Towards a supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory

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The $w_{1+\infty}$ symmetry algebra appears in the Einstein–Yang–Mills theory, proposed recently by Strominger. In this paper, we derive the supersymmetric $w_{1+\infty}$ symmetry by using the known results on the operator product expansions (OPEs) between the graviton, gravitino, gluon, and gluino in the supersymmetric version of the above theory. We calculate the four additional commutator relations between the soft currents explicitly. In addition, we analyze the works of Odake *et al.* and Pope *et al.* and introduce the additional symmetry current that corresponds to the celestial gluino operator. Through this procedure, all seven commutator relations can be connected to the ones associated with the supersymmetric $w_{1+\infty}$ algebra with $SU(N)$ symmetry under the restrictions of wedge modes.

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I. INTRODUCTION

In the tree level Einstein-Yang-Mills theory, the leading operator product expansions (OPEs) on the celestial sphere of conformal primary gravitons and gluons are determined [1]. The structure constants in the right-hand side of these OPEs are given by the Euler beta function of arbitrary weights for the above operators. By analyzing the singular behavior of this function with an appropriate limiting procedure, the OPEs between the conformally soft positive-helicity gluon and graviton operators are obtained and the three corresponding commutator relations are determined [2]. The structure constants in these commutators are quite complex functions of the operator weights and modes. These structures are simplified through absorption of the various gamma functions (which depend on the weights and modes) into each current [3]. The commutator between the gravitons can be interpreted as the wedge subalgebra of $w_{1+\infty}$ algebra [4]. The wedge means that the mode can vary between one minus the weight and the weight minus one. The mode of the graviton contains both half integers and integers while the graviton considered in [4] is an arbitrary integer. The operators in the commutators [2,3] are dependent on the complex coordinate z of the celestial sphere. In [5], the mode expansion for the graviton in the holomorphic sector is performed further and this leads to an additional contour integral during the calculation of the commutator relation. Consequently, we obtain a commutator that is independent of the above z coordinate. Additional details are provided in [6–8] and review papers in [9–11] on the celestial holography.[1]

The w_∞ algebra [4], as an extension of the Virasoro algebra, is the Lie algebra where the currents have the conformal weight (or spin) $s = 2, 3, 4, \ldots, \infty$.[2] By introducing the conformal weight $s = 1$ further into the W_∞ algebra, the $W_{1+\infty}$ algebra is found in [15] and the complete expression is also given by [13]. The structure constants in the W_∞ algebra are different from those in the $W_{1+\infty}$ algebra. After taking the zero limit of a parameter, we obtain the Lie algebra between the currents of weights $s = 1, 2, \ldots, \infty$ having the central extension in the Virasoro sector. This algebra will be denoted as $w_{1+\infty}$ algebra. See also [16]. In this paper, we consider the $w_{1+\infty}$ algebra with a vanishing central term.

In [17], the additional adjoint currents of weights $s = 1, 2, \ldots, \infty$ under the $SU(N)$ are added to the above $W_{1+\infty}$ algebra.[3] We expect that the $w_{1+\infty}$ algebra with $SU(N)$

[1]See also Strominger's talk in strings 2021.

[2]This algebra admits the usual central extension in the Virasoro sector [12] and can also be obtained from the contraction [13] of the W_∞ algebra [12,14], which admits the central extensions for all sectors of arbitrary conformal weights.

[3]An important requirement is that the weight 1 adjoint current should produce the affine $\widehat{SU}(N)$ algebra with a level k and the adjoint currents should transform under the zero mode of the weight 1 adjoint current. After satisfying this requirement and using the Jacobi identities between the currents, two additional commutator relations in addition to the known commutator relation in the $W_{1+\infty}$ algebra are completely fixed. The central charge of the Virasoro current is given by $c = Nk$. In addition, the commutator between the adjoint currents consists of the symmetric tensor d symbol-dependent terms and the (antisymmetric) structure-constant dependent terms.

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symmetry will be obtained after taking the proper limit of a parameter [18]. In general, a central term from the commutator between the weight 1 adjoint currents and a central term of the Virasoro current are present.

The $\mathcal{N}=2$ supersymmetric extension of w_∞ algebra is studied in [19] by generalizing the $\mathcal{N}=2$ superconformal algebra and can be expressed in terms of graded Poisson brackets along the line of [4]. No central term is generated. However, the $\mathcal{N}=2$ supersymmetric extension of W_∞ algebra [20], where the bosonic sector is given by the sum of W_∞ algebra and $W_{1+\infty}$ algebra, exists.[4] Based on the construction of the twisted $\mathcal{N}=2$ superconformal algebra [22,23], the topological W_∞ algebra [24] is obtained by twisting the $\mathcal{N}=2$ supersymmetric extension of W_∞ algebra. No central term is generated and the structure constants appearing in two nontrivial commutator relations are the same.[5]

In this paper, we generalize the results of [2,3] to the supersymmetric Einstein-Yang-Mills theory studied in [26] where the additional nontrivial OPEs are given by those between bosonic and fermionic operators. The work of [6] focuses on similar studies but in a different context: a supersymmetric extension of [2]. The OPEs between the fermionic operators are regular. This is rather unusual because in the conventional conformal field theory they exhibit nontrivial singular behaviors.[6]

By using the known results [26] on the OPEs between the graviton, gravitino, gluon, and gluino associated with the above theory, we would like to describe the supersymmetric $w_{1+\infty}$ symmetry in terms of the celestial conformal field theory.

The four additional commutator relations between the soft currents are calculated explicitly by following the procedures of [2,3] and focusing on the particular mode of the soft currents [see for example (2.4)] along the line of [5]:this is a different viewpoint from [6]. One of the commutators having no $SU(N)$ group indices can be extracted from the supersymmetric topological w_∞ algebra 24]]. The remaining three can be determined by analyzing the previous works ([17] and [24]) further and introducing the additional symmetry current, which corresponds to the celestial gluino operator. The $SU(N)$ symmetry, the supersymmetry and the basic property of the OPE in the two-dimensional conformal field theory are used. Eventually we find that all seven commutator relations can be identified with those in the supersymmetric $w_{1+\infty}$ algebra with wedge modes having the $SU(N)$ symmetry.

In Sec. II, we calculate four commutators for the soft currents. In Sec. III, we present the supersymmetric $w_{1+\infty}$ algebra corresponding to seven commutators from the soft currents. In Sec. IV, we summarize the main result of this study and discuss ideas for future work. In the Appendixes A and B, we repeat the result of [2] and four commutator relations are successively described.

Various works [28–35] have been reviewed. As we will see in Sec. III, the work of [17] and $\mathcal{N}=2$ W_∞ algebra [20] are useful for understanding the structure characterizing the supersymmetric extension of $w_{1+\infty}$ algebra with $SU(N)$. These works are related to the extension of [17] or the $\mathcal{N}=4$ supersymmetric extension of [20]. However, the observation of these algebra in the context of the celestial conformal field theory remains unexplored.

What we have done or added in this paper, compared to the previous works in [5,6], is as follows. In [5], the mode for the graviton current is restricted to the case in which the corresponding transformations do not mix $SL(2,R)_L$ primaries and descendants in the OPE between the graviton currents and matter fields. As described before, these modes are independent of the complex coordinate z after an integration over the holomorphic sector further. As noted by [5], we perform the various OPEs between the soft currents and obtain the commutation relations on these modes. On the other hand, in [6], as mentioned before, the supersymmetric generalization of [2] is obtained. The OPEs between the chiral currents have simple (or first order) poles in the holomorphic sector. The modes are labeled by their transformation under $SL(2,R)_R$ and are dependent of the complex coordinate z. Each chiral current in [2,6] from soft symmetry current is obtained by taking the multiple derivatives in the antiholomorphic sector. By construction, these chiral currents do not depend on the complex coordinate $\bar{z}$. In this paper, we are focusing on the modes which do not depend on both z and $\bar{z}$. Our aim is to follow the procedure of [2] and analyze the description of [3] for the other relevant OPEs between the soft currents in the context of [5].

II. A SUPERSYMMETRIC EINSTEIN-YANG-MILLS THEORY

We will consider the OPEs in [26] and obtain the commutator relations for soft current algebra.

A. A soft current algebra between the graviton and the gluon: A review

From the positive-helicity (conformally primary) graviton operator $G^+_\Delta(z,\bar{z})$ with two-dimensional conformal weight Δ, a family of (conformally) soft positive-helicity graviton current is defined as [2]

[4]By taking the proper limit of the parameter, the corresponding $\mathcal{N}=2$ w_∞ algebra [21], where (anti)commutator relations with central terms comparable to those reported in [19], is obtained.

[5]Through the contraction procedure (introducing new currents with a parameter and taking this parameter to be zero), the right-hand sides of two commutators are simplified (the anticommutator between the fermionic currents vanishes). See also [25].

