Couplings of $\mathcal{N}=4$, $d=1$ mirror supermultiplets

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We construct models of coupled semidynamical (spin) and dynamical mirror multiplets of $\mathcal{N}=4$ supersymmetric mechanics in $d=1$ harmonic superspace. Specifically, we consider a semidynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ coupled to dynamical mirror multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$. Coupling of the multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{1},\mathbf{4},\mathbf{3})$ yields a mirror counterpart of the earlier constructed model implying the Nahm equations for the spin variables with the bosonic component of the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ as an evolution parameter. We also couple the mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ to the mirror semidynamical multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ using chiral $\mathcal{N}=4$ superspace. The models constructed admit a generalization to the $\mathrm{SU}(2|1)$ deformation of $\mathcal{N}=4$, $d=1$ Poincaré supersymmetry.

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I. INTRODUCTION

Diverse models of the supersymmetric quantum mechanics (SQM) as an extreme (one-dimensional) supersymmetric theory provide a good laboratory for studying more ambitious higher-dimensional supersymmetric theories, such as super Yang-Mills and supergravity theories, the higher-spin theories, etc (see, e.g., Ref. [1] for a review). The simplest extended SQM models are associated with $\mathcal{N}=4$, $d=1$ supersymmetry. One of the surprising features of this supersymmetry is the existence of two different types of $\mathcal{N}=4$ supermultiplets which are "mirror" (or "twisted") with respect to each other.

The origin of such a doubling is as follows. The $\mathcal{N}=4$, $d=1$ Poincaré superalgebra reads

$$\{Q^i_\beta, Q^\alpha_j\} = 2\delta^i_j\delta^\alpha_\beta H, \qquad [H, Q^i_\beta] = 0, \tag{1.1}$$

where H is the Hamiltonian which in the superfield (or component) Lagrangian setting is realized as a time derivative. Four supercharges Q^i_β carry the indices of the fundamental representation of the corresponding automorphism $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}$ group ($i=1,\,2$ and $\alpha=1,\,2$). Their permutation as $i,j\leftrightarrow\alpha,\beta$ has no impact on the algebra (1.1). As a result, $\mathcal{N}=4$, $d=1$ supersymmetry possesses two wide classes of the supermultiplets, which differ just by interchanging of these two independent SU(2) factors of the total automorphism group. The mutual interchange of these two SU(2) groups switches ordinary multiplets into mirror ones and vice versa. When limiting only to one type of such multiplets and considering their various invariant actions and interactions, no actual difference from another type can be observed: indeed, all quantities associated with the alternative choice can be reproduced from the initial choice just by substituting the $\mathrm{SU}(2)_\mathrm{L}$ group indices altogether by the appropriate $\mathrm{SU}(2)_\mathrm{R}$ ones. The difference between the two varieties of the multiplets manifests itself only when considering both types of them *simultaneously*.[1]

For a fixed choice of SU(2) [$\mathrm{SU}(2)_\mathrm{L}$ in what follows], a plethora of relevant multiplets was studied in many papers, using the appropriate $\mathcal{N}=4$, $d=1$ superspace approaches in which just this SU(2) invariance is manifest.[2] We will refer to these multiplets as the "ordinary" ones. The best arena for dealing with such multiplets and constructing their interactions is provided by $\mathcal{N}=4$, $d=1$ harmonic superspace [3] involving the harmonic variables which parametrize the coset $\mathrm{SU}(2)_\mathrm{L}/\mathrm{U}(1)_\mathrm{L}$. The second $\mathrm{SU}(2)_\mathrm{R}$ symmetry is realized as a kind of hidden symmetry. On the other hand, in order to put the description of both types of $\mathcal{N}=4$ multiplets on equal footing, the formalism of "biharmonic superspace" was worked out in Ref. [4], with both automorphism SU(2) factors being "harmonized." However, dealing with the two sets of harmonic variables sometimes bears technical complications. So, it would be

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[1]A similar phenomenon takes place for the standard and twisted chiral superfields in 2*D* supersymmetry [2].

[2]The basic $d=1$ superspace technicalities are collected in Appendix A.

advantageous to have a description of the mirror multiplets within the same superspace setting as the more accustomed ordinary $\mathcal{N}=4$ multiplets. The present note is devoted to such an alternative description of mirror multiplets and demonstrating that various interactions between them basically lead to the same component results as those for the ordinary multiplets, modulo the interchange of the $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ automorphism groups mentioned above. We focus on couplings of the dynamical mirror multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$ to the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ considered as semidynamical (or as a "spin multiplet") because the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ was not considered before in such a context. Another reason is that just this kind of coupling admits a rather direct generalization to the case of deformed $\mathcal{N}=4$, $d=1$ supersymmetry associated with the supergroup $\mathrm{SU}(2|1)$ [5–7]. We explicitly present the basic relations of the $\mathrm{SU}(2|1)$ deformed mirror system $(\mathbf{1},\mathbf{4},\mathbf{3})-(\mathbf{3},\mathbf{4},\mathbf{1})$.

II. MIRROR MULTIPLETS

We proceed from the standard $\mathcal{N}=4$, $d=1$ superspace and its simplest harmonic extension described in Appendix A.

An important observation exploited in what follows is that all the standard mirror multiplets with four fermionic physical fields and linear $\mathcal{N}=4$ supersymmetry transformation laws are described by the superfields M which carry no external $\mathrm{SU}(2)_{\mathrm{L}}$ indices [but admit those of $\mathrm{SU}(2)_{\mathrm{R}}$] and satisfy the universal common constraint

$$D^{(i}_{\gamma}D^{j)\gamma}M=0. \tag{2.1}$$

In the harmonic $\mathcal{N}=4$, $d=1$ superspace approach, these superfields are neutral [with respect to the harmonic $\mathrm{U}(1)_{\mathrm{L}}$ charge] and can be defined by the following equivalent constraints:

$$D^{++}M=0,\qquad D^{0}M=0,\qquad D^{+}_{\gamma}D^{+\gamma}M=0. \tag{2.2}$$

The specificity of one or another mirror multiplet manifests itself in the extra constraints one needs to impose on M. Below, we list all $\mathcal{N}=4$ superfield constraints of this kind yielding the complete set of the linear mirror multiplets.

Mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$.—The mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is described by a real superfield X satisfying [8]

$$D^{(i}_{\alpha}D^{j)\alpha}X=0\Leftrightarrow D^{+}_{\alpha}D^{+\alpha}X=0,\qquad D^{++}X=0. \tag{2.3}$$

So, in this simplest case, no any extra constraints are needed.

Mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ *or chiral multiplet*.—The mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by the standard complex chiral $\mathcal{N}=4$, $d=1$ superfield:

$$\bar{D}^{i}Z=0,\qquad \bar{D}^{i}\coloneqq D^{i\alpha=2}=D^{i}_{\alpha=1},$$
$$\Leftrightarrow D^{+\alpha=2}Z=0,\qquad D^{++}Z=0. \tag{2.4}$$

Thus, in the universal description by a superfield M, it is natural to interpret the standard chiral $\mathcal{N}=4$, $d=1$ multiplet as belonging to the mirror type, while the twisted chiral multiplet studied in Refs. [8,9] should be reckoned to the set of ordinary multiplets.

Mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$.—The mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is described by a triplet superfield $V^{\alpha\beta}$ ($V^{\alpha\beta}=V^{\beta\alpha}$, $\overline{(V^{\alpha\beta})}=-V_{\alpha\beta}$) satisfying

$$D^{i(\alpha}V^{\beta\gamma)}=0\Leftrightarrow D^{+(\alpha}V^{\beta\gamma)}=0,\quad D^{++}V^{\alpha\beta}=0. \tag{2.5}$$

Mirror multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$.—The mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet is described by a quartet superfield $Y^{\alpha A}$ ($A=1,2$) that satisfies the constraints

$$D^{i(\alpha}Y^{\beta)A}=0,\qquad \overline{(Y^{\alpha A})}=Y_{\alpha A}. \tag{2.6}$$

Their equivalent harmonic superspace form is

$$D^{+(\alpha}Y^{\beta)A}=0,\qquad D^{++}Y^{\alpha A}=0. \tag{2.7}$$

Mirror multiplet $(\mathbf{0},\mathbf{4},\mathbf{4})$.—In contrast to the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$, the mirror multiplet $(\mathbf{0},\mathbf{4},\mathbf{4})$ is described by a fermionic superfield $\Psi^{\alpha A}$ ($A=1,2$) [10]:

$$D^{i(\alpha}\Psi^{\beta)A}=0,\qquad \overline{(\Psi^{\alpha A})}=\Psi_{\alpha A}$$
$$\Rightarrow D^{+(\alpha}\Psi^{\beta)A}=0,\qquad D^{++}\Psi^{\alpha A}=0. \tag{2.8}$$

All its bosonic components are auxiliary fields.

The component solutions of the constraints for the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$, $(\mathbf{2},\mathbf{4},\mathbf{2})$, and $(\mathbf{3},\mathbf{4},\mathbf{1})$ are given by Eqs. (4.1), (5.1), and (3.1), respectively. Solutions for the remaining two multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$ are presented in Appendix B.

One can check that all superfields listed above indeed satisfy the common constraint (2.1). For $(\mathbf{1},\mathbf{4},\mathbf{3})$, it is obvious. For the rest of supermultiplets, Eq. (2.1) is recovered as a result of action of the appropriate covariant derivative on the basic constraints. For an instructive example, we perform this exercise for the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$:

$$D^{(j}_{\alpha}[D^{i)(\alpha}V^{\beta\gamma)}]=0\Rightarrow D^{(j}_{\alpha}D^{i)\alpha}V^{\beta\gamma}=0. \tag{2.9}$$

It is also worth pointing out that the chirality constraint (2.4) is also valid for some of the superfields describing the mirror multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$, $(\mathbf{4},\mathbf{4},\mathbf{0})$, and $(\mathbf{0},\mathbf{4},\mathbf{4})$, as a part of the full sets of their constraints,

$$D^{i2}V^{22}=0,\qquad D^{i2}Y^{2A}=0,\qquad D^{i2}\Psi^{2A}=0. \tag{2.10}$$

The remaining constraints relate these chiral superfields to other (nonchiral) superfields forming a given $\mathcal{N}=4$, $d=1$ supermultiplet. This property allows one to construct the interacting Lagrangians as the proper superpotentials. In Sec. V, such an interaction is given for the coupled mirror $(\mathbf{2},\mathbf{4},\mathbf{2})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ supermultiplets.

