Quantum vacuum fluctuations and the principle of virtual work in inhomogeneous backgrounds

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We discuss several aspects of the stress-energy tensor for a quantum scalar field in an inhomogeneous background, the latter being modeled by a variable mass. Using a perturbative approach, dimensional regularization, and adiabatic subtraction, we present all-order formal expressions for the stress-energy tensor. Importantly, we provide an explicit proof of the principle of virtual work for Casimir forces, taking advantage of the conservation law for the renormalized stress-energy tensor. We discuss also discontinuity-induced divergences. For the particular case of planar inhomogeneities, we corroborate the perturbative results with a WKB-inspired expansion.

I. INTRODUCTION

Energy densities, stresses, and forces are produced by vacuum fluctuations of the electromagnetic field when a body is immersed in a medium. In this context, the immersion of bodies in homogeneous media was already considered in the seminal works [1,2], in which the so-called Lifshitz formula was derived. This formula is able to describe the force between two flat and parallel interphases that separate three different homogeneous media.

More recently, there have been efforts to define the stress-energy (SE) tensor for a quantum field in a generalized Lifshitz configuration, i.e., in a situation in which the media are characterized by spacetime-dependent electromagnetic properties. Taking into account that the renormalization originally proposed by Lifshitz *et al.* [1] does not work in such a case, the problem has been considered by several authors [3–10]. Despite the different methods, models, and particular subtractions (at the level of either the Green's functions or the SE tensor), several questions are still open.

To discuss some of them, we will consider a toy model that consists of a quantum scalar field interacting with a classical field, in such a way that the quantum field acquires a variable mass. The evaluation of the vacuum expectation values (VEVs) will be performed using a perturbative approach in the variable mass. For the renormalization, we will follow a standard approach, based on dimensional regularization and adiabatic subtraction.

Observe first that similar theories have been analyzed in several contexts. In particular, much attention has been devoted to quantum fields in curved spacetimes [11,12], for which there is a well-established procedure to obtain the renormalized SE tensor: infinities are absorbed into the bare constants of the theory. It is fairly obvious in this context that the renormalization of the SE tensor's VEV cannot be performed as suggested in Refs. [1,2], i.e., by subtracting local quantities that depend only on the value of the background fields at a given point; it must also involve derivatives of the background field. After absorbing the divergences into the bare constants of the theory, the renormalized SE tensor will be expected to be defined up to local terms, which are determined by the finite part of the counterterms; being local, they will not be relevant in the discussion of Casimir interactions between different bodies.

Notice that some general aspects of the renormalization procedure that we employ have been described in detail in Ref. [13]. However, the calculations were performed to lowest order in the variable mass. To this order, it is possible to describe only the energy density and stresses, but not Casimir forces. Here, we extend those results to arbitrary order. The case of a scalar field with variable mass depending on a single coordinate has also been considered in Ref. [6], in which the authors implement a Pauli-Villars renormalization along with a WKB subtraction. A similar

approach, albeit with the scope of analyzing the limit of Dirichlet boundary conditions for thin surfaces, was followed in Refs. [14–16].

One important aspect that we will discuss is that, if the background field models the presence of several bodies, the Casimir force can be then computed as customarily by taking the derivative of the system's vacuum energy with respect to the position of one of the bodies or, alternatively, integrating the component of the SE tensor which is normal to the surface of that body. We will explicitly prove that both approaches are equivalent, a result known as the principle of virtual work (PVW). In this respect, Ref. [8] contains a discussion of the PVW for a planar configuration. Here, we go beyond planar geometries; moreover, we provide an explicit connection with the PVW and the semiclassical conservation law of the SE tensor.

Previous works reported a "pressure anomaly," which may jeopardize the validity of the PVW [17]. It was later recognized that this anomaly is produced by a particular point-splitting regularization [4]. Instead, our prescription using dimensional regularization along with adiabatic subtraction guarantees the fulfillment of the conservation law and avoids the presence of anomalies. In a recent work [10], it has been pointed out that quantum effects could induce a violation of the classical relation between the divergence of the electromagnetic stress and the gradients of the permeability and permittivity of the inhomogeneous media, inducing a "van der Waals anomaly." We have not found the analog of this anomaly in our model.

The last question that we will tackle is the fact that discontinuous backgrounds generate surface divergences in the VEV of the renormalized SE tensor. For the case of a perfectly conducting interphase, the presence of these divergences was pointed out a long time ago [18]. A discussion of Dirichlet and Neumann boundary conditions can be found in almost every textbook; see, for example, Refs. [19,20] and references therein; more general boundary conditions have been studied in Refs. [21–24]. In our case, we do not impose any kind of boundary conditions; the surface divergences appear just as a consequence of the discontinuities in the background. For this reason, we will call them discontinuity-induced divergences. We will characterize this kind of divergences in a planar inhomogeneous model and discuss its irrelevance in the calculation of Casimir forces. This will be confirmed by nonperturbative calculations, based on a WKB-type approximation discussed in Refs. [3,7] for the case of the electromagnetic field.

The paper is organized as follows. In Sec. II, we introduce our model, which consists of a quantum scalar field ϕ in the presence of a background field σ that provides an inhomogeneous mass term for the quantum field. In Sec. III, we discuss the renormalization of the VEVs $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}^{(\phi)}\rangle$, which is performed using standard techniques of quantum fields under the influence of external conditions. We also discuss the validity of the conservation law of the renormalized SE tensor at the semiclassical level. Section IV describes a perturbative approach for computing the above-mentioned mean values, with particular emphasis in time-independent situations (i.e., when the background field is static). In Sec. V, we prove the validity of the PVW. The conservation law for the SE tensor turns out to be crucial in this context. Several examples are discussed then in Sec, VI, including the surface divergences that appear in the renormalized mean values at the points where the background field is discontinuous. Afterward, in Sec. VII, we reanalyze those surface divergences in the case of planar inhomogeneities within an adiabatic approach. Section VIII contains the main conclusions of our work. Finally, the Appendixes A, B, C, and D describe some further details of the calculations.

We use natural units $\hbar = c = 1$ and metric signature $(+ - - \cdots)$ in a spacetime of dimension D. We define $g = -\det g_{\mu\nu}$, and spatial $(D-1)$-vectors are written in bold $(\mathbf{x})$.

II. MODEL

We will consider a quantum field ϕ interacting with a background classical field σ in the same fashion as in Ref. [13]. The field σ provides a variable mass for ϕ, so the action for both fields on a curved background is given by [11,12]

$$S = \frac{1}{2}\int \mathrm{d}^D x \sqrt{g}\left[\phi_{,\mu}\phi^{,\mu} - \left(m_1^2 + \xi_1 R + \frac{\lambda_1}{2}\sigma^2\right)\phi^2 \right.$$
$$\left. + \sigma_{,\mu}\sigma^{,\mu} - (m_2^2 + \xi_2 R)\sigma^2 - \frac{\lambda_2}{12}\sigma^4\right]. \tag{1}$$

This theory can be considered as a toy model for the electromagnetic field in the presence of an inhomogeneous medium. Of course, to mimic the electric permittivity or the magnetic permeability, one could consider alternative models in which the coupling to the background field is through terms that involve spatial or time derivatives of ϕ. However, the action in Eq. (1) will be enough for our purposes.

Even if we are interested in a four-dimensional spacetime, in Eq. (1), we have introduced a dimensional regularization. Moreover, the inclusion of a self-interacting term for the background field, i.e., the one proportional to λ_2, will be crucial for a successful renormalization, as will also be the inclusion of a coupling to the curvature in curved spaces (terms proportional to $\xi_{1,2}$). This will be discussed in detail in Sec. III.

Performing the variation of (1) with respect to both fields, one can obtain the classical field equations, which read

$$\left(\Box + m_1^2 + \xi_1 R + \frac{\lambda_1}{2}\sigma^2\right)\phi = 0, \qquad (2)$$

$$\left(\Box + m_2^2 + \xi_2 R + \frac{\lambda_1}{2}\phi^2\right)\sigma + \frac{\lambda_2}{6}\sigma^3 = 0. \qquad (3)$$

Additionally, we can compute the classical SE tensor. Since we have written the action on a curved spacetime, we can compute it as customarily done through

$$T_{\mu\nu} := \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu}^{(\sigma)} + T_{\mu\nu}^{(\phi)}, \qquad (4)$$

performing a split that will be useful in the following discussion:

$$T_{\mu\nu}^{(\sigma)} := (1 - 2\xi_2)\sigma_{,\mu}\sigma_{,\nu} + \left(2\xi_2 - \frac{1}{2}\right)\eta_{\mu\nu}\sigma_{,\rho}\sigma^{,\rho}$$
$$- 2\xi_2\sigma\sigma_{,\mu\nu} + 2\xi_2\eta_{\mu\nu}\sigma\Box\sigma + \frac{\eta_{\mu\nu}}{2}\left(m_2^2 + \frac{\lambda_2}{12}\sigma^2\right)\sigma^2, \quad (5)$$

$$T_{\mu\nu}^{(\phi)} := (1 - 2\xi_1)\phi_{,\mu}\phi_{,\nu} + \left(2\xi_1 - \frac{1}{2}\right)\eta_{\mu\nu}\phi_{,\rho}\phi^{,\rho} - 2\xi_1\phi\phi_{,\mu\nu}$$
$$+ 2\xi_1\eta_{\mu\nu}\phi\Box\phi + \frac{\eta_{\mu\nu}}{2}\left(m_1^2 + \frac{\lambda_1}{2}\sigma^2\right)\phi^2. \qquad (6)$$

After performing the variation, we have set $g_{\mu\nu} = \eta_{\mu\nu}$, since we will not be interested in discussing the interaction with a curved spacetime; we will work in flat spacetime throughout the rest of the paper.

Two remarks are in order. First of all, $T_{\mu\nu}^{(\phi)}$ corresponds to the SE tensor of a free field with variable mass $M^2 = m_1^2 + \frac{\lambda_1}{2}\sigma^2$, in agreement with the picture that we have described before. Second, the full SE tensor $T_{\mu\nu}$ is of course conserved classically, while one can easily check that[1]

$$\partial^\mu T_{\mu\nu}^{(\phi)} = \frac{\lambda_1}{4}\partial_\nu\sigma^2\phi^2. \qquad (7)$$

We now consider the semiclassical version of the theory, in which the field ϕ is of quantum nature while σ is treated classically. Then, the classical expression (2) is promoted to the Heisenberg equation associated to the quantum operator ϕ. On the other side, the evolution equation for the background field is obtained by taking the VEV of the classical Eq. (3),

$$\left(\Box + m_2^2 + \frac{\lambda_1}{2}\langle\phi^2\rangle\right)\sigma + \frac{\lambda_2}{6}\sigma^3 = 0. \qquad (8)$$

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[1] To simplify the notation, we will adopt the notation $\partial_\nu\sigma^2 := \partial_\nu(\sigma^2)$.