[6]A previous study [27] has presented a supersymmetric extension of the w_∞ algebra, by generalizing the $\mathcal{N}=1$ superconformal algebra. However, we can reduce the supersymmetry to lower symmetry by using the twisting procedure employed in [24] where the anticommutator relation between the fermionic currents vanishes.

$$H^k(z,\bar{z}) = \lim_{\varepsilon\to 0}\varepsilon G^+_{k+\varepsilon}(z,\bar{z}), \quad k=2,1,0,-1,-2,\ldots, \quad (2.1)$$

where, the (celestial) left and right conformal weights are given by

$$(h,\bar{h}) = \left(\frac{k+2}{2},\frac{k-2}{2}\right). \quad (2.2)$$

The additional factor of ε in (2.1) is necessary for canceling the pole of the beta function appearing in the original OPE coefficient.

Taking the holomorphic and antiholomorphic expansions for the above soft graviton current we obtain:

$$H^k(z,\bar{z}) = \sum_{n=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_n(z)}{\bar{z}^{n+\frac{k-2}{2}}} = \sum_m \sum_{n=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_{m,n}}{z^{m+\frac{k+2}{2}}\bar{z}^{n+\frac{k-2}{2}}}. \quad (2.3)$$

Rather than using the $H^k_n(z)$ mode, which depends on the holomorphic coordinate z, we further expand the current with respect to the holomorphic mode m in order to obtain the closed algebra with the $SL(2,R)_R$ generators [5]. The operator $H^k_{m,n}$ is therefore independent of z and $\bar{z}$. In this work, we focus on the case where the mode m is equal to $(1-h)$ together with (2.2)

$$\hat{H}^k_n \equiv H^k_{m=1-h,n}. \quad (2.4)$$

This will lead to $\frac{1}{z}$ dependence for particular terms of (2.3). As usual, the mode in (2.4) can be expressed in terms of the following contour integral

$$\hat{H}^k_n = \oint_{|z|<\varepsilon} \frac{dz}{2\pi i} z^{1-\frac{k+2}{2}+\frac{k+2}{2}-1} \oint_{|\bar{z}|<\varepsilon} \frac{d\bar{z}}{2\pi i} \bar{z}^{n+\frac{k-2}{2}-1} H^k(z,\bar{z}), \quad (2.5)$$

where, we intentionally express the power of z explicitly in the integrand. This can be easily checked (2.5) by substituting the relation (2.3) into the right-hand side of (2.5).

We can repeat the computation performed in [2]. For the calculation of $[\hat{H}^p_m, \hat{H}^q_n]$, we should perform the contour integrals over $z_1, \bar{z}_1, z_2$, and $\bar{z}_2$ with the OPE $H^k(z_1,\bar{z}_1)H^l(z_2,\bar{z}_2)$ in addition to some powers of $\bar{z}_1$ and $\bar{z}_2$. The three contour integrals (except for the coordinate z_2) can be done exactly without any modification. From this procedure, we are left with the contour integral over z_2 acting on the $\sum_p \frac{H^{k+l}_{p,m+n}}{z_2^{p+\frac{k+l+2}{2}}}$ as well as the mode- and weight-dependent terms. This leads to $\hat{H}^{k+l}_{m+n}$ being generated by the $\frac{1}{z_2}$ factor. We present the explicit commutator relation between the soft graviton currents in Appendix A. Similarly, the commutator between the soft gluon currents and the commutator between the soft graviton current and the soft gluon current can be determined [see Appendix (A1)].

In [3], the absorption of the mode- and weight-dependent terms appearing in the right-hand side denominator of the commutator relation are systematically investigated. The corresponding factors in the numerator can then be absorbed in the soft current of the right side. In other words, we have [3,5]

$$\hat{w}^p_n \equiv \frac{1}{\kappa}(p-n-1)!(p+n-1)!\hat{H}^{-2p+4}_n,$$
$$\hat{J}^{q,a}_m \equiv (q-m-1)!(q+m-1)!\hat{R}^{3-2q,a}_m, \quad (2.6)$$

where, κ is the gravitational coupling constant and the index a is an adjoint index of $SU(N)$.

Therefore, with the help of (2.6), we have[7]

$$[\hat{w}^p_m, \hat{w}^q_n] = [m(q-1)-n(p-1)]\hat{w}^{p+q-2}_{m+n},$$
$$[\hat{J}^{p,a}_m, \hat{J}^{q,b}_n] = -if^{ab}_c \hat{J}^{p+q-1,c}_{m+n},$$
$$[\hat{w}^p_m, \hat{J}^{q,a}_n] = [m(q-1)-n(p-1)]\hat{J}^{p+q-2,a}_{m+n}. \quad (2.7)$$

As observed in [3,5], each $\hat{w}^q$ is associated with a finite number of modes $1-q\le n\le q-1$, which provide $(2q-1)$ dimensional closed algebra, and $\hat{w}^{q=2}$ serves as a $SL(2,R)_R$ generator. Here, q is the positive half integer value $q=1,\frac{3}{2},2,\frac{5}{2},\cdots$. We fix $p=2$ in the first equation of (2.7), and this implies that the nth mode of a weight q transforms as a primary under the $SL(2,R)_R$ generator $\hat{w}^2_m$. From the third equation of (2.7), we check that the nth mode of a weight q transforms as a primary under the $\hat{w}^2_m$ and q runs over $q=1,\frac{3}{2},2,\frac{5}{2},\cdots$ as previously mentioned. The first equation of (2.7) includes the wedge subalgebra of $w_{1+\infty}$ algebra [4]. We will provide the corresponding description in the conventional conformal field theory outlined in the subsequent section.

B. Further soft current algebra in the presence of gravitino and gluino currents

We continue our calculation of the soft current algebra and account for the occurrence of fermionic currents.

1. The commutator between the graviton and the gravitino

From the positive-helicity (conformally primary) gravitino operator $\mathcal{O}_{\Delta,+\frac{3}{2}}(z,\bar{z})$ with two-dimensional conformal weight Δ, a family of (conformally) soft positive-helicity gravitino current is defined as [6]

$$I^k(z,\bar{z}) = \lim_{\varepsilon\to 0}\varepsilon\mathcal{O}_{k+\varepsilon,+\frac{3}{2}}(z,\bar{z}), \qquad k=\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\ldots, \quad (2.8)$$

[7]Equations (2.7), (3.6), and (3.8) reported in [3] correspond to (2.7).

where the (celestial) left and right conformal weights are given by[8]

$$(h, \bar{h}) = \left(\frac{k+\frac{3}{2}}{2}, \frac{k-\frac{3}{2}}{2}\right). \tag{2.9}$$

The OPE of the conformal primary graviton and the conformal primary gravitino of arbitrary weights is given as follows [26]:

$$\begin{aligned} &\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2) \\ &= -\frac{\frac{\kappa}{2}}{z_{12}} \sum_{n=0}^{\infty} B\left(\Delta_1 - 1 + n, \Delta_2 - \frac{1}{2}\right) \\ &\quad \times \frac{\bar{z}_{12}^{n+1}}{n!} \bar{\partial}^n \mathcal{O}_{\Delta_1+\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2) + \cdots. \end{aligned} \tag{2.10}$$

The abbreviated parts contain the left-conformal descendants. In the corresponding expression of [26], the $\bar{z}_{12}$ in the numerator is moved into the inside of the summation in (2.10). We express the OPE with the bosonic operators located at a position of $(z_1, \bar{z}_1)$, rather than at $(z_2, \bar{z}_2)$, in order to use the previous relations on the finite sum.

The OPE between the soft positive-helicity graviton (2.1) and the soft positive-helicity gravitino (2.8) can be described by

$$\begin{aligned} H^k(z_1,\bar{z}_1) I^l(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{n=0}^{1-k} \begin{pmatrix} 1-n+\frac{1}{2}-k-l \\ \frac{1}{2}-l \end{pmatrix} \\ \times \frac{\bar{z}_{12}^{n+1}}{n!} \bar{\partial}^n I^{k+l}(z_2,\bar{z}_2) + \cdots. \end{aligned} \tag{2.11}$$

The bracket denotes a binomial coefficient.[9] Consider the maximum value for the dummy variable n in the summation of (2.11) from the infinite sum in (2.10). This value stems from the fact that the highest power of $\bar{z}_1$ in (2.3) is $(2-k)$. We then obtain the relation $\partial_{\bar{z}_1}^{(3-k)} H^k(z_1,\bar{z}_1) = 0$ and in the right-hand side of (2.11), the highest power of $\bar{z}_{12}$ should also be $(2-k)$.

