025+0.035+0.011}_{-0.019-0.026-0.008}$	$0.897^{+0.230+0.092+0.203}_{-0.148-0.085-0.039}$
$B^0 \rightarrow (K^+K^-)(K^+K^-)$	$1.345^{+0.121+0.373+0.286}_{-0.030-0.338-0.263}$	$0.286^{+0.048+0.222+0.028}_{-0.045-0.141-0.044}$	$1.631^{+0.169+0.595+0.314}_{-0.075-0.479-0.307}$

$(1.12 \pm 0.21) \times 10^{-6}$, where the region of the $\pi\pi$ invariant mass is $0.4 < \omega_{\pi\pi} < 1.6$ GeV [125]. It is seen that the $\mathcal{B} = (1.06^{+0.67}_{-0.44}) \times 10^{-7}$ in Eq. (63) is almost 1 order of magnitude smaller than the experimental value $\mathcal{B} = (1.12 \pm 0.21) \times 10^{-6}$, as well as the previous three-body PQCD result $\mathcal{B} = (2.35^{+4.06}_{-1.13}) \times 10^{-6}$ [126]. We have found that the contribution of scalar resonance $f_0(980)$ relies on the final-state invariant mass range strongly, since it has a wide decay width. For example, we recalculate the branching ratio of the four-body decay $\mathcal{B}(B_s^0 \rightarrow \phi f_0(980) \rightarrow (K^+ K^-)(K^+ K^-)) = 8.43 \times 10^{-7}$ by enlarging the KK invariant mass range of the resonance $f_0(980)$ from $[m_\phi - 30 \text{ MeV}, m_\phi + 30 \text{ MeV}]$ to $[2m_K, m_B - m_\phi]$. According to Eq. (63), the relevant branching ratio of the three-body decay $B_s^0 \rightarrow \phi(f_0(980) \rightarrow \pi^+ \pi^-)$ is estimated to be 4.89×10^{-6} , which becomes comparable with the experimental data.

The double S -wave decays $B_{(s)}^0 \rightarrow f_0(980)f_0(980)$ have already been systematically studied in the two-body framework within the PQCD approach [127,128]. Taking the $B_s^0 \rightarrow f_0(980)f_0(980)$ decay as an example, we can roughly estimate $\mathcal{B}(B_s^0 \rightarrow f_0(980)f_0(980)) = (1.43^{+1.29}_{-0.91}) \times 10^{-7}$ from the four-body decay $B_s^0 \rightarrow f_0(980)f_0(980) \rightarrow (K^+ K^-)(K^+ K^-)$ in Table III on the basis of Eq. (61). It is worthwhile to note that the estimation $\mathcal{B} = (1.43^{+1.29}_{-0.91}) \times 10^{-7}$ is much smaller than the previous two PQCD results $\mathcal{B}(B_s^0 \rightarrow f_0(980)f_0(980)) = (2.66^{+1.08}_{-0.85}) \times 10^{-4}$ [127] and $\mathcal{B}(B_s^0 \rightarrow f_0(980)f_0(980)) = (5.31^{+1.74}_{-1.39}) \times 10^{-4}$ [128]. Strictly speaking, the narrow width approximation is actually not fully justified since such an approximation has its scope of application. As pointed out in Refs. [129,130], the narrow width approximation should be corrected by including finite-width effects for the broad scalar intermediate state. The two-body result extracted from the four-body branching ratio may suffer from a large uncertainty due to the finite-width effects of the scalar resonance. In addition,

as stated above, the S -wave contributions show a strong dependence on the range of the KK invariant mass. We hope that the future LHCb and Belle II experiments can perform a direct measurement on four-body decays $B_s^0 \rightarrow \phi f_0(980) \rightarrow (K^+ K^-)(K^+ K^-)$ and $B_s^0 \rightarrow f_0(980) f_0(980) \rightarrow (K^+ K^-)(K^+ K^-)$.

C. Branching ratios and polarization fractions of two-body $B_{(s)}^0 \rightarrow \phi\phi$ decays

With the narrow width approximation in Eq. (61), the branching ratios of two-body $B_{(s)}^0 \rightarrow \phi\phi$ decays are extracted in Table V. The polarization fractions of the two-body $B_{(s)}^0 \rightarrow \phi\phi$ decays together with two relative phases calculated in this work are also listed in Table V. For a comparison, we display the updated predictions in the QCDF [26] and the previous predictions in the PQCD approach [28,29], SCET [31], and FAT [32]. Experimental results for branching ratios, polarization fractions, and relative phases are taken from PDG 2020 [106].

It is obvious that most of the theoretical predictions of $B_s^0 \rightarrow \phi\phi$ decay are consistent well with experiments within errors. In Ref. [29], the authors kept the additional power corrections related to the ratio $r_V^2 = m_V^2/m_B^2$ (m_V and m_B denote the masses of the vector and B mesons, respectively). By including the r_V^2 term, their result $\mathcal{B}(B_s^0 \rightarrow \phi\phi) = (16.7^{+4.9}_{-3.8}) \times 10^{-6}$ is about twice smaller than $\mathcal{B}(B_s^0 \rightarrow \phi\phi) = (35.3^{+18.7}_{-12.3}) \times 10^{-6}$ [28] and closer to the experimental data $\mathcal{B}(B_s^0 \rightarrow \phi\phi) = (18.7 \pm 1.5) \times 10^{-6}$ [106]. As stated in Refs. [53,54], the branching ratios in the quasi-two-body mechanism show their dependence on the invariant masses of the final-state meson pairs. In this work, the factors $\eta_{1,2} = \omega_{1,2}^2/m_B^2$ are equal to $r_V^2 = m_V^2/m_B^2$ in Ref. [29] when $\omega_{1,2} = m_\phi$. In addition, the partonic kinematic variables have been refined to take into account finite masses of final-state mesons, which can suppress the branching ratio of the $B_s^0 \rightarrow \phi\phi$ decay effectively.

TABLE V. Branching ratios, polarization fractions, and relative phases for the two-body $B_{(s)}^0 \rightarrow \phi\phi$ decays. For comparison, we also list the results from PQCD [28,29], QCDF [26], SCET [31], and FAT [32]. The world averages of experimental data are taken from PDG 2020 [106]. The sources of the theoretical errors are the same as in Table III but added in quadrature.

Modes	$\mathcal{B}(10^{-6})$	f_0 [%]	$f_\perp$ [%]	$\phi_\parallel$ [rad]	$\phi_\perp$ [rad]
$B_s^0 \rightarrow \phi\phi$	$18.1^{+8.3}_{-6.1}$	$38.2^{+7.2}_{-7.6}$	$30.9^{+3.6}_{-2.7}$	$1.86^{+0.17}_{-0.24}$	1.85 ± 0.18
PQCD-I [28]	$35.3^{+18.7}_{-12.3}$	$61.9^{+4.4}_{-4.5}$	$17.4^{+1.8}_{-1.9}$	$1.3^{+0.2}_{-0.1}$	$1.3^{+0.2}_{-0.1}$
PQCD-II [29]	$16.7^{+4.9}_{-3.8}$	$34.7^{+8.9}_{-7.1}$	$31.6^{+3.5}_{-4.4}$	2.01 ± 0.23	$2.00^{+0.24}_{-0.21}$
QCDF [26]	$16.7^{+11.6}_{-9.1}$	36^{+23}_{-18}	...	...	...
SCET [31]	19.0 ± 6.5	51.0 ± 16.4	22.2 ± 9.9	2.41 ± 0.62	2.54 ± 0.62
FAT [32]	26.4 ± 7.6	39.7 ± 16.0	31.2 ± 8.9	2.53 ± 0.28	2.56 ± 0.27
Data	18.7 ± 1.5	37.8 ± 1.3	29.2 ± 0.9	2.56 ± 0.06	2.82 ± 0.19
$B^0 \rightarrow \phi\phi$	$0.016^{+0.005}_{-0.004}$	$98.9^{+0.1}_{-0.7}$	$0.02^{+0.01}_{-0.00}$	$2.38^{+0.21}_{-0.10}$	$4.39^{+0.21}_{-0.27}$
PQCD [29]	$0.012^{+0.006}_{-0.005}$	97 ± 1	0.05 ± 0.02	$3.26^{+0.20}_{-0.14}$	3.50 ± 0.17
Data	< 0.027	...	...	...	...