Additionally, the SE tensor of the full semiclassical system is

$$\langle T_{\mu\nu}\rangle = T_{\mu\nu}^{(\sigma)} + \langle T_{\mu\nu}^{(\phi)}\rangle, \qquad (9)$$

given that $T_{\mu\nu}^{(\phi)}$ was defined so that it contains all the terms involving the quantum field ϕ. Thus, the main objects to analyze the vacuum fluctuations are $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}^{(\phi)}\rangle$, the latter being relevant to consider Casimir forces and self-energies. Both of them are divergent quantities; as we will see in the following section, the classical action for the field σ is needed to absorb the divergences into the bare constants of the theory during the renormalization process, after which we obtain a finite and unique expression for the SE tensor (up to finite local terms). Additionally, we will show in Sec. III A that, using the usual prescription, Eq. (7) is valid at the quantum level when the classical quantities are replaced by the corresponding VEVs.

III. RENORMALIZATION AND CONSERVATION LAW

The theory of quantum fields in curved spacetimes can be renormalized using a precise covariant procedure [11,12]. As was shown in Refs. [13,25], the case of a quantum field with a variable mass can be treated in an analogous way; we will briefly review it in the following.

As customarily in theories with four spacetime dimensions, we can define the renormalized quantities as

$$\langle\phi^2\rangle_{\rm ren} := \langle\phi^2\rangle - \langle\phi^2\rangle_{\rm ad2},$$
$$\langle T_{\mu\nu}^{(\phi)}\rangle_{\rm ren} := \langle T_{\mu\nu}^{(\phi)}\rangle - \langle T_{\mu\nu}^{(\phi)}\rangle_{\rm ad4}, \qquad (10)$$

where the VEVs $\langle T_{\mu\nu}^{(\phi)}\rangle_{\rm ad4}$ and $\langle\phi^2\rangle_{\rm ad2}$ are constructed using the Schwinger-DeWitt expansion (SDWE) up to fourth and second adiabatic orders, respectively. Notice that the counting of the adiabatic order includes not only the number of derivatives but also the mass dimensions; for example, a term with j derivatives of σ^2 is of adiabatic order $j + 2$ [25]. After the subtraction in Eq. (10), the divergences in the adiabatic VEVs are to be absorbed into the bare constants of the theory, so we end up with finite renormalized constants and VEVs.

As said above, the adiabatic contributions involve the computation of the SDWE. For the Feynman propagator (G_F) of a scalar field with mass m, one obtains [26]

$$G_F^{SD}(x,x') = \int_0^\infty \frac{{\rm d}s}{(4\pi{\rm i}s)^{D/2}}e^{-\frac{{\rm i}\sigma_S(x,x')}{2s}-{\rm i}(m^2-{\rm i}\epsilon)s}\sum_{j\geq 0}({\rm i}s)^j\Omega_j(x,x'), \qquad (11)$$

where D is the number of spacetime dimensions and $\sigma_S(x,x')$ is Synge's world function, that in flat space is just $\sigma_S(x,x') = (x-x')^2/2$. The functions $\Omega_j(x,x')$ are

defined by a set of recursive equations that follow from imposing the equation for the propagator, i.e.,

$$\left[\Box + m_1^2 + \frac{\lambda}{2}\sigma^2(x) - i\epsilon\right]G_F(x, x') = -\delta^4(x - x'). \quad (12)$$

Their general form is well known; denoting with square brackets the coincidence limit of these functions and their derivatives, it can be shown that, for the action in Eq. (1) in flat spacetime, the first functions read[2] [27,28]

$$\Omega_0(x, x') = 1, \quad [\Omega_1] = -\frac{\lambda_1}{2}\sigma^2,$$

$$[\Omega_{1,\mu\nu}] = -\frac{\lambda_1}{6}\sigma^2_{,\mu\nu}, \quad [\Omega_2] = \frac{\lambda_1}{12}\Box\sigma^2 + \frac{\lambda_1^2}{8}\sigma^4. \quad (13)$$

This expansion can be modified by including the full variable mass M^2 into the exponent of the SDWE in Eq. (11) (see Ref. [25] and more recently Refs. [29,30]). With this modification, we will have an expansion analog to expression (11) involving new functions $\tilde{\Omega}_j(x, x')$; the latter do not contain powers of M^2 but only powers of its derivatives. Although this expansion could be used in principle,[3] it does not provide additional help in the following computations and will not be followed here.

Coming back to the computation of the renormalized quantities, the adiabatic VEVs can be computed by recasting all the expressions in terms of the imaginary part of Feynman's Green's function, which satisfies

$$\mathrm{Im}(G_F(x, x')) = -\frac{1}{2}\langle\{\phi(x), \phi(x)\}\rangle. \quad (14)$$

A direct computation shows that the explicit expressions are (see Ref. [25] and footnote 2 in the current paper)

$$\langle\phi^2\rangle = -\mathrm{Im}[G_F], \quad (15)$$

$$\langle T^{(\phi)}_{\mu\nu}\rangle = -\mathrm{Im}\left\{-[G_{F,\mu\nu}] + \left(\frac{1}{2} - \xi_1\right)[G_F]_{,\mu\nu}\right.$$
$$\left. + \left(\xi_1 - \frac{1}{4}\right)\eta_{\mu\nu}\Box[G_F]\right\}. \quad (16)$$

In these expressions, one can replace the SDWE (11) for the propagator and obtain the adiabatic expansion of the desired quantities up to the appropriate order.

In particular, the coincidence limit of the two-point function for a field with variable mass as in Eq. (1) is therefore given by

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[2]It should be understood that $\Omega_{1,\mu\nu} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\Omega_1(x, x')$. Notice that Ref. [27] works with a Euclidean signature.
[3]One should appropriately modify the discussion in Sec. III A.

$$\langle\phi^2\rangle_{\mathrm{ad2}} = \frac{1}{(4\pi)^{D/2}}\left[\frac{m_1}{\mu}\right]^{D-4}$$
$$\times \left\{m_1^2\Gamma\left(1 - \frac{D}{2}\right) - \frac{\lambda_1}{2}\sigma^2\Gamma\left(2 - \frac{D}{2}\right)\right\}, \quad (17)$$

where μ is an arbitrary scale with dimensions of mass that is introduced in the renormalization process. Both terms diverge as $D \to 4$, and their subtraction will be enough to obtain a finite result in (10). The fact that the divergences can be absorbed into the bare constants of the theory can be seen by inserting expression (17) into the semiclassical equation for σ, i.e., Eq. into (8). Indeed, writing

$$m_2^2 =: m_{2R}^2 + \delta m_2^2,$$
$$\lambda_2 =: \lambda_{2R} + \delta\lambda_2, \quad (18)$$

we obtain the counterterms

$$\delta m_2^2 = -\frac{\lambda_1 m_1^2}{16\pi^2(D - 4)} + \Delta m_1^2,$$
$$\delta\lambda_2 = -\frac{3\lambda_1^2}{16\pi^2(D - 4)} + \Delta\lambda_2, \quad (19)$$

where Δm_2^2 and $\Delta\lambda_2$ are finite contributions that relate different renormalization schemes (they vanish in the minimal subtraction scheme).

We now consider the evaluation of the SE tensor in our semiclassical theory. The expression for its VEV up to fourth adiabatic order reads

$$\langle T^{(\phi)}_{\mu\nu}\rangle_{\mathrm{ad4}} = \frac{1}{(4\pi)^{D/2}}\left[\frac{m_1}{\mu}\right]^{D-4}$$
$$\times \left\{-\frac{\eta_{\mu\nu}}{2}m_1^4\Gamma\left(-\frac{D}{2}\right) + \frac{\lambda_1}{4}m_1^2\sigma^2\eta_{\mu\nu}\Gamma\left(1 - \frac{D}{2}\right)\right.$$
$$+ \Gamma\left(2 - \frac{D}{2}\right)\left[-\frac{\lambda_1^2}{16}\eta_{\mu\nu}\sigma^4\right.$$
$$\left.\left. + \frac{\lambda_1}{2}\left(\xi_1 - \frac{1}{6}\right)(\sigma^2_{,\mu\nu} - \eta_{\mu\nu}\Box\sigma^2)\right]\right\}. \quad (20)$$

Comparing Eq. (20) with Eq. (5), one can show that the σ-dependent divergences can be absorbed using the same counterterms given in Eq. (19) and including a counterterm for ξ_2, the latter needed to absorb the divergence proportional to $(\xi_1 - 1/6)$. The term independent of σ^2 will just renormalize the cosmological constant (or a bare constant in the classical potential for the background field) and will play no role in our considerations. All these terms depend on the arbitrary scale μ that has been introduced in the renormalization process; the arbitrariness is resolved by using experimental data to fix the involved couplings. Therefore, we have a precise procedure for defining the

renormalized SE tensor for the quantum field ϕ in an inhomogeneous background σ.

Since in the following sections we will deal with massless fields, let us recall that then one can simply trade $m_1^{D-4} \to \mu_2^{D-4}$ in Eqs. (20) and (17), setting the other powers of m_1 to zero. The new scale μ_2 is arbitrary but appears only in quotients with μ; for convenience, we can set it to $\mu_2 \equiv e^{-1}\mu$.

Before concluding this section, two last remarks are in order. First, our starting action in Eq. (1) belongs to a theory on curved spacetime. This choice was motivated in part to emphasize that the problem of quantum fields in inhomogeneous backgrounds can be addressed using well-known techniques of quantum fields on curved spaces. However, this point is not crucial from a computational point of view. An alternative route is to start with a theory in Minkowski spacetime and compute the SE tensor using Noether's theorem; afterward, one may add the terms proportional to ξ_1 and ξ_2 in Eqs. (5) and (6) using the fact that Noether's theorem does not constrain them. In any case, note that, while it is not necessary to add the terms proportional to ξ_1 in the SE tensor of the quantum field ϕ, the introduction of the classical terms proportional to ξ_2 is essential to renormalize the theory, even if $\xi_1 = 0$.

The second remark is about regularization. Dimensional regularization in curved spacetimes has been sometimes criticized because of the arbitrariness in the choice of the extra dimensions, that could describe an arbitrary manifold.[4] This concern is not relevant in the present work since, as we have already mentioned, the calculations can be performed entirely in flat spacetime. In spite of this, and for the sake of completeness, in Sec. VI E, we will show how to reobtain some of our results using heat kernel techniques and working in $D = 4$.

