Conformal correlators in the critical $O(N)$ vector model

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We calculate a set of conformal correlators in the critical $O(N)$ vector model in $2 < d < 6$ dimensions. We focus on the correlators involving the Hubbard-Stratonovich field s, and its composite form s^2. In the process, we report a number of new calculations of diagrams involving the composite s^2 operator. Through the calculation of the $\langle s^2 s^2 s\rangle$ three-point function, we shed new light on a conjectured $s \to -s$ symmetry in the s sector of the critical $O(N)$ vector model in $d = 3$.

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I. INTRODUCTION

The critical $O(N)$ vector model with quartic interaction and the critical $U(n)$ Gross-Neveu (GN) model are some of the most well-studied interacting conformal field theories. These CFTs are free in even space-time dimensions, which allows one to study them perturbatively via the ϵ-expansion in the Wilson-Fisher regime [1–6]. In three dimensions, these models are strongly coupled and are not accessible to the perturbative treatment, but prove to be of great interest from the standpoint of understanding the behavior of quantum systems at criticality. Other methods, such as the $1/N$ expansion [7–17] and the conformal bootstrap [18–22] are frequently used to study such CFTs, and have recently been undergoing an active development.

The interest in these critical models in $d = 3$ dimensions is greatly amplified by their relevance in the context of the $3d/4d$ holographic duality, where they are described by the dual higher-spin theories in the AdS bulk [23].[1] When coupled to the Chern-Simons field, the fundamental scalar and fermionic matter exhibit interesting Bose/Fermi dualities, and its holographic dual, in turn, has been conjectured to be an interpolation between type A and type B Vasiliev higher-spin theories [27–29]. One salient feature following from the holographic correspondence is that for the scalar current A_0 in the bulk, dual to the Hubbard-Stratonovich field s of the critical vector model on the boundary, the cubic interaction A_0^3 is absent in Vasiliev's theory [24,25]. This motivates one to explore if the boundary CFT possesses an emergent discrete $\mathbb{Z}_2$ symmetry $s \to -s$, beyond large N in the singlet sector.

In the deep UV regime, the Gross-Neveu model in $2 \leq d \leq 4$ dimensions reaches a fixed point. This fixed point can be studied perturbatively in the vicinity of $d = 2$ dimensions (where the model is asymptotically free [2]), as well as in the vicinity of $d = 4$ dimensions (where the model is critically equivalent to the Gross-Neveu-Yukawa model [4]). In general d, this interacting CFT describes dynamics of the fermions ψ^i and the Hubbard-Stratonovich scalar field σ, and its action is given by

$$S_{\mathrm{G.N}} = \int d^d x \left(\bar{\psi}^i \gamma^\mu \partial_\mu \psi^i + \frac{1}{\sqrt{N}} \sigma \bar{\psi}^i \psi^i\right). \qquad (1.1)$$

The Gross-Neveu model action (1.1) is invariant with respect to the discrete $\mathbb{Z}_2$ symmetry [2,4,30]

$$\begin{aligned}&(x^1, \ldots, x^{a-1}, x^a, x^{a+1}, \ldots, x^d)\\ &\quad \to (x^1, \ldots, x^{a-1}, -x^a, x^{a+1}, \ldots, x^d),\\ &\quad \sigma \to -\sigma, \qquad \psi \to \gamma_a \psi, \qquad \bar{\psi} \to -\bar{\psi}\gamma_a, \end{aligned} \qquad (1.2)$$

for any given $a = 1, \ldots, d$. In particular, (1.2) implies $\bar{\psi}\psi \to -\bar{\psi}\psi$, and as a result the interaction term $\sigma\bar{\psi}^i\psi^i$ in the action (1.1) is left invariant.

Furthermore, the symmetry (1.2) uniquely fixes the structure of some of the correlation functions in the critical GN model. For instance, while conformal symmetry allows two possible structures of the three-point correlation functions involving one scalar and two fermionic fields [31,32], the $\mathbb{Z}_2$ symmetry (1.2) can further select which one of these two structures is allowed, as can be seen on the following example [33]:

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[1]See also [24–26] and references therein for some of the earlier works.

$$\langle\bar{\psi}(x_1)\psi(x_2)\sigma(x_3)^{2k+1}\rangle \sim \frac{\gamma_\mu x_{13}^\mu \gamma_\nu x_{32}^\nu}{|x_{12}|^{d-2-2k}(|x_{13}||x_{23}|)^{2k+2}}, \quad (1.3)$$

$$\langle\bar{\psi}(x_1)\psi(x_2)\sigma(x_3)^{2(k+1)}\rangle \sim \frac{\gamma_\mu x_{12}^\mu}{|x_{12}|^{d-2(k+1)}(|x_{13}||x_{23}|)^{2(k+1)}}, \quad (1.4)$$

where $k = 0, 1, 2, \ldots$. While (1.3), (1.4) are leading order in $1/N$, two such distinct structures will persist to all orders in the $1/N$ expansion.

One can also study manifestations of the symmetry (1.2) in the singlet sector of the critical GN model. This is particularly relevant from the standpoint of the holographic duality, which provides a prescription for evaluation of the correlation function of the $U(n)$ singlets via the dual description of the gauge degrees of freedom in the AdS bulk. An immediate consequence of the symmetry (1.2) is that the correlation functions involving an odd number of the Hubbard-Stratonovich fields σ vanish. The simplest example of this statement is given by the triviality of the three-point function

$$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\rangle = 0. \quad (1.5)$$

The relation (1.5) can be explicitly verified by evaluating the corresponding Feynman diagrams [16,33].

In this paper, we study the critical $O(N)$ vector model with the action

$$S = \int d^d x \left(\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi_i + \frac{1}{\sqrt{N}}s\phi^i\phi^i\right), \quad (1.6)$$

describing dynamics of the fundamental fields ϕ^i and the Hubbard-Stratonovich field s. At first glance, the model (1.6) does not seem to possess any such remarkable discrete $\mathbb{Z}_2$ symmetry due to the purely bosonic nature of the fields ϕ_i. However, a conjecture was put forth in [34] regarding the existence of such a symmetry in the $O(N)$ vector models in $d = 3$, based on conformal bootstrap calculation of the three-point function $\langle sss\rangle$ at the leading order in $1/N$ expansion. It is our aim to shed further light on the fate of this conjecture at both the leading and next-to leading order in $1/N$ in the model (1.6). Inspired by the GN model, we will search for manifestations of this symmetry in certain conformal correlators. We will be mostly interested in the $d = 3$ case, but majority of our calculations will be carried out in general d.

The CFT action (1.6) describes the critical behavior of the $O(N)$ vector model with quartic interaction at its IR fixed point, and the non-linear sigma model at its UV fixed point, for $2 < d < 4$ [4]. When $4 < d < 6$ the model (2.1) describes a UV fixed point of the $O(N)$ vector model with quartic interaction, and is conjectured to describe the IR fixed point of certain vector model with cubic coupling. The latter statement is supported by a perturbative calculation in $d = 6 - \epsilon$ dimensions up to quartic order [5,6,35].

The counterpart of the transformation (1.2) in the $O(N)$ vector model (1.6) would only act on the Hubbard-Stratonovich field s [34],

$$s \to -s. \quad (1.7)$$

While the transformation (1.7) is clearly not a symmetry of the action (1.6), it has been suggested in [34] that such a symmetry might emerge in the $d = 3$ dimensional quantum theory among the correlation functions involving only the s fields. Interestingly, [34] pointed out that the three-point function vanishes at the leading order in the $1/N$ expansion,

$$\langle s(x_1)s(x_2)s(x_3)\rangle|_{d=3} = 0 + \mathcal{O}\left(\frac{1}{N^{3/2}}\right), \quad (1.8)$$

and suggested to explain it by conjecturing the symmetry (1.7). Notice that (1.8) is not valid when $d \neq 3$, in stark contrast with the GN case, where the $\mathbb{Z}_2$ symmetry holds for any d. In [36] the $\langle sss\rangle$ correlation function was calculated at the next-to-leading order in the $1/N$ expansion. Remarkably, [36] demonstrated that the sub-leading correction to the three-point function $\langle sss\rangle$ also vanishes in $d = 3$,

$$\langle s(x_1)s(x_2)s(x_3)\rangle|_{d=3} = 0 + \mathcal{O}\left(\frac{1}{N^{5/2}}\right). \quad (1.9)$$

This further raises the question of whether the symmetry (1.7) is valid also upto the first sub-leading order in $1/N$.

The proposed symmetry, however, is fundamentally different than its counterpart (1.2) in the GN model for a number of reasons. Primary among them is that the symmetry is suggested to be present only in $d = 3$. However, more importantly, the conjectured symmetry transformation (1.7) is not respected by any correlation functions involving the fundamental scalar ϕ_i. As a simple example, one can notice that the correlation function $\langle\phi\phi s\rangle$ is non-vanishing in $d = 3$ [34], although it is odd w.r.t. the transformation (1.7). In fact, it was originally suggested in [34] that the symmetry (1.7), if established, would have to be confined to the sub-sector of the theory, involving only the s fields. This is very unlike the GN model where correlations involving the fundamental ψ_i field also respect the discrete symmetry (1.2), as we reviewed above on the example of the correlation functions (1.3), (1.4).

We then intend to study the transformation of the three-point correlation functions involving the Hubbard-Stratonovich field s only. To test the proposed symmetry transformation (1.7) one needs to study correlation functions involving an odd number of the fields s. Above we have discussed that the three-point function of the s field vanishes up to the next-to-leading order in $1/N$ expansion. A natural next step is to study three-point functions involving five of the fields s, which requires one to deal with the composite operators s^2 or s^3.

Specifically, the objective of this paper is to calculate the three-point correlation function $\langle s^2 s^2 s\rangle$. While we establish that this correlator vanishes at the leading order in the $1/N$ expansion, our main result is that its next-to-leading order correction is in fact nonzero, and therefore does not respect the conjectured symmetry (1.7).

In the process we obtain a number of new results, which can be used for other calculations in the $O(N)$ vector model. Some of the new expressions obtained in this paper involve the s^2ss conformal triangle at the next-to-leading order in $1/N$ in general d, the $\langle s^2ss\rangle$ three-point function in $d=3$, and several self-energy diagrams, which we believe have not been reported in the literature before.

A peculiar feature of the composite operators such as s^n, $n=2,3,\ldots$ is that they fall beyond the scope of the primary/descendant dichotomy of the CFT operators. At the same time, the concept of a scaling dimension for the operators s^n remains well defined, and the scale invariance is expected to hold for the correlation functions involving these operators [37]. The corresponding two-point functions $\langle s^n s^n\rangle$ exhibit power-law behavior, and in the $1/N$ expansion the anomalous dimensions of these scaling operators can be calculated [38,39].[2]

An important manifestation of the nonprimary nature of the composite operator s^2 is given by its mixture with the descendant operator $\partial^2 s$ [40,41]. The true primary operator is given by

$$\mathcal{O}=s^2+\alpha(d)\partial^2 s,\qquad \alpha(d)=\frac{1}{4(d-6)C_s}c(d),\quad (1.10)$$

$$c(d)=\frac{1}{\sqrt{N}}\frac{8^{d-1}\Gamma(5-d)\Gamma(\frac{d-1}{2})^3\sin(\frac{\pi d}{2})^2\sin(\pi d)}{\pi^{\frac{9}{2}}d\Gamma(\frac{d}{2}-2)}+\mathcal{O}\left(\frac{1}{N^{3/2}}\right),\quad (1.11)$$

where we used the leading-order correlator [40,41]

$$\langle s(x)^2 s(0)\rangle=\frac{c}{|x|^6},\quad (1.12)$$

and demanded that

$$\langle\mathcal{O}(x)s(0)\rangle=0.\quad (1.13)$$

An immediate consequence of the mixing (1.12) is the apparent breaking of the $s\to -s$ symmetry. However, the mixing coefficient c (1.11) vanishes at the leading order in $1/N$, $c|_{d=3}=0+\mathcal{O}(\frac{1}{N^{3/2}})$.[3] In particular, for the purpose of exploring the $s\to -s$ symmetry in $d=3$ we can ignore the mixing (1.12), at the considered order of the $1/N$ expansion. It would be interesting to explore the next-to-leading order corrections to the coefficient c, and in particular uncover the fate of the mixing correlator (1.12) in the three-dimensional $O(N)$ vector model.

While s^2 is a well-defined scaling operator, and the two-point function $\langle s^2 s^2\rangle$ has the conformally covariant form, the higher point correlation functions involving s^2 are no longer constrained by the requirement of symmetry under the full conformal group. In particular, nonconformal terms contributing to the three-point functions can be traced back to the mixing (1.10),

$$\langle s^2(x)s(y)s(z)\rangle=\langle\mathcal{O}(x)s(y)s(z)\rangle-\alpha\partial_x^2\langle s(x)s(y)s(z)\rangle,\quad (1.14)$$

$$\begin{aligned}\langle s^2(x)s^2(y)s(z)\rangle&=\langle\mathcal{O}(x)O(y)s(z)\rangle-\alpha\partial_x^2\langle s(x)O(y)s(z)\rangle\\&\quad-\alpha\partial_y^2\langle O(x)s(y)s(z)\rangle\\&\quad+\alpha^2\partial_x^2\partial_y^2\langle s(x)s(y)s(z)\rangle.\end{aligned}\quad (1.15)$$

Only the first terms in the right-hand side of (1.14), (1.15) have the form of conformal three-point functions. For the purpose of this paper it is sufficient to focus on those conformally covariant terms only: we will establish in particular that $\langle\mathcal{O}(x)O(y)s(z)\rangle|_{d=3}\neq 0$ at the next-to-leading order in $1/N$ expansion. This is sufficient to argue that the $s\to -s$ symmetry is broken.[4] This is consistent with expectations from conformal bootstrap calculations at finite N [42].

The rest of this paper is organized as follows. In Sec. II we set up our notations and review known results in the critical $O(N)$ vector model that will be relevant for the purposes of this paper. We also reformulate the result of [36] for the $\langle sss\rangle$ correlation function in terms of the sss conformal triangle, representing the cubic effective vertex at the next-to-leading order in $1/N$. In Sec. III we derive the $\mathcal{O}ss$ conformal triangle in general d, and calculate the $\langle s^2ss\rangle$ three-point function in $d=3$ at the next-to-leading order in $1/N$. In Sec. IV we derive the $\langle s^2s^2s\rangle$ correlation function in $d=3$. We demonstrate that while the leading-order $3d$ correlation function vanishes, its $1/N$ correction is nontrivial. Motivated by this result, in Sec. V we then explore whether the conjectured symmetry (1.7) is an artifact of the large-N limit. We discuss our results in Sec. VI.

[2] In particular case of a two-dimensional CFT the scale symmetry generally leads to the full conformal symmetry.

[3] In fact, $c|_{d=2,3,4}=0+\mathcal{O}(\frac{1}{N^{3/2}})$.

[4] The nonconformal contributions in the right-hand side of (1.15) can be studied separately, because they have a different functional dependence on the coordinates of the operators in general d, and more importantly vanish in $d=3$.