The corresponding commutator relation is given as[10]

$$\begin{aligned} [\hat{H}^k_m, \hat{I}^l_n] = &\oint_{|\bar{z}_1|<\varepsilon} \frac{d\bar{z}_1}{2\pi i} \bar{z}_1^{m+\frac{k-2}{2}-1} \oint_{|\bar{z}_2|<\varepsilon} \frac{d\bar{z}_2}{2\pi i} \bar{z}_2^{n+\frac{l-\frac{3}{2}}{2}-1} \\ &\times \oint_{|z_{12}|<\varepsilon} \frac{dz_1}{2\pi i} \oint_{|z_2|<\varepsilon} \frac{dz_2}{2\pi i} H^k(z_1,\bar{z}_1) I^l(z_2,\bar{z}_2). \end{aligned} \tag{2.12}$$

The $\bar{z}_1$ and z_1 contours receive unequal treatment due to the singular term of $\frac{1}{z_{12}}$ in (2.11). This can be compared with the approach of [5] that allows equal treatment of these contours. After inserting the relation (2.11) into the relation (2.12), four contour integrals are generated. The contour integral over the z_1 coordinate with a factor $\frac{1}{2\pi i}$ where the corresponding integrand is $\frac{1}{z_{12}}$ is simply one.[11] The contour integral over the coordinate is then given as: z_2 selects $\hat{I}^{k+l}_{m+n}$. From this we obtain:

$$\begin{aligned} &[\hat{H}^k_m, \hat{I}^l_n] \\ &= -\frac{\kappa}{2} \frac{(-1)^{m+\frac{k}{2}} (-m-n-\frac{k+l-\frac{3}{2}}{2})!}{(\frac{1}{2}-l)!(\frac{2-k}{2}-m)!} \\ &\quad \times \sum_{s=-m-\frac{k}{2}}^{1-k} \frac{(-1)^s (s+1)(\frac{3}{2}-s-k-l)!}{(1-s-k)!(s+m+\frac{k}{2})!(-m-n-\frac{k+l-\frac{3}{2}}{2}-s)!} \\ &\quad \times \hat{I}^l_{m+n}. \end{aligned} \tag{2.13}$$

As we expected, this intermediate result is the same as that presented in [2] when we replace l with $(l+\frac{1}{2})$. See also the first identity appearing in the footnote 28. Currently, obtaining a closed-form expression of the finite sum over the dummy variable s is challenging. however, we can take the expression obtained from [2] (or previously mentioned first identity) and substitute several values for the modes and weights into the relation. This allows expression of the above finite sum in terms of gamma functions.

The above result (2.13) can be reduced to

$$\begin{aligned} &[\hat{H}^k_m, \hat{I}^l_n] \\ &= \frac{\kappa}{2}\left[m\left(\frac{3}{2}-l\right) - n(2-k)\right] \\ &\quad \times \frac{(\frac{2-k}{2}-m+\frac{\frac{3}{2}-l}{2}-n-1)!(\frac{2-k}{2}+m+\frac{\frac{3}{2}-l}{2}+n-1)!}{(\frac{2-k}{2}-m)!(\frac{\frac{3}{2}-l}{2}-n)!(\frac{2-k}{2}+m)!(\frac{\frac{3}{2}-l}{2}+n)!} \\ &\quad \times \hat{I}^{k+l}_{m+n}. \end{aligned} \tag{2.14}$$

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[8] By performing the supersymmetric Ward identities [26] successively, the sub-subleading graviton $(\Delta = -1)$ leads to the subleading gravitino $(\Delta = -\frac{1}{2})$. Now the latter provides the subleading graviton $(\Delta = 0)$ which enables us to obtain the leading gravitino $(\Delta = \frac{1}{2})$. Finally, we arrive at the leading graviton $(\Delta = 1)$ from the latter. Then we should exclude $k = 2$ for graviton and $k = \frac{3}{2}$ for gravitino as soft currents [6].

[9] Note that the right-hand side of (2.11) looks very similar to equation (3.5) of [2] in the sense that after we replace l with $l+\frac{1}{2}$ we obtain (2.11).

[10] The mode expansion is given as follows: $I^l(z,\bar{z}) = \sum_{n=\frac{l-\frac{3}{2}}{2}}^{\frac{\frac{3}{2}-l}{2}} \frac{I^l_n(z)}{\bar{z}^{n+\frac{l-\frac{3}{2}}{2}}} = \sum_m \sum_{n=\frac{l-\frac{3}{2}}{2}}^{\frac{\frac{3}{2}-l}{2}} \frac{I^l_{m,n}}{z^{m+\frac{l+\frac{3}{2}}{2}} \bar{z}^{n+\frac{l-\frac{3}{2}}{2}}}$ with (2.9) and $\hat{I}^l_n \equiv I^l_{1-\frac{l+\frac{3}{2}}{2},n}$.

[11] The contour integral over the $\bar{z}_1$ coordinate can be obtained by using the Appendix (A.7) identity of [2]. Moreover, the subsequent contour integral over the $\bar{z}_2$ coordinate can be obtained by using the identity presented in Appendix (A.9) of [2].

We observe that the numerical mode- and weight-dependent factor in the right-hand side of (2.14) is the same as the factor included in $[\hat{H}^k_m, \hat{H}^l_n]$ where the weight l is replaced by $(l+\frac{1}{2})$. Further details are provided in Appendix (B1).

As done in (2.6), we would like to absorb the denominator of (2.14) into the currents by changing the weights with mode-dependent parts. Together with the first relation of (2.6), we introduce the following similar quantity

$$\hat{G}^q_n \equiv \frac{1}{\kappa}(q-n-1)!(q+n-1)!\hat{I}^{\frac{7}{2}-2q}_n. \tag{2.15}$$

The above commutator can then be summarized as follows:

$$[\hat{w}^p_m, \hat{G}^q_n] = [m(q-1)-n(p-1)]\hat{G}^{p+q-2}_{m+n}. \tag{2.16}$$

The n-th mode of a weight q in (2.16) transforms as a primary under the $\hat{w}^2_m$. Furthermore, q runs over $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ and its mode n varies as $1-q \le n \le q-1$. The m, n, p and q dependence observed here is the same as that included in (2.7).

2. The commutator between the gluon and the gluino

From the positive-helicity (conformally primary) gluino operator $\mathcal{O}_{\Delta,+\frac{1}{2}}(z,\bar{z})$ with two-dimensional conformal weight Δ, a family of (conformally) soft positive-helicity gluino current is defined as [6]:

$$L^{k,a}(z,\bar{z}) = \lim_{\varepsilon\to 0}\varepsilon\,\mathcal{O}^a_{k+\varepsilon,+\frac{1}{2}}(z,\bar{z}), \qquad k = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots, \tag{2.17}$$

where, the (celestial) left and right conformal weights are given by

$$(h,\bar{h}) = \left(\frac{k+\frac{1}{2}}{2}, \frac{k-\frac{1}{2}}{2}\right). \tag{2.18}$$

The OPE of the (conformal primary) gluon and the (conformal primary) gluino of arbitrary weights is given as follows [26]:

$$\begin{aligned}&\mathcal{O}^a_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}^b_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\\ &= \frac{-if^{ab}_c}{z_{12}}\sum_{n=0}^{\infty} B\left(\Delta_1-1+n, \Delta_2-\frac{1}{2}\right)\\ &\quad\times\frac{\bar{z}^n_{12}}{n!}\bar{\partial}^n\mathcal{O}^c_{\Delta_1+\Delta_2-1,+\frac{1}{2}}(z_2,\bar{z}_2)+\cdots.\end{aligned} \tag{2.19}$$

The regular term δ^{ab} reported in [26] is neglected here.

The OPE between the soft positive-helicity gluon and the soft positive-helicity gluino (2.17) can be summarized as follows:

$$\begin{aligned}R^{k,a}(z_1,\bar{z}_1)L^{l,b}(z_2,\bar{z}_2) &= \frac{-if^{ab}_c}{z_{12}}\sum_{n=0}^{1-k}\binom{1-n+\frac{1}{2}-k-l}{\frac{1}{2}-l}\\ &\quad\times\frac{\bar{z}^n_{12}}{n!}\bar{\partial}^n L^{k+l-1,c}(z_2,\bar{z}_2)+\cdots.\end{aligned} \tag{2.20}$$

The structure constant f^{ab}_c is associated with the $SU(N)$. Note that the infinite sum presented in (2.19) is reduced to the finite sum due to the fact that $\partial^{(2-k)}_{\bar{z}_1}R^{k,a}(z_1,\bar{z}_1) = 0$.[12] The corresponding commutator relation[13] can then be written in terms of

$$\begin{aligned}[\hat{R}^{k,a}_m, \hat{L}^{l,b}_n] &= \oint_{|\bar{z}_1|<\varepsilon}\frac{d\bar{z}_1}{2\pi i}\bar{z}_1^{m+\frac{k-1}{2}-1}\oint_{|\bar{z}_2|<\varepsilon}\frac{d\bar{z}_2}{2\pi i}\bar{z}_2^{n+\frac{l-\frac{1}{2}}{2}-1}\\ &\quad\times\oint_{|z_{12}|<\varepsilon}\frac{dz_1}{2\pi i}\oint_{|z_2|<\varepsilon}\frac{dz_2}{2\pi i}\\ &\quad\times R^{k,a}(z_1,\bar{z}_1)L^{l,b}(z_2,\bar{z}_2).\end{aligned} \tag{2.21}$$