A. Wess-Zumino action

The Wess-Zumino (WZ-)type actions (analytic superpotentials) were defined in Ref. [3] for the ordinary multiplets as integrals over the analytic superspace,

$$S'_{\rm WZ}=\int d\zeta^{--}_{(\rm A)}\mathcal{L}^{++},\qquad D^{+\alpha}\mathcal{L}^{++}=0. \tag{2.11}$$

Here, $\mathcal{L}^{++}$ is an analytic function of harmonic analytic superfields and harmonic variables. Such a construction is admissible for the ordinary multiplets $(\mathbf{0},\mathbf{4},\mathbf{4})$, $(\mathbf{3},\mathbf{4},\mathbf{1})$, and $(\mathbf{4},\mathbf{4},\mathbf{0})$, which are described by analytic superfields additionally constrained by the proper harmonic conditions involving the analyticity-preserving harmonic derivative D^{++}.

WZ actions for the mirror superfields have the same formulation in the relevant mirror (analytic) harmonic superspace forming a subspace in the biharmonic superspace. However, in this paper, we prefer to construct WZ actions for mirror multiplets in the standard (ordinary) harmonic superspace. One of the merits of this construction is that it allows a deformation to SU(2|1) supersymmetry [11].

So, we are going to consider an alternative construction of WZ action for mirror multiplets in the (ordinary) analytic harmonic superspace $\{\zeta_{(\rm A)}\}$. Since mirror superfields carry no external harmonic charges, the only way to compensate the negative harmonic charge -2 of the invariant measure $d\zeta^{--}_{(\rm A)}$ is to include the charged objects, viz., the covariant derivatives and/or superspace coordinates. We will try the simplest option

$$S_{\rm WZ}=\int d\zeta^{--}_{(\rm A)}\theta^+_\alpha D^+_\beta L^{\alpha\beta}, \tag{2.12}$$

where $L^{\alpha\beta}$ is a triplet function ($L^{\alpha\beta}=L^{\beta\alpha}$) of mirror superfields that satisfies

$$D^0L^{\alpha\beta}=0,\qquad D^{++}L^{\alpha\beta}=0,\qquad D^+_\gamma D^{+\gamma}L^{\alpha\beta}=0. \tag{2.13}$$

The last quadratic constraint secures the analyticity of the Lagrangian density, $D^{+\gamma}(D^+_\beta L^{\alpha\beta})=0$, and hence the invariance of WZ action (2.12),

$$\delta S_{\rm WZ}=\int d\zeta^{--}_{(\rm A)}\epsilon^+_\alpha D^+_\beta L^{\alpha\beta}=\int d\zeta^{--}_{(\rm A)}D^{++}(\epsilon^-_\alpha D^+_\beta L^{\alpha\beta})=0, \tag{2.14}$$

where we represented $\epsilon^+_\alpha=D^{++}\epsilon^-_\alpha$ and integrated by parts with respect to D^{++}.

Note that we could start from the superfield function $L^{\alpha\beta}$ having a singlet part L ($L=\varepsilon_{\alpha\beta}L^{\alpha\beta}$) and still satisfying the same constraints (2.13). This part can be discarded because the relevant action is vanishing:

$$\int d\zeta^{--}_{(\rm A)}\theta^+_\alpha D^{+\alpha}L=\int d\zeta^{--}_{(\rm A)}D^{++}(\theta^-_\alpha D^{+\alpha}L-L)=0,$$
$$D^{+\gamma}(\theta^-_\alpha D^{+\alpha}L-L)=0. \tag{2.15}$$

III. SPIN MIRROR MULTIPLET (3, 4, 1)

In this section, we treat the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as semidynamical and construct its general WZ action.

The constraints (2.5) are solved by

$$V^{\alpha\beta}=v^{\alpha\beta}+\theta^{-(\alpha}\chi^{i\beta)}u^+_i-\theta^{+(\alpha}\chi^{i\beta)}u^-_i-2i\theta^{-(\alpha}\theta^+_\gamma\dot v^{\beta)\gamma}$$
$$+\theta^{-(\alpha}\theta^{+\beta)}C-i\theta^{+\gamma}\theta^+_\gamma\theta^{-(\alpha}\dot\chi^{i\beta)}u^-_i, \tag{3.1}$$

where

$$\overline{(v^{\alpha\beta})}=-v_{\alpha\beta},\qquad \overline{(\chi^{k\alpha})}=-\chi_{k\alpha},\qquad \overline{(C)}=C. \tag{3.2}$$

The component fields transform under $\mathcal{N}=4$, $d=1$ supersymmetry as

$$\delta v^{\alpha\beta}=\epsilon^{i(\alpha}\chi^{\beta)}_i,\qquad \delta\chi^{i\alpha}=2i\epsilon^i_\beta\dot v^{\alpha\beta}-\epsilon^{i\alpha}C,$$
$$\delta C=-i\epsilon_{i\alpha}\dot\chi^{i\alpha},\qquad \overline{(\epsilon^{i\alpha})}=-\epsilon_{i\alpha}. \tag{3.3}$$

Let us first construct a Fayet-Iliopoulos term as the simplest example of WZ action with $L^{\alpha\beta}\sim V^{\alpha\beta}$:

$$S_{\rm FI}=\frac{b}{3}\int d\zeta^{--}_{(\rm A)}\theta^+_\alpha D^+_\beta V^{\alpha\beta}=\int dt\mathcal{L}_{\rm FI},\qquad \mathcal{L}_{\rm FI}=bC. \tag{3.4}$$

A less trivial WZ action for $V^{\alpha\beta}$ is constructed according to the prescription (2.12) as

$$S_{\rm WZ}=\int dt\,\mathcal{L}_{\rm WZ}=\int d\zeta^{--}_{(\rm A)}\theta^+_\alpha D^+_\beta L^{\alpha\beta}(V). \tag{3.5}$$

The zero-order component of the θ expansion of the last constraint in (2.13) imposes three-dimensional Laplace equation on the Lagrangian density $L^{\alpha\beta}(v)$:

$$\Delta_{(3)}L^{\alpha\beta}(v)=0,\qquad \Delta_{(3)}=\partial^{\gamma\delta}\partial_{\gamma\delta},\qquad \partial^{\gamma\delta}=\partial/\partial v_{\gamma\delta}. \tag{3.6}$$

In components, using the solution (3.1) for $V^{(\alpha\beta)}$, we obtain

$$\mathcal{L}_{\rm WZ}=C\mathcal{U}+i\dot v^{\alpha\beta}\mathcal{A}_{\alpha\beta}+\frac{1}{2}\mathcal{R}^{\alpha\beta}\chi^i_\alpha\chi_{i\beta}, \tag{3.7}$$

with

$$\mathcal{U}(v) = \partial^{\alpha\beta} L_{\alpha\beta}(v),$$
$$\mathcal{A}_{\alpha\beta}(v) = \varepsilon_{\alpha\gamma}\partial^{\gamma\delta} L_{\beta\delta}(v) + \varepsilon_{\beta\gamma}\partial^{\gamma\delta} L_{\alpha\delta}(v),$$
$$\mathcal{R}^{\alpha\beta}(v) = \partial^{\alpha\gamma}\partial^{\beta\delta} L_{\gamma\delta}(v). \tag{3.8}$$

Taking into account the constraint (3.6), one finds that the quantities defined in (3.8) satisfy the conditions

$$\partial_{\alpha\beta}\mathcal{U} = \mathcal{R}_{\alpha\beta}, \qquad \Delta_{(3)}\mathcal{U} = \Delta_{(3)}\mathcal{R}_{\alpha\beta} = 0,$$
$$\partial_{\alpha\beta}\mathcal{A}_{\gamma\delta} - \partial_{\gamma\delta}\mathcal{A}_{\alpha\beta} = \varepsilon_{\alpha\gamma}\mathcal{R}_{\beta\delta} + \varepsilon_{\beta\delta}\mathcal{R}_{\alpha\gamma}. \tag{3.9}$$

Swapping, in the Lagrangian (3.7) and the constraints (3.9), the indices α, β and i, j, we obtain just WZ Lagrangian for the ordinary multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ constructed in Ref. [3]. Thus, the above formulas yield the correct form of the WZ action for the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$.

Eliminating the fermionic fields in (3.7) by their equations of motion, we pass to the Hamiltonian system with

$$H = \lambda^{\alpha\beta}\pi_{\alpha\beta} - C\mathcal{U}, \tag{3.10}$$

where $\lambda^{\alpha\beta}$ and C are treated as Lagrange multipliers. The second-class Hamiltonian constraints of the system are then given by

$$\pi_{\alpha\beta} = p_{\alpha\beta} - i\mathcal{A}_{\alpha\beta} \approx 0, \qquad \mathcal{U} \approx 0. \tag{3.11}$$

Note that the last constraint is a secondary one for the primary constraint $p_C \approx 0$.