Therefore, our result agrees well with that from the updated PQCD [29].

The rare decay $B^0 \rightarrow \phi\phi$ can occur only via penguin annihilation topology in the SM. The predicted branching ratio is very small at (10^{-8}), which makes it sensitive to any new physics contributions. The current experiment gives the upper limit: $\mathcal{B}(B^0 \rightarrow \phi\phi) < 2.7 \times 10^{-8}$ at 90% C.L. [13], so the more accurate experimental results are needed to test the theory. It is observed that the branching fraction of the $B^0 \rightarrow \phi\phi$ decay is much smaller than that of the $B_s^0 \rightarrow \phi\phi$ decay by almost 3 orders. There are two main reasons: the one is that the $B^0 \rightarrow \phi\phi$ governed by the $b \rightarrow d$ transition is highly suppressed by the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{td}/V_{ts}|^2 \sim 0.05$; the other is that $B^0 \rightarrow \phi\phi$ belongs to the pure annihilation decay. As is known, the contributions from the annihilation diagrams [Figs. 2(e)–2(h)] are always power suppressed compared to the factorizable emission diagrams [Figs. 2(a) and 2(b)] in the PQCD approach. What is more, there exists a big cancellation between the two factorizable annihilation diagrams Figs. 2(e) and 2(f) for the contributions from the $(V-A)(V-A)$ operators, especially when two final-state mesons are identical, such as $B^0 \rightarrow \phi\phi$.

For the charmless B decays, it is naively expected that the helicity amplitudes H_i (with helicity $i = 0, -, +$) satisfy the hierarchy pattern

$$H_0 : H_- : H_+ = 1 : \frac{\Lambda_{\text{QCD}}}{m_b} : \left(\frac{\Lambda_{\text{QCD}}}{m_b} \right)^2, \quad (64)$$

which are related to the spin amplitudes ($A_0, A_{\parallel}, A_{\perp}$) in Appendix A by

$$A_0 = H_0, \quad A_{\parallel} = \frac{H_+ + H_-}{\sqrt{2}}, \quad A_{\perp} = \frac{H_+ - H_-}{\sqrt{2}}. \quad (65)$$

The above hierarchy relation satisfies the expectation in the factorization assumption that the longitudinal polarization should dominate based on the quark helicity analysis [10,11]. In sharp contrast to these expectations, roughly equal longitudinal and transverse components are found in measurements of $B \rightarrow K^*\phi$ and $B \rightarrow K^*\rho$ decays [12,14–16]. Measurements of the low longitudinal polarization fraction in $B_s^0 \rightarrow \phi\phi$ by CDF [19] and LHCb [13,20,22] indicate a large transverse polarization. The longitudinal polarization fraction $f_0 = 0.381 \pm 0.007 \pm 0.012$ of the $B_s^0 \rightarrow \phi\phi$ decay has been reported by LHCb recently [13], where the first uncertainty is statistical and the second is systematic. This shows that the scaling behavior shown in Eq. (64) is violated. The interest in the polarization in penguin transition, such as $b \rightarrow s$ decay $B_s^0 \rightarrow \phi\phi$, is motivated by its potential sensitivity to physics beyond the SM.

As shown in Table V, we obtain the longitudinal polarization fraction $f_0 = (38.2_{-7.6}^{+7.2})\%$ of the $B_s^0 \rightarrow \phi\phi$

decay with the updated Gegenbauer moments of two-meson DAs, which is consistent with the previous PQCD calculation [29] and those from QCDF [26], SCET [31], and FAT [32] within uncertainties. In the PQCD approach, the large transverse polarization fraction can be interpreted on the basis of the chirally enhanced annihilation diagrams, especially the $(S-P)(S+P)$ penguin annihilation, introduced by the QCD penguin operator O_6 [131], which is originally introduced in Ref. [132]. A special feature of the $(S-P)(S+P)$ penguin annihilation operator is that the light quarks in the final states are not produced through chiral currents. So, there is no suppression to the transverse polarization caused by the helicity flip. Then the polarization fractions satisfy $f_0 \approx f_T$, with $f_T = f_{\parallel} + f_{\perp}$.

For the $B^0 \rightarrow \phi\phi$ decay, the longitudinal polarization contribution is dominant, which is consistent with the recent updated PQCD calculation [29] and also verified in Ref. [133]. As clarified before, the contributions from the factorizable annihilation diagrams [Figs. 2(e) and 2(f)] are canceled by each other because of the current conservation. Hence, the terms such as $F_{a\phi}^{LL,0(\parallel)}$ and $F_{a\phi}^{LR,0(\parallel)}$ in Eq. (A1) are exactly equal to zero, while the only left parts $F_{a\phi}^{LL,\perp}$ and $F_{a\phi}^{LR,\perp}$ for the factorizable emission diagrams are power suppressed. For the nonfactorizable annihilation diagrams [Figs. 2(g) and 2(h)], the longitudinal parts give the leading and dominant contributions, and other terms related to parallel ($M_{a\phi}^{LL,\parallel}, M_{a\phi}^{SP,\parallel}$) and perpendicular ($M_{a\phi}^{LL,\perp}, M_{a\phi}^{SP,\perp}$) components are all power suppressed. Thus, the total transverse contributions are actually negligible, leading to $f_0 \sim 1$ as shown in Table V.

The relative phases $\phi_{\parallel}$ and $\phi_{\perp}$ of the $B_{(s)}^0 \rightarrow \phi\phi$ decays are also studied in the present work as shown in Table V. In fact, two relative phases derived from the decay amplitudes $A_{0,\parallel,\perp}$ in Eq. (40) are dependent on the invariant mass $\omega_{1,2}$. We fix $\omega_{1,2} = m_{\phi}$ in our calculation for comparison with the two-body analysis. For the $B_s^0 \rightarrow \phi\phi$ decay, the differences between the two previous PQCD results [28,29] are mainly attributed to the treatment of the terms in the decay amplitude proportional to the ratio $r_V^2 = m_V^2/m_B^2$, which has been neglected in Ref. [28]. While in our calculations, the factors $\eta_{1,2} = \omega_{1,2}^2/m_B^2$ given in Eq. (3) become equal to $r_V^2 = m_V^2/m_B^2$ in Ref. [29] when $\omega_{1,2} = m_{\phi}$. Therefore, our new four-body computation $\phi_{\parallel} = (1.86_{-0.24}^{+0.17})$ rad and $\phi_{\perp} = (1.85 \pm 0.18)$ rad agree with the updated PQCD results $\phi_{\parallel} = (2.01 \pm 0.23)$ rad and $\phi_{\perp} = (2.00_{-0.21}^{+0.24})$ rad [29] within errors. It is obvious that our predictions of the relative phases are smaller than those of the SCET [31] and FAT [32] calculations as well as the experimental data [106]. As stressed above, we use the two decay constants $f_{\phi}^{(T)}$ of the intermediate particle to determine the ratio $F_P^{\perp}(\omega^2)/F_P^{\parallel}(\omega^2) \approx (f_{\phi}^T/f_{\phi})$. To be

TABLE VI. The dependencies of the $\phi_{\parallel}$ (rad) and $\phi_{\perp}$ (rad) on β (rad) at $\omega_{1,2} = m_{\phi}$ in the $B_s^0 \rightarrow \phi\phi$ decay, where $\beta \in [1.0, 1.6]$ rad.