After we calculate the contour integrals over z_1, $\bar{z}_1$,$\bar{z}_2$, and z_2 successively, we obtain the following intermediate result from (2.21)

$$\begin{aligned}[\hat{R}^{k,a}_m, \hat{L}^{l,b}_n] &= -if^{ab}_c\frac{(-1)^{m+\frac{k-1}{2}}(\frac{1-k}{2}-m+\frac{\frac{1}{2}-l}{2}-n)!}{(\frac{1}{2}-l)!(\frac{1-k}{2}-m)!}\\ &\quad\times\sum_{s=-m+\frac{1-k}{2}}^{1-k}\frac{(-1)^s(\frac{3}{2}-s-k-l)!}{(1-s-k)!(s+m+\frac{k-1}{2})!(\frac{1-k}{2}-m+\frac{\frac{1}{2}-l}{2}-n-s)!}\hat{L}^{k+l-1,c}_{m+n}.\end{aligned} \tag{2.22}$$

[12]As observed in a previous subsection, this OPE (2.20) looks very similar to Eq. (2.7) of [2] and the binomial coefficient where l is replaced by $(l+\frac{1}{2})$ becomes the above expression.

[13]The mode expansion is given by $L^{l,b}(z,\bar{z}) = \sum_{n=-\frac{l-\frac{1}{2}}{2}}^{\frac{\frac{1}{2}-l}{2}}\frac{L^{l,b}_n(z)}{\bar{z}^{n+\frac{l-\frac{1}{2}}{2}}} = \sum_m\sum_{n=-\frac{l-\frac{1}{2}}{2}}^{\frac{\frac{1}{2}-l}{2}}\frac{L^{l,b}_{m,n}}{z^{m+\frac{l+\frac{1}{2}}{2}}\bar{z}^{n+\frac{l-\frac{1}{2}}{2}}}$ with (2.18) and $\hat{L}^{l,b}_n \equiv L^{l,b}_{1-\frac{l+\frac{1}{2}}{2},n}$.

See also the second identity appearing in the footnote 28.[14] Although observation of the closed form in terms of gamma functions is challenging, we can check the corresponding identity by applying several values for the modes and weights. This yields:

$$[\hat{R}^{k,a}_m, \hat{L}^{l,b}_n] = -i f^{ab}_c \frac{(\frac{1-k}{2} - m + \frac{\frac{1}{2}-l}{2} - n)!(\frac{1-k}{2} + m + \frac{\frac{1}{2}-l}{2} + n)!}{(\frac{1-k}{2} - m)!(\frac{\frac{1}{2}-l}{2} - n)!(\frac{1-k}{2} + m)!(\frac{\frac{1}{2}-l}{2} + n)!} \times \hat{L}^{k+l-1,c}_{m+n}. \tag{2.23}$$

In order to simplify the commutation relation of (2.23), we introduce

$$\hat{\psi}^{q,b}_n \equiv (q-n-1)!(q+n-1)!\hat{L}^{\frac{5}{2}-2q,b}_n, \tag{2.24}$$

together with the second relation of (2.6). Afterward, we obtain the final commutator relation as follows:

$$[\hat{J}^{p,a}_m, \hat{\psi}^{q,b}_n] = -i f^{ab}_c \hat{\psi}^{p+q-1,c}_{m+n}. \tag{2.25}$$

We will observe, in the subsequent subsection, that the nth mode of a weight q in (2.25) transforms as a primary under the $\hat{w}^2_m$. In addition, q runs over $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ and its mode n varies as $1 - q \le n \le q - 1$, as in a previous case. Note that the weight of the right-hand side of (2.25) is given by $(p + q - 1)$ as in the second case of (2.7).

3. The commutator between the gluon and the gravitino

The OPE of the (conformal primary) gluon and the (conformal primary) gravitino of arbitrary weights is given as follows [26]:

$$\mathcal{O}^a_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\sum_{n=0}^{\infty} B\left(\Delta_1 + n, \Delta_2 - \frac{1}{2}\right)\frac{\bar{z}^n_{12}}{n!}\bar{\partial}^n \mathcal{O}^a_{\Delta_1+\Delta_2,+\frac{1}{2}} \times (z_2,\bar{z}_2) + \cdots. \tag{2.26}$$

Note that a gluino occurs in the right-hand side of this OPE. The OPE between the soft positive-helicity gluon and the soft positive-helicity gravitino can be obtained from:

$$R^{k,a}(z_1,\bar{z}_1)I^l(z_2,\bar{z}_2) = -\frac{\frac{\kappa}{2}}{z_{12}}\sum_{n=0}^{1-k}\begin{pmatrix} -n + \frac{1}{2} - k - l \\ \frac{1}{2} - l \end{pmatrix}\frac{\bar{z}^{n+1}_{12}}{n!} \times \bar{\partial}^n L^{k+l,a}(z_2,\bar{z}_2) + \cdots. \tag{2.27}$$

The corresponding commutator relation can be obtained from (2.27) with various contour integrals. In the expression reported in (2.26), the additional factor $\bar{z}_{12}$ is joined in the inside of the summation presented in (2.27). As in a previous case, by determining the power of $\bar{z}_1$ and $\bar{z}_2$ in the integrand via the conformal weights of currents, we obtain the following expression:

$$[\hat{R}^{k,a}_m, \hat{I}^l_n] = \oint_{|\bar{z}_1|<\epsilon}\frac{d\bar{z}_1}{2\pi i}\bar{z}_1^{m+\frac{k-1}{2}-1}\oint_{|\bar{z}_2|<\epsilon}\frac{d\bar{z}_2}{2\pi i}\bar{z}_2^{n+\frac{l-\frac{3}{2}}{2}-1}\oint_{|z_{12}|<\epsilon}\frac{dz_1}{2\pi i}\oint_{|z_2|<\epsilon}\frac{dz_2}{2\pi i}R^{k,a}(z_1,\bar{z}_1)I^l(z_2,\bar{z}_2). \tag{2.28}$$

By substituting the OPE in (2.27) into (2.28) and performing each contour integral successively, we obtain the following intermediate result, which consists of the finite sum with the mode and weight-dependent overall factor,

$$[\hat{R}^{k,a}_m, \hat{I}^l_n] = -\frac{\kappa}{2}\frac{(-1)^{1+m+\frac{k-1}{2}}(-m-n-\frac{k+l-\frac{1}{2}}{2})!}{(\frac{1}{2}-l)!(\frac{1-k}{2}-m)!} \times \sum_{s=-1-m+\frac{1-k}{2}}^{1-k}\frac{(-1)^s(s+1)(\frac{1}{2}-s-k-l)!}{(-s-k)!(1+s+m+\frac{k-1}{2})!(-m-n-\frac{k+l-\frac{1}{2}}{2}-s)!}\hat{L}^{k+l,a}_{m+n}. \tag{2.29}$$

Note that the above finite sum with l replaced by $(l - \frac{1}{2})$ in (2.29) is calculated in [2] to determine the commutator between the soft gluon and soft graviton..[15] See also the first identity appearing in the footnote 28. We obtain the final result by using the explicit form, which includes various gamma functions in the fractional form,

$$[\hat{R}^{k,a}_m, \hat{I}^l_n] = \frac{\kappa}{2}\left[m\left(\frac{3}{2}-l\right) - n(1-k)\right] \times \frac{(\frac{1-k}{2} - m + \frac{\frac{3}{2}-l}{2} - n - 1)!(\frac{1-k}{2} + m + \frac{\frac{3}{2}-l}{2} + n - 1)!}{(\frac{1-k}{2} - m)!(\frac{\frac{3}{2}-l}{2} - n)!(\frac{1-k}{2} + m)!(\frac{\frac{3}{2}-l}{2} + n)!}\hat{L}^{k+l,a}_{m+n}. \tag{2.30}$$

[14] As mentioned before, the above finite sum in (2.22) can be read off from the analysis of the equation presented in Appendix (A.8) of [2] where the replacement of l and $(l + \frac{1}{2})$ is addressed.

[15] The corresponding identity reported in [2] is given as: $\sum_{s=-1-m-\frac{k-1}{2}}^{1-k}\frac{(-1)^s(s+1)(1-s-k-l)!}{(1-s-k)!(s+m+\frac{k}{2})!(-m-n-\frac{k+l-2}{2}-s)!} = [-m(2-l) + n(1-k)]\frac{(1-l)!(\frac{1-k}{2}+m+\frac{2-l}{2}+n-1)!}{(-1)^{1+m+\frac{k-1}{2}}(\frac{2-l}{2}-n)!(\frac{1-k}{2}+m)!(\frac{2-l}{2}+n)!}$.