The matrix formed by Poisson brackets of the constraints (3.11) is not degenerate,

$$\det\begin{pmatrix} \{\pi_{\alpha\beta},\pi_{\gamma\delta}\}_{\rm PB} & \{\pi_{\alpha\beta},\mathcal{U}\}_{\rm PB} \\ \{\mathcal{U},\pi_{\gamma\delta}\}_{\rm PB} & 0 \end{pmatrix} \neq 0, \tag{3.12}$$

and hence we can pass to Dirac brackets. Calculating the inverse matrix of these constraints, we find the Dirac brackets in the form

$$\{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i\,(\varepsilon_{\alpha\gamma}\mathcal{R}_{\beta\delta} + \varepsilon_{\beta\delta}\mathcal{R}_{\alpha\gamma})}{2\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}}. \tag{3.13}$$

One can check that

$$\{v_{\alpha\beta},\mathcal{U}\} = \frac{i(\mathcal{R}_{\beta\gamma}\mathcal{R}^{\gamma}_{\alpha} + \mathcal{R}_{\alpha\gamma}\mathcal{R}^{\gamma}_{\beta})}{2\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}} = 0. \tag{3.14}$$

The constraint $\mathcal{U} \approx 0$ kills 1 degree of freedom in the triplet $v^{\alpha\beta}$, so this triplet effectively describes a two-dimensional surface embedded in $\mathbb{R}^3$.

A. Noncommutative plane

Let us consider the simplest solution of the Laplace equation $\Delta_{(3)}\mathcal{U} = 0$,

$$\mathcal{U} = \frac{c-y}{2}, \quad c = \text{const}, \tag{3.15}$$

where

$$v_{12} = y, \qquad v_{11} = -\sqrt{2}u, \qquad v_{22} = \sqrt{2}\bar{u}. \tag{3.16}$$

It corresponds to the following choice of the triplet function $L^{\alpha\beta}$:

$$L^{11} = 0, \quad L^{22} = 0, \quad L^{12} = \frac{1}{4}(y^2 - u\bar{u} - 2cy). \tag{3.17}$$

The relevant Lagrangian is then written as

$$\mathcal{L}_{\rm WZ} = \frac{i}{2}(u\dot{\bar{u}} - \dot{u}\,\bar{u}) + \frac{C}{2}(c-y) - \frac{1}{4}\chi^i_1\chi_{i2}. \tag{3.18}$$

It is straightforward to check that it is invariant off shell under the following $\mathcal{N}=4$ supersymmetry transformations (3.3). The Lagrangian (3.18) is none other than $\mathcal{N}=4$, $d=1$ supersymmetrization of the $d=1$ WZ Lagrangian describing the lowest level of the planar Landau model (see, e.g., Ref. [12] for a review). In particular, besides the standard phase $\mathrm{U}(1)_R$ transformations, it is invariant under the so-called magnetic translations

$$\delta u = \lambda, \qquad \delta\bar{u} = \bar{\lambda}, \tag{3.19}$$

with λ being a complex parameter. It is worth noting that the analogous system for the ordinary multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is described by the action (2.11) with $\mathcal{L}^{++} \sim V^{++} + c^{--}(V^{++})^2$, where the analytic superfield V^{++} satisfies the constraint $D^{++}V^{++} = 0$ and $c^{--} = c^{ik}u^-_i u^-_k$. Fixing $\mathrm{SU}(2)_{\rm L}$ frame as $c^{11} = c^{22} = 0$, $c^{12} \neq 0$, and making the appropriate redefinitions, one arrives at the WZ Lagrangian (3.18) with the swapped $\mathrm{SU}(2)_{\rm L,R}$ indices.[3]

The matrix of the second-class constraints in this case takes the very simple nondegenerate form

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}. \tag{3.20}$$

The Dirac brackets are

$$\{u,\bar{u}\} = i, \qquad \{y,u\} = 0, \qquad \{y,\bar{u}\} = 0. \tag{3.21}$$

The complex field u describes a noncommutative plane in $\mathbb{R}^3$, while the third coordinate (component) y, perpendicular to this plane, takes the constant value $y = c$.

[3] It is curious that the magnetic translations (3.19) are realized in the ordinary description as $\delta V^{++} = \lambda^{ik}u^+_i u^+_k$, $c^{ik}\lambda_{ik} = 0$. There is the corresponding superfield realization of these transformations in the mirror description, too.

In Ref. [13], the fuzzy sphere solution was considered [for the ordinary $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet] as a solution of the three-dimensional Laplace equation:

$$\mathcal{U}\sim\frac{1}{\sqrt{y^2+2u\bar{u}}},\qquad(\partial_y^2+2\partial_u\partial_{\bar{u}})\mathcal{U}=0.\qquad(3.22)$$

The noncommutative plane was not considered, so here we fill this gap. The noncommutative plane is the planar limit of the fuzzy sphere. We choose the suitable solution by shifting the center of the sphere as

$$\mathcal{U}=\frac{1}{2}\left[c+R-\frac{R^2}{\sqrt{(y-R)^2+2u\bar{u}}}\right],\qquad(3.23)$$

with R being the radius. In the limit $R\to\infty$, we recover the plane solution (3.15).

Note that an actual effect of considering the $\mathcal{N}=4$, $d=1$ WZ Lagrangians in the present context is manifested while coupling them to the matter $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet, where these Lagrangians give rise to additional on-shell potential terms and Yukawa-type couplings (see the next section).

IV. MIRROR SYSTEM (3, 4, 1)–(1, 4, 3) AND NAHM EQUATIONS

For an instructive example, we consider the simplest coupling of the semidynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ and the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$. In fact, we consider the same model as the one constructed in Ref. [13], but in terms of mirror superfields. Swapping α, β and i, j indices, we reproduce its Lagrangian and Nahm equations associated with its Hamiltonian formulation. In the end, we will consider a deformation to SU(2|1) supersymmetry.

A. Dynamical mirror multiplet (1, 4, 3)

The duality between two $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets was studied in Ref. [14] and, later on, in Ref. [8]. It was shown there that, inserting the constraints (2.3) into the invariant action with a superfield Lagrangian multiplier and integrating the superfield X out, we obtain the action and constraint for the ordinary multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ described by the former superfield Lagrangian multiplier. In our terminology, the mirror $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet is described just by the superfield X with the constraints (2.3).

Solving this constraint, we obtain

$$X=x-\theta^-_\alpha\psi^{i\alpha}u^+_i+\theta^+_\alpha\psi^{i\alpha}u^-_i+\theta^-_{(\alpha}\theta^+_{\beta)}A^{\alpha\beta}$$
$$+i\theta^-_\alpha\theta^{+\alpha}\dot{x}+i\theta^{+\alpha}\theta^+_\alpha\theta^-_\beta\dot{\psi}^{i\beta}u^-_i,\qquad(4.1)$$

where

$$\overline{(x)}=x,\qquad\overline{(\psi^{i\alpha})}=\psi_{i\alpha},\qquad\overline{(A^{\alpha\beta})}=-A_{\alpha\beta}.\qquad(4.2)$$

Supersymmetry transformations are

$$\delta x=\epsilon_{i\alpha}\psi^{i\alpha},\qquad\delta\psi^{i\alpha}=\epsilon^i_\beta A^{\alpha\beta}+i\epsilon^{i\alpha}\dot{x},\qquad\delta A^{\alpha\beta}=2i\epsilon^{i(\alpha}\dot{\psi}^{\beta)}_i.\qquad(4.3)$$

The kinetic Lagrangian for the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is given by the superfield action [8]

$$S_{\rm kin}=\frac{1}{2}\int d\zeta_{\rm H}\,f(X)=\int dt\,\mathcal{L}_{\rm kin}.\qquad(4.4)$$

The component Lagrangian reads

$$\mathcal{L}_{\rm kin}=g\left[\frac{\dot{x}^2}{2}+\frac{i}{2}\psi^{i\alpha}\dot{\psi}_{i\alpha}-\frac{A^{\alpha\beta}A_{\alpha\beta}}{4}\right]-\frac{1}{4}g'A^{\alpha\beta}\psi^i_\alpha\psi_{i\beta}$$
$$-\frac{1}{24}g''\psi^i_\alpha\psi_{i\beta}\psi^{j\alpha}\psi^\beta_j,\qquad(4.5)$$

where $g\coloneqq g(x)=f''(x)$.

The relevant Fayet-Iliopoulos term is defined as

$$S_{\rm FI}=b^{\alpha\beta}\int d\zeta^{--}_{\rm (A)}\theta^+_\alpha D^+_\beta X=\int dt\mathcal{L}_{\rm FI},$$
$$\mathcal{L}_{\rm FI}=b^{\alpha\beta}A_{\alpha\beta}.\qquad(4.6)$$

B. Couplings and total Lagrangian

The total Lagrangian is a sum of three Lagrangians:

$$\mathcal{L}_{\rm tot}=\mathcal{L}_{\rm kin}+\mathcal{L}_{\rm WZ}+\mathcal{L}_{\rm int}.\qquad(4.7)$$

The kinetic and WZ Lagrangians are given by (4.5) and (3.7). We could add the Fayet-Iliopoulos Lagrangians (3.4) and (4.6), but they bring only potential terms and therefore have no impact on the structure of brackets, which is our main subject here. The Lagrangian $\mathcal{L}_{\rm int}$. describes an interaction of the mirror $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplets,

$$S_{\rm int}=\int dt\mathcal{L}_{\rm int}=\frac{\mu}{2}\int d\zeta^{--}_{\rm A}h^{++},\qquad(4.8)$$

where h^{++} is analytic. From Ref. [13] we know that the interaction term in the ordinary case involves both dynamical and semidynamical superfields linearly. Obviously, the same should be true for their mirror counterparts X and $V^{\alpha\beta}$. Supposing this, we find that the correct ansatz for h^{++} is

$$h^{++}=\theta^{+\alpha}V_{\alpha\beta}(D^{+\beta}X)+\frac{1}{3}\theta^{+\alpha}X(D^{+\beta}V_{\alpha\beta})$$
$$+\frac{1}{3}\theta^-_\gamma\theta^{+\gamma}(D^{+\alpha}V_{\alpha\beta})(D^{+\beta}X),$$
$$D^{+\gamma}h^{++}=0,\quad D^{++}h^{++}\neq0.\qquad(4.9)$$