II. SETUP

In this section we set the notations to this paper and review known results from the literature. We study the critical $O(N)$-invariant vector model with the action

$$S = \int d^d x \left(\frac{1}{2} (\partial\phi)^2 + \frac{1}{\sqrt{N}} s\phi^2 \right) + S_{\text{c.t.}}, \tag{2.1}$$

describing dynamics of the multiplet ϕ^i, $i = 1, \ldots, N$ of the real-valued scalar fields, and the Hubbard-Stratonovich field s.[5] In the action (2.1) we have also incorporated the counterterm induced by the wave-function renormalization of the fields ϕ, s; see [36] for a recent review. Specifically, to regularize the divergent conformal graphs, we add a small shift δ to the internal s lines [43]. At the end of the calculation, the limit $\delta \to 0$ is taken, and the finite part is extracted. At the same time, the divergent terms are removed by the wave-function renormalization

$$\phi \to \sqrt{1 + \frac{2\gamma_\phi}{\delta}}\phi, \qquad s \to \sqrt{1 + \frac{2\gamma_s}{\delta}} s. \tag{2.2}$$

In particular, the transformation (2.2) induces a vertex counterterm

$$\frac{1}{\sqrt{N}}\phi^2 s \to \frac{1}{\sqrt{N}}\phi^2 s + S_{\text{c.t.}}, \quad S_{\text{c.t.}} = \frac{2\gamma_\phi + \gamma_s}{\delta} \frac{1}{\sqrt{N}} \phi^2 s, \tag{2.3}$$

which we implicitly account for to remove divergencies.

The ϕ field propagator is given by

$$\langle \phi(x)\phi(0) \rangle = \frac{C_\phi (1 + A_\phi) \mu^{-2\gamma_\phi}}{|x|^{2(\Delta_\phi + \gamma_\phi)}}, \tag{2.4}$$

where μ is an arbitrary RG scale, Δ_ϕ is the free scaling dimension, and γ_ϕ is the anomalous dimension, given by [7,8,34]

$$\Delta_\phi = \frac{d}{2} - 1, \tag{2.5}$$

$$\gamma_\phi = \frac{1}{N} \frac{2^d \sin(\frac{\pi d}{2}) \Gamma(\frac{d-1}{2})}{\pi^{3/2} (d-2) d \Gamma(\frac{d}{2} - 2)} + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{2.6}$$

leading in $1/N$ amplitude is given by

$$C_\phi = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{\frac{d}{2}}}, \tag{2.7}$$

and subleading correction to the amplitude is [44]

$$A_\phi = \left(\frac{d}{2-d} - \frac{2}{d} \right) \gamma_\phi + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{2.8}$$

The Feynman rule corresponding to the propagator (2.4) is

$$0 \; \overset{2\Delta_\phi}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet} \; x \quad = \frac{C_\phi}{|x|^{2\Delta_\phi}}\,.$$

In a general conformal graph we will also use the following Feynman rule for internal lines with unit amplitudes:

$$0 \; \overset{2a}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet} \; x \quad = \frac{1}{|x|^{2a}}\,.$$

Since the action (2.1) is quadratic in ϕ, the corresponding path integral is Gaussian, and can be performed explicitly, resulting in the effective action for s formally written as

$$S_{\text{eff}} = \frac{N}{2} \int d^d x \, \text{Tr} \log \left(\partial^2 - \frac{2}{\sqrt{N}} s \right). \tag{2.9}$$

Expanding the logarithm, we obtain

$$S_{\text{eff}} = -C_\phi^2 \int d^d x_{1,2} \frac{s(x_1)s(x_2)}{|x_{12}|^{2(d-2)}} + \frac{4C_\phi^3}{3\sqrt{N}} \int d^d x_{1,2,3} \frac{s(x_1)s(x_2)s(x_3)}{(|x_{12}||x_{13}||x_{23}|)^{d-2}} + \cdots, \tag{2.10}$$

where ellipsis stand for vertices of higher order in s.

From the quadratic term in the action (2.10) we obtain the propagator for the Hubbard-Stratonovich field s,

$$\langle s(x)s(0) \rangle = \frac{C_s}{|x|^{2\Delta_s}}, \tag{2.11}$$

where

$$\Delta_s = 2, \tag{2.12}$$

$$C_s = \frac{2^d \Gamma(\frac{d-1}{2}) \sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}} \Gamma(\frac{d}{2} - 2)}. \tag{2.13}$$

The corresponding Feynman rule is

$$0 \; \overset{2\Delta_s}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet} \; x \quad = \frac{C_s}{|x|^{2\Delta_s}}\,.$$

The loop corrections to the Hubbard-Stratonovich propagator result in

$$\langle s(x)s(0) \rangle = \frac{C_s (1 + A_s) \mu^{-2\gamma_s}}{|x|^{2(\Delta_s + \gamma_s)}}, \tag{2.14}$$

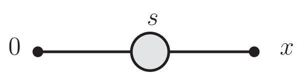
[5] Here and in what follows we skip keeping track of the $O(N)$ indices where it does not cause confusion.

where [7,8,34,44][6]

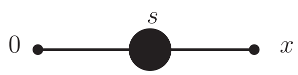
$$\gamma_s = 4\left(d + \frac{6}{d-4} + 1\right)\gamma_\phi + \mathcal{O}\left(\frac{1}{N^2}\right), \qquad (2.15)$$

$$A_s = 2\gamma_\phi\left(\frac{d(d-3)+4}{4-d}\left(H_{d-3} + \pi\cot\left(\frac{\pi d}{2}\right)\right) + \frac{8}{(d-4)^2} + \frac{2}{d-2} + \frac{2}{d} - 2d - 1\right) + \mathcal{O}\left(\frac{1}{N^2}\right). \qquad (2.16)$$

We will also use the Feynman rules corresponding to the dressed propagator

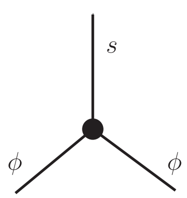


$$= \frac{C_s\,(1+A_s)\,\mu^{-2\gamma_s}}{|x|^{2(\Delta_s+\gamma_s)}},$$



$$= \frac{C_s\,\mu^{-2\gamma_s}}{|x|^{2(\Delta_s+\gamma_s)}}.$$

Higher order terms in the action (2.10) can be represented by conformal graphs with internal ϕ lines and the interaction vertex



$$= -\frac{2}{\sqrt{N}}.$$

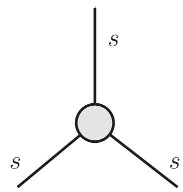
In particular, the second term in the right-hand side of (2.10) gives the leading $\mathcal{O}(1/N^{1/2})$ order sss vertex. Subleading corrections to this vertex can be written down in terms of the corresponding Polyakov's [19] conformal triangle

$$S_{\text{eff}} \supset \frac{Z_{sss}}{\sqrt{N}}\int d^d x_{1,2,3}\,\frac{s(x_1)s(x_2)s(x_3)}{(|x_{12}||x_{13}||x_{23}|)^{d-2-\gamma_s}}\mu^{3\gamma_s}. \qquad (2.17)$$

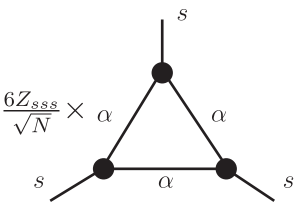
Here the amplitude of the conformal triangle Z_{sss} and the anomalous dimension γ_s are assumed to be expanded to the desired power in $1/N$,

$$Z_{sss} = Z_{sss}^{(0)}(1 + \delta Z_{sss}). \qquad (2.18)$$

The Feynman rule corresponding to the vertex/conformal triangle (2.17) is given by



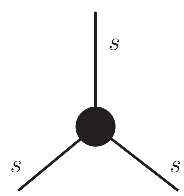
$$= -\frac{6Z_{sss}}{\sqrt{N}} \times$$



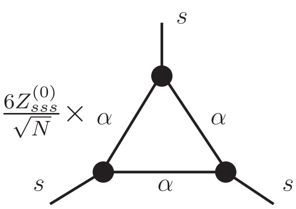
where we denoted

$$\alpha = d - 2 - \gamma_s. \qquad (2.19)$$

We will also use the following notation for the sss conformal triangle with the leading order amplitude only:



$$= -\frac{6Z_{sss}^{(0)}}{\sqrt{N}} \times$$



Comparing the leading order coefficients of the cubic vertex in (2.10) and (2.17), we obtain the leading order amplitude $Z_{sss}^{(0)}$ of the conformal triangle (2.18)

$$Z_{sss}^{(0)} = \frac{4}{3}C_\phi^3. \qquad (2.20)$$

To calculate the subleading contribution δZ_{sss} to the sss conformal triangle, we use the expression for the three-point function

$$\langle s(x_1)s(x_2)s(x_3)\rangle = \frac{C_{sss}^{(0)}(1+\delta C_{sss})}{(|x_{12}||x_{13}||x_{23}|)^{\Delta_s+\gamma_s}}\mu^{-3\gamma_s}, \qquad (2.21)$$

where [34][7]

$$C_{sss}^{(0)} = N\left(-\frac{2}{\sqrt{N}}\right)^3 C_\phi^3 C_s^3\, U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^2 U(1,2,d-3) \qquad (2.22)$$

$$= -\frac{1}{\sqrt{N}}\frac{8^{d-1}\sin^3(\frac{\pi d}{2})\Gamma(3-\frac{d}{2})\Gamma(\frac{d-1}{2})^3}{\pi^{9/2}\Gamma(d-3)}, \qquad (2.23)$$

and δC_{sss} was found in [36] at the next-to-leading order in $1/N$[8]

[6]In this paper we use H_n to denote the nth harmonic number.

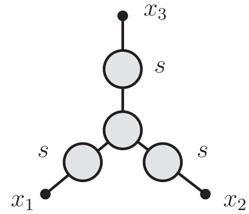
[7]See Appendix A for our conventions.

[8]In notations of [36] we have $\delta C_{sss} = W_{s^3} + 3A_s/2$. This is because W_{s^3} was defined in [36] as a relative correction to the amplitude of the three-point function of the normalized fields s, related to the non-normalized fields by the transformation $s \to \sqrt{C_s(1+A_s)}s$. Appearance of the additional term $3A_s/2$ is clear from such a transformation.

$$\delta C_{sss} = 3W_{\phi\phi s} + f + W_3 + W_4 + \frac{3A_s}{2}, \tag{2.24}$$

$$\begin{aligned}
W_{\phi\phi s} &= \gamma_\phi\left(\frac{d(d-3)+4}{4-d}\left(H_{d-3}+\pi\cot\left(\frac{\pi d}{2}\right)\right)\right. \\
&\quad \left. + \frac{16}{(d-4)^2} + \frac{6}{d-4} + \frac{2}{d-2} - 2d + 3\right), \\
f &= \gamma_\phi \frac{6(d-1)(d-2)}{d-4}\left(H_{d-4} - \frac{2}{d-4} + \pi\cot\left(\frac{\pi d}{2}\right)\right), \\
W_3 &= \gamma_\phi \frac{2d(d-2)(d-3)}{(d-4)^2}\left(6\psi^{(1)}\left(\frac{d}{2}-1\right) - \psi^{(1)}(d-3)\right. \\
&\quad \left. - \frac{\pi^2}{6} - H_{d-4}\left(H_{d-4} + 2\pi\cot\left(\frac{\pi d}{2}\right)\right)\right), \\
W_4 &= \gamma_\phi \frac{3d(d-2)(\pi^2 - 6\psi^{(1)}(\frac{d}{2}-1))}{4(d-4)},
\end{aligned}$$

where $\psi^{(1)}$ is the first derivative of the digamma function, and H_n is the nth harmonic number, γ_ϕ is given by (2.6), and A_s is given by (2.16). On the other hand, the $\langle sss\rangle$ three-point function can be obtained by attaching three s legs to the sss conformal triangle, and integrating over its vertices:



$$= \frac{C^{(0)}_{sss}(1+\delta C_{sss})}{(|x_{12}||x_{13}||x_{23}|)^{\Delta_s+\gamma_s}}\,.$$

Then we can express

$$\delta Z_{sss} = \delta C_{sss} - 3A_s - R_{sss}, \tag{2.25}$$

where R_{sss} is obtained by expansion of the factor

$$\begin{aligned}
&U\left(2+\gamma_s, \frac{d-\gamma_s}{2}-1, \frac{d-\gamma_s}{2}-1\right)^2 \\
&\quad\times U\left(2+\gamma_s, 1+\frac{\gamma_s}{2}, d-3-\frac{3\gamma_s}{2}\right) \\
&= u^{(0)}_{sss}\left(1+R_{sss}+\mathcal{O}\left(\frac{1}{N^2}\right)\right)
\end{aligned} \tag{2.26}$$

originating from taking the integrals over the vertices of the sss conformal triangle,

$$u^{(0)}_{sss} = \frac{8\pi^{\frac{3d}{2}}\Gamma(3-\frac{d}{2})}{(d-4)^4\Gamma(d-4)}, \tag{2.27}$$

$$R_{sss} = \frac{1}{N}\frac{6\sin(\frac{\pi d}{2})\Gamma(d)((d-4)H_{d-4} - 2d + \pi(d-4)\cot(\frac{\pi d}{2}) + 10)}{\pi(d-4)\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2}+1)}. \tag{2.28}$$

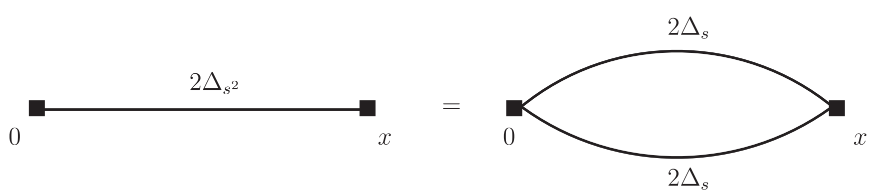
Finally notice that

$$Z^{(0)}_{sss} = -\frac{C^{(0)}_{sss}}{6C_s^3 u^{(0)}_{sss}}, \tag{2.29}$$

which agrees with (2.20).

III. $\langle s^2 ss\rangle$

At the leading order the $\langle s^2 s^2\rangle$ propagator is given by



$$= \frac{C_{s^2}}{|x|^{2\Delta_{s^2}}}\,.$$

Here the leading order scaling dimension of the composite operator s^2 is given by

$$\Delta_{s^2} = 4, \tag{3.1}$$

and the leading order propagator amplitude is

$$C_{s^2} = \frac{1}{2} 2^2 C_s^2 = 2C_s^2, \tag{3.2}$$

where $1/2$ is the symmetry factor, and each factor of 2 comes from the degeneracy of each of the two s^2 insertions. Notice that s^2 is not a conformal primary operator, as it mixes with the nonprimary $\partial^2 s$ at the subleading order in $1/N$ [40,41].[9]

The loop corrections to the s^2 propagator result in

$$\langle s(x)^2 s(0)^2 \rangle = \frac{C_{s^2}(1 + A_{s^2})\mu^{-2\gamma_{s^2}}}{|x|^{2(\Delta_{s^2}+\gamma_{s^2})}}, \tag{3.3}$$

where [38,39]

$$\gamma_{s^2} = -4(d-1)^2 \gamma_\phi + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{3.4}$$

The composite operator s^2 is renormalized according to

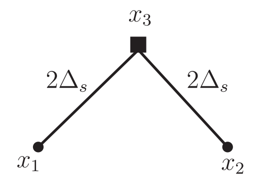
$$s^2 \to \sqrt{1 + \frac{2\gamma_{s^2}}{\delta}} s^2, \tag{3.5}$$

analogously to (2.2).[10]

Conformal contributions to the $\langle s^2 ss \rangle$ three-point function take the form[11]

$$\langle s(x_1)s(x_2)s^2(x_3)\rangle \supset \frac{C^{(0)}_{s^2ss}(1 + \delta C_{s^2ss})}{|x_{12}|^{2\gamma_s-\gamma_{s^2}}(|x_{13}||x_{23}|)^{4+\gamma_{s^2}}} \mu^{-2\gamma_s-\gamma_{s^2}}. \tag{3.6}$$

Here the amplitude of the leading order diagram



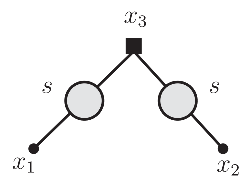
$$= \frac{C^{(0)}_{s^2ss}}{(|x_{13}||x_{23}|)^4}$$

is given by

$$C^{(0)}_{s^2ss} = 2C_s^2, \tag{3.7}$$

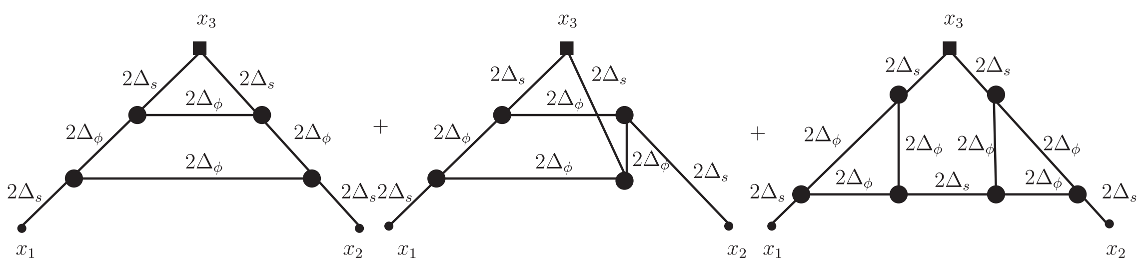
where again the factor of 2 comes from the degeneracy of the s^2 insertion.