From Eqs. (2.6), (2.15), and (2.24), the above commutation relation (2.30) is

$$[\hat{J}^{p,a}_m, \hat{G}^q_n] = [m(q-1) - n(p-1)]\hat{\psi}^{p+q-2,a}_{m+n}. \qquad (2.31)$$

The m, n, p and q dependence here is the same as that occurring in (2.7).

4. The commutator between the graviton and the gluino

The OPE of the (conformal primary) graviton and the (conformal primary) gluino of arbitrary weights is given by [26]

$$\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^a_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\sum_{n=0}^{\infty} B\left(\Delta_1 - 1 + n, \Delta_2 + \frac{1}{2}\right)\frac{\bar{z}^n_{12}}{n!} \times \bar{\partial}^n \mathcal{O}^a_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) + \cdots. \qquad (2.32)$$

The OPE between the soft positive-helicity graviton and the soft positive-helicity gluino can be obtained from:

$$H^k(z_1,\bar{z}_1)L^{l,a}(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{n=0}^{1-k}\begin{pmatrix} 1-n-\frac{1}{2}-k-l \\ -\frac{1}{2}-l \end{pmatrix}\frac{\bar{z}^{n+1}_{12}}{n!}\bar{\partial}^n L^{k+l,a}(z_2,\bar{z}_2) + \cdots. \qquad (2.33)$$

The finite terms from (2.32) survive in the conformally soft limit.[16] Again the commutator between the two currents can be determined as follows:

$$[\hat{H}^k_m, \hat{L}^{l,a}_n] = \oint_{|\bar{z}_1|<\varepsilon}\frac{d\bar{z}_1}{2\pi i}\bar{z}_1^{m+\frac{k-2}{2}-1}\oint_{|\bar{z}_2|<\varepsilon}\frac{d\bar{z}_2}{2\pi i}\bar{z}_2^{n+\frac{l-\frac{1}{2}}{2}-1} \times \oint_{|z_{12}|<\varepsilon}\frac{dz_1}{2\pi i}\oint_{|z_2|<\varepsilon}\frac{dz_2}{2\pi i}H^k(z_1,\bar{z}_1)L^{l,a}(z_2,\bar{z}_2). \qquad (2.34)$$

We arrive at the intermediate result for the commutator, from (2.33) and (2.34),

$$[\hat{H}^k_m, \hat{L}^{l,a}_n] = -\frac{\kappa}{2}\frac{(-1)^{m+\frac{k}{2}}(-m-n-\frac{k+l-\frac{1}{2}}{2})!}{(-\frac{1}{2}-l)!(\frac{2-k}{2}-m)!}\sum_{s=-m-\frac{k}{2}}^{1-k}\frac{(-1)^s(s+1)(\frac{1}{2}-s-k-l)!}{(1-s-k)!(1+s+m+\frac{k-2}{2})!(-m-n-\frac{k+l-\frac{1}{2}}{2}-s)!}\hat{L}^{k+l,a}_{m+n}. \qquad (2.35)$$

We realize that the finite sum in (2.35) appears in (2.13) and by replacing l with $(l+1)$ the latter becomes the former. See also the first identity of the footnote 28. Therefore, we obtain:

$$[\hat{H}^k_m, \hat{L}^{l,a}_n] = \frac{\kappa}{2}\left[m\left(\frac{1}{2}-l\right) - n(2-k)\right]\frac{(\frac{2-k}{2}-m+\frac{\frac{1}{2}-l}{2}-n-1)!(\frac{2-k}{2}+m+\frac{\frac{1}{2}-l}{2}+n-1)!}{(\frac{2-k}{2}-m)!(\frac{\frac{1}{2}-l}{2}-n)!(\frac{2-k}{2}+m)!(\frac{\frac{1}{2}-l}{2}+n)!}\hat{L}^{k+l,a}_{m+n}. \qquad (2.36)$$

By using the Eqs. (2.6) and (2.24), the above commutation relation (2.36) becomes

$$[\hat{w}^p_m, \hat{\psi}^{q,a}_n] = [m(q-1) - n(p-1)]\hat{\psi}^{p+q-2,a}_{m+n}. \qquad (2.37)$$

The nth mode of a weight q in (2.37) transforms as a primary under the $\hat{w}^2_m$. In addition, q runs over $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$ and its mode n varies as $1-q \le n \le q-1$ as previously stated.

5. Summary of this subsection

We collect the previous four commutator relations, (2.16), (2.25), (2.31), and (2.37) as follows:

$$[\hat{J}^{p,a}_m, \hat{\psi}^{q,b}_n] = -if^{ab}_{\ \ c}\hat{\psi}^{p+q-1,c}_{m+n},$$
$$[\hat{w}^p_m, \hat{G}^q_n] = [m(q-1) - n(p-1)]\hat{G}^{p+q-2}_{m+n},$$
$$[\hat{J}^{p,a}_m, \hat{G}^q_n] = [m(q-1) - n(p-1)]\hat{\psi}^{p+q-2,a}_{m+n},$$
$$[\hat{w}^p_m, \hat{\psi}^{q,a}_n] = [m(q-1) - n(p-1)]\hat{\psi}^{p+q-2,a}_{m+n}. \qquad (2.38)$$

Therefore, we obtain all seven commutator relations given by (2.7) and (2.38). We have checked that the graded Jacobi identities containing commutator or anticommutator between four currents are satisfied by using the definition of $(-1)^{AC}[X^A,[X^B,X^C\}\} + \text{cycl.perm.} = 0$ where X^A denotes a current. The factor $(-1)^{AC}$ gives us -1 for the fermionic currents X^A and X^C and 1 for the other three cases. Subsequently, we will describe the corresponding supersymmetric $w_{1+\infty}$ algebra in the conventional conformal field theory.[17]

[16] From Eq. (4.2) of [2], by replacing the l with $(l+\frac{1}{2})$, the binomial coefficient becomes the above-mentioned value reported in (2.33).

[17] Three vanishing anticommutator relations, $\{\hat{\psi}^{p,a}_m, \hat{\psi}^{q,b}_n\} = 0$, $\{\hat{\psi}^{p,a}_m, \hat{G}^q_n\} = 0$, and $\{\hat{G}^p_m, \hat{G}^q_n\} = 0$ from the corresponding regular OPEs in [26] must be considered.

III. A SUPERSYMMETRIC $w_{1+\infty}$ SYMMETRY

We describe the supersymmetric $w_{1+\infty}$ algebra in order to understand the symmetry associated with the supersymmetric Einstein-Yang-Mills theory discussed in the previous section.

A. A $w_{1+\infty}$ algebra with $SU(N)$ symmetry

Odake and Sano [17] introduced the affine current $J^{q,a}$ with level k, which has weight $q = 1, 2, \ldots$ and an adjoint index $a = 1, 2, \ldots, (N^2-1)$ of $SU(N)$ in addition to the current w^p where $p = 1, 2, \ldots$, into the $W_{1+\infty}$ algebra [15]. The results revealed that the commutators between the currents are determined as follows[18]:

$$\begin{aligned}
[W^p_m, W^q_n] &= \sum_{r\geq 2,\mathrm{even}}^{p+q-1} \lambda^{r-2} g^{p-2,q-2}_{r-2}(m,n) W^{p+q-r}_{m+n} \\
&\quad + \delta^{pq}\delta_{m+n,0}\lambda^{2(p-2)} c_{p-2}(m), \\
[W^p_m, J^{q,a}_n] &= \sum_{r\geq 2,\mathrm{even}}^{p+q-1} \lambda^{r-2} g^{p-2,q-2}_{r-2}(m,n) J^{p+q-r,a}_{m+n}, \\
[J^{p,a}_m, J^{q,b}_n] &= \frac{i}{2} f^{ab}_c \sum_{r\geq 1,\mathrm{odd}}^{p+q-1} \lambda^{r-2} g^{p-2,q-2}_{r-2}(m,n) J^{p+q-r,c}_{m+n} \\
&\quad + \delta^{pq}\delta^{ab}\delta_{m+n,0}\lambda^{2(p-2)} k_{p-2}(m) \\
&\quad + \sum_{r\geq 2,\mathrm{even}}^{p+q-1} \lambda^{r-2} g^{p-2,q-2}_{r-2}(m,n) \\
&\quad \times \left(d^{ab}_c J^{p+q-r,c}_{m+n} + \frac{1}{N}\delta^{ab} W^{p+q-r}_{m+n} \right). \qquad (3.1)
\end{aligned}$$

This is referred to as the $\widehat{SU}(N)_k$ $W_{1+\infty}$ algebra.[19] The dummy variable r is even or odd.[20]