β	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$\phi_{\parallel}$	2.07	2.18	2.34	2.55	2.90	3.59	4.36
$\phi_{\perp}$	2.03	2.13	2.26	2.46	2.82	3.49	4.35

honest, we have omitted the phase difference between the two form factors $F_P^{\perp}(\omega^2)$ and $F_P^{\parallel}(\omega^2)$ due to the limited studies on the form factor $F_P^{\perp}(\omega^2)$. We have found that the gap between our predictions and the measurements of two relative phases can be resolved effectively by introducing an additional phase β in the above approximate equation

$$\frac{F_P^{\perp}(\omega^2)}{F_P^{\parallel}(\omega^2)} \approx \frac{f_{\phi}^T}{f_{\phi}} e^{i\beta}. \quad (66)$$

In Table VI, we have examined the dependencies of two relative phases $\phi_{\parallel,\perp}$ on β ($\beta \in [1.0, 1.6]$ rad) and found that $\phi_{\parallel} = 2.55$ rad and $\phi_{\perp} = 2.46$ rad, obtained for $\beta = 1.3$ rad, are well matched to the data $\phi_{\parallel} = (2.56 \pm 0.06)$ rad and $\phi_{\perp} = (2.82 \pm 0.19)$ rad within errors. Nonetheless, it is not appropriate for us to include the parameter β in the present fit due to the limited data. For the pure annihilation decay $B^0 \rightarrow \phi\phi$, however, the new four-body predictions $\phi_{\perp} = (4.39^{+0.21}_{-0.27})$ rad and $\phi_{\parallel} = (2.38^{+0.21}_{-0.10})$ rad are quite different from those of two-body results $\phi_{\parallel} = (3.26^{+0.20}_{-0.14})$ rad and $\phi_{\perp} = (3.50 \pm 0.17)$ rad [29]. The main reason is that we have kept track of the additional higher power corrections related to the momenta fraction x_B , which has been ignored in Ref. [29]. We have reexamined the two phases $\phi_{\parallel,\perp}$ without the contributions from x_B : $\phi_{\parallel} = 2.67$ rad and $\phi_{\perp} = 2.75$ rad, which are similar to the two-body analysis. It should be stressed that the contributions from the annihilation diagrams

[Figs. 2(e)–2(h)] are of higher power themselves for a pure annihilation decay mode without chiral enhancement, such as $B^0 \rightarrow \phi\phi$. In that case, the terms proportional to x_B in the amplitudes are not negligible and should be reserved in the calculations. Anyway, all these theoretical predictions need to be further tested in the future when more data are available.

D. Triple product asymmetries in $B_{(s)}^0 \rightarrow (K^+ K^-)(K^+ K^-)$ decays

In the involved neutral intermediate states $B_s^0 \rightarrow \phi\phi$ and $B^0 \rightarrow \phi\phi$ modes, each helicity amplitude involves the same single weak phase in the SM. This results in $\mathcal{A}_{T\text{-true}}^i = 0$ due to the vanishing weak phase difference. The true TPAs for these neutral modes are thus predicted to be zero in the SM as shown in Table VII. If such asymmetries are observed experimentally, it is probably to signify the presence of new physics. On the experimental side, the measurements of TPAs for $B_s^0 \rightarrow \phi\phi \rightarrow (K^+ K^-)(K^+ K^-)$ have been reported by CDF [19] and LHCb Collaborations [20,22] and have shown no evidence of deviations from the SM. The most recent measurements of the true TPAs give [13]

$$\begin{aligned} \mathcal{A}_V &= -0.014 \pm 0.011(\text{stat}) \pm 0.004(\text{syst}), \\ \mathcal{A}_U &= -0.003 \pm 0.011(\text{stat}) \pm 0.004(\text{syst}), \end{aligned} \quad (67)$$

where the first uncertainty is statistical and the second systematic. No evidence for CP violation is found, which is consistent with SM predictions.

The predicted fake TPAs for the $B_{(s)}^0 \rightarrow (K^+ K^-)(K^+ K^-)$ decays are presented in Table VII. As fake TPAs are due to strong phases and require no CP violation, the large fake $\mathcal{A}_{T\text{-fake}}^{1,2}$ simply reflects the importance of the strong final-state phases. The magnitude of $\mathcal{A}_{T\text{-fake}}^1$ for the B_s^0 channel exceeds 10% and reaches 30.4%. The sizable magnitude is mainly enhanced by the strong phase difference between the longitudinal and the perpendicular polarization amplitudes, which is found in Table V. The smallness of $\mathcal{A}_{T\text{-fake}}^2$ is attributed to the suppression from the strong phase

TABLE VII. PQCD predictions for the TPAs (%) of the four-body $B_{(s)}^0 \rightarrow (K^+ K^-)(K^+ K^-)$ decays. The sources of theoretical errors are the same as in Table III but added in quadrature.

Channel	TPAs – 1			
	$\mathcal{A}_T^1$	$\bar{\mathcal{A}}_T^1$	$\mathcal{A}_{T\text{-true}}^1$	$\mathcal{A}_{T\text{-fake}}^1$
$B_s^0 \rightarrow \phi\phi \rightarrow (K^+ K^-)(K^+ K^-)$	$30.38^{+1.16}_{-2.39}$	$-30.38^{+2.39}_{-1.16}$	0	$30.38^{+1.16}_{-2.39}$
$B^0 \rightarrow \phi\phi \rightarrow (K^+ K^-)(K^+ K^-)$	$0.67^{+0.21}_{-0.14}$	$-0.67^{+0.14}_{-0.21}$	0	$0.67^{+0.21}_{-0.14}$
Channel	TPAs – 2			
	$\mathcal{A}_T^2$	$\bar{\mathcal{A}}_T^2$	$\mathcal{A}_{T\text{-true}}^2$	$\mathcal{A}_{T\text{-fake}}^2$
$B_s^0 \rightarrow \phi\phi \rightarrow (K^+ K^-)(K^+ K^-)$	$0.15^{+0.03}_{-0.10}$	$-0.15^{+0.10}_{-0.03}$	0	$0.15^{+0.03}_{-0.10}$
$B^0 \rightarrow \phi\phi \rightarrow (K^+ K^-)(K^+ K^-)$	$-0.11^{+0.02}_{-0.04}$	$0.11^{+0.04}_{-0.02}$	0	$-0.11^{+0.02}_{-0.04}$

difference between the perpendicular and parallel polarization amplitudes, which can be seen in Table V and has been verified by LHCb [13]. Hence, observations of $\mathcal{A}_{\text{T-fake}}^2$ with large values would signal new physics beyond the SM. As mentioned above, the hierarchy in Eq. (64) is numerically not respected by penguin-dominated decays. Thus, final states with large transverse amplitude fractions are favorable for the measurement of TPAs and can provide valuable complementary information on CP violation without requiring the generation of a sizable strong phase difference. Our predictions can be tested in the future.

IV. CONCLUSION

In this work, we have studied the related helicity amplitudes of four-body $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays based on the angular analysis, where the K^+K^- invariant-mass spectrum is dominated by the vector resonance ϕ . The scalar resonance $f_0(980)$ is also contributed in the K^+K^- invariant-mass range. The strong dynamics of the scalar or vector resonance decays into the meson pair is parametrized into the corresponding two-meson distribution amplitude, which has been established in three-body B meson decays and further improved by performing a global fit through combining the measured branching ratios in four-body decays.