At the next-to-leading order the $\langle s^2 ss \rangle$ three-point function is composed of the s propagator corrections to the leading-order $\langle s^2 ss \rangle$ diagram,



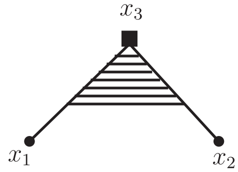
$$= \frac{C^{(0)}_{s^2ss}(1+2A_s)\,\mu^{-4\gamma_s}}{(|x_{13}||x_{23}|)^{4+2\gamma_s}} \tag{3.8}$$

and three diagrams due to the next-to-leading order vertex corrections



(3.9)

We will denote the total of three diagrams in (3.9) as



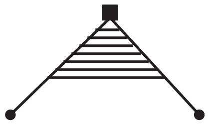
$$= \frac{C^{(0)}_{s^2ss}(1+\delta C_{s^2ss}-2A_s)\,\mu^{2\gamma_s-\gamma_{s^2}}}{(|x_{13}||x_{23}|)^{4+\gamma_{s^2}-2\gamma_s}|x_{12}|^{2\gamma_s-\gamma_{s^2}}} - \frac{C^{(0)}_{s^2ss}}{(|x_{13}||x_{23}|)^4}$$

[9]We will comment more on this in Sec. V in the context of the leading order contribution to the $\langle s^3 ss \rangle$ three-point function.

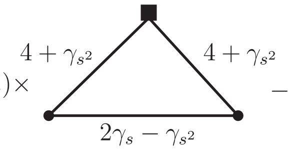
[10]See also [45] for a recent discussion of the analogous renormalization of the composite operators in the Gross-Neveu model.

[11]As mentioned in Introduction, we keep track only of the conformal contributions to the $\langle s^2 ss \rangle$ three-point function. The right-hand side of (3.6) is in fact the three-point function $\langle \mathcal{O} ss \rangle$, where the primary operator $\mathcal{O}$ is given by (1.10).

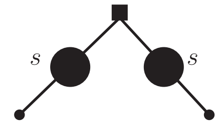
where the amplitude and the exponents have been determined from the requirement that the total adds up to (3.6), and we also subtracted the leading order $\langle s^2 ss\rangle$ three-point function to ensure that the result is purely of the next-to-leading order in $1/N$. We can also rewrite this relation as



$$= C^{(0)}_{s^2ss}(1+\delta C_{s^2ss} - 2A_s)\times$$

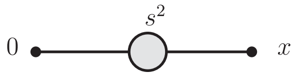


$$-$$



(3.10)

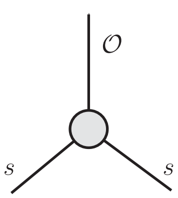
We will use the Feynman rule corresponding to the dressed s^2 propagator



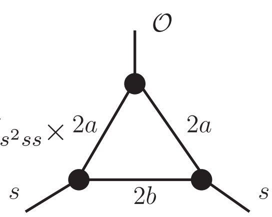
$$= \frac{C_{s^2}\,(1+A_{s^2})\,\mu^{-2\gamma_{s^2}}}{|x|^{2(\Delta_{s^2}+\gamma_{s^2})}}\,.$$

A. $\mathcal{O}ss$ conformal triangle

In this paper we are mostly interested in conformal contributions to the three-point functions involving the composite operator s^2. As reviewed in Sec. I, this operator is not primary, and the true primary operator $\mathcal{O}$ is obtained by mixing with the operator $\partial^2 s$, see (1.10).[12] As a consequence, the conformal contributions to the three-point function $\langle s^2 ss\rangle$ can be obtained from the corresponding conformal triangle $\mathcal{O}ss$, which we intend to derive in this subsection. It is defined by the following diagram:



$$= -2Z_{s^2ss}\times$$



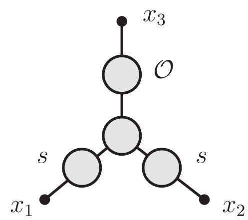
Here we introduced

$$a = \frac{d-\gamma_{s^2}}{2} - 2, \qquad b = \frac{d+\gamma_{s^2}}{2} - \gamma_s. \quad (3.11)$$

We can expand the amplitude of the conformal triangle in $1/N$ as[13]

$$Z_{s^2ss} = Z^{(0)}_{s^2ss}(1+\delta Z_{s^2ss}). \quad (3.12)$$

The conformal triangle is defined so that when we attach full propagators of the $\mathcal{O}$ and s and integrate over the three internal vertices, we obtain the three-point function:



$$= \frac{C^{(0)}_{s^2ss}\,(1+\delta C_{s^2ss})\,\mu^{-\gamma_{s^2}-2\gamma_s}}{(|x_{13}||x_{23}|)^{4+\gamma_{s^2}}\,|x_{12}|^{2\gamma_s-\gamma_{s^2}}}\,.$$

Expanding the integral over the three vertices of the conformal triangle as

$$\begin{aligned} &U\left(4+\gamma_{s^2}, \frac{d-\gamma_{s^2}}{2}-2, \frac{d-\gamma_{s^2}}{2}-2\right) \\ &\quad\times U\left(2+\frac{\gamma_{s^2}}{2}, 2+\gamma_s, d-4-\gamma_s-\frac{\gamma_{s^2}}{2}\right) \\ &\quad\times U\left(\frac{d+\gamma_{s^2}}{2}-\gamma_s, \frac{d-\gamma_{s^2}}{2}-2, 2+\gamma_s\right) \\ &\quad= u^{(0)}_{s^2ss}\left(1+R_{s^2ss}+\mathcal{O}\left(\frac{1}{N^2}\right)\right), \end{aligned} \quad (3.13)$$

where

$$u^{(0)}_{s^2ss} = -N\frac{\pi^{\frac{3d}{2}+2}\csc^2(\frac{\pi d}{2})\Gamma(\frac{d}{2}+1)}{3(d-8)(d-3)\Gamma(d-4)\Gamma(d+1)}, \quad (3.14)$$

$$\begin{aligned} R_{s^2ss} &= \frac{1}{N}\frac{d\Gamma(d)}{6\pi(d-8)(d-6)(d-4)(d-2)\Gamma(\frac{d}{2}+1)^2} \\ &\quad\times\Bigg((-3(d-8)(d-6)(d-4)(d-2)((d-7)d+8)H_{d-5} + d(d(d(d(d(5d-143)+1550)-8252)+23096) \\ &\quad -32768)+19200)\sin\left(\frac{\pi d}{2}\right) - 3\pi(d-8)(d-6)(d-4)(d-2)((d-7)d+8)\cos\left(\frac{\pi d}{2}\right)\Bigg), \end{aligned} \quad (3.15)$$

[12]This mixing disappears in $d=3$ dimensions at the leading order in $1/N$.

[13]Here Z_{s^2ss} denotes the amplitude of the $\mathcal{O}ss$ conformal triangle.

we then obtain[14]

$$Z^{(0)}_{s^2ss} = -\frac{C^{(0)}_{s^2ss}}{2C_{s^2}C_s^2 u^{(0)}_{s^2ss}}, \tag{3.16}$$

$$\delta Z_{s^2ss} = \delta C_{s^2ss} - A_{\mathcal{O}} - 2A_s - R_{s^2ss}. \tag{3.17}$$

where A_{s^2} is defined in (3.3), and $A_{\mathcal{O}}$ is a relative correction to the amplitude of the propagator

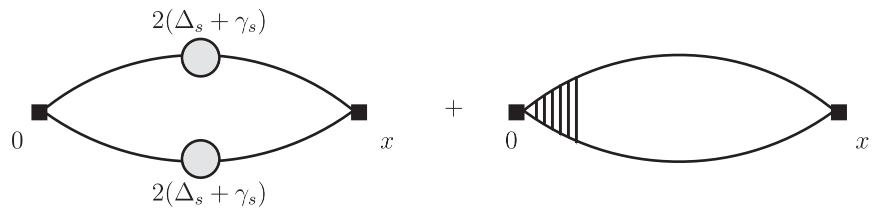
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C_{\mathcal{O}}(1+A_{\mathcal{O}})\mu^{-2\gamma_{\mathcal{O}}}}{|x|^{8+2\gamma_{\mathcal{O}}}}. \tag{3.18}$$

Using (1.10), (1.11), we obtain

$$A_{\mathcal{O}} = A_{s^2} - 24(d-6)(d-8)\frac{C_s}{C_{s^2}}\alpha(d)^2. \tag{3.19}$$

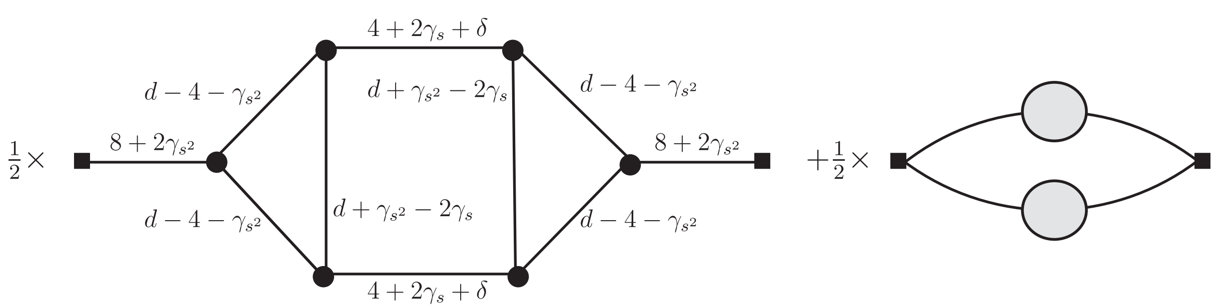
Notice that $A_{s^2}|_{d=3} = A_{\mathcal{O}}|_{d=3}$.

Up to the next-to-leading order in $1/N$, the $\langle\mathcal{O}\mathcal{O}\rangle$ propagator is given by



$$= \frac{C_{s^2}(1+A_{\mathcal{O}})\mu^{-2\gamma_{s^2}}}{|x|^{2(\Delta_{s^2}+\gamma_{s^2})}}.$$

Here the second term stands for dressing one of the $\mathcal{O}ss$ subdiagrams of the leading-order $\langle\mathcal{O}\mathcal{O}\rangle$ diagram, i.e., incorporating three diagrams introduced in (3.9). In fact, using the $\mathcal{O}ss$ conformal triangle we can rewrite the total of the diagrams contributing to $\langle\mathcal{O}\mathcal{O}\rangle$ up to the next-to-leading order as[15]



Notice that here the first diagram in fact contains two $\mathcal{O}ss$ conformal triangles. To compensate for such a double counting, however, we multiplied it by the factor of $1/2$. This diagram already contains the leading order $\langle\mathcal{O}\mathcal{O}\rangle$ diagram, as well as its $1/N$ corrections obtained by dressing of the internal s lines. However, since it is multiplied by the factor of $1/2$, we need to add another one-half of the leading diagram with the corrected propagators. Finally, notice that the first diagram is divergent. To regularize it, we introduced a small shift δ to the dressed internal s lines [43].

Contribution of the second diagram is given by

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \supset C_{s^2}\frac{1}{|x|^8}\left(\frac{1}{2} + A_s - 2\gamma_s\log(\mu|x|)\right) \tag{3.20}$$

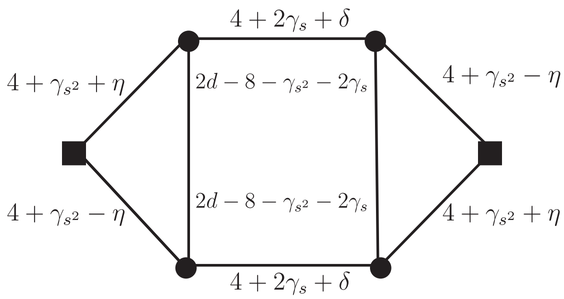
while contribution of the first diagram is

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \supset \frac{1}{2}(C_{s^2}(1+A_{\mathcal{O}})C_s(1+A_s)(-2) \times Z^{(0)}_{s^2ss}(1+\delta Z_{s^2ss}))^2 V(\delta), \tag{3.21}$$

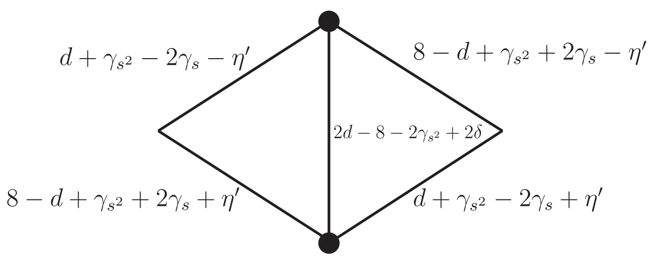
where $V(\delta)$ is obtained by integrating over the internal vertices of the diagram. To find the latter we first integrate over the leftmost and the rightmost vertices, resulting in

[14]For the leading order propagator amplitudes we have $C_{\mathcal{O}} = C_{s^2}$. At the next-to-leading order the anomalous dimensions also agree, $\gamma_{\mathcal{O}} = \gamma_{s^2}$.

[15]Such a method of calculation of conformal triangles was first proposed in [45], where it was applied to determine the s^2ss conformal triangle in the Gross-Neveu model.