By taking the new currents as

$$W^p_m \to w^p_m, \qquad J^{p,a}_m \to \lambda J^{p,a}_m, \qquad (3.2)$$

and taking the limit $\lambda \to 0$, the resulting algebra from (3.1) can be described as

$$\begin{aligned}
[w^p_m, w^q_n] &= g^{p-2,q-2}_0(m,n) w^{p+q-2}_{m+n} + \delta^{p,2}\delta^{q,2}\delta_{m+n,0} c_0(m), \\
[w^p_m, J^{q,a}_n] &= g^{p-2,q-2}_0(m,n) J^{p+q-2,a}_{m+n}, \\
[J^{p,a}_m, J^{q,b}_n] &= \frac{i}{2} f^{ab}_c g^{p-2,q-2}_{-1}(m,n) J^{p+q-1,c}_{m+n} \\
&\quad + \delta^{p,1}\delta^{q,1}\delta_{m+n,0} k_{-1}(m), \qquad (3.3)
\end{aligned}$$

where the structure constants are $g^{p-2,q-2}_0(m,n) = m(q-1) - n(p-1)$ and $g^{p-2,q-2}_{-1}(m,n) = \frac{1}{2}$. In addition, the central terms are given by $c_0(m) = \frac{1}{12} m(m^2-1)c$ and $k_{-1}(m) = \frac{m}{16} k$. Note that the central charge c is given by $c = Nk$. Further details are provided in [18]. Consider $k = 0$ and rescaling the current $J^{q,a}$ with (assuming that the structure constants are the same as those mentioned in the previous section) the $-\frac{1}{4}$ factor. In this case, the above algebra (3.3) with wedge modes

$$\begin{aligned}
[w^p_m, w^q_n] &= [m(q-1) - n(p-1)] w^{p+q-2}_{m+n}, \\
[J^{p,a}_m, J^{q,b}_n] &= -i f^{ab}_c J^{p+q-1,c}_{m+n}, \\
[w^p_m, J^{q,a}_n] &= [m(q-1) - n(p-1)] J^{p+q-2,a}_{m+n}, \qquad (3.4)
\end{aligned}$$

coincides with the one in (2.7) when hats are included. In general, no restrictions are imposed on the modes in (3.4) and the weights p and q are positive integers $p, q = 1, 2, \ldots$. Note that in (2.7), the modes can be half integers. The three commutators in (3.4) are equivalent to the following OPEs in the antiholomorphic sector (by decomposing the usual mode expansions)

$$\begin{aligned}
w^p(\bar{z}_1) w^q(\bar{z}_2) &= \frac{(p+q-2)}{(\bar{z}_1-\bar{z}_2)^2} w^{p+q-2}(\bar{z}_2) \\
&\quad + \frac{(p-1)}{(\bar{z}_1-\bar{z}_2)} \bar{\partial} w^{p+q-2}(\bar{z}_2) + \cdots, \\
J^{p,a}(\bar{z}_1) J^{q,b}(\bar{z}_2) &= \frac{-i f^{ab}_c}{(\bar{z}_1-\bar{z}_2)} J^{p+q-1,c}(\bar{z}_2) + \cdots, \\
w^p(\bar{z}_1) J^{q,a}(\bar{z}_2) &= \frac{(p+q-2)}{(\bar{z}_1-\bar{z}_2)^2} J^{p+q-2,a}(\bar{z}_2) \\
&\quad + \frac{(p-1)}{(\bar{z}_1-\bar{z}_2)} \bar{\partial} J^{p+q-2,a}(\bar{z}_2) + \cdots. \qquad (3.5)
\end{aligned}$$

According to [36], through field redefinitions of the currents, the weight 1 current in the $W_{1+\infty}$ algebra is decoupled and the W_∞ algebra is generated by the remaining currents. Moreover, the so-called w_N algebra, which is a truncation of the w_∞ algebra, is introduced in [37].

B. A supersymmetric topological w_∞ algebra

The $\mathcal{N}=2$ supersymmetric W_∞ algebra [20] can be twisted to provide a topological W_∞ algebra. By taking one of the fermionic generators as the nilpotent BRST charge, the corresponding nontrivial commutator relations are obtained as follows:

$$\begin{aligned}
[\hat{V}^p_m, \hat{V}^q_n] &= \sum_{l\geq 2}^{p+q-2} \hat{g}^{p-2,q-2}_{l-2}(m,n) \hat{V}^{p+q-l}_{m+n}, \\
[\hat{V}^p_m, G^q_{n+\frac{1}{2}}] &= \sum_{l\geq 2}^{p+q-\frac{3}{2}} \hat{g}^{p-2,q-2}_{l-2}(m,n) G^{p+q-l}_{m+n+\frac{1}{2}}. \qquad (3.6)
\end{aligned}$$

[18] Previously, the weight was represented by $p+2$ (or $p+\frac{3}{2}$) rather than an arbitrary p and we account for this shift properly everywhere.

[19] The relations (3.1) correspond to equations (5), (6), and (7) of [17]. Their V^{i-2} and $W^{j-2,a}$ correspond to our W^i and $J^{i,a}$, respectively, in this paper.

[20] In that work, they take the first relation in (3.1) from [15] and make an ansatz for the remaining two with arbitrary coefficients, which can be determined using various Jacobi identities.

Here, no central term is considered. The weight of G^q is given by $q = \frac{3}{2}, \frac{5}{2}, \cdots$ while the weight of $\hat{V}^p$ is given by $p = 2, 3, \ldots$ where $p = 1$ is excluded. The bosonic currents $\hat{V}^p_m$ in (3.6) are given by two kinds of bosonic currents in [20].[21] By construction, the bosonic current $\hat{V}^p$ is obtained through linear combination of the bosonic current V^p of W_∞ algebra and the bosonic current $\tilde{V}^p$ of $W_{1+\infty}$ algebra..[22] Note that the structure constants in two commutators are the same.

As in the case of [24], the relevant contraction is described by introducing $v^p_m \to \lambda^{p-2}\hat{V}^p_m$ and $G^p_m \to \lambda^{p-2}G^p_m$ and taking the limit $\lambda \to 0$ along the line of (3.2).[23] Moreover,

$$[v^p_m, v^q_n] = \hat{g}_0^{p-2,q-2}(m,n) v^{p+q-2}_{m+n} = [m(q-1) - n(p-1)] v^{p+q-2}_{m+n},$$
$$[v^p_m, G^q_{n+\frac{1}{2}}] = \hat{g}_0^{p-2,q-2}(m,n) G^{p+q-2}_{m+n+\frac{1}{2}} = [m(q-1) - n(p-1)] G^{p+q-2}_{m+n+\frac{1}{2}}, \quad (3.7)$$

where the nontrivial structure constant in (3.7) can be obtained[24] with the help of some formulas in [20]

$$\begin{aligned} \hat{g}_0^{p-2,q-2}(m,n) &= \frac{(2pq - 3p + 2q^2 - 8q + 8)(-2pn - p + 2qm - 3m + 2n + 1)}{2(2q-3)(2p+2q-5)} \\ &\quad + \frac{(2pq - 3p + 2q^2 - 8q + 7)(-2pn - p + 2qm - 3m + 2n + 1)}{2(2q-3)(2p+2q-5)} \\ &\quad - \frac{2(p-2)(p+m-1)}{(2p-3)}\left(-\frac{1}{4}\right) + \frac{2(p-1)(p+m-1)}{(2p-3)}\left(\frac{1}{4}\right) \\ &= m(q-1) - n(p-1). \end{aligned} \quad (3.8)$$

Under the v^2, the weights of v^p and G^q are $p = 2, 3, 4, \ldots$ and $q = 2, 3, 4, \ldots$ respectively. During the twisting procedure, the original weights of q in the current G^q is shifted by $\frac{1}{2}$. In (3.7), the mode of the fermionic current is given by the half integers (NS sector). This can be seen from the relation (2.15) by taking $(n + \frac{1}{2})$ rather than n in the left-hand side. Further details are provided in [18]. In terms of the OPEs, the following relations corresponding to (3.7) and (3.8) are satisfied, similar to the case of (3.5),

$$v^p(\bar{z}_1) v^q(\bar{z}_2) = \frac{(p+q-2)}{(\bar{z}_1 - \bar{z}_2)^2} v^{p+q-2}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1 - \bar{z}_2)} \bar{\partial} v^{p+q-2}(\bar{z}_2) + \cdots,$$
$$v^p(\bar{z}_1) G^q(\bar{z}_2) = \frac{(p+q-2)}{(\bar{z}_1 - \bar{z}_2)^2} G^{p+q-2}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1 - \bar{z}_2)} \bar{\partial} G^{p+q-2}(\bar{z}_2) + \cdots. \quad (3.9)$$

From the first equation, we can check the corresponding commutator relation in (3.7). That is, in the expression of $[v^p_m, v^q_n] = \oint_{|\bar{z}_1|<\epsilon} \frac{d\bar{z}_1}{2\pi i} \bar{z}_1^{n+q-1} \oint_{|\bar{z}_{12}|<\epsilon} \frac{d\bar{z}_2}{2\pi i} \bar{z}_2^{m+p-1} v^p(\bar{z}_1) v^q(\bar{z}_2)$, this leads to $(m+p-1)(p+q-2) - (p-1)(m+n+p+q-2) = m(q-1) - n(p-1)$. For the second relation of (3.9) corresponding to the second relation in (3.7), the dummy variable undergoes a shift during the mode expansion of the fermionic current, contrary to the case of the bosonic current.