Now, one can directly check that the action is invariant:

$$\delta S_{\rm int} = \frac{\mu}{2}\int d\zeta_{\rm A}^{--}\delta h^{++} = \frac{\mu}{2}\int d\zeta_{\rm A}^{--} D^{++}\delta h = 0,$$
$$D^{+\gamma}\delta h = 0,$$
$$\delta h = \Big[\epsilon^{-\alpha}V_{\alpha\beta}(D^{+\beta}X) + \frac{1}{3}\epsilon^{-\alpha}X(D^{+\beta}V_{\alpha\beta}) + \frac{1}{3}\epsilon^{-}_{\gamma}\theta^{-\gamma}(D^{+\alpha}V_{\alpha\beta})(D^{+\beta}X)\Big]. \tag{4.10}$$

The component Lagrangian is found to be

$$\mathcal{L}_{\rm int} = \frac{\mu}{2}(xC + A^{\alpha\beta}v_{\alpha\beta} - \psi^{i\alpha}\chi_{i\alpha}). \tag{4.11}$$

Eliminating the auxiliary fields $\chi^{i\alpha}$ and $A^{\alpha\beta}$ by their equations of motion, we obtain the total Lagrangian:

$$\mathcal{L}_{\rm tot} = g\Big[\frac{\dot{x}^2}{2} + \frac{i}{2}\psi^{i\alpha}\dot{\psi}_{i\alpha}\Big] + \frac{\mu^2 v^{\alpha\beta}v_{\alpha\beta}}{4g} - \frac{\mu g' v_{\alpha\beta}}{4g}\psi^{i\alpha}\psi^{\beta}_{i} + i\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta} - \frac{\mu^2\mathcal{R}^{\alpha\beta}\psi^{i}_{\alpha}\psi_{i\beta}}{4\mathcal{R}^{\gamma\delta}\mathcal{R}_{\gamma\delta}} - \frac{1}{24}\Big[g'' - \frac{3(g')^2}{2g}\Big]\psi^{i}_{\alpha}\psi_{i\beta}\psi^{j\alpha}\psi^{\beta}_{j} + C\Big(\frac{\mu x}{2} + \mathcal{U}\Big). \tag{4.12}$$

C. Nahm equations

The Hamiltonian corresponding to (4.12) reads

$$H = \frac{p^2}{2g} - \frac{\mu^2 v^{\alpha\beta}v_{\alpha\beta}}{4g} + \frac{\mu g' v_{\alpha\beta}}{4g}\psi^{i\alpha}\psi^{\beta}_{i} + \frac{\mu^2\mathcal{R}^{\alpha\beta}\psi^{i}_{\alpha}\psi_{i\beta}}{4\mathcal{R}^{\gamma\delta}\mathcal{R}_{\gamma\delta}} + \frac{1}{24}\Big[g'' - \frac{3(g')^2}{2g}\Big]\psi^{i}_{\alpha}\psi_{i\beta}\psi^{j\alpha}\psi^{\beta}_{j} + \lambda^{\alpha\beta}\pi_{\alpha\beta} - C\Big(\frac{\mu x}{2} + \mathcal{U}\Big) + \tilde{\lambda}^{i\alpha}\tilde{\pi}_{i\alpha}. \tag{4.13}$$

The relevant Hamiltonian constraints are

$$\pi_{\alpha\beta} = p_{\alpha\beta} - i\mathcal{A}_{\alpha\beta} \approx 0, \quad h = \frac{\mu x}{2} + \mathcal{U} \approx 0, \quad \tilde{\pi}_{i\alpha} = p_{i\alpha} + \frac{i}{2}g\psi_{i\alpha}. \tag{4.14}$$

We observe that the $\mathcal{N}=4$ supersymmetric coupling to the mirror dynamical multiplet modifies the previous constraint $\mathcal{U}\approx 0$ as

$$h = \mathcal{U} + \frac{\mu x}{2} \approx 0. \tag{4.15}$$

It relates 1 degree of freedom of the spin variables $v^{\alpha\beta}$ to the dynamical bosonic field x.

The Dirac brackets are calculated as

$$\{x, p\} = 1, \qquad \{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i(\varepsilon_{\alpha\gamma}\mathcal{R}_{\beta\delta} + \varepsilon_{\beta\delta}\mathcal{R}_{\alpha\gamma})}{2\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}},$$
$$\{p, v_{\alpha\beta}\} = \frac{\mu\mathcal{R}_{\alpha\beta}}{2\mathcal{R}^{\lambda\mu}\mathcal{R}_{\lambda\mu}}, \qquad \{\psi^{i\alpha}, \psi_{j\beta}\} = -\frac{i}{g}\delta^{i}_{j}\delta^{\alpha}_{\beta},$$
$$\{p, \psi^{i\alpha}\} = \frac{1}{2g}g'\psi^{i\alpha}. \tag{4.16}$$

In complete analogy with the results of Ref. [13] for the ordinary multiplets, the triplet of spin variables $v^{\alpha\beta}$ describes two-dimensional surface in $\mathbb{R}^3$ defined by the equations:

$$\{v_{\alpha\beta}, v_{\gamma\delta}\} = \frac{i}{\mu}(\varepsilon_{\alpha\gamma}\{p, v_{\beta\delta}\} + \varepsilon_{\beta\delta}\{p, v_{\alpha\gamma}\}). \tag{4.17}$$

These are just famous Nahm equations [15],[4] and they can be put in the standard form as

$$\{p, v_c\} = \frac{1}{2}\varepsilon_{abc}\{v_a, v_b\}, \qquad v_{\alpha\gamma} \to \frac{v_a}{\mu}, \quad a = 1,2,3. \tag{4.18}$$

Here, x plays the role of evolution parameter, and p appears as a derivation with respect to the latter. Thus, we obtained a model equivalent to the model constructed earlier in Ref. [13]. To establish the exact equivalence, we need to interchange the SU(2) indices as $i, j \leftrightarrow \alpha, \beta$.

Let us consider as an example Nahm equations for the noncommutative plane (3.15). The constraint (4.15) implies that

$$y = \mu x + c. \tag{4.19}$$

We obtain the same Dirac brackets (3.21) for the spin variables. The relevant Nahm equations are written as

$$\{u, \bar{u}\} = \frac{i}{\mu}\{y, p\} = \frac{i}{\mu}\partial_x y, \qquad \{y, \bar{u}\} = -\frac{i}{\mu}\{\bar{u}, p\} = 0,$$
$$\{y, u\} = \frac{i}{\mu}\{u, p\} = 0, \tag{4.20}$$

where the perpendicular coordinate y is directly related to the dynamical component x.

The resume of this subsection is that the $\mathcal{N}=4$, $d=1$ supersymmetric coupling of the mirror dynamical $(\mathbf{1},\mathbf{4},\mathbf{3})$ and semidynamical $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplets reveals no new features compared to its analog for the ordinary multiplets of this kind. All the results, suggestions and conjectures of Ref. [13] apply for the mirror multiplets as well. In particular, just the Nahm equations of the type discussed

[4]To be more exact, it is some generalization of them (see, e.g., Ref. [16] and references therein).

above ensure the correct closure of $\mathcal{N}=4$ supercharges and the Hamiltonian in both the classical and the quantum cases.

D. SU(2|1) supersymmetry

We limit our consideration of the deformed SU(2|1), $d=1$ supersymmetry by the component level, following Refs. [11,17,18]. As was shown in Ref. [11], deformed multiplets and their mirror counterparts cease to be equivalent after such a deformation. In particular, WZ Lagrangians for the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ can be constructed only if it belongs to the mirror type. A similar situation is expected for the multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$. Until now, we were able to construct self-consistent SU(2|1) invariant WZ Lagrangians only for the mirror multiplets.

For a start, the centrally extended superalgebra $su(2|1)\oplus u(1)$ is defined by the following nonvanishing (anti)commutator[5]:

$$\begin{aligned}
&\{Q^i_\beta, Q^\alpha_j\} = 2\delta^i_j\delta^\alpha_\beta(H-mF) - 2m(\sigma_3)^\alpha_\beta I^i_j,\\
&[I^i_j, I^k_l] = \delta^k_j I^i_l - \delta^i_l I^k_j,\\
&[I^i_j, Q^{k\alpha}] = \delta^k_j Q^{i\alpha} - \frac{1}{2}\delta^i_j Q^{k\alpha}, \qquad [F, Q^{i\alpha}] = \frac{1}{2}(\sigma_3)^\alpha_\beta Q^{i\beta},\\
&[H, Q^i_\beta] = 0, \qquad [H,F]=0,\\
&[H, I^i_j] = 0, \qquad [I^i_j, F] = 0.
\end{aligned} \tag{4.21}$$

Here, σ_3 is the standard Pauli matrix,

$$(\sigma_3)^1_1 = -(\sigma_3)^2_2 = 1, \tag{4.22}$$

and the Hamiltonian H is treated as a central charge operator commuting with all other generators. The superalgebra (4.21) contains additional bosonic generators I^i_j and F which form the subalgebra $su(2)_\mathrm{L}\oplus u(1)_\mathrm{R}$. Hence, the equivalence between ordinary and mirror multiplets cannot be valid for SU(2|1) supersymmetry since swapping of the $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ indices yields a different superalgebra.

We skip details of solving superfield constraints and proceed to the component transformations and Lagrangians.