Here we have introduced an auxiliary parameter η [7,8,46] shifting exponents of some of the lines. One can easily see that the diagram is an even function of η,[16] and as a result choosing $\eta = \mathcal{O}(\delta)$ will not affect the value of the diagram in the limit $\delta \to 0$. We will take advantage of this fact by setting $\eta = \delta$, which will render two of the vertices unique. Integrating over those vertices we obtain the diagram:



Here we introduced yet another auxiliary shift η', such that the resulting diagram is an even function of η'.[17] Consequently choosing $\eta' = \delta$ we will not change the value of the diagram in the $\delta \to 0$ limit, while this will make the topmost vertex unique. Completing the last two integrals we obtain for the total:

$$\begin{aligned}V(\delta) = &\frac{1}{2} U\left(4+\gamma_{s^2}, \frac{d-\gamma_{s^2}}{2}-2, \frac{d-\gamma_{s^2}}{2}-2\right)^2 \\ &\times U\left(2+\gamma_s+\frac{\delta}{2}, d-4-\gamma_s-\frac{\gamma_{s^2}}{2}, 2+\frac{\gamma_{s^2}-\delta}{2}\right)^2 \\ &\times U\left(d-4-\gamma_{s^2}+\delta, \frac{d+\gamma_{s^2}-\delta}{2}-\gamma_s,\right. \\ &\left.\frac{\gamma_{s^2}-d-\delta}{2}+4+\gamma_s\right) U\left(\frac{d}{2}+\delta, \frac{d}{2}+\delta, -2\delta\right) \\ &\times \frac{\mu^{-2\delta}}{|x|^{8+2\gamma_{s^2}+2\delta}}, \end{aligned} \tag{3.22}$$

where $1/2$ is the symmetry factor of the diagram. Expanding the product of the U functions around $\delta = 0$ and $N = \infty$ we obtain

$$\begin{aligned}V(\delta) &= v_0\left(1+\frac{\gamma_{s^2}-2\gamma_s}{\delta}+\delta v\right)\frac{\mu^{-2\delta}}{|x|^{8+2\gamma_{s^2}+2\delta}} \\ &= v_0(1+\delta v+(4\gamma_s-4\gamma_{s^2})\log(\mu|x|))\frac{1}{|x|^8}, \end{aligned} \tag{3.23}$$

where we subtracted the divergent part using the s^2ss counterterm, and[18]

$$v_0 = \frac{C_{s^2}}{(C_{s^2}C_s(-2)Z^{(0)}_{s^2ss})^2}, \tag{3.24}$$

$$\begin{aligned}\delta v = &2\gamma_s(\gamma-1)+\gamma_{s^2}-\frac{8\gamma_{s^2}}{3}+\frac{2(2\gamma_s+\gamma_{s^2})}{d-8} \\ &-\frac{4(d-7)(2\gamma_s-\gamma_{s^2})}{(d-8)(d-6)}+\pi(2\gamma_s+\gamma_{s^2})\cot\left(\frac{\pi d}{2}\right) \\ &+(2\gamma_s+\gamma_{s^2})\psi^{(0)}(d-4). \end{aligned} \tag{3.25}$$

The corresponding contribution to the two-point function is then

$$\begin{aligned}\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \supset C_{s^2}&\left(\frac{1}{2}+A_{\mathcal{O}}+A_s+\delta Z_{s^2ss}+\frac{\delta v}{2}\right. \\ &\left.+(2\gamma_s-2\gamma_{s^2})\log(\mu|x|)\right)\frac{1}{|x|^8}. \end{aligned} \tag{3.26}$$

Combining (3.20), (3.26) we obtain

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = C_{s^2}\left(1+A_{\mathcal{O}}+2A_s+\delta Z_{s^2ss}+\frac{\delta v}{2}\right)\frac{1}{|x|^{8+2\gamma_{s^2}}}. \tag{3.27}$$

Consequently

$$\delta Z_{s^2ss} = -2A_s - \frac{\delta v}{2} \tag{3.28}$$

For the purpose of calculating correction δC_{s^2ss} to the amplitude of the three-point function (3.6), we can now use Eq. (3.17). Expanding around $d = 3$, we notice that the singular parts of two terms δZ_{s^2ss} and R_{s^2ss} contributing to δC_{s^2ss} cancel each other out:

$$\delta Z_{s^2ss} = -\frac{1}{N}\frac{64}{3\pi^2(d-3)}+\mathcal{O}((d-3)^0), \tag{3.29}$$

[16]One can see this by noticing that $\eta \to -\eta$ can be undone by swapping vertices of integration related by mirror reflection in the horizontal axes.

[17]This can be seen by renaming the vertices of integration $x_{1,2}$ as $x_1 \to x - x_2$, $x_2 \to x - x_1$. We refer the reader to [46] for the detailed explanation of this method of calculating similar diagrams.

[18]Here γ is Euler's constant.

$$R_{s^2ss} = \frac{1}{N}\frac{64}{3\pi^2(d-3)} + \mathcal{O}((d-3)^0). \tag{3.30}$$

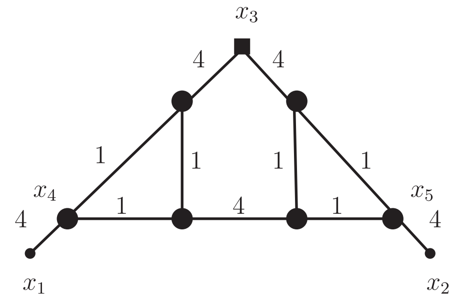
Below in Sec. III B we will calculate the value of δC_{s^2ss} in $d = 3$. Using (3.17), (3.28) we then can solve for the value of the A_{s^2} in $3d$ (recall that that $A_{s^2}|_{d=3} = A_{\mathcal{O}}|_{d=3}$),

$$A_{\mathcal{O}} = \delta C_{s^2ss} - R_{s^2ss} + \frac{\delta v}{2}. \tag{3.31}$$

B. $\langle s^2 ss\rangle$ in 3d

In this subsection we will calculate the $\langle s^2 ss\rangle$ three-point function in $d = 3$ dimensions. The key simplification of considering specifically the three-dimensional case is that in $3d$ the third vertex correction diagram in (3.9) does not contribute to the three-point function. In fact, in three dimensions it exhibits a vanishing contribution both to the anomalous dimensions exponents of the three-point function (3.6), as well as to its overall amplitude.

We have established this as follows. First, one can easily extract only divergent contributions of the diagrams (3.9) in any dimension d.[19] Setting then $d = 3$ one can see that the third diagram in (3.9) does not contribute any divergence in three dimensions. Independently, one can verify that the contributions of the diagram (3.8) and the first two diagrams in (3.9) to the total anomalous dimensions, entirely account for the anomalous dimensions structure of the three-point function (3.6). With this result in mind, we conclude that the third diagram in (3.9) is finite in $3d$, and proceed to its evaluation without the need to introduce regulators:

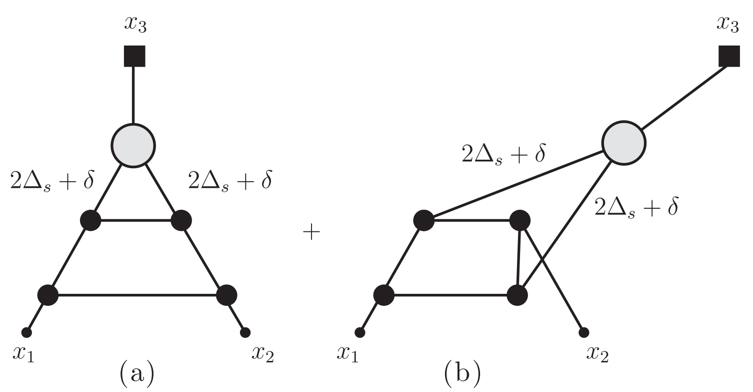


By taking the unique integrals over $x_{4,5}$ one can see that the resulting diagram becomes proportional to $\delta^{(3)}(x_{13})\delta^{(3)}(x_{23})$,[20] and is therefore identically zero for the three-point function, that is defined for noncoincident points only.

We proceed to the calculation of the contributions of the diagram (3.8) and the first two diagrams in (3.9) to the $\langle s^2 ss\rangle$. We performed this calculation in any d, so we keep the dimension general, and set $d = 3$ in the very end. We have the following equation:

$$(3.6) = (3.8) + (3.9). \tag{3.33}$$

Following [45] we replace the $s^2 ss$ subdiagram in the first two diagrams in (3.9) with the $s^2 ss$ conformal triangle. Such a procedure creates two internal s propagators, which we regularize by adding a small shift δ to their exponent:



$$\subset \langle s(x_3)^2 s(x_1) s(x_2)\rangle. \tag{3.34}$$

[19]Since these are divergent diagrams, ordinarily one proceeds by regularizing them via a small correction δ added to the exponent of the internal s field propagators [43]. However, for the purpose of extracting the singular contributions/anomalous dimensions only, a simpler calculation can be performed, see, e.g., [33,47]. In such an approach one carries out all of the integrals explicitly, and replaces the logarithmically divergent integral with $S_d \log\mu$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of $d-1$-dimensional sphere. The total of the anomalous dimensions exponents of the considered diagram is then read off from the coefficient in front of the $\log\mu$ term.

[20]One can see that using the inverse propagator relation

$$\int d^d x_3 \frac{1}{|x_{13}|^{2\Delta}|x_{23}|^{2(d-\Delta)}} = \pi^d A(\Delta)A(d-\Delta)\delta^{(d)}(x_{12}), \tag{3.32}$$

applied for $d = 3$, $\Delta = 1$.

We proceed by integrating both sides of (3.33) over x_3. Due to (3.6) on the left-hand side of (3.33) we obtain

$$\begin{aligned}\text{L.H.S.} &= C^{(0)}_{s^2ss}(1+\delta C_{s^2ss})U\left(2+\frac{\gamma_{s^2}}{2},2+\frac{\gamma_{s^2}}{2},d-4-\gamma_{s^2}\right)\frac{\mu^{-\gamma_{s^2}-2\gamma_s}}{|x_{12}|^{8-d+2\gamma_s+\gamma_{s^2}}}\\ &= 2C_s^2 U(2,2,d-4)(1+\delta C_{s^2ss}+h_1)\frac{\mu^{-\gamma_{s^2}-2\gamma_s}}{|x_{12}|^{8-d+2\gamma_s+\gamma_{s^2}}},\end{aligned} \tag{3.35}$$

$$h_1 = -\frac{1}{N}\frac{2^{d+3}(d-1)\sin(\frac{\pi d}{2})\Gamma(\frac{d+1}{2})(-\frac{2}{d-6}+\pi\cot(\frac{\pi d}{2})+\psi^{(0)}(d-4)+\gamma-1)}{\pi^{3/2}(d-2)d\Gamma(\frac{d}{2}-2)}. \tag{3.36}$$

Next, using (3.8) we obtain the following contribution due to the tree-level diagram with dressed s propagators:

$$\begin{aligned}\int d^dx_3\frac{C^{(0)}_{s^2ss}(1+2A_s)\mu^{-4\gamma_s}}{(|x_{13}||x_{23}|)^{4+2\gamma_s}} &= 2C_s^2(1+2A_s)U(2+\gamma_s,2+\gamma_s,d-4-2\gamma_s)\frac{\mu^{-4\gamma_s}}{|x_{12}|^{8-d+4\gamma_s}}\\ &= 2C_s^2(1+2A_s+h_2)U(2,2,d-4)\frac{\mu^{-4\gamma_s}}{|x_{12}|^{8-d+4\gamma_s}},\end{aligned} \tag{3.37}$$

$$h_2 = \frac{1}{N}\frac{8\sin(\frac{\pi d}{2})\Gamma(d)((d-6)H_{d-5}-d+\pi(d-6)\cot(\frac{\pi d}{2})+4)}{\pi(d-6)\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2}+1)}. \tag{3.38}$$

Integrating diagrams (a), (b) in Fig. (3.34) over x_3 we obtain

$$\int d^dx_3\frac{\langle s(x_3)^2 s(x_1)s(x_2)\rangle}{2C_s^2U(2,2,d-4)} \supset (h_{a,b}-\omega_{a,b}\log(\mu|x_{12}|))\frac{1}{|x_{12}|^{8-d}}. \tag{3.39}$$

Analogously, integrating the third diagram in (3.9) we obtain

$$\int d^dx_3\frac{\langle s(x_3)^2 s(x_1)s(x_2)\rangle}{2C_s^2U(2,2,d-4)} \supset (h_c-\omega_c\log(\mu|x_{12}|))\frac{1}{|x_{12}|^{8-d}}. \tag{3.40}$$

For our purposes, we only need to know

$$h_c(d=3)=0, \qquad \omega_c(d=3)=0, \tag{3.41}$$

as we established above. Integrating over vertices of the $\mathcal{O}ss$ conformal triangle in the diagrams (a), (b) in Fig. (3.34), and collecting the δ-dependent factors, we obtain

$$\begin{aligned}&A\left(2+\frac{\delta}{2}\right)A\left(\frac{d-\delta}{2}-2\right)U\left(2+\frac{\delta}{2},2+\frac{\delta}{2},d-4-\delta\right)\\ &\quad= A(2)A\left(\frac{d}{2}-2\right)U(2,2,d-4)(1+r\delta),\\ &r = -\frac{2}{d-6}+\pi\cot\left(\frac{\pi d}{2}\right)+\psi^{(0)}(d-4)+\gamma-1.\end{aligned} \tag{3.42}$$

In the process we dropped the $1/N$ corrections due to the anomalous dimensions, which are subleading in $1/N$.[21] At the same time, using (3.16), (3.7), the leading order factors can be assembled into the leading order amplitude $C^{(0)}_{s^2ss}=2C_s^2$. The factor of $1+r\delta$ will be important below for the calculation of the finite correction δC_{s^2ss} to the amplitude of the three-point function $\langle s^2ss\rangle$. Integrating over the remaining four vertices of the (a) in Fig. (3.34) we obtain

$$\begin{aligned}&\int d^dx_3\frac{\langle s(x_3)^2 s(x_1)s(x_2)\rangle}{2C_s^2U(2,2,d-4)}\\ &\supset NC_s^2C_\phi^4(1+r\delta)\left(-\frac{2}{\sqrt{N}}\right)^4U\left(3+\delta,\frac{d}{2}-1,\frac{d}{2}-2-\delta\right)\\ &\quad\times U\left(\frac{d}{2}-1,2+\delta,\frac{d}{2}-1-\delta\right)U\left(\frac{d}{2}+\delta,2,\frac{d}{2}-2-\delta\right)\\ &\quad\times U(2,2+\delta,d-4-\delta)\frac{\mu^{-2\delta}}{x^{8-d+2\delta}}.\end{aligned} \tag{3.43}$$

Expanding it in δ we obtain

$$\omega_a = \frac{d-4}{2(d-1)}(2\gamma_s-\gamma_{s^2}), \tag{3.44}$$

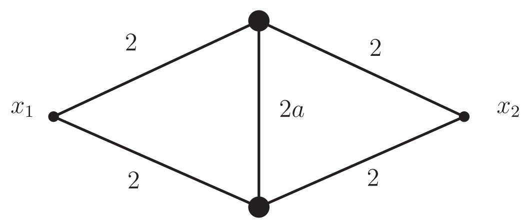
[21]The U functions generated due to integrals over the vertices of the $\mathcal{O}ss$ conformal triangle, contain the factor $A(\frac{d+\gamma_{s^2}}{2}-\gamma_s)$. We have expanded it in $1/N$ and kept the leading $\mathcal{O}(N)$ term.

$$h_a = \frac{1}{N}\frac{4(d-3)\Gamma(4-\frac{d}{2})\Gamma(d-2)\sin^2(\frac{\pi d}{2})}{\pi^2(d-6)^2(d-2)\Gamma(\frac{d}{2})}\left(d(5d-32) - 4\pi(d-6)(d-2)\cot\left(\frac{\pi d}{2}\right)\right.$$
$$\left. + 28 - 4(d-6)(d-2)H_{d-5}\right). \tag{3.45}$$