The branching ratios of four-body $B_{(s)}^0 \rightarrow (K^+K^-)(K^+K^-)$ decays are presented with the updated P -wave two-kaon distribution amplitudes. We have extracted the two-body $B_{(s)}^0 \rightarrow \phi\phi$ branching ratios from the results for the corresponding four-body decays under the narrow-width approximation and shown the polarization fractions and relative phases of the decay channels. The obtained two-body branching ratios agree well with previous theoretical studies in the two-body framework within errors. The predicted hierarchy pattern for the longitudinal

polarization fractions in the $B_{(s)}^0 \rightarrow \phi\phi$ decays is in agreement with the data.

Since the triple product asymmetries are helpful to discover physics beyond the standard model, we perform an angular analysis and estimate the triple product asymmetries on four-body $B_{(s)}^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decays. The true TPAs of four-body $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decays are predicted to be zero due to the vanishing weak phase difference, which is consistent with the experiments. The prediction of fake TPA $\mathcal{A}_{\text{T-fake}}^1$ of $B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ reaches 30% in magnitude, which reflects the importance of the strong final-state phases and can be tested in the future. We also make predictions of TPAs for the $B^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decay and wait for the confrontation with future data.

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APPENDIX: DECAY AMPLITUDES

In this Appendix, we present the PQCD factorization formulas for the amplitudes of the considered four-body hadronic B meson decays:

- (i) $B \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)$ decay modes ($h = 0, \parallel, \perp$)

$$\begin{aligned} \sqrt{2}A_h(B^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)) = & -\frac{2G_F}{\sqrt{2}}V_{tb}^*V_{td} \left[\left(C_3 + \frac{C_4}{3} - \frac{C_9}{2} - \frac{C_{10}}{6} \right) F_{a\phi}^{LL,h} + \left(C_4 - \frac{C_{10}}{2} \right) M_{a\phi}^{LL,h} + \left(C_6 - \frac{C_8}{2} \right) M_{a\phi}^{SP,h} \right. \\ & \left. + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6} \right) F_{a\phi}^{LR,h} \right], \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} \sqrt{2}A_h(B_s^0 \rightarrow \phi\phi \rightarrow (K^+K^-)(K^+K^-)) = & -\frac{2G_F}{\sqrt{2}}V_{tb}^*V_{ts} \left[\left(C_5 - \frac{C_7}{2} \right) (M_{e\phi}^{LR,h} + M_{a\phi}^{LR,h}) + \frac{4}{3} \left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2} \right) (F_{e\phi}^{LL,h} + F_{a\phi}^{LL,h}) \right. \\ & + \left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2} \right) (M_{e\phi}^{LL,h} + M_{a\phi}^{LL,h}) + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6} \right) (F_{e\phi}^{LR,h} + F_{a\phi}^{LR,h}) \\ & \left. + \left(C_6 - \frac{C_8}{2} \right) (M_{e\phi}^{SP,h} + M_{a\phi}^{SP,h}) + \left(C_6 + \frac{C_5}{3} - \frac{C_8}{2} - \frac{C_7}{6} \right) F_{a\phi}^{SP,h} \right], \end{aligned} \quad (\text{A2})$$

(ii) $B \rightarrow f_0(980)\phi \rightarrow (K^+K^-)(K^+K^-)$ decay modes

$$A(B^0 \rightarrow f_0\phi \rightarrow (K^+K^-)(K^+K^-)) = -\frac{G_F}{\sqrt{2}}V_{tb}^*V_{td}\left[\left(C_4 - \frac{C_{10}}{2}\right)(M_{a\phi}^{LL} + M_{af_0}^{LL}) + \left(C_3 + \frac{C_4}{3} - \frac{C_9}{2} - \frac{C_{10}}{6}\right)(F_{a\phi}^{LL} + F_{af_0}^{LL}) + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6}\right)(F_{a\phi}^{LR} + F_{af_0}^{LR}) + \left(C_6 - \frac{C_8}{2}\right)(M_{a\phi}^{SP} + M_{af_0}^{SP})\right], \quad (A3)$$

$$\begin{aligned} A(B_s^0 \rightarrow f_0\phi \rightarrow (K^+K^-)(K^+K^-)) = & -\frac{2G_F}{\sqrt{2}}V_{tb}^*V_{ts}\left[\left(C_5 - \frac{C_7}{2}\right)(M_{e\phi}^{LR} + M_{a\phi}^{LR} + M_{ef_0}^{LR} + M_{af_0}^{LR}) \right. \\ & + \frac{4}{3}\left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2}\right)(F_{ef_0}^{LL} + F_{a\phi}^{LL} + F_{af_0}^{LL}) \\ & + \left(C_6 - \frac{C_8}{2}\right)(M_{ef_0}^{SP} + M_{af_0}^{SP} + M_{e\phi}^{SP} + M_{a\phi}^{SP}) \\ & + \left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2}\right)(M_{e\phi}^{LL} + M_{a\phi}^{LL} + M_{ef_0}^{LL} + M_{af_0}^{LL}) \\ & + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6}\right)(F_{ef_0}^{LR} + F_{a\phi}^{LR} + F_{af_0}^{LR}) \\ & \left. + \left(C_6 + \frac{C_5}{3} - \frac{C_8}{2} - \frac{C_7}{6}\right)(F_{e\phi}^{SP} + F_{a\phi}^{SP} + F_{af_0}^{SP})\right], \quad (A4) \end{aligned}$$

(iii) $B \rightarrow f_0(980)f_0(980) \rightarrow (K^+K^-)(K^+K^-)$ decay modes

$$\sqrt{2}A(B^0 \rightarrow f_0f_0 \rightarrow (K^+K^-)(K^+K^-)) = -\frac{2G_F}{\sqrt{2}}V_{tb}^*V_{td}\left[\left(C_3 + \frac{C_4}{3} - \frac{C_9}{2} - \frac{C_{10}}{6}\right)F_{af_0}^{LL} + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6}\right)F_{af_0}^{LR} + \left(C_4 - \frac{C_{10}}{2}\right)M_{af_0}^{LL} + \left(C_6 - \frac{C_8}{2}\right)M_{af_0}^{SP}\right], \quad (A5)$$

$$\begin{aligned} \sqrt{2}A(B_s^0 \rightarrow f_0f_0 \rightarrow (K^+K^-)(K^+K^-)) = & -\frac{2G_F}{\sqrt{2}}V_{tb}^*V_{ts}\left[\frac{4}{3}\left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2}\right)F_{af_0}^{LL} + \left(C_5 + \frac{C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6}\right)F_{af_0}^{LR} \right. \\ & + \left(C_6 - \frac{C_8}{2}\right)(M_{ef_0}^{SP} + M_{af_0}^{SP}) + \left(C_5 - \frac{C_7}{2}\right)(M_{ef_0}^{LR} + M_{af_0}^{LR}) \\ & + \left(C_6 + \frac{C_5}{3} - \frac{C_8}{2} - \frac{C_7}{6}\right)(F_{ef_0}^{SP} + F_{af_0}^{SP}) \\ & \left. + \left(C_3 + C_4 - \frac{C_9}{2} - \frac{C_{10}}{2}\right)(M_{ef_0}^{LL} + M_{af_0}^{LL})\right], \quad (A6) \end{aligned}$$

where $G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant and the V_{ij} 's are the Cabibbo-Kobayashi-Maskawa matrix elements. The superscripts LL , LR , and SP refer to the contributions from $(V-A) \otimes (V-A)$, $(V-A) \otimes (V+A)$, and $(S-P) \otimes (S+P)$ operators, respectively. The explicit formulas for the factorizable emission (annihilation) contributions $F_{e(a)}$ and the nonfactorizable emission (annihilation) contributions $M_{e(a)}$ from Fig. 2 can be obtained easily in Ref. [76].