C. A supersymmetric $w_{1+\infty}$ algebra with $SU(N)$ symmetry

By examining the construction of two previous subsections, we realize that the currents w^p_m in (3.3) are equivalent to the currents v^p_m in (3.7) up to w^1_m current.[25] The currents $\hat{V}^p_m$ in (3.6) consist of four parts and two of these parts have no singular OPEs with $J^{q,a}_n$ in (3.1). The remaining two terms have nontrivial OPEs with $J^{q,a}_n$. One of these OPEs is exactly the same as the W^p_m and the other is a W^{p-1}_m term. The W^{p-1}_m term provides no contribution after the above-mentioned contraction procedure ($W^p_m \to \lambda^{p-2} w^p_m$ and $J^{p,a}_m \to \lambda^p J^{p,a}_m$) is performed.

In this subsection, we would like to construct a supersymmetric $w_{1+\infty}$ algebra with $SU(N)$ symmetry, which

[21] The explicit relation can be found in Eq. (8) of [24].

[22] Furthermore, the structure constants $\hat{g}_{l-2}^{p-2,q-2}(m,n)$ appearing in (3.6) are given by Eq. (11) reported in [24].

[23] Their $\hat{v}^i$ and g^j correspond to our v^{i-2} and G^{j-2} in this paper.

[24] By calculating the four terms in Eq. (11) of [24], we obtain (3.7) which occurs in Eq. (42) in [24].

[25] By decoupling the w^1_m current with field redefinitions or inserting the v^1_m current from the analysis performed in [36], we can realize the same number of currents.

contains the previous algebras (3.4) and (3.7). We search for this extended algebra in a minimal manner. In other words, we introduce the extra currents minimally. So far, we have the currents w_m^p, G_m^p, and $J_m^{p,a}$. We must also determine the superpartner of $J_m^{p,a}$ (G_m^p is the superpartner of w_m^p).

Let us consider the construction that yields the OPE between $J^{p,a}$ and G^q. We expect that a fermionic current with an adjoint index a of $SU(N)$ will occur in the right-hand side of this OPE. In other words, the right-hand side of this OPE should contain the superpartner of $J^{p,a}$ having an adjoint index a. We have previously considered the OPE structure in the context of w_∞ algebra. That is, the OPE between the primary current with weight p and the current with weight q consists of both the second-order pole and the first-order pole. The relative coefficients between these are completely fixed We must now consider one unknown structure constant appearing in the second-order pole.

Subsequently, we express the following ansatz, in the antiholomorphic sector,

$$J^{p,a}(\bar{z}_1)G^q(\bar{z}_2) = \frac{(p+q-2)}{(\bar{z}_1-\bar{z}_2)^2}\psi^{p+q-2,a}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1-\bar{z}_2)}\bar{\partial}\psi^{p+q-2,a}(\bar{z}_2) + \cdots. \quad (3.10)$$

The coefficient appearing in the second-order pole can be taken from the normalizations in [14,38]. Furthermore, we note that the relative coefficient $\frac{(p-1)}{(p+q-2)}$ can be determined from the formula $\frac{\bar{h}_p-\bar{h}_q+\bar{h}_{p+q-2}}{2\bar{h}_{p+q-2}}$ with each weight $\bar{h}_p = p$, $\bar{h}_q = q$ and $\bar{h}_{p+q-2} = (p+q-2)$.[26] We obtain the corresponding commutator relation from (3.10), by using the procedure reported in (3.9), That is,

$$[J_m^{p,a}, G_n^q] = [m(q-1)-n(p-1)]\psi_{m+n}^{p+q-2,a}. \quad (3.11)$$

Subsequently, we obtain the OPEs between $\psi^{q,b}$ and its superpartner $J_m^{p,a}$ and the currents w_m^p. From the OPE between the affine currents, we generalize this to the following OPE that lacks a central term

$$J^{p,a}(\bar{z}_1)\psi^{q,b}(\bar{z}_2) = \frac{-if^{ab}_c}{(\bar{z}_1-\bar{z}_2)}\psi^{p+q-1,c}(\bar{z}_2) + \cdots. \quad (3.12)$$

In terms of the commutator, we obtain

$$[J_m^{p,a}, \psi_n^{q,b}] = -if^{ab}_c\psi_{m+n}^{p+q-1,c}. \quad (3.13)$$

As usual, from the Jacobi identities between the generalized affine (bosonic and fermionic) currents, the sign of the right-hand side of (3.13) can be fixed by using the Jacobi identity between the structure constants.

For the final OPE we would like to construct, we expect that the OPE obtained will be similar to the OPE reported in (3.10). The right-hand side of the OPE between $w^p(\bar{z}_1)\psi^{q,a}(\bar{z}_2)$ should contain the fermionic current having an adjoint index a. Then we obtain the following OPE

$$w^p(\bar{z}_1)\psi^{q,a}(\bar{z}_2) = \frac{(p+q-2)}{(\bar{z}_1-\bar{z}_2)^2}\psi^{p+q-2,a}(\bar{z}_2) + \frac{(p-1)}{(\bar{z}_1-\bar{z}_2)}\bar{\partial}\psi^{p+q-2,a}(\bar{z}_2) + \cdots. \quad (3.14)$$

The corresponding commutator relation, obtained by following the procedure reported in (3.9), can be expressed as follows:

$$[w_m^p, \psi_n^{q,a}] = [m(q-1)-n(p-1)]\psi_{m+n}^{p+q-2,a}. \quad (3.15)$$

This is a natural generalization in the sense that the fermionic current $\psi^{q,a}$ is the primary weight q under the stress energy tensor w^2.

Therefore, four additional OPEs, (3.10), (3.12), (3.14) and the second OPE in (3.9) corresponding to (3.11), (3.13), (3.15), and the second relation of (3.7), must be considered. They, under the wedge modes, correspond to the ones in (2.38) when hats are included. In this correspondence, we assume that the weight of the current w^p should be generalized to include the $p=1$ case. We can determine whether the graded Jacobi identities between four currents are satisfied.[27]

D. Appearance of a supersymmetric $w_{1+\infty}$ symmetry in the context of the celestial conformal field theory

Then we can compare the symmetry involved in the supersymmetric Einstein-Yang-Mills theory with the symmetry we described in the context of the conventional conform field theory, by using the following field correspondences,

$$\hat{w}^p \leftrightarrow w^p, \quad \hat{J}^{p,a} \leftrightarrow J^{p,a}, \quad \hat{G}^p \leftrightarrow G^p, \quad \hat{\psi}^{p,a} \leftrightarrow \psi^{p,a}. \quad (3.16)$$

In (3.16), the hatted currents have the $\frac{1}{z}$ terms in the holomorphic and antiholomorphic mode decomposition as in (2.3) and (2.4) while the unhatted ones have the standard antiholomorphic mode decomposition with unrestricted modes. The weights in the hatted currents are given by positive integers or half integers while the weights in the unhatted

[26] If we assume additional currents, then these currents will appear in other singular terms up to the central term at the $(p+q)$ th order pole. The expression for the relative coefficients can be found in [39,40].

[27] Three vanishing anticommutator relations are also considered. These are $\{\psi_m^{p,a}, \psi_n^{q,b}\} = 0$ associated with vanishing of the central term in the generalization of the affine current algebra, $\{G_m^p, G_n^q\} = 0$ from the result of topological w_∞ algebra, and $\{\psi_m^{p,a}, G_n^q\} = 0$. One realization for the algebra in this paper is given by the following expressions $w_m^p = iy^{p-2}e^{imx}[(p-1)\frac{\partial}{\partial x} - imy\frac{\partial}{\partial y}]$, $J_m^{q,a} = -it^a y^{q-1}e^{imx}$, $G_m^p = \theta w_m^p$ and $\psi_m^{q,a} = \theta J_m^{q,a}$ where there are no $\frac{\partial}{\partial\theta}$ terms from the analysis in [18].

currents are given by positive integers. After we impose the wedge modes onto the unhatted currents, we observe that the supersymmetric $w_{1+\infty}$ symmetry with $SU(N)$ can be described by the celestial conformal field theory.