For the dynamical mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$, the deformation of the transformation laws (4.3) amounts to

$$\begin{aligned}
&\delta x = \epsilon_{i\alpha}\psi^{i\alpha},\\
&\delta A^{\alpha\beta} = 2\epsilon^{i(\alpha}[i\dot{\psi}^{\beta)}_i + m(\sigma_3)^{\beta)}_\gamma\psi^\gamma_i] + m(\sigma_3)^{\alpha\beta}\epsilon_{i\gamma}\psi^{i\gamma},\\
&\delta\psi^{i\alpha} = \epsilon^i_\beta A^{\alpha\beta} + i\epsilon^{i\alpha}\dot{x}.
\end{aligned} \tag{4.23}$$

The deformed kinetic Lagrangian invariant under these transformations is as follows:

$$\begin{aligned}
\mathcal{L}_{\rm kin} = {} & g\left[\frac{\dot{x}^2}{2} + \frac{i}{2}\psi^{i\alpha}\dot{\psi}_{i\alpha} - \frac{A^{\alpha\beta}A_{\alpha\beta}}{4} - \frac{m}{2}(\sigma_3)^\alpha_\beta\psi^{i\beta}\psi_{i\alpha}\right]\\
&- \frac{1}{4}g' A^{\alpha\beta}\psi^i_\alpha\psi_{i\beta} + \frac{m}{2}f'(\sigma_3)^{\alpha\beta}A_{\alpha\beta}\\
&- \frac{1}{24}g''\psi^i_\alpha\psi_{i\beta}\psi^{j\alpha}\psi^\beta_j.
\end{aligned} \tag{4.24}$$

The transformations (3.3) of the spin multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ are deformed as

$$\begin{aligned}
&\delta v^{\alpha\beta} = \epsilon^{i(\alpha}\chi^{\beta)}_i, \qquad \delta C = -i\epsilon_{i\alpha}\dot{\chi}^{i\alpha},\\
&\delta\chi^{i\alpha} = 2\epsilon^i_\beta[i\dot{v}^{\alpha\beta} + m(\sigma_3)^{(\alpha}_\gamma v^{\beta)\gamma}] - \epsilon^{i\alpha}C.
\end{aligned} \tag{4.25}$$

The deformed WZ Lagrangian is then given by

$$\begin{aligned}
\mathcal{L}_{\rm WZ} = {} & C\mathcal{U} + i\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta} + \frac{1}{2}\mathcal{R}^{\alpha\beta}\chi^i_\alpha\chi_{i\beta}\\
&+ m(\sigma_3)^\alpha_\beta v^{\beta\gamma}\mathcal{A}_{\alpha\gamma},
\end{aligned} \tag{4.26}$$

where the quantities $\mathcal{U},\mathcal{A}_{\alpha\beta}$, and $\mathcal{R}^{\alpha\beta}$ are still defined according to Eqs. (3.8) and (3.9), with the "prepotential" $L^{\alpha\beta}$ satisfying the three-dimensional Laplace equation (3.6). The SU(2|1) invariance requires the deformed Lagrangian (4.26) to be invariant under $\mathrm{U}(1)_\mathrm{R}$ symmetry,[6] which imposes additional conditions on (4.26):

$$\begin{aligned}
&m(\sigma_3)^\gamma_\lambda v^{\delta\lambda}\mathcal{R}_{\gamma\delta} = 0,\\
&m[(\sigma_3)^\lambda_\delta v^{\delta\gamma}\partial_{\gamma\lambda}\mathcal{A}_{\alpha\beta} + (\sigma_3)^\gamma_\alpha\mathcal{A}_{\beta\gamma}] = 0.
\end{aligned} \tag{4.27}$$

The necessity of these conditions for invariance of the Lagrangian (4.26) under the deformed transformations (4.25) can be directly checked.

[5] The deformed supercharges originally defined in Ref. [17] correspond to $Q^i \coloneqq Q^{i1}$, $\bar{Q}_j \coloneqq -Q_{j1}$.

[6] If we pass to the Hamiltonian $\tilde{H} \coloneqq H - mF$ [18], the $\mathrm{U}(1)_\mathrm{R}$ generator F becomes an external automorphism generator, and we can withdraw the condition (4.27). Passing to the new basis requires a redefinition of the component fields as

$$\begin{aligned}
&v^{\alpha\beta} \to \frac{1}{2}[v^{\alpha\gamma}e^{im(\sigma_3)^\beta_\gamma} + v^{\gamma\beta}e^{im(\sigma_3)^\alpha_\gamma}], \quad \chi^{i\alpha}\to\chi^{i\gamma}e^{\frac{i}{2}m(\sigma_3)^\alpha_\gamma}, \quad C\to C,\\
&A^{\alpha\beta} \to \frac{1}{2}[A^{\alpha\gamma}e^{im(\sigma_3)^\beta_\gamma} + A^{\gamma\beta}e^{im(\sigma_3)^\alpha_\gamma}], \quad \psi^{i\alpha}\to\psi^{i\gamma}e^{\frac{i}{2}m(\sigma_3)^\alpha_\gamma}, \quad x\to x.
\end{aligned}$$

In the new basis, the Lagrangian (4.26) gets undeformed, and the Lagrangian (4.11) stays undeformed, whereas the conditions (4.27) can be preserved (in this case, the Lagrangian is invariant under the external F automorphisms) or dismissed [in this case, no extra U(1) invariance is present]. In the second case, the inverse transformation to the original variables would yield a generalization of the Lagrangian (4.26) by some t-dependent terms breaking the invariance under time translations (with H as the corresponding generator). The requirement of the absence of such terms leads, once again, to Eqs. (4.27).

Surprisingly, the interaction term (4.11) is invariant under the deformed transformations (4.23) and (4.25) as it stands; i.e., it stays undeformed.

Finally, the total Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\rm tot} &= g\left[\frac{\dot{x}^2}{2}+\frac{i}{2}\psi^{i\alpha}\dot{\psi}_{i\alpha}-\frac{m}{2}(\sigma_3)^\alpha_\beta\psi^{i\beta}\psi_{i\alpha}\right]+\frac{1}{4g}[\mu v^{\alpha\beta}+mf'(\sigma_3)^{\alpha\beta}][\mu v_{\alpha\beta}+mf'(\sigma_3)_{\alpha\beta}]\\ &\quad-\frac{g'}{4g}[\mu v_{\alpha\beta}+mf'(\sigma_3)_{\alpha\beta}]\psi^{i\alpha}\psi^\beta_i+i\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta}-\frac{\mu^2\mathcal{R}^{\alpha\beta}\psi^i_\alpha\psi_{i\beta}}{4\mathcal{R}^{\gamma\delta}\mathcal{R}_{\gamma\delta}}+m(\sigma_3)^\alpha_\beta v^{\beta\gamma}\mathcal{A}_{\alpha\gamma}\\ &\quad-\frac{1}{24}\left[g''-\frac{3(g')^2}{2g}\right]\psi^i_\alpha\psi_{i\beta}\psi^{j\alpha}\psi^\beta_j+C\left(\frac{\mu x}{2}+\mathcal{U}\right).\end{aligned}\tag{4.28}$$

After passing to the Hamiltonian formalism, the brackets (4.16) and Nahm equations (4.17) keep their form. This is due to the fact that new terms $\sim m$ and $\sim m^2$ appear without time derivatives; i.e., they all are potential terms.

The SU(2|1) supercharges for the simplest free Lagrangian corresponding to $f = x^2/2$ and $g = 1$ are written as

$$Q^{i\alpha} = i\,p\,\psi^{i\alpha}+[\mu\, v^\alpha_\gamma+mx(\sigma_3)^\alpha_\gamma]\,\psi^{i\gamma}.\tag{4.29}$$

The bracket for the fermionic fields is simplified to

$$\{\psi^{i\alpha},\psi_{j\beta}\} = -i\delta^i_j\delta^\alpha_\beta.\tag{4.30}$$

Taking into account this bracket, we obtain that[7]

$$\begin{aligned}\{Q^i_\beta,Q^\alpha_j\}_{\rm cl} &= -i\delta^i_j\delta^\alpha_\beta\left(p^2-\frac{1}{2}[\mu v^{\gamma\delta}+mx(\sigma_3)^{\gamma\delta}][\mu v_{\gamma\delta}+mx(\sigma_3)_{\gamma\delta}]+\mu\{p,v^{\gamma\delta}\}\psi^k_\gamma\psi_{k\delta}\right)-\frac{i}{2}\delta^i_j m(\sigma_3)^\delta_\gamma\psi^{k\gamma}\psi_{k\delta}+im(\sigma_3)^\alpha_\beta\psi^{i\gamma}\psi_{j\gamma}\\ &\quad+\underline{i\mu\{p,v^{\alpha\delta}\}\psi^i_\delta\psi_{j\beta}+i\mu\{p,v_{\beta\gamma}\}\psi^{i\alpha}\psi^\gamma_j-\mu^2\{v_{\beta\gamma},v^{\alpha\delta}\}\psi^{i\gamma}\psi_{j\delta}}.\end{aligned}\tag{4.31}$$

The underlined expression vanishes due to the Nahm equations (4.17). Thus, the supercharges close on the following bosonic generators:

$$\begin{aligned}H-mF &= \frac{p^2}{2}-\frac{1}{4}[\mu v^{\alpha\beta}+m\,x(\sigma_3)^{\alpha\beta}][\mu v_{\alpha\beta}+m\,x(\sigma_3)_{\alpha\beta}]\\ &\quad+\frac{\mu}{2}\{p,v^{\alpha\beta}\}\psi^k_\alpha\psi_{k\beta}+\frac{m}{4}(\sigma_3)^\alpha_\beta\psi^{k\beta}\psi_{k\alpha},\\ I^i_j &= \frac{1}{2}\psi^{i\alpha}\psi_{j\alpha}.\end{aligned}\tag{4.32}$$