Integrating over the remaining four vertices of the (b) in Fig. (3.34) we obtain

$$\int d^d x_3 \frac{\langle s(x_3)^2 s(x_1) s(x_2)\rangle}{2C_s^2 U(2,2,d-4)} \supset \frac{1}{2} N C_s^2 C_\phi^4 (1+r\delta)\left(-\frac{2}{\sqrt{N}}\right)^4 U\left(\frac{d}{2}-1,\frac{d}{2}-1,2\right)^2 \frac{\mathrm{sk}(\frac{d}{2}+\delta)\mu^{-2\delta}}{|x_{12}|^{8-d+2\delta}}, \tag{3.46}$$

where $1/2$ is the symmetry factor, and we used the special kite diagram



$$= \frac{\mathrm{sk}(a)}{|x_{12}|^{8+2a-2d}}.$$

For general a, the value of the special kite diagram can be found in [48].[22] However, since we are only interested in expansion around $\delta = 0$ and retaining only singular and finite terms, it is sufficient to use expression (A5) for $\alpha_{1,2,3,4} = 1$, $\alpha_5 = d/2 + \delta$, and expand around $\delta = 0$. While the second and third diagrams in the right-hand side of (A5) are straightforward to calculate using the propagator merging relation, the first diagram can be calculated by inserting a point into the diagonal propagator, splitting it into two propagators with the exponents $2d-4$ and $2+2\delta$. Taking the unique integral we will obtain the diagram equal to $F(\frac{d}{2}-1,\frac{d}{2}-1)$, where we dropped corrections linear in δ, since the diagram is finite. Assembling everything together, we obtain

$$\mathrm{sk}\left(\frac{d}{2}+\delta\right) = \frac{f_1}{\delta} + f_2 + \mathcal{O}(\delta), \tag{3.47}$$

$$f_1 = \frac{2(d-6)\pi^{d+1}\csc(\frac{\pi d}{2})}{(d-2)\Gamma(d-3)}, \tag{3.48}$$

$$f_2 = \frac{\pi^d}{2}\left(\frac{4\pi\csc(\frac{\pi d}{2})}{(d-2)^2\Gamma(d-3)}\left(-2(d-5)d + \pi(d-6)(d-2)\cot\left(\frac{\pi d}{2}\right) - 4 + (d-6)(d-2)H_{d-4}\right)\right.$$
$$\left. + (d-4)\cos\left(\frac{\pi d}{2}\right)\Gamma(3-d)\left(\pi^2 - 6\psi^{(1)}\left(\frac{d}{2}-1\right)\right)\right). \tag{3.49}$$

Using it in (3.46) and expanding it in δ we obtain

$$\omega_b = \frac{d-6}{2(d-1)}(2\gamma_s - \gamma_{s^2}), \tag{3.50}$$

$$h_b = \frac{1}{N}\frac{4^{d-3}\sin(\frac{\pi d}{2})\Gamma(\frac{d-1}{2})^2}{\pi^2(d-2)^2\Gamma(d-3)\Gamma(d-1)}\left(-4(3-d)(d-2)^2\Gamma(d-3)\left(\gamma(d-6) - d + \pi(d-6)\cot\left(\frac{\pi d}{2}\right)\right.\right.$$
$$\left. + (d-6)\psi^{(0)}(d-4) + 4\right) + \Gamma(d-1)\left(-8(d-5)d - 16 + 4(d-6)(d-2)H_{d-4} + 4\pi(d-6)(d-2)\cot\left(\frac{\pi d}{2}\right)\right.$$
$$\left.\left. + \frac{(d-4)(d-2)^2\sin(\pi d)\Gamma(3-d)\Gamma(d-3)(\pi^2 - 6\psi^{(1)}(\frac{d}{2}-1))}{2\pi}\right)\right). \tag{3.51}$$

[22]See Eq. (22) therein; notice that each integral over internal vertex in [48] is multiplied by $1/(2\pi)^d$, so we need to multiply that expression by $(2\pi)^{2d}$ to adjust it to our conventions.

Using (3.37), (3.44), (3.50) in $d = 3$ we obtain the total anomalous dimensions term $-(\gamma_{s^2} + 2\gamma_s)\log(\mu|x_{12}|)$, in agreement with (3.33), (3.35). At the same time, using (3.33), (3.36), (3.38), (3.45), (3.51) we obtain

$$\delta C_{s^2ss} = -h_1 + h_2 + 2A_s + h_a + h_b + h_c. \tag{3.52}$$

Evaluating in $d = 3$ and using (3.31) gives[23]

$$A_{s^2}(d=3) = \delta C_{s^2ss}(d=3) = \frac{1}{N}\left(\frac{176}{9\pi^2} - 1\right). \tag{3.53}$$

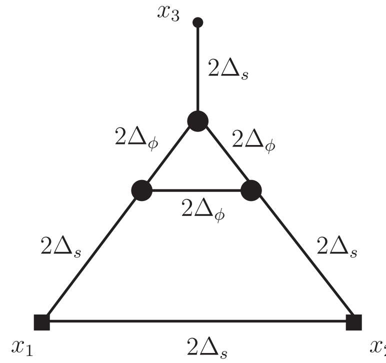
IV. $\langle s^2 s^2 s\rangle$

In this section we are going to calculate the $\langle s^2 s^2 s\rangle$ three-point function at the next-to-leading order in the $1/N$ expansion. Conformal contributions to this correlation function have the form[24]

$$\langle s^2(x_1)s^2(x_2)s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+\delta C_{s^2s^2s})\mu^{-\gamma_s - 2\gamma_{s^2}}}{|x_{12}|^{6+2\gamma_{s^2}-\gamma_s}(|x_{13}||x_{23}|)^{2+\gamma_s}}. \tag{4.1}$$

Our goal is to find the amplitude correction $\delta C_{s^2s^2s}$.

At the leading $\mathcal{O}(1/\sqrt{N})$ order in $1/N$ the $\langle s^2 s^2 s\rangle$ three-point function is determined by the diagram



$$= \frac{C^{(0)}_{s^2s^2s}}{|x_{12}|^6(|x_{13}||x_{23}|)^2}$$

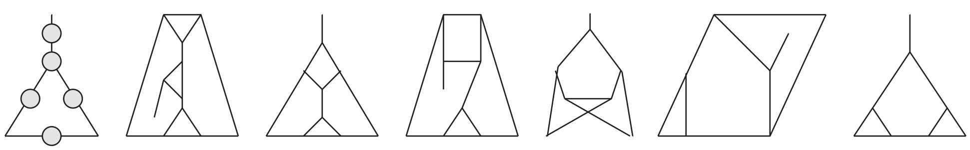
where the leading order amplitude is given by[25]

$$C^{(0)}_{s^2s^2s} = 2^2 C^{(0)}_{sss} C_s. \tag{4.2}$$

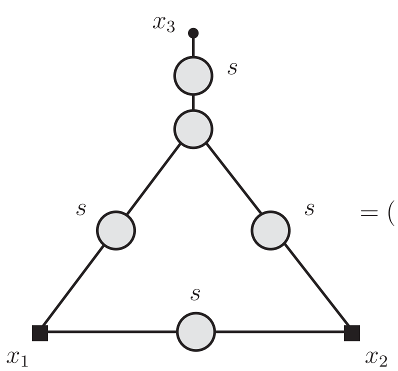
This amplitude vanishes in $3d$, owing to the fact that $C^{(0)}_{sss}(d=3) = 0$.

A. Next-to-leading order

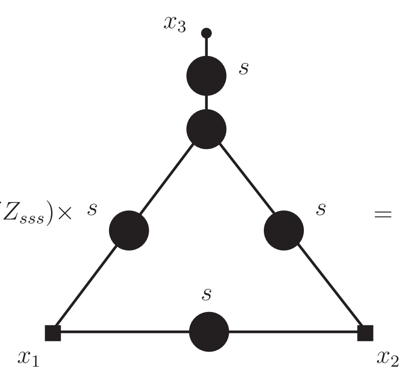
We now proceed to calculation of the $\langle s^2 s^2 s\rangle$ three-point function at the next-to-leading order in the $1/N$ expansion. The contributing diagrams are



In what follows, we will provide a detailed calculation of each of these diagrams. Let us begin by considering the contributing diagrams which are obtained by incorporating the $1/N$ corrections to the s propagators and the and the sss subdiagram of the leading order $\langle s^2 s^2 s\rangle$ diagram:



$$= (1 + 4A_s + \delta Z_{sss})\times$$



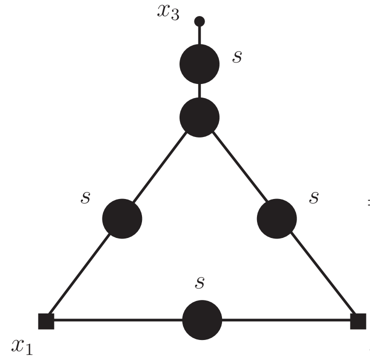
$$= \frac{C^{(0)}_{s^2s^2s}(1+v_1)\mu^{-5\gamma_s}}{|x_{12}|^{6+3\gamma_s}(|x_{13}||x_{23}|)^{2+\gamma_s}} \tag{4.3}$$

[23]Recall that $h_c(d=3) = 0$, due to (3.41).

[24]We skip keeping track of the nonconformal contributions due to mixing of s^2 and $\partial^2 s$, which in particular vanish in $d = 3$.

[25]In such diagrams the factor of 2^2 is a degeneracy factor due to the composite operators s^2.

Here we have used the sss conformal triangle to integrate the diagram



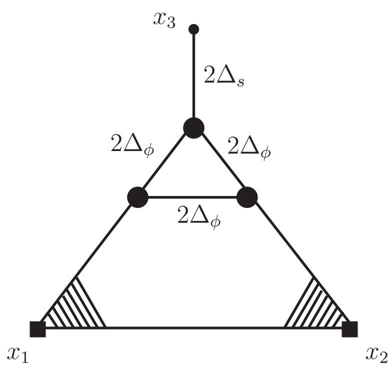
$$= \frac{C^{(0)}_{s^2s^2s}(1+R_{sss})\mu^{-5\gamma_s}}{|x_{12}|^{6+3\gamma_s}(|x_{13}||x_{23}|)^{2+\gamma_s}} \tag{4.4}$$

obtaining

$$v_1 = 4A_s + \delta Z_{sss} + R_{sss} = \delta C_{sss} + A_s, \tag{4.5}$$

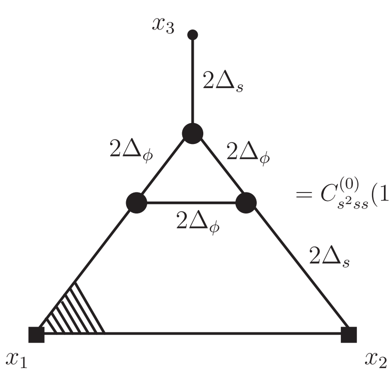
where in the last equality we used (2.25).

Next we consider $1/N$ corrections to the s^2ss subdiagrams of the leading order $\langle s^2 s^2 s\rangle$. We will demonstrate below that these diagrams have the form

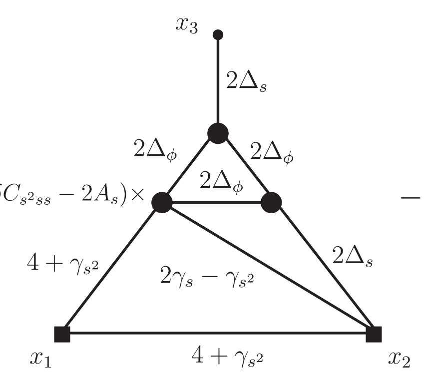


$$= \frac{C^{(0)}_{s^2s^2s}(1+v_2+v_3)\mu^{4\gamma_s-2\gamma_{s^2}}}{|x_{12}|^{6+2\gamma_{s^2}-4\gamma_s}(|x_{13}||x_{23}|)^2} - \frac{C^{(0)}_{s^2s^2s}}{|x_{12}|^6(|x_{13}||x_{23}|)^2} \tag{4.6}$$

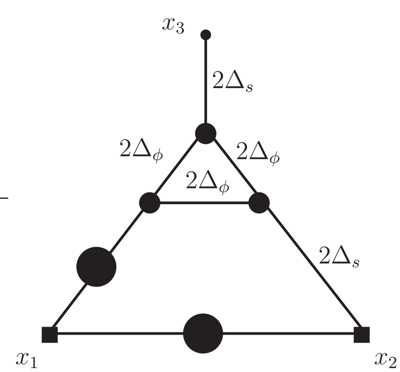
We proceed to calculating diagrams involving corrections to the left- and the right-hand s^2ss subdiagrams separately. The total of these contributions is additive, due to linearization of the next-to-leading $1/N$ corrections. Using (3.10) we can express



$= C^{(0)}_{s^2ss}(1+\delta C_{s^2ss} - 2A_s)\times$

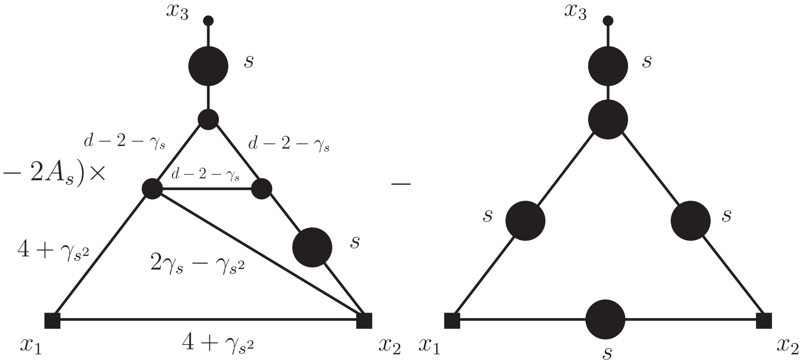


$-$



By linearizing over the $1/N$ corrections to the exponents of propagators of diagrams with identical leading-order skeleton structure, we can further rewrite this expression as

$$-\frac{8C_\phi^3}{\sqrt{N}}C^{(0)}_{s^2ss}(1+\delta C_{s^2ss}-2A_s)\times$$



Notice that the first diagram is completely integrable, giving

$$\langle s(x_1)^2 s(x_2)^2 s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+\delta C_{s^2ss}-2A_s+v_2)\mu^{-\gamma_{s^2}-3\gamma_s}}{|x_{12}|^{6+\gamma_{s^2}+\gamma_s}|x_{13}|^{2-\gamma_s+\gamma_{s^2}}|x_{23}|^{2-\gamma_{s^2}+3\gamma_s}}, \tag{4.7}$$

where

$$v_2 = \frac{1}{N}\frac{\Gamma(d)((3(d-4)(d-2)H_{d-4}+d((d-15)d+60)-60)\sin(\frac{\pi d}{2})+3\pi(d-4)(d-2)\cos(\frac{\pi d}{2}))}{\pi(d-4)\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})}. \tag{4.8}$$

is obtained by the expansion of the U functions generated during the integration over the unique internal vertices

$$U\left(\frac{d-\gamma_s}{2}-1,\frac{d-\gamma_s}{2}-1,2+\gamma_s\right)U\left(1+\frac{\gamma_s}{2},2+\gamma_s,d-3-\frac{3\gamma_s}{2}\right)U\left(\frac{d-\gamma_s}{2}-1,2+\frac{\gamma_{s^2}}{2},\frac{d+\gamma_s-\gamma_{s^2}}{2}-1\right), \tag{4.9}$$

while the second diagram is given by (4.4), for the total of

$$\langle s(x_1)^2 s(x_2)^2 s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+\delta C_{s^2ss}-2A_s+v_2-R_{sss})\mu^{-\gamma_{s^2}+2\gamma_s}}{|x_{12}|^{6+\gamma_{s^2}-2\gamma_s}|x_{13}|^{2-2\gamma_s+\gamma_{s^2}}|x_{23}|^{2-\gamma_{s^2}+2\gamma_s}} - \frac{C^{(0)}_{s^2s^2s}}{|x_{12}|^6(|x_{13}||x_{23}|)^4}, \tag{4.10}$$