- [1] G. Valencia, Angular correlations in the decay $B \rightarrow VV$ and CP violation, *Phys. Rev. D* **39**, 3339 (1989).
- [2] B. Kayser, Kinematically nontrivial CP violation in beauty decay, *Nucl. Phys. B, Proc. Suppl.* **13**, 487 (1990).
- [3] A. Datta and D. London, Triple-product correlations in $B \rightarrow V_1 V_2$ decays and new physics, *Int. J. Mod. Phys. A* **19**, 2505 (2004).
- [4] M. Gronau and J. L. Rosner, Triple-product asymmetries in K , $D_{(s)}$, and $B_{(s)}$ decays, *Phys. Rev. D* **84**, 096013 (2011).
- [5] A. Datta, M. Duraisamy, and D. London, Searching for new physics with B -decay fake triple products, *Phys. Lett. B* **701**, 357 (2011).
- [6] A. Datta, M. Duraisamy, and D. London, New physics in $\bar{b} \rightarrow \bar{s}$ transitions and the $B_{d,s}^0 \rightarrow V_1 V_2$ angular analysis, *Phys. Rev. D* **86**, 076011 (2012).
- [7] B. Bhattacharya, A. Datta, M. Duraisamy, and D. London, Searching for new physics with $\bar{b} \rightarrow \bar{s}$ $B_s^0 \rightarrow V_1 V_2$ penguin decays, *Phys. Rev. D* **88**, 016007 (2013).
- [8] G. Durieux and Y. Grossman, Probing CP violation systematically in differential distributions, *Phys. Rev. D* **92**, 076013 (2015).
- [9] S. K. Patra and A. Kundu, CPT violation and triple-product correlations in B decays, *Phys. Rev. D* **87**, 116005 (2013).
- [10] A. Ali, J. G. Körner, G. Kramer, and J. Willrodt, Non-leptonic weak decays of bottom mesons, *Z. Phys. C* **1**, 269 (1979).
- [11] M. Suzuki, Final-state interactions and s -quark helicity conservation in $B \rightarrow J/\psi K^*$, *Phys. Rev. D* **64**, 117503 (2001).
- [12] R. Aaij *et al.* (LHCb Collaboration), Measurement of polarization amplitudes and CP asymmetries in $B^0 \rightarrow \phi K^*(892)^0$, *J. High Energy Phys.* **05** (2014) 069.
- [13] R. Aaij *et al.* (LHCb Collaboration), Measurement of CP violation in the $B_s^0 \rightarrow \phi\phi$ decay and search for the $B^0 \rightarrow \phi\phi$ decay, *J. High Energy Phys.* **12** (2019) 155.
- [14] K. F. Chen *et al.* (Belle Collaboration), Measurement of Branching Fractions and Polarization in $B \rightarrow \phi K^{(*)}$ Decays, *Phys. Rev. Lett.* **91**, 201801 (2003).
- [15] B. Aubert *et al.* (BABAR Collaboration), Time-dependent and time-integrated angular analysis of $B \rightarrow \phi K_S \pi^0$ and $\phi K^\pm \pi^\mp$, *Phys. Rev. D* **78**, 092008 (2008).
- [16] J. P. Lees *et al.* (BABAR Collaboration), B^0 meson decays to $\rho^0 K^{*0}$, $f_0 K^{*0}$, and $\rho^- K^{*+}$, including higher K^* resonances, *Phys. Rev. D* **85**, 072005 (2012).
- [17] R. Itoh *et al.* (Belle Collaboration), Studies of CP Violation in $B \rightarrow J/\psi K^*$ Decays, *Phys. Rev. Lett.* **95**, 091601 (2005).
- [18] B. Aubert *et al.* (BABAR Collaboration), Measurement of decay amplitudes of $B \rightarrow J/\psi K^*, \psi(2S) K^*$, and $\chi_{c1} K^*$ with an angular analysis, *Phys. Rev. D* **76**, 031102 (2007).
- [19] T. Aaltonen *et al.* (CDF Collaboration), Measurement of Polarization and Search for CP Violation in $B_s^0 \rightarrow \phi\phi$ Decays, *Phys. Rev. Lett.* **107**, 261802 (2011).
- [20] R. Aaij *et al.* (LHCb Collaboration), Measurement of the polarization amplitudes and triple product asymmetries in the $B_s^0 \rightarrow \phi\phi$ decay, *Phys. Lett. B* **713**, 369 (2012).
- [21] R. Aaij *et al.* (LHCb Collaboration), First Measurement of the CP -Violating Phase in $B_s^0 \rightarrow \phi\phi$ Decays, *Phys. Rev. Lett.* **110**, 241802 (2013).
- [22] R. Aaij *et al.* (LHCb Collaboration), Measurement of CP violation in $B_s^0 \rightarrow \phi\phi$ decays, *Phys. Rev. D* **90**, 052011 (2014).
- [23] R. Aaij *et al.* (LHCb Collaboration), Measurement of CP asymmetries and polarisation fractions in $B_s^0 \rightarrow K^{*0} \bar{K}^{*0}$ decays, *J. High Energy Phys.* **07** (2015) 166.
- [24] R. Aaij *et al.* (LHCb Collaboration), Study of the $B^0 \rightarrow \rho(770)^0 K^*(892)^0$ decay with an amplitude analysis of $B^0 \rightarrow (\pi^+ \pi^-)(K^+ \pi^-)$ decays, *J. High Energy Phys.* **05** (2019) 026.
- [25] M. Beneke, J. Rohrer, and D. S. Yang, Branching fractions, polarisation and asymmetries of $B \rightarrow VV$ decays, *Nucl. Phys. B* **774**, 64 (2007).
- [26] H. Y. Cheng and C. K. Chua, QCD factorization for charmless hadronic B_s decays revisited, *Phys. Rev. D* **80**, 114026 (2009).
- [27] C. D. Lü, Y. L. Shen, C. Wang, and Y. M. Wang, Enhanced next-to-leading-order corrections to weak annihilation B -meson decays, *arXiv:2202.08073*.
- [28] A. Ali, G. Kramer, Y. Li, C. D. Lü, Y. L. Shen, W. Wang, and Y. M. Wang, Charmless non-leptonic B_s decays to PP , PV and VV final states in the perturbative QCD approach, *Phys. Rev. D* **76**, 074018 (2007).
- [29] Z. T. Zou, A. Ali, C. D. Lü, X. Liu, and Y. Li, Improved estimates of the $B_{(s)} \rightarrow VV$ decays in perturbative QCD approach, *Phys. Rev. D* **91**, 054033 (2015).
- [30] D. C. Yan, X. Liu, and Z. J. Xiao, Anatomy of $B_s \rightarrow VV$ decays and effects of next-to-leading order contributions in the perturbative QCD factorization approach, *Nucl. Phys. B* **935**, 17 (2018).
- [31] C. Wang, S. H. Zhou, Y. Li, and C. D. Lü, Global analysis of charmless B decays into two vector mesons in soft-collinear effective theory, *Phys. Rev. D* **96**, 073004 (2017).
- [32] C. Wang, Q. A. Zhang, Y. Li, and C. D. Lü, Charmless $B_{(s)} \rightarrow VV$ decays in factorization-assisted topological-amplitude approach, *Eur. Phys. J. C* **77**, 333 (2017).
- [33] I. Bediaga, T. Frederico, and O. Lourenço, CP violation and CPT invariance in $B^\pm$ decays with final state interactions, *Phys. Rev. D* **89**, 094013 (2014).
- [34] I. Bediaga and P. C. Magalhães, Final state interaction on $B^+ \rightarrow \pi^- \pi^+ \pi^+$, *arXiv:1512.09284*.
- [35] X. W. Kang, B. Kubis, C. Hanhart, and U. G. Meißner, B_{14} decays and the extraction of $|V_{ub}|$, *Phys. Rev. D* **89**, 053015 (2014).
- [36] M. Gronau and J. L. Rosner, Symmetry relations in charmless $B \rightarrow PPP$ decays, *Phys. Rev. D* **72**, 094031 (2005).
- [37] M. Gronau, U -spin breaking in CP asymmetries in B decays, *Phys. Lett. B* **727**, 136 (2013).
- [38] G. Engelhard, Y. Nir, and G. Raz, $SU(3)$ relations and the CP asymmetry in $B \rightarrow K_S K_S K_S$, *Phys. Rev. D* **72**, 075013 (2005).
- [39] M. Imbeault and D. London, $SU(3)$ breaking in charmless B decays, *Phys. Rev. D* **84**, 056002 (2011).
- [40] D. Xu, G. N. Li, and X. G. He, U -spin analysis of CP violation in B^- decays into three charged light pseudo-scalar mesons, *Phys. Lett. B* **728**, 579 (2014).
- [41] X. G. He, G. N. Li, and D. Xu, $SU(3)$ and isospin breaking effects on $B \rightarrow PPP$ amplitudes, *Phys. Rev. D* **91**, 014029 (2015).