Then, the generator $\tilde{H} \coloneqq H - mF$ can be divided into

$$\begin{aligned}H &= \frac{p^2}{2}-\frac{1}{4}[\mu v^{\alpha\beta}+mx(\sigma_3)^{\alpha\beta}][\mu v_{\alpha\beta}+mx(\sigma_3)_{\alpha\beta}]\\ &\quad+\frac{\mu}{2}\{p,v^{\alpha\beta}\}\psi^k_\alpha\psi_{k\beta}+\frac{m}{2}(\sigma_3)^\alpha_\beta\psi^{k\beta}\psi_{k\alpha}-m(\sigma_3)^\alpha_\beta v^{\beta\gamma}\mathcal{A}_{\alpha\gamma},\\ F &= (\sigma_3)^\alpha_\beta\left[\frac{1}{4}\psi^{k\beta}\psi_{k\alpha}-v^{\beta\gamma}\mathcal{A}_{\alpha\gamma}\right].\end{aligned}\tag{4.33}$$

The term $\sim(\sigma_3)^\alpha_\beta v^{\beta\gamma}\mathcal{A}_{\alpha\gamma}$ enters as a part of both H and F, but it is absent in their combination $\tilde{H} = H - mF$. Because of the presence of this term, the correct commutators of H and F with supercharges are not guaranteed by the Nahm equations and the bracket (4.30) only. One also needs to make use of the whole set of the Dirac brackets (4.16) and to keep in mind the conditions (4.27). Thus, the Nahm equations (4.17) and the fermionic bracket (4.30) alone suffice to provide relations for the $su(2|1)$ superalgebra without central charge [18], in which the bosonic generator $\tilde{H} = H - mF$ plays the role of the Hamiltonian, while the generator F corresponds to the external automorphisms under which the Lagrangian is not obliged to be invariant (see the discussion in footnote 6).

V. COUPLING WITH A CHIRAL MULTIPLET

Here, we construct the superfield and component couplings of the mirror multiplets (**2**, **4**, **2**) and (**3**, **4**, **1**). The first multiplet is dynamical, while the second one is semidynamical. It turns out that the corresponding Lagrangians are formulated most directly in the standard $\mathcal{N} = 4$ superspace and its chiral and antichiral subspaces, without applying to the harmonic formalism. This system is

[7]Here, we deal with the classical (anti)commutators generated by Dirac brackets, when the right-hand sides in the superalgebra (4.21) are multiplied by $-i$.

considered here for the first time, and it can be regarded as the main new result of our paper.

A. Dynamical mirror multiplet (2, 4, 2)

The chiral $\mathcal{N}=4$ superfield as a solution of the constraints (2.4) is written as

$$Z(t_{\mathrm{L}},\theta_i)=z+\sqrt{2}\theta_k\xi^k+\theta_k\theta^k B. \tag{5.1}$$

The relevant off-shell supersymmetry transformations are

$$\delta z=-\sqrt{2}\epsilon_k\xi^k,\quad \delta\xi^i=\sqrt{2}i\bar\epsilon^i\dot z-\sqrt{2}\epsilon^i B,\quad \delta B=-\sqrt{2}i\bar\epsilon_k\dot\xi^k. \tag{5.2}$$

The total action for the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ can involve the kinetic and superpotential parts:

$$S_{(\mathbf{2},\mathbf{4},\mathbf{2})}=\frac{1}{4}\int dt\,d\bar\theta^2\,d\theta^2\,K(Z,\bar Z)+\frac{1}{2}\int dt_{\mathrm{L}}\,d^2\theta\,\mathcal{K}(Z)+\frac{1}{2}\int dt_{\mathrm{R}}\,d^2\bar\theta\,\bar{\mathcal{K}}(\bar Z). \tag{5.3}$$

The corresponding off-shell component Lagrangian reads

$$\begin{aligned}\mathcal{L}_{(\mathbf{2},\mathbf{4},\mathbf{2})}={}&g\left[\dot{\bar z}\,\dot z+\frac{i}{2}(\xi^k\dot{\bar\xi}_k-\dot\xi^k\bar\xi_k)+\bar B B\right]\\&+\frac{i}{2}(\dot{\bar z}\partial_{\bar z}g-\dot z\partial_z g)\xi^k\bar\xi_k+\frac{\bar B}{2}\partial_z g\xi^k\xi_k+\frac{B}{2}\partial_{\bar z}g\bar\xi_k\bar\xi^k\\&+\frac{1}{4}\partial_z\partial_{\bar z}g\xi^i\xi_i\bar\xi_j\bar\xi^j+\bar B\partial_{\bar z}\bar{\mathcal{K}}+B\partial_z\mathcal{K}\\&-\frac{1}{2}\xi_k\xi^k\partial_z\partial_z\mathcal{K}-\frac{1}{2}\bar\xi^k\bar\xi_k\partial_{\bar z}\partial_{\bar z}\bar{\mathcal{K}},\end{aligned} \tag{5.4}$$

where $g\coloneqq g(z,\bar z)=\partial_z\partial_{\bar z}K(z,\bar z)$.

B. Spin mirror multiplet (3, 4, 1) in the chiral superspace

The triplet superfield $V^{\alpha\beta}$ defined in (2.5) can be split into complex and real superfields as

$$V_{12}=Y,\qquad V_{11}=-\sqrt{2}U,\qquad V_{22}=\sqrt{2}\bar U. \tag{5.5}$$

The constraints (2.5) are rewritten as

$$D^i\bar U=0,\quad \bar D_i U=0,\quad \sqrt{2}D_iY=\bar D_i\bar U,\quad \sqrt{2}\bar D_iY=-D_iU, \tag{5.6}$$

where $D^i=D^{i1}$, $\bar D^i=D^{i2}$. Passing to the new basis at the component level

$$\begin{gathered}v_{12}=y,\qquad v_{11}=-\sqrt{2}u,\qquad v_{22}=\sqrt{2}\bar u,\\ \chi^i_1=-2\chi^i,\qquad \chi_{j2}=2\bar\chi_j,\qquad C=C,\\ \epsilon_i\coloneqq\epsilon_{i1},\qquad \bar\epsilon^i=\epsilon^i_2,\end{gathered} \tag{5.7}$$

we rewrite the off-shell transformations (2.5) as

$$\begin{gathered}\delta u=-\sqrt{2}\epsilon_k\chi^k,\quad \delta\bar u=\sqrt{2}\bar\epsilon^k\bar\chi_k,\quad \delta y=\bar\epsilon_k\chi^k+\epsilon^k\bar\chi_k,\\ \delta\chi^i=\sqrt{2}i\bar\epsilon^i\dot u+\frac{\epsilon^i}{2}(C+2i\dot y),\quad \delta\bar\chi_j=-\sqrt{2}i\epsilon_j\dot{\bar u}-\frac{\bar\epsilon_j}{2}(C-2i\dot y),\\ \delta C=2i(\bar\epsilon_k\dot\chi^k-\epsilon^k\dot{\bar\chi}_k).\end{gathered} \tag{5.8}$$

Obviously, the complex superfield U is chiral:

$$U(t_{\mathrm{L}},\theta_i)=u+\sqrt{2}\,\theta_k\chi^k-\frac{1}{2\sqrt{2}}\theta_k\theta^k(C+2i\dot y). \tag{5.9}$$

Since the superfield U is chiral, we can construct a superpotential as a real sum of the integrals over chiral and antichiral subspaces of the $\mathcal{N}=4$, $d=1$ superspace:

$$S_{\mathbf{pot}}=\int dt_{\mathrm{L}}d^2\theta\,\mathcal{M}(U)+\int dt_{\mathrm{R}}\,d^2\bar\theta\,\bar{\mathcal{M}}(\bar U). \tag{5.10}$$

It results in a WZ-type Lagrangian

$$\mathcal{L}_{\mathrm{pot}}=-\left[\sqrt{2}i\dot y\,(\partial_u\mathcal{M}-\partial_{\bar u}\bar{\mathcal{M}})+\frac{C}{\sqrt{2}}(\partial_u\mathcal{M}+\partial_{\bar u}\bar{\mathcal{M}})+\chi_k\chi^k\partial_u\partial_u\mathcal{M}+\bar\chi^k\bar\chi_k\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{M}}\right], \tag{5.11}$$

which in fact coincides with a particular choice of (3.7), with

$$\mathcal{U}(u,\bar u)=-\frac{1}{\sqrt{2}}[\partial_u\mathcal{M}(u)+\partial_{\bar u}\bar{\mathcal{M}}(\bar u)]. \tag{5.12}$$

Thus, the superpotential term can be ignored, since it is already present in (3.7).

C. Interaction

The interaction term for chiral superfields is also written as a superpotential:

$$S_{\mathrm{int}}=\frac{\mu}{2}\int dt_{\mathrm{L}}\,d^2\theta\,\mathcal{F}(Z,U)+\frac{\mu}{2}\int dt_{\mathrm{R}}\,d^2\bar\theta\,\bar{\mathcal{F}}(\bar Z,\bar U). \tag{5.13}$$

The component Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{\rm int} = \mu\Big[&\bar{B}\partial_{\bar z}\bar{\mathcal{F}} + B\partial_z\mathcal{F} - \frac{i\dot y}{\sqrt2}(\partial_u\mathcal{F} - \partial_{\bar u}\bar{\mathcal{F}}) \\
&- \frac{C}{2\sqrt2}(\partial_u\mathcal{F} + \partial_{\bar u}\bar{\mathcal{F}}) - \chi_k\xi^k\partial_u\partial_z\mathcal{F} - \frac12\xi_k\xi^k\partial_z\partial_z\mathcal{F} \\
&- \frac12\chi_k\chi^k\partial_u\partial_u\mathcal{F} - \bar\chi^k\bar\xi_k\partial_{\bar u}\partial_{\bar z}\mathcal{F} \\
&- \frac12\bar\xi^k\bar\xi_k\partial_{\bar z}\partial_{\bar z}\mathcal{F} - \frac12\bar\chi^k\bar\chi_k\partial_{\bar u}\partial_{\bar u}\mathcal{F}\Big]. \qquad (5.14)
\end{aligned}
$$

Note that the interaction Lagrangian $\mathcal{L}_{\rm int}$. contains a term $\sim\dot y$; i.e., it can be formally called the interacting WZ Lagrangian.