Analogously, correcting the right-hand s^2ss subdiagram in (4.6) we obtain

$$\langle s(x_1)^2 s(x_2)^2 s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+\delta C_{s^2ss}-2A_s+v_3-R_{sss})\mu^{-\gamma_{s^2}+2\gamma_s}}{|x_{12}|^{6+\gamma_{s^2}-2\gamma_s}|x_{13}|^{2-\gamma_{s^2}+2\gamma_s}|x_{23}|^{2-2\gamma_s+\gamma_{s^2}}} - \frac{C^{(0)}_{s^2s^2s}}{|x_{12}|^6(|x_{13}||x_{23}|)^4}, \tag{4.11}$$

where

$$v_3 = v_2. \tag{4.12}$$

Combining (4.3), (4.10), (4.11), we obtain

$$\langle s(x_1)^2 s(x_2)^2 s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+w_0)\mu^{-\gamma_s-2\gamma_{s^2}}}{|x_{12}|^{6+2\gamma_{s^2}-\gamma_s}(|x_{13}||x_{23}|)^{2+\gamma_s}}, \tag{4.13}$$

where we denoted

$$w_0 = v_1 + 2v_2 + 2\delta C_{s^2ss} - 4A_s - 2R_{sss}. \tag{4.14}$$

Using (4.5) we obtain

$$w_0 = \delta C_{sss} - 3A_s + 2(\delta C_{s^2ss} + v_2 - R_{sss}). \tag{4.15}$$

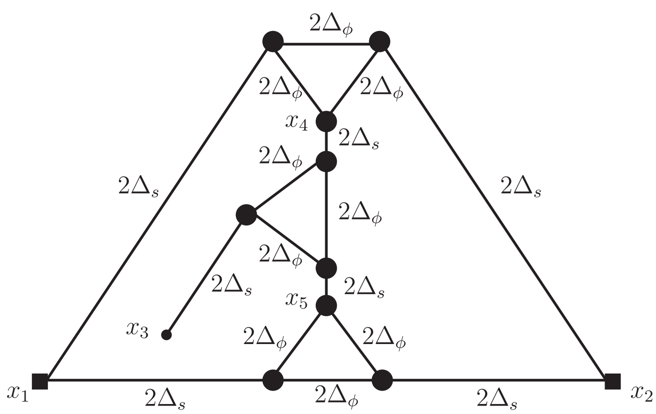
Notice that (4.13) already has the structure required by the conformal symmetry of the $\langle s^2 s^2 s\rangle$ three-point function (4.1). This means that the rest of the diagrams which contribute to the $\langle s^2 s^2 s\rangle$ are finite, as we will confirm explicitly below in this section. Expanding around $d = 3$, we obtain

$$v_2 = \frac{16}{\pi^2}\frac{1}{d-3} + \mathcal{O}((d-3)^0),$$
$$R_{sss} = \frac{16}{\pi^2}\frac{1}{d-3} + \mathcal{O}((d-3)^0). \tag{4.16}$$

We also know that $\langle sss\rangle$ vanishes in $d = 3$, and therefore $\delta C_{sss} = \mathcal{O}((d-3)^0)$. In addition, we know that $A_s = \mathcal{O}((d-3)^0)$. Finally, while we have not calculated δC_{s^2ss}, we know that $C^{(0)}_{s^2ss}\delta C_{s^2ss}$ should be finite in $3d$, and therefore, since $C^{(0)}_{s^2ss}$ is finite, we conclude that δC_{s^2ss} must be finite in $3d$.[26] Then from (4.15) we conclude that

$$w_0 = \mathcal{O}((d-3)^0). \tag{4.17}$$

Consider the following diagram

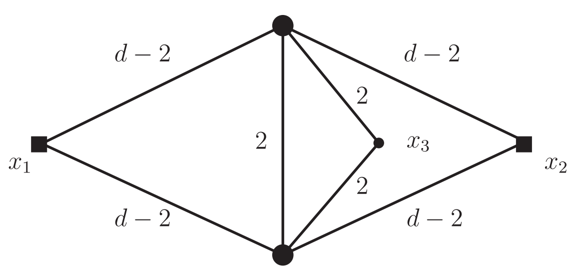


$$= \frac{w_1}{|x_{12}|^6(|x_{13}||x_{23}|)^2}.$$

Integrating over all of the internal vertices except for $x_{4,5}$, and denoting

$$w_1 = 2^2 N^2\left(-\frac{2}{\sqrt{N}}\right)^6 C^{(0)}_{sss} C^6_\phi C^4_s U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^2 U(1,2,d-3)^2 \hat{w}_1, \tag{4.18}$$

we obtain



$$= \frac{\hat{w}_1}{|x_{12}|^{2d-6}(|x_{13}||x_{23}|)^2}.$$

Integrating both sides of the last diagrammatic equation with respect to x_2 we obtain

$$\hat{w}_1 = \frac{\mathrm{ChT}(1,1)}{U(1,2,d-3)} \tag{4.19}$$

where ChT is given by [49,50][27]

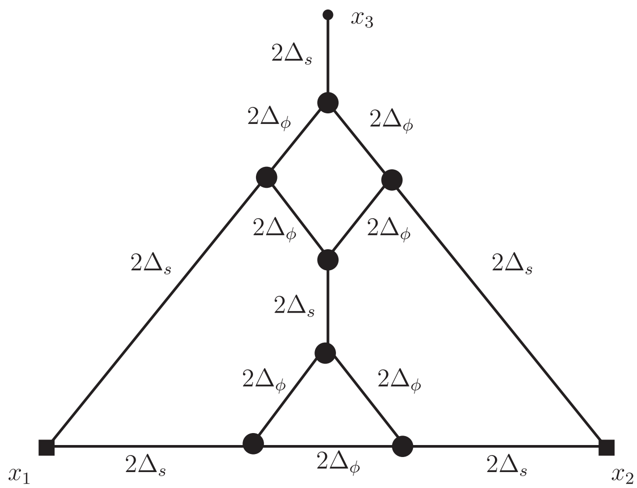
[26]Actually we calculated δC_{s^2ss} in $3d$ and showed explicitly that it is finite.
[27]See Appendix B for details.

$$\mathrm{ChT}(\alpha,\beta) = \frac{\pi^d\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2}-1)\Gamma(d-2)}\left(\frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}+\alpha-2)}{(1-\beta)(\alpha+\beta-2)\Gamma(2-\alpha)\Gamma(\alpha)} + \frac{\Gamma(\frac{d}{2}-\beta)\Gamma(\frac{d}{2}+\beta-2)}{(1-\alpha)(\alpha+\beta-2)\Gamma(2-\beta)\Gamma(\beta)} + \frac{\Gamma(\frac{d}{2}-\alpha-\beta+1)\Gamma(\frac{d}{2}+\alpha+\beta-3)}{(\alpha-1)(\beta-1)\Gamma(-\alpha-\beta+3)\Gamma(\alpha+\beta-1)}\right). \tag{4.20}$$

Combining (4.18), (4.19), we obtain

$$w_1 = -\frac{1}{N^{3/2}}2^{4d-3}(d-3)^6\pi^{-\frac{d}{2}-9}\sin^7\left(\frac{\pi d}{2}\right)\Gamma\left(3-\frac{d}{2}\right)^3\Gamma\left(\frac{d-3}{2}\right)^4\left(\pi^2-6\psi^{(1)}\left(\frac{d}{2}-1\right)\right). \tag{4.21}$$

Next, consider the diagram



$$= \frac{w_2}{|x_{12}|^6(|x_{13}||x_{23}|)^2}$$

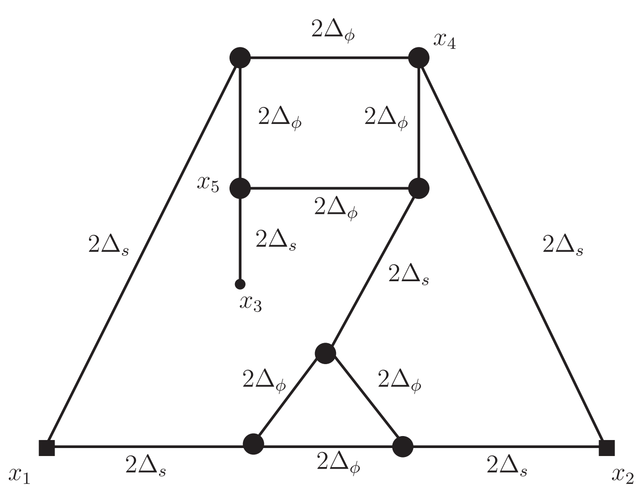
It is straightforward to find w_2 by integrating both sides of this diagrammatic equation over x_3:

$$w_2 = 2^2N\left(-\frac{2}{\sqrt{N}}\right)^4 C_{sss}^{(0)}C_\phi^4C_s^3U\left(\frac{d}{2}-1,\frac{d}{2}-1,2\right)^3U(2,2,d-4). \tag{4.22}$$

Simplifying this expression, we obtain

$$w_2 = \frac{1}{N^{3/2}}\frac{4^{3d-2}\sin^6(\frac{\pi d}{2})\Gamma(2-\frac{d}{2})\Gamma(4-\frac{d}{2})\Gamma(\frac{d-1}{2})^6}{\pi^9\Gamma(d-3)^2}. \tag{4.23}$$

The other contributing diagram is given by

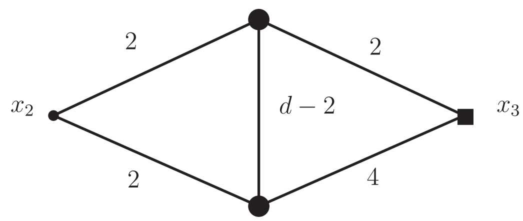


$$= \frac{w_3}{|x_{12}|^6(|x_{13}||x_{23}|)^2}.$$

To calculate this diagram, we proceed by integrating over the sss subdiagram, followed by the integral over the $x_{4,5}$ vertices. Integrating both sides of the resulting diagrammatic equation over x_1, we obtain

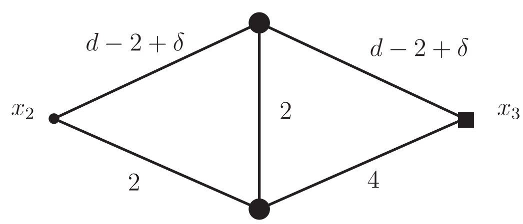
$$w_3 = 2^2 N \left(-\frac{2}{\sqrt{N}}\right)^4 C_{sss}^{(0)} C_\phi^4 C_s^3 U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^2 \hat{w}_3, \tag{4.24}$$

where $\hat{w}_3$ is determined by



$$= \frac{\hat{w}_3}{|x_{23}|^{8-d}}.$$

To calculate this self-energy diagram, we split the diagonal propagator into two merging propagators, with the exponents $2d-4$ and 2, while dividing the diagram by $U(\frac{d}{2}-1, \frac{d}{2}-1, 2)$. Integrating over the unique topmost vertex produces the factor of $U(\frac{d}{2}-1, \frac{d}{2}-1, 2)$, resulting in



$$= \frac{\hat{w}_3(\delta)}{|x_{23}|^{4+2\delta}}.$$

Here we introduced an auxiliary regulator δ, which we will eventually set to zero.[28] Applying the integration by parts identity to the topmost vertex (with the exponents $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \frac{d+\delta}{2} - 1$), we can solve for $\hat{w}_3(\delta)$ by calculating four diagrams, three of which can be easily calculated using the propagator merging relations, while the fourth one is the Fourier dual of the ChT diagram. Assembling everything together, and taking the limit $\delta \to 0$, we obtain

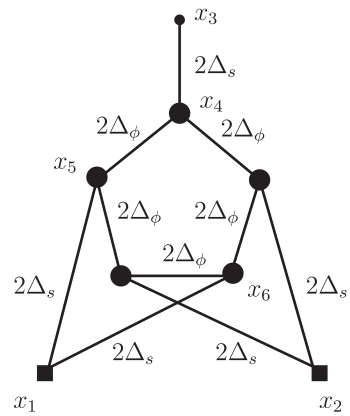
$$\hat{w}_3 = \frac{\pi^{d+1}\left(-\pi^2(d-4)^2 + \frac{8(d((d-4)d-8)+36)}{(d-2)^2} + 6(d-4)^2 \psi^{(1)}(\frac{d}{2})\right)}{4(d-4)\Gamma(d-2)\sin(\frac{\pi d}{2})}. \tag{4.25}$$

Plugging this into (4.24) we obtain

$$w_3 = \frac{1}{N^{3/2}} \frac{2^{5d-7}\sin^5(\frac{\pi d}{2})\Gamma(3-\frac{d}{2})\Gamma(\frac{d-1}{2})^5}{\pi^{15/2}\Gamma(d-3)\Gamma(\frac{d}{2})^2}\left(\pi^2(d^2-6d+8)^2 - 6(d^2-6d+8)^2\psi^{(1)}\left(\frac{d}{2}\right) - 8(d((d-4)d-8)+36)\right). \tag{4.26}$$

[28]The regulator is auxiliary because the diagram is in fact finite in the $\delta \to 0$ limit.

Next, consider the first nonplanar pentagon-based diagram

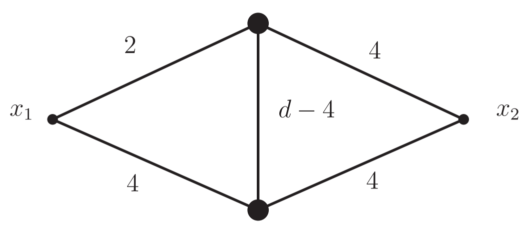


$$= \frac{w_4}{|x_{12}|^6(|x_{13}||x_{23}|)^2}.$$

To calculate this diagram, we integrate over the internal vertex x_4, and then integrate both sides of this diagrammatic equation over x_3, followed by integration over the vertices $x_{5,6}$. This will give

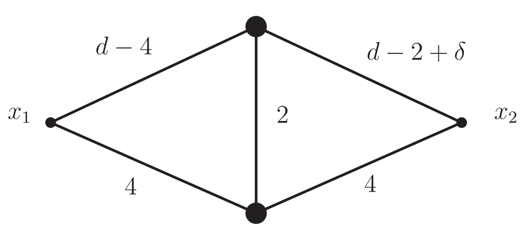
$$w_4 = 2^2 N\left(-\frac{2}{\sqrt{N}}\right)^5 C_\phi^5 C_s^5 U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^3 \hat{w}_4, \tag{4.27}$$

where $\hat{w}_4$ is determined by



$$= \frac{\hat{w}_4}{|x_{12}|^{10-d}}.$$

To find $\hat{w}_4$, we split the diagonal propagator with the exponent $d-4$ into two propagators with the exponents $2d-6$ and 2, while dividing the diagram by $U(1,2,d-3)$. This will make the topmost vertex unique, integrating over which will produce the factor of $U(1,2,d-3)$, resulting in