A. Semiclassical conservation law

It is well known that the renormalization procedure may induce anomalies in the quantum theory, which may be caused by the regularization and/or the corresponding subtractions. Typical examples are the nonconservation of the chiral current for massless fermions in the presence of background gauge fields [31] and the trace anomaly for conformal fields in curved spaces, first discovered in Ref. [32] and lately revisited in relation with Weyl fermions [33,34]. We will now show that the conservation law in Eq. (7) remains valid after the quantization of ϕ, if we replace the classical quantities with the corresponding renormalized VEVs given by expression (10):

$$\partial^\mu \langle T_{\mu\nu}^{(\phi)} \rangle_{\mathrm{ren}} = \frac{\lambda_1}{4} \partial_\nu \sigma^2 \langle \phi^2 \rangle_{\mathrm{ren}}. \quad (21)$$

To see this, we rewrite the above equation as

$$\partial^\mu (\langle T_{\mu\nu}^{(\phi)} \rangle - \langle T_{\mu\nu}^{(\phi)} \rangle_{\mathrm{ad4}}) = \frac{\lambda_1}{4} \partial_\nu \sigma^2 (\langle \phi^2 \rangle - \langle \phi^2 \rangle_{\mathrm{ad2}}), \quad (22)$$

where all calculations are performed in D dimensions. As dimensional regularization is covariant, one expects the regularized mean values $\langle T_{\mu\nu}^{(\phi)} \rangle$ and $\langle \phi^2 \rangle$ to satisfy the conservation law. Indeed, from Eqs. (15) and (16), one can check this explicitly using the expression for the propagator, which is of course valid in D dimensions. Moreover, computing the derivative of $\langle T_{\mu\nu}^{(\phi)} \rangle_{\mathrm{ad4}}$ in Eq. (20), we straightforwardly obtain

$$\partial^\mu \langle T_{\mu\nu}^{(\phi)} \rangle_{\mathrm{ad4}} = \frac{\lambda_1}{4} \partial_\nu \sigma^2 \langle \phi^2 \rangle_{\mathrm{ad2}}, \quad (23)$$

so neither the regularization nor the subtraction breaks the conservation law at the quantum level for ϕ. Therefore, Eq. (21) is valid. This will be crucial in the discussion of the principle of virtual work in Sec. V.

IV. PERTURBATIVE APPROACH

We will now obtain explicit expressions for the renormalized VEVs $\langle T_{\mu\nu}^{(\phi)} \rangle_{\mathrm{ren}}$ and $\langle \phi^2 \rangle_{\mathrm{ren}}$, using a perturbative expansion in powers of λ_1. We will start studying Feynman's propagator, since those VEVs can be obtained from it as shown in Eqs. (15) and (16). For simplicity, we will consider the massless case ($m_1^2 \equiv 0$) and replace $\lambda_1 \to \lambda$, so using the customary "$i\epsilon$" prescription Feynman's propagator satisfies

$$\left[\Box + \frac{\lambda}{2}\sigma^2(x) - i\epsilon \right] G_F(x, x') = -\delta^4(x - x'). \quad (24)$$

Solving this equation perturbatively in λ, we obtain

$$G_F(x, x') = G_F^{(0)}(x, x') + G_F^{(1)}(x, x') + \cdots, \quad (25)$$

defining $G_F^{(0)}(x, x')$ as the usual free propagator and the contribution of order λ^n as

$$G_F^{(n)}(x, x') := \frac{\lambda^n}{2^n} \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \sigma^2(x_1) \cdots$$
$$\times \sigma^2(x_n) G_F^{(0)}(x, x_n) \cdots G_F^{(0)}(x_1, x'). \quad (26)$$

This is a notation that we will employ frequently in the following: the order n contribution in λ of a given quantity will be denoted by adding a superscript (n). Coming back to (26), we can recast it by expressing every free propagator in momentum space:

$$G_F^{(n)}(x,x') = \frac{\lambda^n}{2^n(2\pi)^{4n+4}} \int d^4x_1 \cdots d^4x_n d^4q_1 \cdots d^4q_n \sigma^2(x_1) \cdots \sigma^2(x_n)$$

$$\times e^{-iq_n \cdot (x_n - x_{n-1})} \cdots e^{-iq_1 \cdot (x_1 - x')} \int \frac{d^D s}{\mu^{D-4}} \frac{e^{-is(x-x')}}{s^2(s+q_1)^2 \cdots (s+q_n)^2}. \quad (27)$$

Notice that we have implemented dimensional regularization only in the internal momentum s, introducing as usual an arbitrary scale μ with dimensions of mass; to shorten the notation, we have omitted the $i\epsilon$ terms in the propagators. An immediate consequence of this result is that

$$\langle \phi^2 \rangle^{(n)} = -\frac{\lambda^n}{2^n(2\pi)^{4n+4}} \mathrm{Im}\left[\int d^4x_1 \cdots d^4x_n d^4q_1 \cdots d^4q_n \sigma^2(x_1) \cdots \sigma^2(x_n) e^{-iq_n \cdot (x_n - x_{n-1})} \cdots e^{-iq_1 \cdot (x_1 - x)} \mathcal{I}(q_1, ..., q_n) \right], \quad (28)$$

where we have introduced the tensorial integrals[5]

$$\mathcal{I}^{\mu_1, \mu_2 \cdots}(q_1, ..., q_n)$$
$$:= \int \frac{d^D s}{\mu^{D-4}} \frac{s^{\mu_1} s^{\mu_2} \cdots}{s^2(s+q_1)^2(s+q_2)^2 \cdots (s+q_n)^2}. \quad (29)$$

The computation of these integrals can be done in various ways, the most famous one being probably the Veltman-Passarino reduction method [35] (see also Refs. [33,36,37]).

An analogous expansion for the ϕ contribution to the SE tensor can be obtained. Inserting the nth-order expression for the propagator into Eq. (16) and dropping the (ϕ) superscript, one can find

$$\langle T_{\mu\nu} \rangle^{(n)} = \frac{\lambda}{4} \eta_{\mu\nu} \sigma^2(x) \langle \phi^2 \rangle^{(n-1)} - \frac{\lambda^n}{2^n(2\pi)^{4n+4}} \mathrm{Im}\left\{ \int d^4x_1 \cdots d^4q_1 \cdots \sigma^2(x_1) \cdots \sigma^2(x_n) \right.$$

$$\left. \times e^{-iq_n \cdot (x_n - x_{n-1})} \cdots e^{-iq_1 \cdot (x_1 - x)} \left[(\mathcal{I}_{\rho\sigma} + \mathcal{I}_\rho q_{1\sigma}) \left(\eta_\mu^\rho \eta_\nu^\sigma - \frac{1}{2} \eta^{\rho\sigma} \eta_{\mu\nu} \right) - 2\xi \mathcal{I}_\rho q_{1\sigma} (\eta_\mu^\rho \eta_\nu^\sigma - \eta^{\rho\sigma} \eta_{\mu\nu}) \right] \right\}, \quad (30)$$

where it will be understood that the arguments in the $\mathcal{I}$ tensorial integrals, when missing, are all the involved momenta $q_1, ..., q_n$.

From now on, we will assume that the background field is time independent. In that case, integrating over all space, we find an expression for the total vacuum energy, E, that reads

$$E^{(n)} = -\frac{\lambda^n}{2^n(2\pi)^{3n+1}} \mathrm{Im}\left[\int d^3\mathbf{x}_1 \cdots d^3\mathbf{q}_2 \cdots \sigma^2(\mathbf{x}_1) \cdots \sigma^2(\mathbf{x}_n) \right.$$

$$\left. \times e^{-i\mathbf{q}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1})} \cdots e^{-i\mathbf{q}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_1)} \left(\mathcal{I}_{00} - \frac{1}{2} \mathcal{I}^\rho{}_\rho \right) \right]_{q_1^\mu = 0} + \frac{\lambda}{4} \int d^3\mathbf{x} \sigma^2(\mathbf{x}) \langle \phi^2 \rangle^{(n-1)}(\mathbf{x}). \quad (31)$$

This expression can be further simplified. First of all, the term involving $\mathcal{I}^\rho{}_\rho$ cancels with the one proportional to $\langle \phi^2 \rangle^{(n-1)}$. Second, $\mathcal{I}_{00}$ can be recast, integrating by parts in the zeroth component of the internal momentum s, using the symmetry of the integrand in the variables q_i and rewriting the result in terms of $\langle \phi^2 \rangle^{(n-1)}$. This leads to the following master formula for the time-independent case:

$$E^{(n)} = \frac{\lambda}{4n} \int d^3\mathbf{x} \sigma^2(\mathbf{x}) \langle \phi^2 \rangle^{(n-1)}(\mathbf{x}). \quad (32)$$

It is important to notice that, with our renormalization prescription, Eq. (32) remains valid when replacing the regularized quantities by the renormalized ones.

In the next sections, we will derive explicit expressions for all the relevant physical quantities at first and second orders in λ, along with some illustrative examples. Before doing this, we would like to stress some general properties of the preceding results.

In Sec. III, we have discussed the divergences' structure of the VEVs $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$. The ones that should be renormalized are at most quadratic in the coupling constant λ, and therefore we will be able to reproduce them in a second-order perturbative approach. After subtracting the appropriate adiabatic expansions, the renormalized VEVs

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[5]The integral $\mathcal{I}$ without indices should be understood with a factor 1 in the integrand's numerator of (29).

will be determined up to local terms whose dependence on σ is that of the counterterms. Since they are local, they are not relevant in the computation of Casimir forces between different bodies; in other words, Casimir forces will have no undeterminacy. However, if one were interested in self-energies, then one should use experiments to fix the otherwise free parameter μ.

One subtle point is that there could be additional divergences. First, they could be generated by discontinuities in the background field σ^2 or its derivatives. These are the scalar counterparts of those arising near a perfect conductor, which depend on the local geometry of the surface [18]. In these situations, one should be careful to give the right interpretation of the conservation equation (21). We will describe this kind of divergences in Sec. VI.

We would also like to point out that, due to the fact that we are using massless propagators, one could encounter infrared divergences at higher orders in λ. To avoid these divergences, one could consider massive propagators. That will be the case if the field ϕ is massive. Alternatively, for a massless field, one can perform the perturbative expansion around the average of σ^2 over all space ($\bar{\sigma}^2$). If the latter is nonvanishing, one can write the equation for the propagator as

$$\left[\Box + \frac{\lambda}{2}\bar{\sigma}^2 - i\epsilon + \frac{\lambda}{2}(\sigma^2(x) - \bar{\sigma}^2)\right]G_F(x, x') = -\delta^4(x - x')$$

(33)

and perform the expansion with a free propagator of mass $\bar{m}^2 = \frac{\lambda}{2}\bar{\sigma}^2$. This corresponds to a resummation of the perturbative results, that will show a nonanalytic dependence with $\bar{m}^2$. In both cases, the corresponding perturbative expressions can be obtained just by replacing in Eqs. (26)–(32) the free massless propagators by massive ones.

Finally, we would like to point out that the perturbative approach should be modified when considering a time-dependent background field. Indeed, the solution to Eq. (24) is the matrix element

$$G_F(x, x') = i\frac{\langle 0_{IN}|T(\phi(x)\phi(x'))|0_{OUT}\rangle}{\langle 0_{IN}|0_{OUT}\rangle},$$

(34)

which involves the initial and final vacuum states, not the mean value $\langle 0_{IN}|T(\phi(x)\phi(x'))|0_{IN}\rangle$. The same remark applies to the other VEVs in this section. This situation can be amended, following a procedure inspired in the Schwinger-Keldysh formalism [38], by computing perturbatively the generalized Green's function

$$G_{\mathcal{C}}(x, x) = i\langle 0_{IN}|T_{\mathcal{C}}(\phi(x)\phi(x'))|0_{IN}\rangle,$$

(35)

where $T_{\mathcal{C}}$ is the temporal ordering along a closed temporal contour $\mathcal{C}$. This is beyond the scope of the present paper.
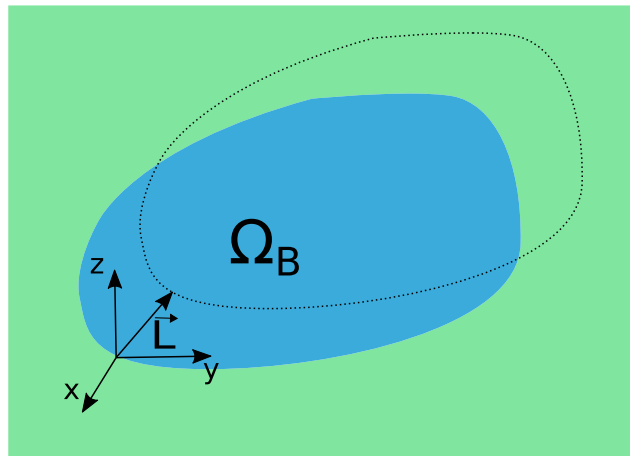

FIG. 1. The body B, that initially occupies the volume Ω_B, is virtually displaced by the vector **L**.