The total Lagrangian is a sum of (3.7), (5.4), and (5.14):

$$
\mathcal{L}_{\rm total} = \mathcal{L}_{(\mathbf{2},\mathbf{4},\mathbf{2})} + \mathcal{L}_{\rm WZ} + \mathcal{L}_{\rm int}. \qquad (5.15)
$$

The function $\mathcal{F}(z,u)$ can start with the holomorphic parts $\mathcal{F}_1(z)$ and $\mathcal{F}_2(u)$. However, their contributions are identical to those from the corresponding parts of (5.4) and (5.11). So, they have been already accounted for by $\mathcal{L}_{(\mathbf{2},\mathbf{4},\mathbf{2})}$ and (3.7). Keeping this in mind, we assume that such parts are absent in the interaction Lagrangian.

For simplicity, when passing to the Hamiltonian formulation, we will limit our consideration to the bosonic constraints:

$$
\begin{aligned}
&\pi_u = p_u + \sqrt2 i\mathcal{A}_u \approx 0, \quad \pi_{\bar u} = p_{\bar u} - \sqrt2 i\mathcal{A}_{\bar u} \approx 0, \\
&\pi_y = p_y - i\mathcal{A}_y + \frac{i\mu}{\sqrt2}[\partial_u\mathcal{F}(z,u) - \partial_{\bar u}\bar{\mathcal{F}}(\bar z,\bar u)] \approx 0, \\
&h = \mathcal{U}(y,u,\bar u) - \frac{\mu}{2\sqrt2}[\partial_u\mathcal{F}(z,u) + \partial_{\bar u}\bar{\mathcal{F}}(\bar z,\bar u)] \approx 0. \qquad (5.16)
\end{aligned}
$$

Here, the last constraint imposes a more complicated relation between the dynamical complex boson z and the semidynamical triplet $(y,u,\bar u)$.

The matrix of the constraints (5.16) is defined as

$$
\begin{pmatrix}
0 & \{\pi_u,\pi_{\bar u}\}_{\rm PB} & \{\pi_u,\pi_y\}_{\rm PB} & \{\pi_u,h\}_{\rm PB} \\
\{\pi_{\bar u},\pi_u\}_{\rm PB} & 0 & \{\pi_{\bar u},\pi_y\}_{\rm PB} & \{\pi_{\bar u},h\}_{\rm PB} \\
\{\pi_y,\pi_u\}_{\rm PB} & \{\pi_y,\pi_{\bar u}\}_{\rm PB} & 0 & \{\pi_y,h\}_{\rm PB} \\
\{h,\pi_u\}_{\rm PB} & \{h,\pi_{\bar u}\}_{\rm PB} & \{h,\pi_y\}_{\rm PB} & 0
\end{pmatrix}. \qquad (5.17)
$$

Calculating its inverse (see Appendix C), we obtain the following Dirac brackets:

$$
\begin{aligned}
&\{z,p_z\} = 1, \qquad \{p_z,y\} = -\frac{\mu\partial_u\partial_z\mathcal{F}\partial_y\mathcal{U}}{2\sqrt2(\partial\mathcal{U})^2}, \\
&\{p_z,u\} = -\frac{\mu\partial_u\partial_z\mathcal{F}}{\sqrt2(\partial\mathcal{U})^2}\left(\partial_{\bar u}\mathcal{U} - \frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt2}\right), \\
&\{\bar z,p_{\bar z}\} = 1, \qquad \{p_{\bar z},y\} = -\frac{\mu\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}\partial_y\mathcal{U}}{2\sqrt2(\partial\mathcal{U})^2}, \\
&\{p_{\bar z},\bar u\} = -\frac{\mu\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}}{\sqrt2(\partial\mathcal{U})^2}\left(\partial_u\mathcal{U} - \frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt2}\right), \\
&\{p_z,p_{\bar z}\} = -\frac{i\mu^2\partial_u\partial_z\mathcal{F}\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}\partial_y\mathcal{U}}{2(\partial\mathcal{U})^2}, \\
&\{u,\bar u\} = -\frac{i\partial_y\mathcal{U}}{2(\partial\mathcal{U})^2}, \\
&\{y,u\} = -\frac{i}{2(\partial\mathcal{U})^2}\left(\partial_{\bar u}\mathcal{U} - \frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt2}\right), \\
&\{y,\bar u\} = \frac{i}{2(\partial\mathcal{U})^2}\left(\partial_u\mathcal{U} - \frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt2}\right), \\
&(\partial\mathcal{U})^2 = \left[\partial_y\mathcal{U}\partial_y\mathcal{U} + 2\left(\partial_{\bar u}\mathcal{U} - \frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt2}\right)\right. \\
&\qquad\left.\times\left(\partial_u\mathcal{U} - \frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt2}\right)\right]. \qquad (5.18)
\end{aligned}
$$

One can make use of the identity

$$
\{p_z,y\}\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}} = \{p_{\bar z},y\}\partial_u\partial_z\mathcal{F} \qquad (5.19)
$$

in order to simplify (5.18) to the form

$$
\begin{aligned}
&\{z,p_z\} = 1, \qquad \{\bar z,p_{\bar z}\} = 1, \\
&\{p_z,p_{\bar z}\} = \frac{i}{\sqrt2}\mu(\{p_z,y\}\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}} + \{p_{\bar z},y\}\partial_u\partial_z\mathcal{F}), \\
&\{p_z,y\} = -\frac{i}{\sqrt2}\mu\{u,\bar u\}\partial_u\partial_z\mathcal{F}, \\
&\{p_{\bar z},y\} = -\frac{i}{\sqrt2}\mu\{u,\bar u\}\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}, \\
&\{p_z,u\} = -\sqrt2 i\mu\{y,\bar u\}\partial_u\partial_z\mathcal{F}, \\
&\{p_{\bar z},\bar u\} = \sqrt2 i\mu\,\{y,\bar u\}\,\partial_{\bar u}\partial_{\bar z}\bar{\mathcal{F}}. \qquad (5.20)
\end{aligned}
$$

These are a generalization of the Nahm equations (4.17) with a complex evolution parameter z.

The simplest noncommutative plane solution (3.21) requires that

$$\partial_u \mathcal{U}(y,u,\bar{u}) - \frac{\mu}{2\sqrt{2}}\partial_u\partial_u\mathcal{F}(u,z) = 0,$$
$$\partial_{\bar{u}} \mathcal{U}(y,u,\bar{u}) - \frac{\mu}{2\sqrt{2}}\partial_{\bar{u}}\partial_{\bar{u}}\bar{\mathcal{F}}(\bar{u},\bar{z}) = 0 \Rightarrow$$
$$\Rightarrow \partial_u\mathcal{U} = 0, \qquad \partial_{\bar{u}}\mathcal{U} = 0,$$
$$\partial_u\partial_u\mathcal{F} = 0, \qquad \partial_{\bar{u}}\partial_{\bar{u}}\bar{\mathcal{F}} = 0. \tag{5.21}$$

It yields (3.15) and fixes the function $\mathcal{F}$ as

$$\mathcal{F}(z,u) = u\mathcal{S}(z), \qquad \bar{\mathcal{F}}(\bar{z},\bar{u}) = \bar{u}\,\bar{\mathcal{S}}(\bar{z}). \tag{5.22}$$

So, we obtain

$$\{z,p_z\} = 1, \qquad \{\bar{z},p_{\bar{z}}\} = 1,$$
$$\{p_z,p_{\bar{z}}\} = 2i\mu^2\partial_z\mathcal{S}\partial_{\bar{z}}\bar{\mathcal{S}}, \qquad \{u,\bar{u}\} = i. \tag{5.23}$$

The third spin variable y is now represented as a function of the dynamical boson z:

$$y = c - \frac{\mu}{\sqrt{2}}[\mathcal{S}(z) + \bar{\mathcal{S}}(\bar{z})]. \tag{5.24}$$

VI. CONCLUSIONS

In this paper, we elucidated the distinction between ordinary and mirror multiplets of $\mathcal{N}=4$, $d=1$ supersymmetry. Mirror multiplets are described by superfields carrying no external $SU(2)_L$ indices and satisfying the common constraint (2.1) as a consequence of their basic constraints linear in the covariant spinor derivatives. According to this general definition, the standard chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ belongs to the mirror type.

We considered the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as semidynamical and constructed its action (3.5) in the analytic harmonic superspace as a particular case of the general WZ action for mirror multiplets (2.12). We coupled it to the dynamical mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ and constructed their interaction (4.8) in the analytic harmonic superspace. We obtained a mirror analog of the model studied in Ref. [13] and considered its deformation to $SU(2|1)$ supersymmetry.