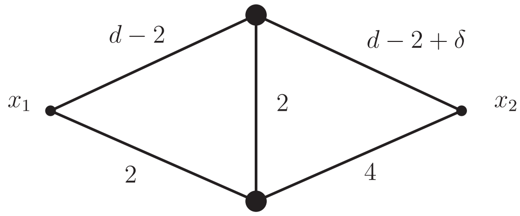


$$= \frac{\hat{w}_4(\delta)}{|x_{12}|^{4+\delta}}.$$

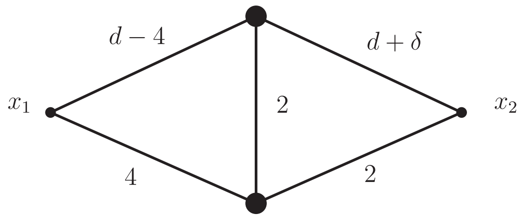
Here we introduced an auxiliary regulator δ, which allows us to apply the integration by parts relation to the topmost vertex (with the exponents $\alpha_1 = 1$, $\alpha_2 = \frac{d}{2}-2$, $\alpha_3 = \frac{d+\delta}{2}-1$), resulting in

$$\begin{aligned}\hat{w}_4 = &\frac{\frac{d}{2}-2}{1-\frac{\delta}{2}}\left(U(2,2,d-4)U\left(\frac{d}{2}-1, \frac{d+\delta}{2}-1, 2-\frac{\delta}{2}\right) - c_1\right)\\ &+\frac{\frac{d+\delta}{2}-1}{1-\frac{\delta}{2}}\left(U(2,2,d-4)U\left(\frac{d}{2}-2, \frac{d+\delta}{2}, 2-\frac{\delta}{2}\right) - c_2\right),\end{aligned} \tag{4.28}$$

where we denoted



$$= \frac{c_1}{|x_{12}|^{4+\delta}},$$



$$= \frac{c_2}{|x_{12}|^{4+\delta}}.$$

The diagram c_1 is in fact finite, so we can set $\delta = 0$. In fact, it is equal to the diagram $\hat{w}_3$, calculated above,

$$c_1 = \lim_{\delta\to 0} \hat{w}_3. \tag{4.29}$$

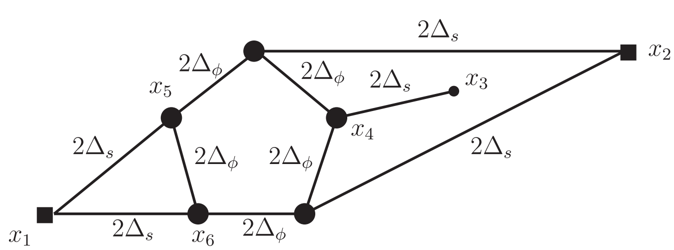
To calculate c_2, we apply the integration by parts relation to the topmost vertex (with the exponents $\alpha_1 = 1$, $\alpha_2 = \frac{d}{2} - 2$, $\alpha_3 = \frac{d+\delta}{2}$), which will express that diagram in terms of a sum of four diagrams. Three of these diagrams can be easily calculated using the propagator merging relation, while the fourth diagram is given by the Fourier transform of the ChT diagram. Assembling everything together, we obtain

$$\begin{aligned} c_2 = {} & \frac{4(d-6)\pi^{d+1}\csc(\frac{\pi d}{2})}{(d-2)\delta\Gamma(d-3)} + \frac{\pi^d}{12}\Bigg(-\frac{4(\pi^2(d-4)(d-2) - 6(d-4)d + 24)\cos(\frac{\pi d}{2})\Gamma(4-d)}{(d-2)^2} \\ & + \frac{\pi\csc(\frac{\pi d}{2})\Gamma(d-2)}{(d-4)\Gamma(d-1)^2}\left(-18(d^2-6d+8)^2\psi^{(1)}\left(\frac{d}{2}\right) + \pi^2(d-2)d(d-4)^2 \right. \\ & \left. - 12(d(d((d-13)d+64)-140)+120)\right)\Bigg). \end{aligned} \tag{4.30}$$

Combining (4.27), (4.28), (4.29), (4.30), we obtain

$$w_4 = -\frac{1}{N^{3/2}} \frac{32^{d-1}\Gamma(\frac{d-1}{2})^5 \sin^5(\frac{\pi d}{2})}{\pi^{15/2}\Gamma(\frac{d}{2}-2)^2\Gamma(\frac{d-2}{2})}\left(-\frac{8\cos(\frac{\pi d}{2})\Gamma(7-d)}{d^2-9d+20} - \frac{\pi(d-4)\csc(\frac{\pi d}{2})(-8d - 6\psi^{(1)}(\frac{d}{2}-2) + \pi^2 + 48)}{\Gamma(d-2)}\right). \tag{4.31}$$

The other nonplanar diagram based on a pentagon effective vertex for s is given by

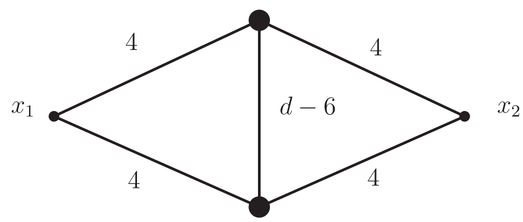


$$= \frac{w_5}{|x_{12}|^4(|x_{13}||x_{23}|)^2}.$$

To calculate this diagram, we integrate over x_4, followed by integration of both sides of this diagrammatic equation over x_3, followed by integration over x_5, x_6. This will give

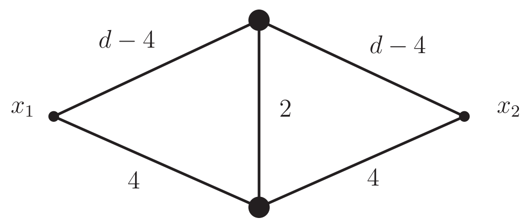
$$w_5 = 2^2 N\left(-\frac{2}{\sqrt{N}}\right)^5 C_\phi^5 C_s^5 U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^2 U\left(\frac{d}{2}-1, \frac{d}{2}-2, 3\right)\hat{w}_5, \tag{4.32}$$

where $\hat{w}_5$ is determined by



$$= \frac{\hat{w}_5}{|x_{12}|^{10-d}}.$$

To determine $\hat{w}_5$, we split the diagonal propagator with the exponent $d-6$ into two, with the exponents $2d-8$ and 2, while dividing the diagram by $U(1, 3, d-4)$. This will make the topmost vertex unique, integrating over which will produce the factor of $U(2, 2, d-4)$, resulting in



$$= \frac{\tilde{w}_5}{|x_{12}|^2}.$$

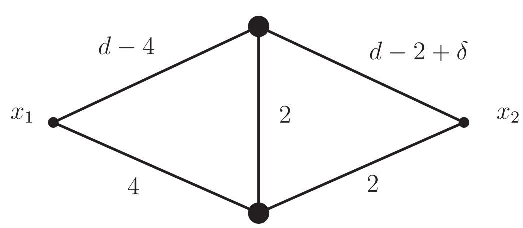
Here we denoted

$$\hat{w}_5 = \frac{U(2, 2, d-4)}{U(1, 3, d-4)} \tilde{w}_5. \tag{4.33}$$

To find $\tilde{w}_5$ we will apply the integration by parts relation to the topmost vertex (with the exponents $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \frac{d}{2} - 2$), resulting in

$$\tilde{w}_5 = \frac{d-4}{4} \left(2U(2, 2, d-4) U\left(\frac{d}{2} - 1, \frac{d}{2} - 2, 3 \right) - 2c_3 \right), \tag{4.34}$$

where c_3 is determined by



$$= \frac{c_3(\delta)}{|x_{12}|^{2+\delta}}.$$

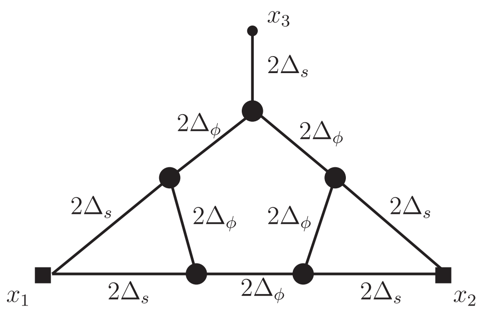
Here we introduced an auxiliary regulator δ, assuming that in the end we will take the limit $c_3 = \lim_{\delta \to 0} c_3(\delta)$. To find $c_3(\delta)$ we apply the integration by parts relation to the topmost vertex (with the exponents $\alpha_1 = 1$, $\alpha_2 = \frac{d+\delta}{2} - 1$, $\alpha_3 = \frac{d}{2} - 2$), which will express that diagram in terms of a sum of four diagrams. Three of these diagrams can be straightforwardly integrated using the propagator merging relation, while the fourth one is given by the Fourier transform of the ChT diagram. Assembling everything together, we obtain

$$c_3 = -\frac{\pi^{d+1} \csc(\frac{\pi d}{2})(\pi^2 (d-4)^2 - 8d - 6(d-4)^2 \psi^{(1)}(\frac{d}{2} - 1) + 24)}{4((d-4)\Gamma(d-2))}. \tag{4.35}$$

Putting together (4.32), (4.33), (4.34), (4.35), we obtain

$$w_5 = \frac{1}{N^{3/2}} \frac{2^{4d-1} \sin^4(\frac{\pi d}{2})\Gamma(\frac{d-1}{2})^4(-\pi^2(d-4)^2 + 4(d-3)(d-2) + 6(d-4)^2\psi^{(1)}(\frac{d}{2}-1))}{\pi^6(d-4)^2\Gamma(\frac{d}{2}-2)^4}. \tag{4.36}$$

The last diagram based on the pentagon effective vertex for s is planar:



$$= \frac{w_6}{|x_{12}|^4(|x_{13}||x_{23}|)^2}.$$

This diagram is straightforward to calculate by applying the uniqueness and the propagator merging relations, as well as integrating both sides over x_3, yielding

$$w_6 = 2^2 N\left(-\frac{2}{\sqrt{N}}\right)^5 C_\phi^5 C_s^5 U\left(\frac{d}{2}-1, \frac{d}{2}-1, 2\right)^3 \times U\left(\frac{d}{2}-1, \frac{d}{2}-2, 3\right) U(2,3,d-5). \tag{4.37}$$

Simplifying it, we obtain

$$w_6 = -\frac{1}{N^{3/2}} \frac{2^{5d-3}\sin^5(\frac{\pi d}{2})\Gamma(5-\frac{d}{2})\Gamma(\frac{d-1}{2})^5}{\pi^{15/2}(d-6)^3\Gamma(\frac{d}{2}-1)^2\Gamma(d-6)}. \tag{4.38}$$

Combining everything together, we obtain

$$C^{(0)}_{s^2s^2s}\delta C_{s^2s^2s} = C^{(0)}_{s^2s^2s} w_0 + \sum_{a=1}^{6} w_a \tag{4.39}$$

The only nonvanishing contributions in $3d$ are given by

$$w_4(d=3) = \frac{1}{N^{3/2}}\frac{512}{\pi^6}, \quad w_5(d=3) = \frac{1}{N^{3/2}}\frac{256}{\pi^6}. \tag{4.40}$$

This implies

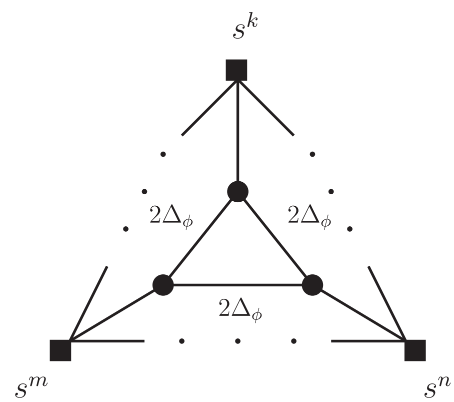
$$C^{(0)}_{s^2s^2s}\delta C_{s^2s^2s}(d=3) = \frac{1}{N^{3/2}}\frac{768}{\pi^6}. \tag{4.41}$$

V. FATE OF THE EMERGENT $\mathbb{Z}_2$ SYMMETRY AT LARGE N

In the previous section, we demonstrated that while the leading $\mathcal{O}(1/N^{1/2})$ order amplitude of the $\langle s^2 s^2 s\rangle$ three-point function vanishes in $d=3$ dimensions, its next-to-leading $\mathcal{O}(1/N^{3/2})$ order correction is nonvanishing. This implies that the conjectured $s \to -s$ symmetry of correlation functions in the s sector of the theory is violated at the first subleading order in $1/N$.

Instead, in this section we intend to discuss whether the statement of the $s \to -s$ symmetry has a more general validity at the leading order in $1/N$. Specifically, we are going to focus on the three-point correlation functions $\langle s^k s^m s^n\rangle$, with $k+m+n = 2l+3$, where k, m, n are positive integers and $l = 0, 1, 2, \ldots$[29] This question is particularly relevant from the standpoint of the holographic correspondence, which is usually limited for technical reasons to the leading $1/N$ order calculations in the bulk [23–26].

Consider a subset of the $\langle s^k s^m s^n\rangle$ three-point functions that at the leading order in $1/N$ expansion are determined by the diagram



Here we symbolically denoted the possible contractions of the s lines with dots. Since $k+m+n-3 = 2l$ is even

[29] In order to confirm the existence of $\mathbb{Z}_2$ symmetry at the leading order, it is necessary to consider general n-point functions. This section does not intend to prove this symmetry, rather to provide further evidence for its existence by computing three-point functions.

valued, we will always be able to connect the s lines which did not go into the vertices of the ϕ triangle. To avoid tadpole loops of the s propagators we need to impose the following triangle inequalities

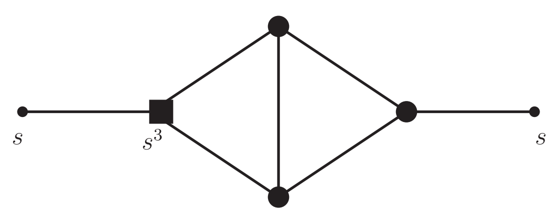
$$k \leq m + n - 1, \qquad k \geq m \geq n \tag{5.1}$$

The resulting diagram will be given by[30]

$$\langle s^m(x_1) s^n(x_2) s^k(x_3) \rangle \simeq \frac{C_{sss}^{(0)} C_s^l}{|x_{12}|^{2(m+n-k)} |x_{13}|^{2(m+k-n)} |x_{23}|^{2(k+n-m)}}. \tag{5.2}$$

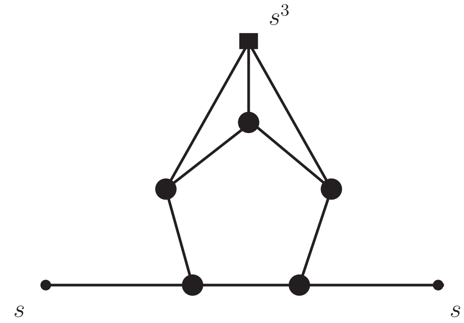
Since $C_{sss}^{(0)}(d=3) = 0$, this leading $\mathcal{O}(1/N^{1/2})$ diagram (5.2) vanishes in $3d$.

In case when the triangle inequalities (5.1) are not satisfied, the leading order behavior of the $\langle s^k s^m s^n \rangle$ three-point correlation functions demands a separate treatment. For instance, the $\mathcal{O}(1/N^{1/2})$ diagram contributing to $\langle s^3 s s \rangle$ is given by[31]



Here we recognize the subdiagram representing the leading order contribution to the $\langle s^2 s \rangle$ correlator, given by (1.12) [40,41]. In particular, the above diagram, contributing at the leading order to the $\langle s^3 s s \rangle$ three-point function, is nonzero in general d, but vanishes in $d = 3$, due to the property $c|_{d=3} = 0$ of the leading order mixing coefficient (1.11). This means that the three-point function $\langle s^3 s s \rangle$ also respects the $s \to -s$ symmetry at the leading order in $1/N$ expansion.

As a result, the leading order contributions to the $\langle s^3 s s \rangle$ in $d = 3$ dimensions can only begin at the order $\mathcal{O}(1/N^{3/2})$. Among the contributing diagrams there are planar and nonplanar graphs consisting of the pentagon of the ϕ lines, such as



Notice, that this diagram needs to be regularized. It is an interesting open problem to finish the calculation of the three-point function $\langle s^3 s s \rangle$, which is nontrivial even at the $\mathcal{O}(1/N^{3/2})$ order.