IV. CONCLUSIONS AND OUTLOOK

We have identified the soft current algebra involved in the supersymmetric Einstein-Yang-Mills theory with the supersymmetric $w_{1+\infty}$ algebra where the subalgebra involves contraction of the (i) $\widehat{SU}(N)_k$ $W_{1+\infty}$ algebra and (ii) topological W_∞ algebra. Moreover, three additional OPEs (or commutator relations) are required for realizing the above soft algebra.

In this paper, we have considered the supersymmetric $w_{1+\infty}$ algebra in an abstract manner without presenting any (free field) realization. We expect that the bosonic $w_{1+\infty}$ algebra can be generalized to the $W_{1+\infty}$ algebra at the quantum level, and hence the quantum version of the supersymmetric $w_{1+\infty}$ algebra we have obtained must be determined. In other words, two subalgebras, with quantum versions that are known before we consider contractions, occur in our findings. Determining whether the full quantum version of the (classical) supersymmetric $w_{1+\infty}$ algebra exists is an open problem. Description of this algebra by the $\mathcal{N}=1$ supersymmetric theory may be challenging. In addition, consideration of the $\mathcal{N}=2$ supersymmetric W_∞ algebra and addition of the bosonic and fermionic affine currents with the help of the twisting procedure may be manageable. In the final check, the Jacobi identity will be used to fix the unknown structure constants.

Consider the case where the conformal weights of the right-hand side of the OPE are less than or equal to the sum of the conformal weights of the left-hand side of the OPE. In this case, the OPE between the currents in the W_∞ algebra contains other high-spin currents.

In other words, in this work, we have identified OPEs where the nontrivial singular terms are given by both the second- and the first-order poles in the antiholomorphic sector. We still face the challenge of identifying all the currents appearing in the singular terms higher than the third-order pole associated with the corresponding (supersymmetric) Einstein-Yang-Mills theory.

So far, we have considered the $\mathcal{N}=1$ supersymmetric Einstein-Yang-Mills theory. Examining the $\mathcal{N}=2$ version of this theory may also be an open problem. Are there any OPEs between the $\mathcal{N}=2$ soft currents in the celestial conformal field theory which correspond to the known $\mathcal{N}=2$ supersymmetric W_∞ algebra (and its variants)? However, A $\mathcal{N}=4$ supersymmetric high-spin algebra exists, as mentioned in the Introduction, and determining whether this algebra is described by the corresponding celestial conformal field theory is an open problem.

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APPENDIX A: A SOFT CURRENT ALGEBRA I

For convenience, we present the result of [2] as follows[28]:

$$
\begin{aligned}
[\hat{R}^{k,a}_m, \hat{R}^{l,b}_n] &= -i f^{ab}_c \frac{(\frac{1-k}{2}-m+\frac{1-l}{2}-n)!(\frac{1-k}{2}+m+\frac{1-l}{2}+n)!}{(\frac{1-k}{2}-m)!(\frac{1-l}{2}-n)!(\frac{1-k}{2}+m)!(\frac{1-l}{2}+n)!}\hat{R}^{k+l-1,c}_{m+n},\\
[\hat{H}^k_m, \hat{H}^l_n] &= \frac{\kappa}{2}[m(2-l)-n(2-k)]\frac{(\frac{2-k}{2}-m+\frac{2-l}{2}-n-1)!(\frac{2-k}{2}+m+\frac{2-l}{2}+n-1)!}{(\frac{2-k}{2}-m)!(\frac{2-l}{2}-n)!(\frac{2-k}{2}+m)!(\frac{2-l}{2}+n)!}\hat{H}^{k+l}_{m+n},\\
[\hat{H}^k_m, \hat{R}^{l,a}_n] &= \frac{\kappa}{2}[m(1-l)-n(2-k)]\frac{(\frac{2-k}{2}-m+\frac{1-l}{2}-n-1)!(\frac{2-k}{2}+m+\frac{1-l}{2}+n-1)!}{(\frac{2-k}{2}-m)!(\frac{1-l}{2}-n)!(\frac{2-k}{2}+m)!(\frac{1-l}{2}+n)!}\hat{R}^{k+l,a}_{m+n}.
\end{aligned}
\tag{A1}
$$

[28]We present two identities on the finite sums $\sum_{s=-m-\frac{k}{2}}^{1-k}\frac{(-1)^s(s+1)(2-s-k-l)!}{(1-s-k)!(s+m+\frac{k}{2})!(-m-n-\frac{k+l-2}{2}-s)!} = [-m(2-l)+n(2-k)]\times\frac{(1-l)!(\frac{2-k}{2}+m+\frac{2-l}{2}+n-1)!}{(-1)^{m+\frac{k}{2}}(\frac{2-l}{2}-n)!(\frac{2-k}{2}+m)!(\frac{2-l}{2}+n)!}$ and $\sum_{s=-m+\frac{1-k}{2}}^{1-k}\frac{(-1)^s(2-s-k-l)!}{(1-s-k)!(s+m+\frac{k-1}{2})!(\frac{1-k}{2}-m+\frac{1-l}{2}-n-s)!} = \frac{(1-l)!(\frac{1-k}{2}+m+\frac{1-l}{2}+n)!}{(-1)^{m+\frac{k-1}{2}}(\frac{1-l}{2}-n)!(\frac{1-k}{2}+m)!(\frac{1-l}{2}+n)!}$, which are used in [2]. We can check, for example, the first identity inside a mathematica (after introducing the function `function[k1_, l1_, m1_, n1_]` as a difference between the left hand and the right hand of identity) as follows: `result = Table[0, {k1, -10, 1}, {l1, -10, 1}, {m1, -10, 10}, {n1, -10, 10}]`. After that we perform `Do[result[[k1, l1, m1, n1]] = FullSimplify[function[k1, l1, m1, n1]]; Print["result[", k1, l1, m1, n1, "] == ", result[[k1, l1, m1, n1]]], {k1,-3,-1}, {l1,-3,-1}, {m1, 1, 3}, {n1, 1, 3}];`. Then we obtain the zeros for relevant and allowed `k1`, `l1`, `m1` and `n1`.

APPENDIX B: A SOFT CURRENT ALGEBRA II

The four additional commutator relations appearing in (2.23), (2.14), (2.30), and (2.36), are summarized as follows:

$$
\begin{aligned}
[\hat{R}^{k,a}_m, \hat{L}^{l,b}_n] &= -i f^{ab}_c \frac{(\frac{1-k}{2} - m + \frac{\frac{1}{2}-l}{2} - n)!(\frac{1-k}{2} + m + \frac{\frac{1}{2}-l}{2} + n)!}{(\frac{1-k}{2} - m)!(\frac{\frac{1}{2}-l}{2} - n)!(\frac{1-k}{2} + m)!(\frac{\frac{1}{2}-l}{2} + n)!} \hat{L}^{k+l-1,c}_{m+n}, \\
[\hat{H}^k_m, \hat{I}^l_n] &= \frac{\kappa}{2}\left[m\left(\frac{3}{2} - l\right) - n(2-k)\right] \frac{(\frac{2-k}{2} - m + \frac{\frac{3}{2}-l}{2} - n - 1)!(\frac{2-k}{2} + m + \frac{\frac{3}{2}-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)!(\frac{\frac{3}{2}-l}{2} - n)!(\frac{2-k}{2} + m)!(\frac{\frac{3}{2}-l}{2} + n)!} \hat{I}^{k+l}_{m+n}, \\
[\hat{R}^{k,a}_m, \hat{I}^l_n] &= \frac{\kappa}{2}\left[m\left(\frac{3}{2} - l\right) - n(1-k)\right] \frac{(\frac{1-k}{2} - m + \frac{\frac{3}{2}-l}{2} - n - 1)!(\frac{1-k}{2} + m + \frac{\frac{3}{2}-l}{2} + n - 1)!}{(\frac{1-k}{2} - m)!(\frac{\frac{3}{2}-l}{2} - n)!(\frac{1-k}{2} + m)!(\frac{\frac{3}{2}-l}{2} + n)!} \hat{L}^{k+l,a}_{m+n}, \\
[\hat{H}^k_m, \hat{L}^{l,a}_n] &= \frac{\kappa}{2}\left[m\left(\frac{1}{2} - l\right) - n(2-k)\right] \frac{(\frac{2-k}{2} - m + \frac{\frac{1}{2}-l}{2} - n - 1)!(\frac{2-k}{2} + m + \frac{\frac{1}{2}-l}{2} + n - 1)!}{(\frac{2-k}{2} - m)!(\frac{\frac{1}{2}-l}{2} - n)!(\frac{2-k}{2} + m)!(\frac{\frac{1}{2}-l}{2} + n)!} \hat{L}^{k+l,a}_{m+n}.
\end{aligned}
\tag{B1}
$$

Here, we present the nonsimplified versions of these expressions.

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