We constructed, for the first time, the coupling of the semidynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ to the dynamical mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ in the chiral $\mathcal{N}=4$, $d=1$ superspace. We calculated Dirac brackets for the spin variables, dynamical fields, and their momenta. These brackets accomplish a kind of generalization of Nahm equations associated with the previously considered $(\mathbf{3},\mathbf{4},\mathbf{1}) - (\mathbf{1},\mathbf{4},\mathbf{3})$ system, such that the complex $d=1$ field z plays now the role of complex evolution parameter. It would be interesting to study this model in more detail, in particular to calculate its supercharges and to pass to its quantum version. Also, the possible deformed $SU(2|1)$ version of this new system seems to deserve an attention.[8]

The most intriguing question for further study is whether it is possible to construct $\mathcal{N}=4$ SQM models involving interactions between the multiplets, which are mirror to each other. Perhaps, the biharmonic formalism of Ref. [4] augmented with the consideration in the present paper could help to advance toward this goal.[9]

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APPENDIX A: BASICS OF $\mathcal{N}=4$, $d=1$ SUPERSPACE

The coordinates of $\mathcal{N}=4$, $d=1$ superspace $\zeta \coloneqq \{t,\theta^{i\alpha}\}$ transform under $\mathcal{N}=4$ supersymmetry as

$$\delta\theta^{i\alpha} = \epsilon^{i\alpha}, \qquad \delta t = -i\epsilon^{i\alpha}\theta_{i\alpha},$$
$$\overline{(\theta^{i\alpha})} = -\theta_{i\alpha}, \qquad \overline{(\epsilon^{i\alpha})} = -\epsilon_{i\alpha}, \tag{A1}$$

where $\epsilon^{i\alpha}$ is a quartet of the corresponding Grassmann parameters. The covariant derivatives are defined as

$$D^{i\alpha} = \frac{\partial}{\partial\theta_{i\alpha}} + i\theta^{i\alpha}\partial_t. \tag{A2}$$

The coordinates of the left-handed chiral subspace $\zeta_L \coloneqq \{t_L,\theta_i\}$ are related to the previously defined ones as

$$t_L \coloneqq t - i\theta^{i1}\theta_{i1}, \qquad \theta_i \coloneqq \theta_{i1}. \tag{A3}$$

They transform as

$$\delta\theta_i = \epsilon_i, \qquad \delta t_L = 2i\bar{\epsilon}^k\theta_k, \qquad \overline{(\epsilon_i)} = \bar{\epsilon}^i. \tag{A4}$$

1. Harmonic superspace

We perform harmonization over the indices corresponding to $SU(2)_L$:

$$t_{(A)} = t - \frac{i}{2}\theta^i_\alpha\theta^{j\alpha}(u^+_i u^-_j + u^+_j u^-_i),$$
$$\theta^\pm_\alpha \coloneqq \theta^i_\alpha u^\pm_i, \qquad u^+_i u^-_j - u^+_j u^-_i = \varepsilon_{ij}. \tag{A5}$$

Then, the harmonic superspace is defined by

[8] The $SU(2|1)$-invariant system $(\mathbf{4},\mathbf{4},\mathbf{0}) - (\mathbf{2},\mathbf{4},\mathbf{2})$, with both multiplets being mirror, was constructed in Ref. [19].

[9] To date, only one example of such a system with nontrivial self-interaction is known [4].

$$\zeta_{\rm H} \coloneqq \{t_{\rm (A)}, \theta^{\pm}_{\alpha}, u^{\pm}_i\}. \tag{A6}$$

Its coordinates transform as

$$\delta\theta^{\pm}_{\alpha} = \epsilon^{\pm}_{\alpha}, \quad \delta u^{\pm}_i = 0, \quad \delta t_{\rm (A)} = 2i\epsilon^{-\alpha}\theta^{+}_{\alpha}, \quad \epsilon^{\pm}_{\alpha} \coloneqq \epsilon^{i}_{\alpha}u^{\pm}_i. \tag{A7}$$

The measure of integration over the full harmonic superspace is defined as

$$d\zeta_{\rm H} \coloneqq \frac{1}{4}du\, dt_{\rm (A)}\, d\theta^{+}_{\alpha} d\theta^{+\alpha}\, d\theta^{-}_{\beta}\, d\theta^{-\beta}. \tag{A8}$$

The harmonic superspace involves the analytic harmonic subspace parametrized by the reduced coordinate set

$$\zeta_{\rm (A)} \coloneqq \{t_{\rm (A)}, \theta^{+}_{\alpha}, u^{\pm}_i\}, \tag{A9}$$

which is closed under the transformations (A7).

We use the standard notation for covariant derivatives

$$D^{+\alpha} = \frac{\partial}{\partial\theta^{-}_{\alpha}},$$
$$D^{++} = \partial^{++} - i\theta^{+}_{\alpha}\theta^{+\alpha}\frac{\partial}{\partial t_{\rm (A)}} + \theta^{+}_{\alpha}\frac{\partial}{\partial\theta^{-}_{\alpha}},$$
$$D^{0} = \partial^{0} + \theta^{+}_{\alpha}\frac{\partial}{\partial\theta^{+}_{\alpha}} - \theta^{-}_{\alpha}\frac{\partial}{\partial\theta^{-}_{\alpha}}, \tag{A10}$$

where the partial harmonic derivatives are

$$\partial^{\pm\pm} \coloneqq u^{\pm}_i\frac{\partial}{\partial u^{\mp}_i}, \qquad \partial^{0} \coloneqq u^{+}_i\frac{\partial}{\partial u^{+}_i} - u^{-}_i\frac{\partial}{\partial u^{-}_i},$$
$$[\partial^{++}, \partial^{--}] = \partial^{0}, \qquad [\partial^{0}, \partial^{\pm\pm}] = \pm 2\partial^{\pm\pm}. \tag{A11}$$

According to these definitions, the invariant integration measure of the analytic subspace $d\zeta^{--}_{\rm (A)}$ is related to $d\zeta_{\rm H}$ as

$$d\zeta^{--}_{\rm (A)} \coloneqq \frac{1}{2}du\, dt_{\rm (A)}\, d\theta^{+}_{\alpha} d\theta^{+\alpha},$$
$$d\zeta_{\rm H} = \frac{1}{2}d\zeta^{--}_{\rm (A)} D^{+}_{\alpha} D^{+\alpha}. \tag{A12}$$

APPENDIX B: COMPONENT SOLUTIONS FOR THE MIRROR MULTIPLETS (4, 4, 0) AND (0, 4, 4)

Mirror multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$.—The constraints (2.7) are solved by

$$Y^{\alpha A} = y^{\alpha A} + \theta^{-\alpha}\psi^{iA}u^{+}_i - \theta^{+\alpha}\psi^{iA}u^{-}_i + 2i\theta^{+}_{\beta}\theta^{-\alpha}\dot{y}^{\beta A} + i\theta^{+}_{\beta}\theta^{+\beta}\theta^{-\alpha}\dot{\psi}^{iA}u^{-}_i. \tag{B1}$$

The components transform as

$$\delta y^{\alpha A} = -\epsilon^{\alpha}_{k}\psi^{kA}, \qquad \delta\psi^{iA} = 2i\epsilon^{i\gamma}\dot{y}^{A}_{\gamma}. \tag{B2}$$

The corresponding SU(2|1) deformed solution, and the transformation properties are given in Ref. [11].

Mirror multiplet $(\mathbf{0},\mathbf{4},\mathbf{4})$.—The solution of Eq. (2.8) for the fermionic superfield Ψ^{iA} is written as

$$\Psi^{\alpha A} = \psi^{\alpha A} + \theta^{-\alpha}D^{iA}u^{+}_i - \theta^{+\alpha}D^{iA}u^{-}_i + 2i\theta^{+}_{\beta}\theta^{-\alpha}\dot{\psi}^{\beta A} + i\theta^{+}_{\beta}\theta^{+\beta}\theta^{-\alpha}\dot{D}^{iA}u^{-}_i. \tag{B3}$$

The transformation properties of the component fields are

$$\delta\psi^{\alpha A} = -\epsilon^{\alpha}_{k}D^{kA}, \qquad \delta D^{iA} = 2i\epsilon^{i\gamma}\dot{\psi}^{A}_{\gamma}. \tag{B4}$$

APPENDIX C: MATRICES OF SECOND-CLASS CONSTRAINTS

The matrix (5.17) in the explicit form reads

$$\begin{pmatrix} 0 & -2i\partial_y\mathcal{U} & 2i\partial_u\mathcal{U} - \frac{i\mu\partial_u\partial_u\mathcal{F}}{\sqrt{2}} & -\partial_u\mathcal{U} + \frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt{2}} \\ 2i\partial_y\mathcal{U} & 0 & -2i\partial_{\bar u}\mathcal{U} + \frac{i\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{\sqrt{2}} & -\partial_{\bar u}\mathcal{U} + \frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}} \\ -2i\partial_u\mathcal{U} + \frac{i\mu\partial_u\partial_u\mathcal{F}}{\sqrt{2}} & 2i\partial_{\bar u}\mathcal{U} - \frac{i\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{\sqrt{2}} & 0 & -\partial_y\mathcal{U} \\ \partial_u\mathcal{U} - \frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt{2}} & \partial_{\bar u}\mathcal{U} - \frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}} & \partial_y\mathcal{U} & 0 \end{pmatrix}. \tag{C1}$$

The corresponding inverse matrix is then calculated to be

$$\frac{1}{(\partial\mathcal{U})^2}\begin{pmatrix} 0 & -\frac{i}{2}\partial_y\mathcal{U} & \frac{i}{2}\partial_{\bar u}\mathcal{U}-\frac{i\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{4\sqrt{2}} & \partial_{\bar u}\mathcal{U}-\frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}} \\ \frac{i}{2}\partial_y\mathcal{U} & 0 & -\frac{i}{2}\partial_u\mathcal{U}+\frac{i\mu\partial_u\partial_u\mathcal{F}}{4\sqrt{2}} & \partial_u\mathcal{U}-\frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt{2}} \\ -\frac{i}{2}\partial_{\bar u}\mathcal{U}+\frac{i\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{4\sqrt{2}} & \frac{i}{2}\partial_u\mathcal{U}-\frac{i\mu\partial_u\partial_u\mathcal{F}}{4\sqrt{2}} & 0 & \partial_y\mathcal{U} \\ -\partial_{\bar u}\mathcal{U}+\frac{\mu\partial_{\bar u}\partial_{\bar u}\bar{\mathcal{F}}}{2\sqrt{2}} & -\partial_u\mathcal{U}+\frac{\mu\partial_u\partial_u\mathcal{F}}{2\sqrt{2}} & -\partial_y\mathcal{U} & 0 \end{pmatrix}. \tag{C2}$$

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