VI. DISCUSSION

In this paper, we set out to calculate the three-point correlation function $\langle s^2 s^2 s \rangle$ in the critical $O(N)$ vector model at the next-to-leading order in the $1/N$ expansion. In the process, we computed the $\mathcal{O}ss$ conformal triangle, following the technique developed in [45] for the analogous calculation in the Gross-Neveu model. Additionally, we determined the finite correction A_{s^2} to the amplitude of the $\langle s^2 s^2 \rangle$ two-point function. This involved calculating an extra diagram, absent in the GN model, that is hard to find in general $2 < d < 6$ using conventional techniques, and is still an open problem. However, it conveniently vanishes in $d = 3$,[32] allowing us to compute the amplitude correction A_{s^2} in $3d$. The computation of the $\langle s^2 s^2 s \rangle$ three-point function required evaluating a new set of self-energy diagrams that were unknown in the literature. We have outlined the steps of all these calculations in detail. Assembling all these components together, we were able to compute the $\langle s^2 s^2 s \rangle$ correlator in $d = 3$ dimensions, that turned out to have a nonzero value.

As was discussed in detail in Sec. V, we explored the $\mathbb{Z}_2$ symmetry $s \to -s$, that was originally conjectured to emerge in the large N limit of the $O(N)$ vector model in $d = 3$. In [36], the symmetry seemed to surprisingly persist in the three-point function $\langle sss \rangle$ up to the next-to-leading order in the $1/N$ expansion. The nonzero value at the next-to-leading order of the $\langle s^2 s^2 s \rangle$ correlator, obtained in this paper, suggests that the emergent $\mathbb{Z}_2$ symmetry is lifted at the subleading orders in $1/N$. Besides this result, the calculation of the conformal triangle $\mathcal{O}ss$ as well as the subdiagrams contributing to the $\langle s^2 s^2 s \rangle$ are important in themselves for other conformal correlators in vector models. For convenience, we summarized the new results in a table and their expression in three dimensions:

[30]We skipped keeping track of numerical symmetry and degeneracy factors, that are unimportant for our purposes.

[31]To lighten up the notation, we skip labeling exponents of the propagators; all of the internal lines are the s lines, except for the ϕ lines in the polygons.

[32]This can be traced back to the triviality of the $\langle sss \rangle$ subdiagram, which appears in that calculation, at the leading order in $1/N$.

Correlator	$d=3$
$\langle s(x_1)s(x_2)s^2(x_3)\rangle \supset \frac{C^{(0)}_{s^2ss}(1+\delta C_{s^2ss})}{\vert x_{12}\vert^{2\gamma_s-\gamma_{s^2}}(\vert x_{13}\vert\vert x_{23}\vert)^{4+\gamma_{s^2}}}\mu^{-2\gamma_s-\gamma_{s^2}}$ $\langle s^2(x)s^2(0)\rangle = \frac{C_{s^2}(1+A_{s^2})\mu^{-2\gamma_{s^2}}}{\vert x\vert^{2(\Delta_{s^2}+\gamma_{s^2})}}$ $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C_{\mathcal{O}}(1+A_{\mathcal{O}})\mu^{-2\gamma_{\mathcal{O}}}}{\vert x\vert^{8+2\gamma_{\mathcal{O}}}}$ ($\mathcal{O}$ is defined in (1.10)	$C^{(0)}_{s^2ss} = \frac{32}{\pi^4}$ $\delta C_{s^2ss} = \frac{1}{N}(\frac{176}{9\pi^2}-1)$ $C_{s^2} = C^{(0)}_{s^2ss}$ $A_{s^2} = \delta C_{s^2ss}$ $\mathcal{O} \to s^2$
$\langle s^2(x_1)s^2(x_2)s(x_3)\rangle \supset \frac{C^{(0)}_{s^2s^2s}(1+\delta C_{s^2s^2s})\mu^{-\gamma_s-2\gamma_{s^2}}}{\vert x_{12}\vert^{6+2\gamma_{s^2}-\gamma_s}(\vert x_{13}\vert\vert x_{23}\vert)^{2+\gamma_s}}$	$C^{(0)}_{s^2s^2s} = 0$ $C^{(0)}_{s^2s^2s}\delta C_{s^2s^2s} = \frac{1}{N^{3/2}}\frac{768}{\pi^6}$
$\langle s^m(x_1)s^n(x_2)s^k(x_3)\rangle \simeq \frac{C^{(0)}_{sss}C^l_s}{\vert x_{12}\vert^{2(m+n-k)}\vert x_{13}\vert^{2(m+k-n)}\vert x_{23}\vert^{2(k+n-m)}}$	for $k\le m+n-1, k\ge m\ge n$ the $\mathcal{O}(N^{-1/2})$ term vanishes

Regarding the emergent symmetry, our results are supportive of the statement that it is exact at large N, as we have illustrated in Sec. V, where we demonstrated its presence for an entire set of correlators involving composite operators in s. This may have important implications for the AdS/CFT correspondence, which states that the critical $O(N)$ vector model in $3d$ is dual to a higher-spin Vasiliev theory on AdS_4. The AdS/CFT mapping suggests that the operator s is dual to the spin zero field A_0 in AdS. Correlations of composite currents in the AdS bulk are hitherto largely undiscussed in the literature to the best of our knowledge. Our CFT results for the composite operators should have direct implication for the dual theory in AdS. However, symmetries of the bulk action alone, known to exist at least at the classical level, can demand some boundary correlators to vanish. For instance, the cubic interaction A_0^3 is absent in the Vasiliev's theory in the bulk [24,25], which translates to the statement that the boundary CFT possesses the $s\to -s$ symmetry. Next, the calculation of subleading corrections to such correlators in the large N CFT has a direct implication on one-loop corrections in the AdS bulk, which are otherwise very hard to compute [51]. At the same time, the symmetry considerations might again indicate whether certain correlators can become nonzero at subleading orders, due to an anomalous symmetry breaking mechanism in the bulk. An example of the latter is furnished by the possible anomalous torsion term generated by loops in the bulk.[33]

ACKNOWLEDGMENTS

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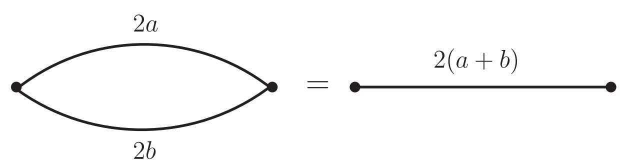
[33]See, e.g., [52] for a holographic description of parity-breaking system via torsion deformation of the AdS bulk. We thank A. Petkou for the discussion and drawing our attention to relevant references.

APPENDIX A: SOME USEFUL IDENTITIES

In this Appendix we review some useful expressions and identities.

Loop diagrams in the position space are additive:



The propagator merging relation has the form

$$\int d^dx_3 \frac{1}{(x_3^2)^a((x_3-x_{12})^2)^b} = U(a,b,d-a-b)\frac{1}{(x_{12}^2)^{a+b-\frac{d}{2}}}, \tag{A1}$$

where we defined

$$U(a,b,c) = \pi^{\frac{d}{2}}A(a)A(b)A(c), \tag{A2}$$

$$A(a) = \frac{\Gamma(\frac{d}{2}-a)}{\Gamma(a)}. \tag{A3}$$

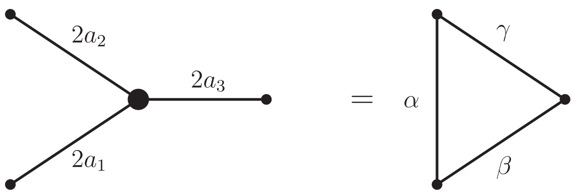
This can be graphically represented as

$\times U(a, b, d - a - b)\,.$

Uniqueness relation for $a_1 + a_2 + a_3 = d$ is written as [53,54]

$$\int d^d x \frac{1}{|x_1 - x|^{2a_1} |x_2 - x|^{2a_2} |x_3 - x|^{2a_3}} = \frac{U(a_1, a_2, a_3)}{|x_{12}|^{d-2a_3} |x_{13}|^{d-2a_2} |x_{23}|^{d-2a_1}}\,. \tag{A4}$$

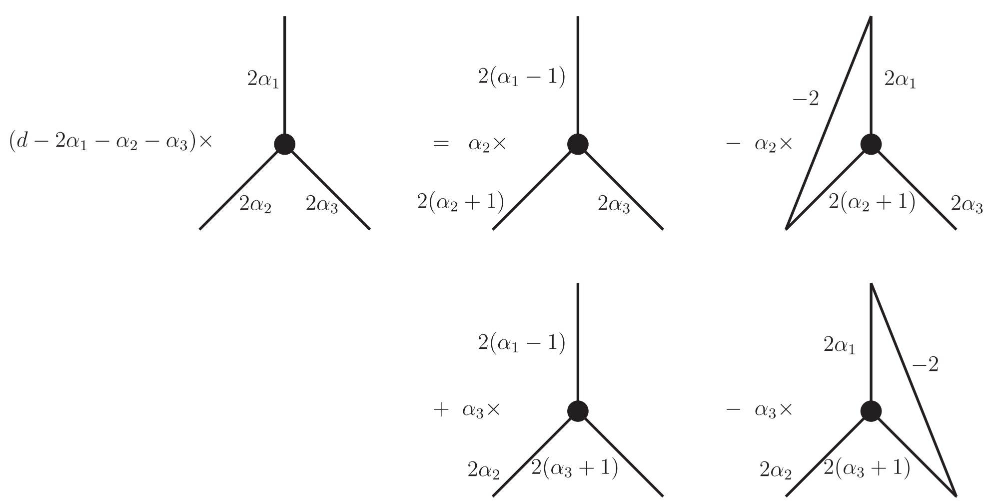
This can be diagrammatically represented as



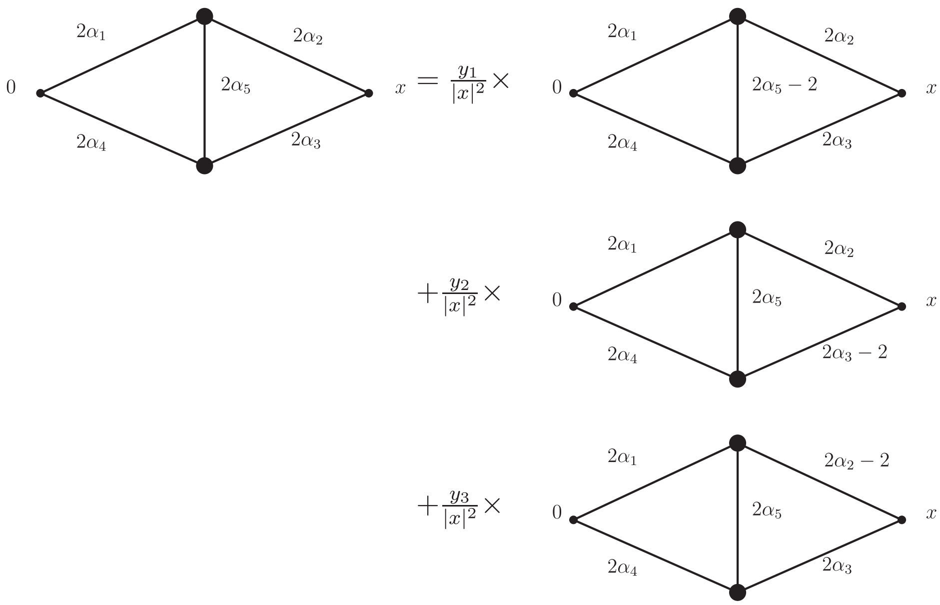
$\times \left(-\frac{2}{\sqrt{N}}\right) U\left(a_1, a_2, a_3\right).$

Here we denoted $\alpha = d - 2a_3$, $\beta = d - 2a_2$, $\gamma = d - 2a_1$.

Integration by parts relation [49,50] read as



We will also find useful the following relation [55][34]:



(A5)

Here

$$y_1 = \frac{(d - s_1)(d - s_2)}{(d - t_2)(t_2 - d/2 - 1)}, \qquad y_2 = \frac{(d - s_2)(D + \alpha_5 - 3d/2 - 1)}{(d - t_2)(t_2 - d/2 - 1)}, \tag{A6}$$

$$y_3 = \frac{(d - s_1)(D + \alpha_5 - 3d/2 - 1)}{(d - t_2)(t_2 - d/2 - 1)}, \tag{A7}$$

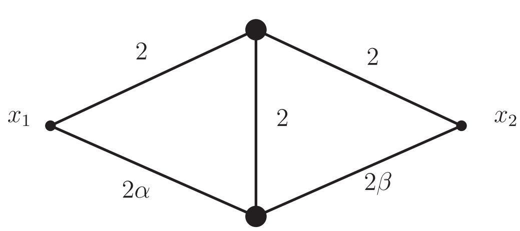
and

$$s_1 = \alpha_1 + \alpha_2 + \alpha_5, \qquad s_2 = \alpha_3 + \alpha_4 + \alpha_5, \tag{A8}$$

$$t_1 = \alpha_1 + \alpha_4 + \alpha_5, \qquad t_2 = \alpha_2 + \alpha_3 + \alpha_5, \qquad D = \sum_{i=1}^{5} \alpha_i. \tag{A9}$$

APPENDIX B: SOME USEFUL DIAGRAMS

Using the integration by parts relation, we can derive [36,49,50]



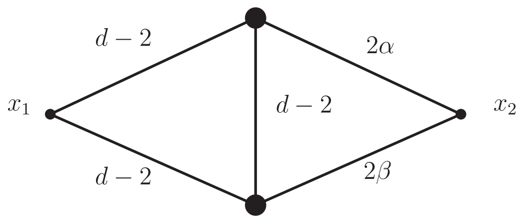
$$= \frac{F(\alpha,\beta)}{|x_{12}|^{6-2d+2\alpha+2\beta}}$$

where

[34]See Fig. 15 therein.

$$F(\alpha,\beta)=\frac{U(d-2,1,1)}{d-2-\alpha-\beta}\left(\alpha\left(U(\alpha+1,\beta,d-\alpha-\beta-1)-U\left(\alpha+1,\beta+2-\frac{d}{2},\frac{3d}{2}-\alpha-\beta-3\right)\right)\right.$$
$$\left.+\beta\left(U(\beta+1,\alpha,d-\alpha-\beta-1)-U\left(\beta+1,\alpha+2-\frac{d}{2},\frac{3d}{2}-\alpha-\beta-3\right)\right)\right). \tag{B1}$$

Performing the Fourier transform we can derive the ChT diagram [49,50]



$$=\frac{\mathrm{ChT}(\alpha,\beta)}{|x_{12}|^{d-6+2\alpha+2\beta}}$$

where ChT is given by (4.20), that we reproduce here for completeness:

$$\mathrm{ChT}(\alpha,\beta)=\frac{\pi^d\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2}-1)\Gamma(d-2)}\left(\frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}+\alpha-2)}{(1-\beta)(\alpha+\beta-2)\Gamma(2-\alpha)\Gamma(\alpha)}\right.$$
$$\left.+\frac{\Gamma(\frac{d}{2}-\beta)\Gamma(\frac{d}{2}+\beta-2)}{(1-\alpha)(\alpha+\beta-2)\Gamma(2-\beta)\Gamma(\beta)}+\frac{\Gamma(\frac{d}{2}-\alpha-\beta+1)\Gamma(\frac{d}{2}+\alpha+\beta-3)}{(\alpha-1)(\beta-1)\Gamma(-\alpha-\beta+3)\Gamma(\alpha+\beta-1)}\right). \tag{B2}$$

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