V. PRINCIPLE OF VIRTUAL WORK

Before we apply our formulas in Sec. VI to some particular configurations, we will provide an explicit proof of the validity of the PVW in this model. To do that, we consider the situation is illustrated in Fig. 1, in which a body is immersed in an inhomogeneous media. Then, we compare the variation of the energy under an infinitesimal displacement of the body B and the integral of the normal component of the SE tensor over the surface of the same body.

Let us denote by Ω_B the volume occupied by the body in the initial position. The body B is characterized by a field $\sigma_B^2(\mathbf{x})$, while the surrounding media corresponds to $\sigma_M^2(\mathbf{x})$. Introducing the characteristic function

$$\chi_\Omega(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \notin \Omega \end{cases},$$

(36)

it is clear that after a translation by a vector **L** the background field becomes

$$\sigma_{\mathbf{L}}^2(\mathbf{x}) = \sigma_B^2(\mathbf{x} - \mathbf{L})\chi_{\Omega_B}(\mathbf{x} - \mathbf{L}) + \sigma_M^2(\mathbf{x})[1 - \chi_{\Omega_B}(\mathbf{x} - \mathbf{L})],$$

(37)

which is different from $\sigma^2(\mathbf{x} - \mathbf{L})$. In such affirmation, we are assuming that the effects of one media on the other, if they exist, can be neglected in the evaluation of response functions. We are also supposing that the function σ_M^2 is defined over all space, independently of the presence of the body B.

We now consider the gradient ∇ of the background field with respect to **x**,

$$\nabla\sigma^2(\mathbf{x}) = \nabla[\sigma_B^2(\mathbf{x})\chi_{\Omega_B}(\mathbf{x}) + \sigma_M^2(\mathbf{x})(1 - \chi_{\Omega_B}(\mathbf{x}))].$$

(38)

Computing the gradient of $\sigma_{\mathbf{L}}^2(\mathbf{x})$ with respect to **L** at zero displacement,

$$\frac{\partial}{\partial \mathbf{L}} \sigma_{\mathbf{L}}^2(\mathbf{x})\Big|_{\mathbf{L}=0} = \frac{\partial}{\partial \mathbf{L}}[\sigma_B^2(\mathbf{x} - \mathbf{L})\chi_{\Omega_B}(\mathbf{x} - \mathbf{L})]$$

$$- \sigma_M^2(\mathbf{x})\frac{\partial}{\partial \mathbf{L}}\chi_{\Omega_B}(\mathbf{x} - \mathbf{L})\Big|_{\mathbf{L}=0}$$

$$= -\nabla[\sigma_B^2(\mathbf{x} - \mathbf{L})\chi_{\Omega_B}(\mathbf{x} - \mathbf{L})]$$

$$+ \sigma_M^2(\mathbf{x})\nabla\chi_{\Omega_B}(\mathbf{x} - \mathbf{L})|_{\mathbf{L}=0}, \qquad (39)$$

it is immediate to see that $\frac{\partial}{\partial \mathbf{L}}\sigma_{\mathbf{L}}^2(\mathbf{x})|_{\mathbf{L}=0}$ is nonvanishing only in the region Ω_B (including the boundary), and in that region,

$$\frac{\partial}{\partial \mathbf{L}}\sigma_{\mathbf{L}}^2(\mathbf{x})\Big|_{\mathbf{L}=0} = -\nabla\sigma^2(\mathbf{x}), \qquad \mathbf{x} \in \Omega_B. \qquad (40)$$

In the time-independent situation, one can compute the energy after a virtual displacement of the body by replacing σ^2 with $\sigma_{\mathbf{L}}^2$ in Eq. (31). Afterward, taking the derivative of the energy with respect to $\mathbf{L}$ and using the symmetry of the integrand, we get

$$-\frac{\partial E^{(n)}}{\partial \mathbf{L}}\Big|_{\mathbf{L}=0} = \frac{1}{2}\frac{\lambda^n}{2^n(2\pi)^{3n+1}}$$

$$\times \text{Im}\Bigg[\int d^3\mathbf{x}_1\cdots d^3\mathbf{q}_2\cdots\frac{\partial\sigma_{\mathbf{L}}^2}{\partial \mathbf{L}}(\mathbf{x}_1)\sigma_{\mathbf{L}}^2(\mathbf{x}_2)\cdots$$

$$\times e^{i\mathbf{q}_n\cdot(\mathbf{x}_n - \mathbf{x}_{n-1})}\cdots e^{i\mathbf{q}_2\cdot(\mathbf{x}_2 - \mathbf{x}_1)}\mathcal{I}(q_2,\cdots)\Bigg]_{q_k^0=0,\mathbf{L}=0}.$$

$$(41)$$

As previously done with the energy, we can rewrite this expression in terms of $\langle \phi^2(\mathbf{x})\rangle^{(n-1)}$ as follows:

$$-\frac{\partial E^{(n)}}{\partial \mathbf{L}}\Big|_{\mathbf{L}=0} = -\frac{\lambda}{4}\int d^3\mathbf{x}\langle \phi^2(\mathbf{x})\rangle^{(n-1)}\frac{\partial\sigma_{\mathbf{L}}^2}{\partial \mathbf{L}}(\mathbf{x})\Big|_{\mathbf{L}=0}. \qquad (42)$$

Recalling from Eq. (40) that $\frac{\partial\sigma_{\mathbf{L}}^2}{\partial \mathbf{L}}(\mathbf{x})|_{\mathbf{L}=0}$ is different from zero only for $\mathbf{x} \in \Omega_B$, we may replace the derivatives with respect to the displacement by minus the gradient and obtain

$$-\frac{\partial E^{(n)}}{\partial \mathbf{L}}\Big|_{\mathbf{L}=0} = \frac{\lambda}{4}\int_{\Omega_B} d^3\mathbf{x}\nabla\sigma^2(\mathbf{x})\langle \phi^2(\mathbf{x})\rangle^{(n-1)}, \qquad (43)$$

where the integral over Ω_B includes possible surface-localized contributions. Comparing this expression with the conservation law (21) of the SE tensor for a static configuration,[6]

$$\partial_i\langle T^i{}_j(\mathbf{x})\rangle^{(n)} = \frac{\lambda}{4}\partial_j\sigma^2(\mathbf{x})\langle \phi^2(\mathbf{x})\rangle^{(n-1)}, \qquad (44)$$

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[6]Latin indices are used for spatial coordinates.

we have therefore

$$-\frac{\partial E^{(n)}}{\partial L^j}\Big|_{\mathbf{L}=0} = \int_{\Omega_B} d^3\mathbf{x}\partial_i\langle T^i{}_j(\mathbf{x})\rangle^{(n)}. \qquad (45)$$

If $\partial_i\langle T^i{}_j(\mathbf{x})\rangle^{(n)}$ is regular enough, one can then prove the PVW by using Gauss's theorem; calling $d^2\Sigma_i$ the positive volume 1-form on $\partial\Omega_B$, we obtain

$$-\frac{\partial E^{(n)}}{\partial L^j}\Big|_{\mathbf{L}=0} = \int_{\partial\Omega_B} d^2\Sigma_i\langle T^i{}_j(\mathbf{x})\rangle^{(n)}. \qquad (46)$$

The extension of the proof to the renormalized VEVs of the SE tensor can be done by showing that the subtracted adiabatic terms satisfy an equation analogous to (46). Notice that if $\partial_i\langle T^i{}_j(\mathbf{x})\rangle^{(n)}$ has surface-localized contributions on $\partial\Omega_B$ then they should be added to the rhs of Eq. (46).

VI. EXAMPLES

A. First-order perturbation theory

The first-order expressions have been previously obtained in Ref. [13]. The divergent parts can be straightforwardly obtained in our formalism by computing the involved scalar integral $\mathcal{I}$; they agree with those predicted by the adiabatic expansions (17) and (20). Furthermore, one can obtain an explicit result for the renormalized quantities,

$$\langle \phi^2\rangle_{\text{ren}}^{(1)} = \frac{\lambda\pi^2}{2(2\pi)^8}\text{Re}\Bigg\{\int d^4q_1 e^{iq_1\cdot x}\tilde{\sigma}^2(q_1)\log\left(-\frac{q_1^2}{\mu^2} - i\epsilon\right)\Bigg\}, \qquad (47)$$

$$\langle T^{\mu\nu}\rangle_{\text{ren}}^{(1)} = -\frac{\lambda}{48(2\pi)^6}\text{Re}\Bigg\{\int d^4q_1 e^{iq_1\cdot x}\tilde{\sigma}^2(q_1)(q_1^\mu q_1^\nu - q_1^2\eta_{\mu\nu})$$

$$\times \log\left(-\frac{q_1^2}{\mu^2} - i\epsilon\right)\Bigg\}$$

$$+ \frac{\lambda}{144(2\pi)^2}[\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box]\sigma^2(x), \qquad (48)$$

where we have made explicit the "$+i\epsilon$" prescription and we have defined a Fourier transform in Minkowski space as

$$\tilde{\sigma}^2(q) := \int d^4x_1 e^{-iq\cdot x_1}\sigma^2(x_1). \qquad (49)$$

Although one could be tempted to cancel the last term in the rhs of Eq. (48) by performing a redefinition of the renormalization scale μ, that would imply the introduction of an additional term in other quantities, such as expression (47). Related to this fact, the choice of μ_2 made in Sec. III is

such that there are no local terms in the expression for $\langle\phi^2\rangle_{\text{ren}}^{(1)}$ (apart from the μ dependent ones).

The first-order approximation for the SE tensor satisfies $\partial_\mu\langle T^{\mu\nu}\rangle_{\text{ren}}^{(1)} = 0$, which is consistent with the conservation law in Eq. (21) up to order λ, given that $\langle\phi^2\rangle_{\text{ren}}^{(0)} = 0$. It could be useful to analyze the eventual gravitational effects of the vacuum fluctuations, when used as a source in the semiclassical Einstein equations [13]. However, the associated vacuum energy $E^{(1)}$ vanishes for static backgrounds and therefore has no relevance in the computation of nondynamical Casimir forces.