- [42] A. Furman, R. Kamiński, L. Leśniak, and B. Loiseau, Long-distance effects and final state interactions in $B \rightarrow \pi\pi K$ and $B \rightarrow K\bar{K}K$ decays, *Phys. Lett. B* **622**, 207 (2005).
- [43] B. E. Bennich, A. Furman, R. Kamiński, L. Leśniak, and B. Loiseau, Interference between $f_0(980)$ and $\rho(770)^0$ resonances in $B \rightarrow \pi^+\pi^-K$ decays, *Phys. Rev. D* **74**, 114009 (2006).
- [44] J. P. Dedonder, A. Furman, R. Kamiński, L. Leśniak, and B. Loiseau, S -, P - and D -wave $\pi\pi$ final state interactions and CP violation in $B^\pm \rightarrow \pi^\pm\pi^\mp\pi^\pm$ decays, *Acta Phys. Pol. B* **42**, 2013 (2011).
- [45] H. Y. Cheng, C. K. Chua, and A. Soni, Charmless three-body decays of B mesons, *Phys. Rev. D* **76**, 094006 (2007).
- [46] H. Y. Cheng and C. K. Chua, Branching fractions and direct CP violation in charmless three-body decays of B mesons, *Phys. Rev. D* **88**, 114014 (2013).
- [47] H. Y. Cheng, C. K. Chua, and Z. Q. Zhang, Direct CP violation in charmless three-body decays of B mesons, *Phys. Rev. D* **94**, 094015 (2016).
- [48] Y. Li, Comprehensive study of $\bar{B}^0 \rightarrow K^0(\bar{K}^0)K^\mp\pi^\pm$ decays in the factorization approach, *Phys. Rev. D* **89**, 094007 (2014).
- [49] Z. H. Zhang, X. H. Guo, and Y. D. Yang, CP violation in $B^\pm \rightarrow \pi^\pm\pi^+\pi^-$ in the region with low invariant mass of one $\pi^+\pi^-$ pair, *Phys. Rev. D* **87**, 076007 (2013).
- [50] R. Klein, T. Mannel, J. Virtob, and K. Keri Vos, CP violation in multibody B decays from QCD factorization, *J. High Energy Phys.* **10** (2017) 117.
- [51] H. Y. Cheng, CP violation in $B^\pm \rightarrow \rho^0\pi^\pm$ and $B^\pm \rightarrow \sigma\pi^\pm$ decays, [arXiv:2005.06080](https://arxiv.org/abs/2005.06080).
- [52] J. J. Qi, Z. Y. Wang, X. H. Guo, Z. H. Zhang, and C. Wang, Study of CP violation in $B^- \rightarrow K^-\pi^+\pi^-$ and $B^- \rightarrow K^-\sigma(600)$ decays in the QCD factorization approach, *Phys. Rev. D* **99**, 076010 (2019).
- [53] W. F. Wang and H. n. Li, Quasi-two-body decays $B \rightarrow K\rho \rightarrow K\pi\pi$ in perturbative QCD approach, *Phys. Lett. B* **763**, 29 (2016).
- [54] Y. Li, A. J. Ma, W. F. Wang, and Z. J. Xiao, Quasi-two-body decays $B_{(s)} \rightarrow P\rho \rightarrow P\pi\pi$ in the perturbative QCD approach, *Phys. Rev. D* **95**, 056008 (2017).
- [55] A. J. Ma, Y. Li, W. F. Wang, and Z. J. Xiao, Quasi-two-body decays $B_{(s)} \rightarrow D(\rho(1450), \rho(1700)) \rightarrow D\pi\pi$ in the perturbative QCD factorization approach, *Phys. Rev. D* **96**, 093011 (2017).
- [56] Y. Li, A. J. Ma, Z. Rui, W. F. Wang, and Z. J. Xiao, Quasi-two-body decays $B_{(s)} \rightarrow Pf_2(1270) \rightarrow P\pi\pi$ in the perturbative QCD approach, *Phys. Rev. D* **98**, 056019 (2018).
- [57] Z. Rui, Y. Li, and H. n. Li, P -wave contributions to $B \rightarrow \psi\pi\pi$ decays in the perturbative QCD approach, *Phys. Rev. D* **98**, 113003 (2018).
- [58] N. Wang, Q. Chang, Y. L. Yang, and J. F. Sun, Study of the $B_s \rightarrow \phi f_0(980) \rightarrow \phi\pi^+\pi^-$ decay with perturbative QCD approach, *J. Phys. G* **46**, 095001 (2019).
- [59] Y. Xing and Z. P. Xing, S -wave contributions to $\bar{B}_s^0 \rightarrow (D^0, \bar{D}^0)\pi^+\pi^-$ in the perturbative QCD framework, *Chin. Phys. C* **43**, 073103 (2019).
- [60] Y. Li, W. F. Wang, A. J. Ma, and Z. J. Xiao, Quasi-two-body decays $B_{(s)} \rightarrow K^*(892)h \rightarrow K\pi h$ in perturbative QCD approach, *Eur. Phys. J. C* **79**, 37 (2019).
- [61] Y. Li, Z. Rui, and Z. J. Xiao, P -wave contributions to $B_{(s)} \rightarrow \psi K\pi$ decays in perturbative QCD approach, *Chin. Phys. C* **44**, 073102 (2020).
- [62] W. F. Wang, J. Chai, and A. J. Ma, Contributions of $K_0^*(1430)$ and $K_0^*(1950)$ in the three-body decays $B \rightarrow K\pi h$, *J. High Energy Phys.* **03** (2020) 162.
- [63] Z. T. Zou, Y. Li, Q. X. Li, and X. Liu, Resonant contributions to three-body $B \rightarrow KKK$ decays in perturbative QCD approach, *Eur. Phys. J. C* **80**, 394 (2020).
- [64] Z. T. Zou, Y. Li, and X. Liu, Branching fractions and CP asymmetries of the quasi-two-body decays in $B_s \rightarrow K^0(\bar{K}^0)K^\pm\pi^\mp$ within PQCD approach, *Eur. Phys. J. C* **80**, 517 (2020).
- [65] Y. Li, D. C. Yan, Z. Rui, L. Liu, Y. T. Zhang, and Z. J. Xiao, S , P and D -wave resonance contributions to $B_{(s)} \rightarrow \eta_c(1S, 2S)K\pi$ decays in the perturbative QCD approach, *Phys. Rev. D* **101**, 016015 (2020).
- [66] Y. Li, D. C. Yan, Z. Rui, and Z. J. Xiao, Resonant contributions to three-body $B_{(s)} \rightarrow [D^{(*)}, \bar{D}^{(*)}]K^+K^-$ decays in the perturbative QCD approach, *Phys. Rev. D* **102**, 056017 (2020).
- [67] Y. Li, D. C. Yan, J. Hua, Z. Rui, and H. n. Li, Global determination of two-meson distribution amplitudes from three-body B decays in the perturbative QCD approach, *Phys. Rev. D* **104**, 096014 (2021).
- [68] Z. Rui and W. F. Wang, S -wave $K\pi$ contributions to the hadronic charmonium B decays in the perturbative QCD approach, *Phys. Rev. D* **97**, 033006 (2018).
- [69] Z. Rui, Y. Li, and W. F. Wang, The S -wave resonance contributions in the B_s^0 decays into $\psi(2S, 3S)$ plus pion pair, *Eur. Phys. J. C* **77**, 199 (2017).
- [70] C. H. Chen and H. n. Li, Three body nonleptonic B decays in perturbative QCD, *Phys. Lett. B* **561**, 258 (2003).
- [71] B. E. Bennich, A. Furman, R. Kamiński, L. Leśniak, B. Loiseau, and B. Moussallam, CP violation and kaon-pion interactions in $B \rightarrow K\pi^+\pi^-$ decays, *Phys. Rev. D* **79**, 094005 (2009).
- [72] J. Virto, Charmless non-leptonic multi-body B decays, *Proc. Sci.*, FPCP2016 (2017) 007 [[arXiv:1609.07430](https://arxiv.org/abs/1609.07430)].
- [73] S. Kränkl, T. Mannel, and J. Virto, Three-body non-leptonic B decays and QCD factorization, *Nucl. Phys. B* **899**, 247 (2015).
- [74] J. J. Qi, Z. Y. Wang, J. Xu, and X. H. Guo, Studying the localized CP violation and the branching fraction of the $\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-$ decay, *Chin. Phys. C* **44**, 103104 (2020).
- [75] J. J. Qi, Z. Y. Wang, and X. H. Guo, Phenomenological studies on the $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S} \rightarrow K^-\pi^+\pi^+\pi^-$ decay, *Chin. Phys. C* **45**, 053104 (2021).
- [76] Z. Rui, Y. Li, and H. n. Li, Four-body decays $B_{(s)} \rightarrow (K\pi)_{S/P}(K\pi)_{S/P}$ in the perturbative QCD approach, *J. High Energy Phys.* **05** (2021) 082.
- [77] Y. Li, D. C. Yan, Z. Rui, and Z. J. Xiao, Study of $B_{(s)} \rightarrow (\pi\pi)(K\pi)$ decays in the perturbative QCD approach, *Eur. Phys. J. C* **81**, 806 (2021).