B. Second-order perturbation theory

The computation at second order in λ is more challenging. After introducing Feynman parameters, we are able to isolate the divergences in the $\mathcal{I}$ integrals and perform the corresponding renormalization; afterward, we obtain the results for the contributions

$$\langle\phi^2\rangle_{\text{ren}}^{(2)} = -\frac{\lambda^2}{2^2(2\pi)^{12}}\text{Re}\left[\int d^4q_1 d^4q_2 \tilde{\sigma}^2(q_1-q_2)\tilde{\sigma}^2(q_2)e^{iq_1\cdot x}\int_0^1\int_0^{1-s_1} ds_2 ds_1 (M_E + i\epsilon)^{-1}\right], \tag{50}$$

$$\langle T^{\mu\nu}\rangle_{\text{ren}}^{(2)} = -\frac{\lambda^2}{16(2\pi)^{10}}\text{Re}\left\{\int_0^1\int_0^{1-s_1} ds_2 ds_1 \int d^4q_1 d^4q_2 \tilde{\sigma}^2(q_1-q_2)\tilde{\sigma}^2(q_2)e^{iq_1\cdot x}\right.$$
$$\times\left[\frac{\eta^{\mu\nu}}{2}\log\left(\frac{M_E}{\mu^2}+i\epsilon\right) + \frac{((s_1-1)q_1+s_2q_2)^\rho(s_1q_1+s_2q_2)^\sigma}{M_E+i0}\left(\eta_\rho^\mu\eta_\sigma^\nu - \frac{1}{2}\eta^{\mu\nu}\eta_{\rho\sigma}\right)\right]\right\}$$
$$+\frac{\lambda}{4}\eta^{\mu\nu}\sigma^2(x)\langle\phi^2\rangle_{\text{ren}}^{(1)} - \frac{3\lambda^2}{2^8\pi^2}\eta^{\mu\nu}\sigma^4(x), \tag{51}$$

where we have defined

$$M_E := s_1(1-s_1)q_1^2 + s_2(1-s_2)q_2^2 - 2s_1s_2(q_1\cdot q_2). \tag{52}$$

A direct computation is arduous, and collinear divergences are always threatening; for a planar geometry, keeping the $i\epsilon$ prescription, one can introduce the basic form factors

$$F_0(q_1,q_2)$$
$$:= q_1\left(q_1\log\left(\frac{q_1^2}{(q_1-q_2)^2}\right) + q_2\log\left(\frac{(q_1-q_2)^2}{q_2^2}\right)\right), \tag{53}$$

$$F_1(q_1,q_2) := q_1^4\log\left(\frac{q_1^2}{(q_1-q_2)^2}\right) + q_1q_2^3\log\left(\frac{(q_1-q_2)^2}{q_2^2}\right), \tag{54}$$

$$F_2(q_1,q_2) := q_1^4\log\left(\frac{q_1^2}{(q_1-q_2)^2}\right) + q_1^2q_2^2\log\left(\frac{(q_1-q_2)^2}{q_2^2}\right), \tag{55}$$

$$F_3(q_1,q_2) := (q_1^2 - q_2q_1 - i\epsilon_1)(q_1q_2 + i\epsilon_2), \tag{56}$$

in which $\epsilon_{1,2}$ are prescription parameters for the Feynman propagator. Using them, we may write a closed expression valid for a planar background field that varies only in the z direction,

$$\langle\phi^2(z)\rangle_{\text{ren}}^{(2)}$$
$$= -\frac{\lambda^2}{2^4(2\pi)^4}\text{Re}\left\{\int d_{q_1}d_{q_2}e^{-iq_1z}\tilde{\sigma}^2(q_1-q_2)\tilde{\sigma}^2(q_2)\frac{F_0}{F_3}\right\}, \tag{57}$$

$$\langle T^{\mu\nu}(z)\rangle_{\text{ren}}^{(2)} = \frac{\lambda\eta^{\mu\nu}}{8}\sigma^2(z)\langle\phi^2\rangle_{\text{ren}}^{(1)} - \frac{\lambda^2}{3\cdot 2^6(2\pi)^2}\eta^{\mu\nu}\sigma^4(z) - \frac{\lambda^2}{3\cdot 2^5(2\pi)^2}\left(\eta^\mu_3\eta^\nu_3 + \frac{1}{2}\eta^{\mu\nu}\right)\sigma^4(z)$$
$$- \frac{\lambda^2}{16(2\pi)^4}\text{Re}\left\{\int dq_1 dq_2 \frac{e^{-iq_1z}}{F_3}\tilde{\sigma}^2(q_1-q_2)\tilde{\sigma}^2(q_2)\left[\left(\eta^\mu_3\eta^\nu_3 + \frac{1}{2}\eta^{\mu\nu}\right)\left(-\frac{1}{3}F_1 + \frac{1}{2}F_2\right) + \frac{\eta^{\mu\nu}}{12}F_1\right]\right\}, \tag{58}$$

in which we are omitting the variables (q_1, q_2) in the form factors F_i. A direct computation shows the conservation law (21) is satisfied at second perturbative order.

If one considers time-independent backgrounds, the expressions become more tractable than in the general case. In particular, the total energy is represented by the simple formula

$$E_{\text{ren}}^{(2)} = \frac{\lambda^2}{2^6 (2\pi)^5} \text{Re}\left[\int d^3\mathbf{q}_1 \tilde{\sigma}^2(\mathbf{q}_1) \tilde{\sigma}^2(-\mathbf{q}_1) \log\left(\frac{\mathbf{q}_1^2}{\mu^2}\right) \right],$$

(59)

where the Fourier transform evaluated at spatial coordinates implies omitting time variables, i.e.,

$$\tilde{\sigma}^2(\mathbf{q}) := \int d^3\mathbf{x}_1 e^{i\mathbf{q}\cdot\mathbf{x}_1} \sigma^2(\mathbf{x}_1).$$

(60)

C. Discontinuity-induced divergences of $\langle\phi^2\rangle_{\text{ren}}$ for a barrier

As explained in Sec. IV, even after the appropriate renormalization procedure has been carried out, both $\langle\phi^2\rangle_{\text{ren}}$ and $\langle T^{\mu\nu}\rangle_{\text{ren}}$ display divergences at the points where the background field σ is discontinuous. Employing the perturbative formalism that was developed in the preceding sections, we can unravel the precise structure of these divergences. We will call them "discontinuity-induced divergences" or "surface divergences," as a way to distinguish them from the divergences that require renormalization, which will be called bulk divergences.

First of all, we will consider a barrier of height $\Delta\sigma$ depending on only just one spatial coordinate

$$\sigma_b^2(z) := \Delta\sigma^2 (\Theta(z-a) - \Theta(z-b)),$$

(61)

where $\Theta(z)$ is the Heaviside function, and we will refer to a and b as the surface of the barrier. The Fourier transform of σ_b^2

$$\tilde{\sigma}_b^2(q) = \Delta\sigma^2 \frac{i(e^{iaq} - e^{ibq})}{q}.$$

(62)

From this expression, one can already appreciate why divergences will occur in $\langle\phi^2\rangle_{\text{ren}}^{(1)}$ for such a background: the convergence for large momenta is only conditionally guaranteed by the oscillatory exponentials. In other words, at those points where the exponents cancel, mild divergences should be present. Indeed, this can be confirmed by replacing σ_b in expression (47), as done in Ref. [13],

$$\langle\phi_b^2\rangle_{\text{ren}}^{(1)} = \frac{\lambda\Delta\sigma^2}{2^5\pi^2} \{-\text{sign}(z-a)[\gamma + \log(\mu|z-a|)]$$
$$+ \text{sign}(z-b)[\gamma + \log(\mu|z-b|)]\},$$

(63)

where γ is the Euler-Mascheroni constant. Even if this expression is divergent at the surface of the barrier, it is local, in the sense that it only depends on the information of the local jump, and integrable, so that one is able to define its mean value over any desired region in space.

One important thing to notice is that, if σ^2 or its derivatives have a finite number of discontinuities,[7] the only types of divergences $\langle\phi^2\rangle_{\text{ren}}^{(1)}$ are those in Eq. (63). Indeed, if the discontinuities appear only in the derivatives of σ^2, then the Fourier transform will contain additional powers of the momentum that will guarantee a nonconditional convergence.

Analogously, if one considers the second- or higher-perturbative orders of $\langle\phi^2\rangle_{\text{ren}}$, a dimensional argument shows that for large momenta the integrand should behave as a power that provides convergence of the integral, cf. Eq. (28).

At this point, the educated reader may be worried about the IR and collinear divergences that we have mentioned in Sec. IV. They will appear in higher-order computations since we are dealing with massless fields; an appropriate regulator should thus be used, or at least the $i\epsilon$ prescription from the Wick rotation should be kept (see a related discussion for the SE tensor in Appendix A). They will also appear in our first-order contribution only if the σ^2 profile decays too slowly at infinity, as is the case of a step function.

D. Divergences in the stress-energy tensor for a barrier

1. First-order computation for a barrier

Consider now the first-order expression (48) for $\langle T^{\mu\nu}\rangle_{\text{ren}}$, focusing for the time being on the nonlocal contribution. If we naively replace the background field with σ_b^2, then we end up with a formally divergent expression, to which a meaning should be ascribed:

$$\langle T_b^{\mu\nu}\rangle_{\text{ren}}^{(1)} = \frac{\lambda\Delta\sigma^2}{48(2\pi)^3} (\eta^{\mu\nu} + \delta^\mu_{\ 3}\delta^\nu_{\ 3})$$
$$\times \text{Im}\left\{ \int dq_1 e^{-iq_1 z} (e^{iaq_1} - e^{ibq_1}) q_1 \log\left(\frac{q_1^2}{\mu^2}\right) \right\}.$$

(64)

In Appendix A, we show that this expression is well defined in the sense of distributions, which is the natural language of quantum field theory (see, for example, Ref. [17] or [39] for a recent discussion in astrophysics). In this section, we will follow a physical approach, introducing an exponential cutoff $c > 0$ in the Fourier transform,

$$\langle T_b^{\mu\nu}\rangle_{\text{ren}}^{(1)} = \frac{\lambda\Delta\sigma^2}{48(2\pi)^3} (\eta^{\mu\nu} + \delta^\mu_{\ 3}\delta^\nu_{\ 3})$$
$$\times \text{Im}\left\{ \int dq_1 e^{-iq_1 z - c|q_1|} (e^{iaq_1} - e^{ibq_1}) q_1 \log\left(\frac{q_1^2}{\mu^2}\right) \right\},$$

(65)

---

[7]Additional divergences may occur in cases where the background field starts oscillating unconstrainedly.

which is tantamount to saying that we have smoothed the discontinuity in the background field. A straightforward computation gives

$$\langle T_b^{\mu\nu}\rangle_{\text{ren}}^{(1)} = -\frac{\lambda\Delta\sigma^2}{48(2\pi)^3}(\eta^{\mu\nu}+\delta^\mu{}_3\delta^\nu{}_3)$$

$$\times \text{Im}\left\{\frac{\log(\mu^2)+2[\log(ia-iz+c)+\gamma-1]}{(-a+ic+z)^2}\right.$$

$$-\frac{\log(\mu^2)+2[\log(-ia+iz+c)+\gamma-1]}{(a+ic-z)^2}$$

$$+\frac{\log(\mu^2)+2[\log(-ib+iz+c)+\gamma-1]}{(b+ic-z)^2}$$

$$\left.-\frac{\log(\mu^2)+2[\log(ib-iz+c)+\gamma-1]}{(-b+ic+z)^2}\right\}. \quad (66)$$

Notice first of all that, due to the tensorial structure, the $\langle T_b^{33}\rangle_{\text{ren}}^{(1)}$ vanishes; the only components that survive are the diagonal terms in the other directions. Second, Eq. (66) means that, as we approach the barrier profile by taking $c\to 0^+$, $\langle T_b^{\mu\nu}\rangle_{\text{ren}}^{(1)}$ should display a bump that resembles a divergence at the surface of the barrier (z_0 may be either a or b in the following formula):

$$\langle T_b^{\mu\nu}\rangle_{\text{ren}}^{(1)}(z)\overset{z\to z_0}{\sim}\text{sign}(z-z_0)(z-z_0)^{-2}(\eta^{\mu\nu}+\delta^\mu{}_3\delta^\nu{}_3). \quad (67)$$

2. Second-order computation for a barrier

The second-order contribution to the SE tensor shares some similarities with the first-order computation of $\langle\phi^2\rangle_{\text{ren}}$. Indeed, a power-counting argument in (51) shows that the integrals involved in the computation are conditionally convergent in the UV as long as we are not evaluating the expressions at the surface of the barrier; at those points, the oscillatory behavior may disappear, and a mild divergence should then occur.

As a particular example, we may analyze the divergent terms for the barrier in Eq. (61). It should be expected that divergences will arise unless some fortuitous cancellations take place, since already the first term, i.e., the one involving $\langle\phi^2\rangle_{\text{ren}}^{(1)}$, is divergent at the surface of the barrier. We leave the lengthy computations to Appendix B, simply stating the result:

$$\langle T_b{}^{\mu\nu}\rangle_{\text{ren}}^{(2)}\big|_{z\to a}$$

$$= -\frac{\lambda^2}{3\cdot 2^4(2\pi)^2}(\Delta\sigma^2)^2\log(|z-a|)(\eta^{\mu\nu}+\delta^\mu{}_3\delta^\nu{}_3)+\cdots. \quad (68)$$

As was the case described in Sec. VI D 1 for the first-order contributions, the tensorial structure implies that $\langle T_b^{33}\rangle_{\text{ren}}^{(2)}$ is finite, while the remaining diagonal components of the SE tensor will display a divergence. In this case, it is an

integrable logarithmic one, and it is of local nature, depending only on the discontinuity of the background field at the corresponding surface.

E. Comparison with a heat kernel approach

One technique that is widely employed in the computation of vacuum energies is the heat kernel. As shown in Sec. III, the divergent contributions to the Green's function (and to the effective action) can be related to the first terms of the heat kernel in the SDWE (or small proper-time expansion). Thus, one may wonder whether they can be employed to rederive the precedent perturbative results.

To begin, notice that in our perturbative discussion we have given explicit results for the finite contributions up to second order in powers of the potential. This is not equivalent to the set of first coefficients in the SDWE but to a resummation of all the coefficients in the heat kernel up to second order in the potential, independently of the number of derivatives involved. Nevertheless, there exists a partial resummation of heat kernel coefficients, the so-called curvature expansion [40,41], that will allow a comparison.

Consider thus a smooth Riemannian manifold with metric $g_{\mu\nu}$; let us call ∇^μ its compatible covariant derivative and define $\square := g_{\mu\nu}\nabla^\mu\nabla^\nu$. The curvature expansion of the heat kernel's diagonal of a Laplacian-type operator is given by [41]

$$e^{-(-\square+Q+m^2)}(x,x;t)$$

$$= \frac{1}{(4\pi t)^{D/2}}e^{-tm^2}\left[1-t\gamma^{(1)}(t\square)Q+\frac{t^2}{2}Q\gamma^{(1)}(t\square)Q\right.$$

$$\left.+\mathcal{O}(R,\nabla(QQ),Q^3)\right], \quad (69)$$

where Q in our case is a smooth function of the coordinates and the coefficient $\gamma^{(1)}(\cdot)$ is defined as

$$\gamma^{(1)}(x) := \int_0^1 d\xi e^{(1-\xi^2)x/4}. \quad (70)$$

Notice that $\mathcal{O}(R,\nabla(QQ))$ means that the expansion (69) does not include terms that are total derivatives and would vanish upon integration in x; given that we work in Minkowski space, we are also neglecting powers of the curvature R.

Recall now that the Euclidean coincident point propagator can be obtained by integration over the proper time. In this scenario, the divergences of the propagator arise because of the small proper-time behavior of the heat kernel; setting $D=4$, we subtract the leading terms up to[8] t^0 and obtain

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[8]Alternatively, to regularize the expression one may introduce a power t^p in Eq. (71) to consider the zeta function regularization, i.e., the complex power of the studied operator. This choice will not affect the conclusions of this section.

$$\langle\phi^2(x)\rangle_{\text{ren}} = \int_0^\infty dt\left[e^{-(-\Box+Q+m^2)}(x,x;t) - \frac{e^{-tm^2}}{(4\pi t)^2}(1-tQ)\right]$$

$$= \frac{\log(-\frac{\Box}{m^2})-2}{16\pi^2}Q + Q\frac{\log(-\frac{m^2}{\Box})}{16\pi^2\Box}Q$$

$$+ \mathcal{O}(\nabla(QQ), Q^3, m^2). \tag{71}$$

A comparison with the expressions that we have obtained in our perturbative expansion is now straightforward. First of all, we should Fourier transform to momentum space, so that under the momentum integral we replace $\Box \to q^2$,

$Q \to \frac{\lambda}{2}\sigma^2(q)$ and $m^2 \to e^{-2}\mu^2$, since the latter is the chosen scale of reference (cf. the discussion in Sec. III). This is enough to see the agreement at the linear level in Q, viz., with (47).

At quadratic order in Q, $\langle\phi^2(x)\rangle$ does not have ultraviolet divergences, and therefore we expect that all methods will produce the same finite results. The comparison is, however, instructive. A subtle point is that expression (71) neglects all the terms that are total derivatives. To understand this, let us consider an Euclidean massive version of our results in Eq. (28),

$$\langle\phi^2(x)\rangle_m^{(2)} := \frac{\lambda^2}{2^2(2\pi)^8}\int d^4q_1 d^4q_2 \tilde{\sigma^2}(q_1)\tilde{\sigma^2}(q_2)e^{iq_+\cdot x}\mathcal{I}_m(-q_+, -q_2), \qquad q_+ := q_1 + q_2, \tag{72}$$

where the new scalar integral $\mathcal{I}_m$ involves massive propagators:

$$\mathcal{I}_m(-q_+, -q_2) := \int \frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+m^2)((q+q_+)^2+m^2)((q+q_2)^2+m^2)}. \tag{73}$$

Neglecting total derivatives in Eq. (72) simply means to set $q_+ \equiv 0$ in Eq. (73). A direct computation gives then

$$\mathcal{I}_m(0, -q_2) = \frac{1}{16\pi^2}\frac{\log(\frac{q_2^2}{m^2})}{q_2^2} + \mathcal{O}(m^2). \tag{74}$$

Finally, replacing this result in expression (72), one can confirm the agreement between our perturbative computation and the resummed heat-kernel expansion for small masses, up to total derivatives.

VII. ADIABATIC APPROACH AND PLANAR INHOMOGENEITIES

Up to this point, we have shown how to to compute physical quantities in a perturbative expansion in powers of σ^2. It is instructive to compare them with the results obtained in other approximations, performing thus a cross-check. In this section, we will employ an adiabatic- or WKB-type approach, in which instead of expanding in powers of σ^2 one performs an expansion in the number of derivatives acting on the background field. Our main goal is to confirm the results of the precedent section regarding the divergences for discontinuous backgrounds.

It will prove useful to introduce a special notation. We will focus on planar inhomogeneities which depend on only one spatial coordinate, which without loss of generality we choose to be x^3 (or simply z for formulas involving only one coordinate). The spacetime coordinates perpendicular to this preferred direction will be denoted as $x^\parallel$, while its spatial subset will be written as $x^\parallel$. As we will see, in order

to be able to perform an adiabatic expansion, we will need to work with a Euclidean signature; we will thus first show how the Euclideanization of our theory proceeds.

A. Stress-energy tensor in terms of the Euclidean propagator

Since we have shown that all the relevant quantities can be written in terms of Feynman's propagator (24), we begin by studying its alternative Euclidean expression. As a first step, we can Fourier transform it in the directions perpendicular to x^3:

$G_F(x, y)$

$$= \int \frac{d\omega}{(2\pi)}\frac{d^{D-2}k^\parallel}{(2\pi)^{D-2}}e^{-i\omega(x^0-y^0)+ik^\parallel\cdot(x^\parallel-y^\parallel)}\mathcal{G}(\omega, k^\parallel; x^3, y^3). \tag{75}$$

Imposing the fact that the background field depends on just the coordinate x^3, the partially Fourier transformed propagator $\mathcal{G}$ (usually called the reduced Green's function) should satisfy the equation[9]

$$\left(-\omega^2 + k^{\parallel 2} - (\partial_x^3)^2 + \frac{\sigma^2(x^3)}{2}\right)\mathcal{G}(\omega, k^\parallel; x^3, y^3)$$

$$= -\delta(x^3, y^3). \tag{76}$$

---

[9]We are setting $\lambda = 1$ with respect to the previous sections. In writing ∂_x^3, we mean the partial derivative in the third direction of the coordinate x.