- [78] C. Q. Zhang, J. M. Li, M. K. Jia, Y. Li, and Z. Rui, *CP*-violating observables in four-body $B \rightarrow \phi(\rightarrow K\bar{K})K^*(\rightarrow K\pi)$ decays, *Phys. Rev. D* **105**, 053002 (2022).
- [79] H. Q. Liang and X. Q. Yu, Study of the four-body decays $B_s^0 \rightarrow N_1 N_2 \rightarrow \pi\pi\pi\pi$ in the perturbative QCD approach, [arXiv:2202.05172](https://arxiv.org/abs/2202.05172).
- [80] D. Müller, D. Robaschik, B. Geyer, F. M. Dittes, and J. Hořejši, Wave functions, evolution equations and evolution kernels from light ray operators of QCD, *Fortschr. Phys.* **42**, 101 (1994).
- [81] M. Diehl, T. Gousset, B. Pire, and O. Teryaev, Probing Partonic Structure in $\gamma^*\gamma \rightarrow \pi\pi$ Near Threshold, *Phys. Rev. Lett.* **81**, 1782 (1998).
- [82] M. Diehl, T. Gousset, and B. Pire, Exclusive production of pion pairs in $\gamma^*\gamma$ collisions at large Q^2 , *Phys. Rev. D* **62**, 073014 (2000).
- [83] Ph. Hägler, B. Pire, L. Szymanowski, and O. V. Teryaev, Pomeron-odderon interference effects in electroproduction of two pions, *Eur. Phys. J. C* **26**, 261 (2002).
- [84] M. V. Polyakov, Hard exclusive electroproduction of two pions and their resonances, *Nucl. Phys.* **B555**, 231 (1999).
- [85] A. G. Grozin, On wave functions of mesonic pairs and mesonic resonances, *Sov. J. Nucl. Phys.* **38**, 289 (1983).
- [86] A. G. Grozin, One- and two-particle wave functions of multihadron systems, *Theor. Math. Phys.* **69**, 1109 (1986).
- [87] S. M. Flatté, On the nature of 0^+ mesons, *Phys. Lett.* **63B**, 228 (1976).
- [88] R. Aaij *et al.* (LHCb Collaboration), Evidence for an $\eta_c(1S)\pi^-$ resonance in $B^0 \rightarrow \eta_c(1S)K^+\pi^-$ decays, *Eur. Phys. J. C* **78**, 1019 (2018).
- [89] Y. Y. Keum, H. n. Li, and A. I. Sanda, Penguin enhancement and $B \rightarrow K\pi$ decays in perturbative QCD, *Phys. Rev. D* **63**, 054008 (2001).
- [90] T. Kurimoto, H. n. Li, and A. I. Sanda, Leading-power contributions to $B \rightarrow \pi, \rho$ transition form factors, *Phys. Rev. D* **65**, 014007 (2001).
- [91] C. D. Lü and M. Z. Yang, B to light meson transition form factors calculated in perturbative QCD approach, *Eur. Phys. J. C* **28**, 515 (2003).
- [92] H. n. Li, QCD aspects of exclusive B meson decays, *Prog. Part. Nucl. Phys.* **51**, 85 (2003) and references therein.
- [93] Z. J. Xiao, W. F. Wang, and Y. Y. Fan, Revisiting the pure annihilation decays $B_s \rightarrow \pi^+\pi^-$ and $B^0 \rightarrow K^+K^-$: The data and the pQCD predictions, *Phys. Rev. D* **85**, 094003 (2012).
- [94] S. Cheng and Z. J. Xiao, The PQCD approach towards to next-to-leading order: A short review, *Front. Phys.* **16**, 24201 (2021).
- [95] A. G. Grozin and M. Neubert, Asymptotics of heavy meson form-factors, *Phys. Rev. D* **55**, 272 (1997).
- [96] Y. L. Yang, L. Lang, X. L. Zhao, J. S. Huang, and J. F. Sun, Reinvestigating the $B \rightarrow PP$ decays by including the contributions from ϕ_{B2} , *Phys. Rev. D* **103**, 056006 (2021).
- [97] V. M. Braun, Y. Ji, and A. N. Manashov, Higher-twist B -meson distribution amplitudes in HQET, *J. High Energy Phys.* **05** (2017) 022.
- [98] H. n. Li, Y. L. Shen, and Y. M. Wang, Resummation of rapidity logarithms in B meson wave functions, *J. High Energy Phys.* **02** (2013) 008.
- [99] J. Hua, H. n. Li, C. D. Lü, W. Wang, and Z. P. Xing, Global analysis of hadronic two-body B decays in the perturbative QCD approach, *Phys. Rev. D* **104**, 016025 (2021).
- [100] Y. Y. Keum, H. n. Li, and A. I. Sanda, Fat penguins and imaginary penguins in perturbative QCD, *Phys. Lett. B* **504**, 6 (2001).
- [101] C. D. Lü, K. Ukai, and M. Z. Yang, Branching ratio and *CP* violation of $B \rightarrow \pi\pi$ decays in perturbative QCD approach, *Phys. Rev. D* **63**, 074009 (2001).
- [102] W. F. Wang, H. n. Li, W. Wang, and C. D. Lü, *S*-wave resonance contributions to the $B_{(s)}^0 \rightarrow J/\psi\pi^+\pi^-$ and $B_s \rightarrow \pi^+\pi^-\mu^+\mu^-$ decays, *Phys. Rev. D* **91**, 094024 (2015).
- [103] Z. Rui, Y. Li, and H. Li, Studies of the resonance components in the B_s decays into charmonia plus kaon pair, *Eur. Phys. J. C* **79**, 792 (2019).
- [104] K. M. Watson, The effect of final state interactions on reaction cross sections, *Phys. Rev.* **88**, 1163 (1952).
- [105] G. Breit and E. Wigner, Capture of slow neutrons, *Phys. Rev.* **49**, 519 (1936).
- [106] P. A. Zyla *et al.* (Particle Data Group), Review of particle physics, *Prog. Theor. Exp. Phys.* **2020**, 083C01 (2020).
- [107] D. V. Bugg, Reanalysis of data on $a_0(1450)$ and $a_0(980)$, *Phys. Rev. D* **78**, 074023 (2008).
- [108] R. Aaij *et al.* (LHCb Collaboration), Measurement of resonant and *CP* components in $\bar{B}_s^0 \rightarrow J/\psi\pi^+\pi^-$ decays, *Phys. Rev. D* **89**, 092006 (2014).
- [109] R. Aaij *et al.* (LHCb Collaboration), Measurement of the resonant and *CP* components in $\bar{B}^0 \rightarrow J/\psi\pi^+\pi^-$ decays, *Phys. Rev. D* **90**, 012003 (2014).
- [110] R. Aaij *et al.* (LHCb Collaboration), Analysis of the resonant components in $\bar{B}^0 \rightarrow J/\psi\pi^+\pi^-$, *Phys. Rev. D* **87**, 052001 (2013).
- [111] A. Pais and S. B. Treiman, Pion phase-shift information from K_{l4} decays, *Phys. Rev.* **168**, 1858 (1968).
- [112] H. Y. Cheng, C. Y. Cheung, W. Dimm, G. L. Lin, Y. C. Lin, T. M. Yan, and H. L. Yu, Heavy quark and chiral symmetry predictions for semileptonic decays $\bar{B} \rightarrow D(D^*)\pi l\bar{\nu}$, *Phys. Rev. D* **48**, 3204 (1993).
- [113] C. Q. Geng and Y. K. Hsiao, Rare $B^- \rightarrow \Lambda\bar{p}\nu\bar{\nu}$ decay, *Phys. Rev. D* **85**, 094019 (2012).
- [114] H. Y. Cheng and X. W. Kang, The semileptonic baryonic decay $D_s^+ \rightarrow p\bar{p}e^+\nu_e$, *Phys. Lett. B* **780**, 100 (2018).
- [115] Y. K. Hsiao and C. Q. Geng, Four-body baryonic decays of $B \rightarrow p\bar{p}\pi^+\pi^-(\pi^+K^-)$ and $\Lambda\bar{p}\pi^+\pi^-(\pi^+K^-)$, *Phys. Lett. B* **770**, 348 (2017).
- [116] P. Lepage and C. Gohlke, gplepage/lqfit: lqfit version 11.7, Zenodo, [http://doi.org/10.5281/zenodo.4037174](https://doi.org/10.5281/zenodo.4037174).
- [117] P. Colangelo, F. De Fazio, and W. Wang, $B_s \rightarrow f_0(980)$ form factors and B_s decays into $f_0(980)$, *Phys. Rev. D* **81**, 074001 (2010).
- [118] P. Colangelo, F. De Fazio, and W. Wang, Nonleptonic B_s to charmonium decays: Analysis in pursuit of determining the weak phase β_s , *Phys. Rev. D* **83**, 094027 (2011).
- [119] S. Cheng and J. M. Shen, $\bar{B}_s \rightarrow f_0(980)$ form factors and the width effect from light-cone sum rules, *Eur. Phys. J. C* **80**, 554 (2020).
- [120] B. Aubert *et al.* (BABAR Collaboration), Dalitz plot analysis of the decay $B^\pm \rightarrow K^\pm K^\pm K^\mp$, *Phys. Rev. D* **74**, 032003 (2006).