If we perform a rotation to Euclidean space, i.e., $\omega_M \to i\omega_E$, we obtain that the Euclidean propagator is a solution of the following differential equation:

$$\left(\omega_E^2 + \mathbf{k}^{\|2} - (\partial_x^3)^2 + \frac{\sigma^2(x^3)}{2}\right)\mathcal{G}_E(\omega_E, \mathbf{k}^{\|}; x^3, y^3)$$
$$= -\delta(x^3, y^3). \tag{77}$$

To compute the propagator, instead of departing from (16), we will use an equivalent expression where a point splitting is kept until the end of the computation:

$$\langle T^{\mu\nu}\rangle(x) = \left(\frac{1}{2}g^{\mu\nu}\left[\partial_{x,\alpha}\partial_y^\alpha - \frac{\sigma^2(x^3)}{2}\right] - \partial_x^\mu\partial_y^\nu\right)$$
$$\times \int \frac{d\omega}{(2\pi)}\frac{d^{D-2}\mathbf{k}^{\|}}{(2\pi)^{D-2}}e^{-i\omega(x^0-y^0)+i\mathbf{k}^{\|}\cdot(\mathbf{x}^{\|}-\mathbf{y}^{\|})}$$
$$\times \text{Im}\{\mathcal{G}(\omega, \mathbf{k}^{\|}; x^3, y^3)\}|_{x=y}. \tag{78}$$

Keeping track of the Euclideanization also in the coordinates, Eq. (78) becomes

$$\langle T^{\mu\nu}\rangle(ix_E^0, \mathbf{x}) = \int \frac{d\omega_E}{(2\pi)}\frac{d^{D-2}\mathbf{k}^{\|}}{(2\pi)^{D-2}}$$
$$\times \left(\frac{1}{2}g^{\mu\nu}\left[\omega_E^2 + \mathbf{k}^{\|2} + \partial_x^3\partial_y^3 + \frac{\sigma^2(x^3)}{2}\right]\right.$$
$$\left.+ \partial'^\mu_{x_E}\partial'^\nu_{y_E}\right)\mathcal{G}_E(\omega_E, \mathbf{k}^{\|}; x^3, y^3)|_{x=y} \tag{79}$$

in terms of the formal vectors

$$\partial'^\mu_{x_E} := (\omega_E, i\mathbf{k}^{\|}, \partial_x^3),$$
$$\partial'^\mu_{y_E} := (-\omega_E, -i\mathbf{k}^{\|}, \partial_y^3). \tag{80}$$

We can further simplify this expression, taking into account that $\mathcal{G}_E$ must be invariant in the $(D-1)$-dimensional space $(\omega_E, \mathbf{k}^{\|})$; performing the corresponding angular integration, we find the desired expression,

$$\langle T^{\mu\nu}\rangle(ix_E^0, \mathbf{x}) = \frac{S_{D-2}}{2\pi^2}\int dk^{\|}(k^{\|})^{D-2}\left\{\frac{1}{2}g^{\mu\nu}\left[\left(1 - \frac{H_{D-2}}{S_{D-2}}\right)k^{\|2} + \partial_x^3\partial_y^3 + \frac{\sigma^2(x^3)}{2}\right]\right.$$
$$\left.+ \delta^\mu_3\delta^\nu_3\left(\partial_x^3\partial_y^3 - \frac{H_{D-2}}{S_{D-2}}k^{\|2}\right)\right\}\mathcal{G}_E(\omega_E, \mathbf{k}^{\|}; x^3, y^3)|_{x=y}, \tag{81}$$

in terms of the $(n-1)$-sphere's hyperarea,

$$S_{n-1} := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{82}$$

and the projection factor

$$H_{n-1} := \int d\Omega_{n-1}\cos^2(\phi_1) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{83}$$

B. Adiabatic technique and planar inhomogeneities

Now that we have recast the relevant expressions in terms of the Euclidean Green's function, we need to compute the latter. In general, the homogeneous version of Eq. (77) will have two linearly independent solutions, which we call $f_\pm$,

$$(-\partial^2 + \omega^2(x))f_\pm(x) = 0, \tag{84}$$

with $\omega^2(x) := \omega_E^2 + \mathbf{k}^{\|2} + \frac{\sigma^2(x^3)}{2}$. One can use them to construct the corresponding Green's function as dictated by the theory of Sturm-Liouville operators,

$$\mathcal{G}_E(k^{\|}; x^3, x'^3) = \frac{1}{[f_+, f_-]}f_+(x_>^3)f_-(x_<^3), \tag{85}$$

where $[f, g]$ is the Wronskian[10] between f and g; additionally, $x_>^3$ $(x_<^3)$ is the greatest (smallest) of the two numbers x^3 and x'^3.

However, in practice, it is not possible to obtain the functions $f_\pm$ explicitly. The adiabatic approach is a way to obtain their expansions in powers of the derivatives of σ^2. In this framework, one begins by proposing the substitution

$$f_\pm(x^3) \to \frac{e^{\mp\int dxW(x^3)}}{\sqrt{2W(x^3)}}, \tag{86}$$

where $W(x^3)$ is the new unknown function. Then, one can propose an expansion of $W(x^3)$ in the number of derivatives and obtain its coefficients recursively. In Appendix C, we show the first coefficients of this expansion.

We will focus on the case of an arbitrary background field, apart from the fact that it is discontinuous only at two planes. These two planes will be defined by the equations[11]

---

[10]We are defining the Wronskian as usual, i.e., $[f, g](x) := f(x)g'(x) - f'(x)g(x)$.

[11]As said before, to simplify the notation, we will write z instead of x^3.

$z = a, b$. A formalism has been developed in previous works to deal with this problem [3,8]. In those articles, it has been shown that $f_\pm$ can be obtained by appropriately gluing the solutions obtained in each single slab of space where the background σ^2 is continuous. In short, we call

$$\sigma^2(z) = \begin{cases} \sigma_1^2(z), & z < a \\ \sigma_2^2(z), & a < z < b, \\ \sigma_3^2(z), & z > b \end{cases} \qquad (87)$$

so that the solutions to the homogeneous equation (84) with σ_i as background are called $e_{i,\pm}$, $i = 1, 2, 3$; the global solutions (with σ as background) are denoted as $e_\pm$. We provide more details in Appendix D.

C. Divergences of the two-point function

To simplify the discussion, we will choose the following convention to fix the constants in the indefinite integrals involved in the adiabatic expansion, cf. Eq. (86):

$$e_{1,\pm} \equiv \frac{e^{\mp \int_a^z W_{\sigma_1} dz}}{\sqrt{2W_{\sigma_1}}}, \quad e_{2,\pm} \equiv \frac{e^{\mp \int_a^z W_{\sigma_2} dz}}{\sqrt{2W_{\sigma_2}}}, \quad e_{3,\pm} \equiv \frac{e^{\mp \int_b^z W_{\sigma_3} dz}}{\sqrt{2W_{\sigma_3}}}. \qquad (88)$$

Of course, these arbitrary constants involved in the WKB expansion will play no role in the Green's function, given that they will cancel out when dividing by the appropriate Wronskians. However, if we consider the convention in (88), the coefficients $A_\pm$, $B_\pm$, $C_\pm$, and $D_\pm$ defined in Appendix D simplify, since then the Wronskians $[e_{i,+}, e_{i,-}] \equiv 1, i = 1, 2, 3$. In particular, employing (88), it is immediate to express the Wronskian $[e_+, e_-]$ (which as in the Sturm-Liouville problems is constant) in terms of different coefficients:

$$[e_+, e_-] = C_+ = (B_- A_+ - A_- B_+) = D_-. \qquad (89)$$

Using this information, we may write the Euclidean reduced Green's function in the following form:

$$\mathcal{G}_E(k^\parallel; z) = \begin{cases} \frac{C_-}{C_+} e_{3,+}^2 + \frac{1}{2W_{\sigma_3}}, & z > b \\ \frac{A_- A_+}{C_+} e_{2,+}^2 + \frac{B_- B_+}{C_+} e_{2,-}^2 + \frac{(B_- A_+ + A_- B_+)}{C_+} \frac{1}{2W_{\sigma_2}}, & a < z < b. \\ \frac{D_+}{C_+} e_{1,-}^2 + \frac{1}{2W_{\sigma_1}}, & z < a \end{cases} \qquad (90)$$

At this point, the intuition tells us which are the divergent terms that require renormalization: they will come from the terms proportional to $(W_{\sigma_i})^{-1}$ because the remaining terms are exponentially damped for large parallel momenta (see the first coefficients of the adiabatic expansion in Appendix C). However, some fortuitous cancellations of

the exponential factors may take place at the surface of the barrier as we will see later.

Before analyzing the divergences, it is better to extract from the Wronskians the polynomial dependence in W_{σ_i}; operationally calling this action "polynomial," we introduce then the definition

$$g_{i,j}^{s_1 s_2} := \text{polynomial}([e_{i,s_1}, e_{j,s_2}]) = \frac{2W_{\sigma_j}(z) W_{\sigma_i}(z)(-s_2 W_{\sigma_j}(z) + s_1 W_{\sigma_i}(z)) + [W_{\sigma_j}(z), W_{\sigma_i}(z)]}{4W_{\sigma_i}(z)^{3/2} W_{\sigma_j}(z)^{3/2}}. \qquad (91)$$

In this way, the coefficients are simplified to

$$A_+ \to g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}}, \qquad B_+ \to -g_{3,2}^{++}(b) e^{-\int_a^b W_{\sigma_2}},$$

$$A_- \to -g_{2,1}^{--}(a), \qquad B_- \to g_{2,1}^{+-}(a),$$

$$C_+ \to g_{2,1}^{+-}(a) g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}} + g_{1,2}^{--}(a) g_{3,2}^{++}(b) e^{-\int_a^b W_{\sigma_2}},$$

$$C_- \to -g_{2,1}^{--}(a) g_{2,3}^{+-}(b) e^{-\int_a^b W_{\sigma_2}} - g_{3,2}^{--}(b) g_{2,1}^{+-}(a) e^{\int_a^b W_{\sigma_2}},$$

$$D_+ \to -g_{2,1}^{++}(a) g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}} - g_{3,2}^{++}(b) g_{1,2}^{+-}(a) e^{-\int_a^b W_{\sigma_2}},$$

$$D_- \to g_{2,1}^{--}(a) g_{2,3}^{++}(b) e^{-\int_a^b W_{\sigma_2}} + g_{2,1}^{+-}(a) g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}}. \qquad (92)$$

1. Renormalization

Now, let us study the divergences that must be renormalized; we will call them bulk divergences. Employing the coefficients written in Eq. (92), in the region $a < z < b$, one notices that

$$
\frac{(B_- A_+ + A_- B_+)}{C_+} \frac{1}{2W_{\sigma_2}}
$$

$$
= \frac{g_{2,1}^{+-}(a) g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}} + g_{2,1}^{--}(a) g_{3,2}^{++}(b) e^{-\int_a^b W_{\sigma_2}}}{g_{2,1}^{+-}(a) g_{3,2}^{+-}(b) e^{\int_a^b W_{\sigma_2}} + g_{1,2}^{--}(a) g_{3,2}^{++}(b) e^{-\int_a^b W_{\sigma_2}}} \frac{1}{2W_{\sigma_2}}
$$

$$
\sim \frac{1}{2W_{\sigma_2}} + \cdots, \tag{93}
$$

up to exponentially decreasing functions for large momenta. This kind of contribution is already explicit in the regions where $z < a$ or $z > b$. An explicit computation in terms of the coefficients given in Appendix C gives an expansion in inverse powers of the parallel momenta,

$$
\frac{C_-}{C_+} e_{3,+}^2 = -\frac{g_{2,1}^{--}(a) g_{2,3}^{+-}(b) e^{-2\int_a^b W_{\sigma_2}} + g_{3,2}^{--}(b) g_{2,1}^{+-}(a)}{g_{2,1}^{+-}(a) g_{3,2}^{+-}(b) + g_{1,2}^{--}(a) g_{3,2}^{++}(b) e^{-2\int_a^b W_{\sigma_2}}} \frac{e^{-2\int_b^z W_{\sigma_3}}}{2W_{\sigma_3}}
$$

$$
= \left\{ \frac{\sigma_3(b)^2 - \sigma_2(b)^2}{16(k^\parallel)^3} + \frac{\sigma_2(b)\sigma_2'(b) - \sigma_3(b)\sigma_3'(b)}{16(k^\parallel)^4} + \frac{1}{32(k^\parallel)^5} [\sigma_2(b)^2 \sigma_3(z)^2 - \sigma_3(b)^2 \sigma_3(z)^2 - \sigma_2'(b)^2 + \sigma_3'(b)^2 \right.
$$

$$
\left. - \sigma_2(b)\sigma_2''(b) + \sigma_3(b)\sigma_3''(b) + \sigma_2(b)^4 - \sigma_3(b)^4] + \cdots \right\} e^{-2\int_b^z W_{\sigma_3}} + \cdots. \tag{96}
$$

The situation is now patent: the exponential decay is guaranteed for any $z \neq b$; however, when $z \to b^+$, the exponent vanishes and gives rise to divergences if the inverse powers of $k^\parallel$ of the expression in (96) are not large enough. Of course, in $D = 4$, the discontinuity-induced divergences of $\langle \phi^2 \rangle_{\text{ren}}$ involve only the $(k^\parallel)^{-3}$ contribution; we have written also the higher-order contributions that will be relevant in the analysis of the SE tensor.