- [121] M. Ablikim *et al.* (BES Collaboration), Evidence for $f_0(980)f_0(980)$ production in χ_{c0} decays, *Phys. Rev. D* **70**, 092002 (2004).
- [122] M. Ablikim *et al.* (BES Collaboration), Partial wave analysis of $\chi_{c0} \rightarrow \pi^+\pi^-K^+K^-$, *Phys. Rev. D* **72**, 092002 (2005).
- [123] R. Aaij *et al.* (LHCb Collaboration), Dalitz plot analysis of $B^0 \rightarrow \bar{D}^0\pi^+\pi^-$ decays, *Phys. Rev. D* **92**, 032002 (2015).
- [124] H. Y. Cheng, C. K. Chua, K. C. Yang, and Z. Q. Zhang, Revisiting charmless hadronic B decays to scalar mesons, *Phys. Rev. D* **87**, 114001 (2013).
- [125] R. Aaij *et al.* (LHCb Collaboration), Observation of the decay $B_s^0 \rightarrow \phi\pi^+\pi^-$ and evidence for $B^0 \rightarrow \phi\pi^+\pi^-$, *Phys. Rev. D* **95**, 012006 (2017).
- [126] Z. T. Zou, L. Yang, Y. Li, and X. Liu, Study of quasi-two-body $B_{(s)} \rightarrow \phi(f_0(980)/f_2(1270) \rightarrow \pi\pi)$ decays in perturbative QCD approach, *Eur. Phys. J. C* **81**, 91 (2021).
- [127] Z. R. Liang and X. Q. Yu, Perturbative QCD predictions for the decay $B_s^0 \rightarrow SS(a_0(980), f_0(980), f_0(500))$, *Phys. Rev. D* **102**, 116007 (2020).
- [128] H. D. Niu, G. D. Li, J. L. Ren, and X. Liu, Perturbative QCD analysis of neutral B -meson decays into $\sigma\sigma, \sigma f_0$ and $f_0 f_0$, *Eur. Phys. J. C* **82**, 177 (2022).
- [129] H. Y. Cheng, C. W. Chiang, and C. K. Chua, Finite-width effects in three-body B decays, *Phys. Rev. D* **103**, 036017 (2021).
- [130] H. Y. Cheng, C. W. Chiang, and C. K. Chua, Width effects in resonant three-body decays: B decay as an example, *Phys. Lett. B* **813**, 136058 (2021).
- [131] H. n. Li and S. Mishima, Polarizations in $B \rightarrow VV$ decays, *Phys. Rev. D* **71**, 054025 (2005).
- [132] A. L. Kagan, Polarization in $B \rightarrow VV$ decays, *Phys. Lett. B* **601**, 151 (2004).
- [133] Y. Li, Investigating the effect of family non-universal Z' boson in $B \rightarrow \phi\phi$ decay, *Phys. Rev. D* **89**, 014003 (2014).