At this point, a direct computation shows the exact form of the divergence,

$$
\int_{k^\parallel \geq 1} \frac{d^{D-1}k^\parallel}{(2\pi)^{D-1}} \left[\frac{\sigma_3(b)^2 - \sigma_2(b)^2}{16(k^\parallel)^3} \right] e^{-2(z-b)k^\parallel}
$$

$$
= -\frac{\Delta\sigma^2(b)}{32\pi^2} (\log(z-b) + \gamma) + \mathcal{O}(z-b), \tag{97}
$$

where $\Delta\sigma^2(y)$ denotes the jump of the background field at y, i.e.,

$$
\frac{1}{2W_{\sigma_i}} = \frac{1}{2(k^\parallel)} - \frac{\sigma_i(z)^2}{8(k^\parallel)^3} + \frac{-2\sigma_i'(z)^2 - 2\sigma_i(z)\sigma_i''(z) + 3\sigma_i(z)^4}{32(k^\parallel)^5}
$$

$$
+ \cdots. \tag{94}
$$

This is enough to compute the bulk divergent terms of the two-point function; indeed, upon integration over the $k^\parallel$-momentum variables, we obtain

$$
\langle \phi^2(z) \rangle^{\text{WKB}} = \int \frac{d^{D-1}k^\parallel}{(2\pi)^{D-1}} \frac{1}{2W_{\sigma_i}}
$$

$$
= \frac{1}{D-4} \frac{\sigma^2(z)}{16\pi^2} + \text{finite terms.} \tag{95}
$$

A direct computation shows that this coincides with both the SDWE adiabatic result in Eq. (17) and the perturbative one.

2. Discontinuity-induced divergences

For simplicity, we will consider just the region where $z > b$; the remaining ones can be worked out in an analogous way. The contribution for large parallel momentum reads

$$
\Delta\sigma^2(y) = \sigma^2(y^+) - \sigma^2(y^-). \tag{98}
$$

Computing the remaining contributions, one obtains a result that coincides with the one obtained in Eq. (63). Notice, however, that in this section our conclusion is not restricted to a given power in σ^2. Then, one can conclude that the only discontinuity-induced divergences present in $\langle \phi^2 \rangle_{\text{ren}}$ are all linear in σ^2.

D. Divergences of the stress-energy tensor

Taking into account Eq. (81), the divergences' structure of the SE tensor can be analyzed in a manner analogous to that for $\langle \phi^2 \rangle$. The only difference is that we additionally need an expansion for the product of derivatives[12] $e_+'(z)e_-'(z)$. The computation is straightforward, albeit lengthy; this can be appreciated already from its structure:

---

[12]There are additional terms involving ultralocal factors that vanish in dimensional regularization.

$$\frac{1}{[e_+, e_-]}e'_+e'_- = \begin{cases} \dfrac{C_-}{C_+}\dfrac{(W'_{\sigma_3}(z)+2W_{\sigma_3}(z)^2)^2}{8W_{\sigma_3}(z)^3}e^{-2\int_b^z W_{\sigma_3}dz} \\[2ex] +\dfrac{W'_{\sigma_3}(z)^2-4W_{\sigma_3}(z)^4}{8W_{\sigma_3}(z)^3}, \qquad\qquad z > b \\[3ex] \dfrac{A_-A_+}{C_+}\dfrac{(W'_{\sigma_2}(z)+2W_{\sigma_2}(z)^2)^2}{8W_{\sigma_2}(z)^3}e^{-2\int_a^z W_{\sigma_2}(z)dz} \\[2ex] -\dfrac{B_-B_+}{C_+}\dfrac{(W'_{\sigma_2}(z)-2W_{\sigma_2}(z)^2)^2}{8W_{\sigma_2}(z)^3}e^{2\int_a^z W_{\sigma_2}(z)dz} \\[2ex] +\dfrac{(A_+B_-+A_-B_+)}{C_+}\dfrac{W'_{\sigma_2}(z)^2-4W_{\sigma_2}(z)^4}{8W_{\sigma_2}(z)^3}, \qquad a < z < b \\[3ex] \dfrac{W'_{\sigma_1}(z)^2-4W_{\sigma_1}(z)^4}{8W_{\sigma_1}(z)^3} \\[2ex] +\dfrac{D_+}{C_+}\dfrac{(W'_{\sigma_1}(z)-2W_{\sigma_1}(z)^2)^2}{8W_{\sigma_1}(z)^3}e^{2\int_a^z W_{\sigma_1}(z)dz}, \qquad z < a \end{cases} \tag{99}$$

1. Renormalization

Equation (81) contains several factors that are not exponentially suppressed for large parallel momentum. The large-$k^\parallel$ expansion of many of them has already been derived in Sec. VII C 1. The only new contribution of this type, inherited from expression (99), can be expanded as

$$\frac{W'_{\sigma_i}(z)^2 - 4W_{\sigma_i}(z)^4}{8W_{\sigma_i}(z)^3} = -\frac{(k^\parallel)}{2} - \frac{\sigma_i(z)^2}{8(k^\parallel)}$$
$$+ \frac{-4\sigma'_i(z)^2 - 4\sigma_i(z)\sigma''_i(z) + \sigma_i(z)^4}{64(k^\parallel)^3}$$
$$+ \cdots. \tag{100}$$

Summing all the contributions, in dimensional regularization, we obtain

$$\frac{C_-}{C_+}\frac{(W'_{\sigma_3}(z) + 2W_{\sigma_3}(z)^2)^2}{8W_{\sigma_3}(z)^3}e^{-2\int_b^z W_{\sigma_3}dz}$$
$$= \left\{ \frac{\sigma_3(b)^2 - \sigma_2(b)^2}{16(k^\parallel)} + \frac{\sigma_2(b)\sigma'_2(b) - \sigma_3(b)\sigma'_3(b)}{16(k^\parallel)^2} + \frac{1}{32(k^\parallel)^3}[-\sigma_2(b)^2\sigma_3(z)^2 + \sigma_3(b)^2\sigma_3(z)^2 - \sigma'_2(b)^2 \right.$$
$$\left. + \sigma'_3(b)^2 - \sigma_2(b)\sigma''_2(b) + \sigma_3(b)\sigma''_3(b) + \sigma_2(b)^4 - \sigma_3(b)^4 + \cdots] \right\}e^{-2\int_b^z W_{\sigma_3}dz} + \cdots. \tag{102}$$

Although the exponent provides the necessary fast decay for large $k^\parallel$, it happens only if $z \neq b$. Replacing in Eq. (81) both the contributions analogous to (102) and the results of Sec. VII C 2, we finally obtain

$$\langle T^{\mu\nu}\rangle^{\text{WKB}}_{z\to z_0} = \frac{(\eta^{\mu\nu} + \delta^\mu_3\delta^\nu_3)}{3\cdot 2^4(2\pi)^2}H(z, z_0), \tag{103}$$

$$\langle T^{\mu\nu}\rangle^{\text{WKB}} = \frac{(\eta^{\mu\nu} + \delta^\mu_3\delta^\nu_3)}{D-4}\frac{1}{3\cdot 2^5(2\pi)^2}(4(\sigma^2)'' + 3\sigma^4)$$
$$+ \text{finite terms}, \tag{101}$$

which agrees with our perturbative computation, as well as with the adiabatic approach in the SDWE framework, cf. Eq. (20).

2. Discontinuity-induced divergences

Boundary divergences arise as in the case of $\langle\phi^2\rangle$, i.e., some exponentially decreasing factors that guarantee the convergence of the integrals for large momenta may disappear at the surface of the barrier. As an example, consider the following term from expression (99), for $z > b$:

where $z_0 = a$, b and we have defined the scalar function

$$H(x, y) := \text{sign}(x - y)\left[\frac{\Delta\sigma^2(y)}{(x-y)^2} - \frac{\Delta((\sigma^2)')(y)}{(x-y)}\right]$$
$$+ \log(|y - x|)[-[\Delta(\sigma^2)(y)]^2$$
$$- \text{sign}(x - y)\Delta((\sigma^2)'')(y)]. \tag{104}$$

In particular, if we restrict ourselves to the case of a barrier, then we reobtain the results (67) and (68). The importance of the expansion (103) resides in the following two facts: in the worst case, divergences are of second order in powers of σ^2, so all of them can be studied by our perturbative expressions in Sec. VI, and they are of local nature, confirming that they will play no role in the computation of Casimir forces.

VIII. CONCLUSIONS

We have employed a perturbative method, together with dimensional regularization and adiabatic renormalization, to prove master formulas for a scalar model in the realm of generalized Lifshitz configurations.

First of all, we have provided a general (perturbative) proof of the validity of the principle of virtual work, showing that in the time-independent situation one can indeed compute the Casimir force exerted on one body in two different ways: either by considering the change in the energy of the system after a virtual displacement of the body or by computing the stresses acting on the latter, cf. Eq. (46). The derivation is valid for arbitrary geometries and to all orders in the perturbation.

A fundamental pillar that allowed the proof was the conservation law that the energy-stress tensor satisfies not only at the classical level but also at the level of renormalized VEVs in the semiclassical theory (quantum for the ϕ field and classical for the background one), as is guaranteed by Eq. (21). This is a highly nontrivial point, since in general the regularization and the renormalization process may break classical laws at any point, introducing the so-called quantum anomalies.

We have also provided master expressions for the nth perturbative order VEVs of the two-point function and the energy-stress tensor. In particular, we have shown that in the static case only $\langle\phi\rangle^{(n-1)}$ is required in order to compute the total energy of the system at order n. Given that the complexity of the calculations increases with the order of the perturbation and is greater for $\langle T^{\mu\nu}\rangle$ than for $\langle\phi^2\rangle$, we believe that such a formula will be extremely useful for evaluating the vacuum energy in concrete examples. Additionally, we have written explicit formulas for all the relevant VEVs at first and second perturbative orders, having computed the relevant form factors for planar configurations.

With the help of those master formulas, we have analyzed in detail the divergences that appear both in $\langle\phi^2\rangle$ and $\langle T^{\mu\nu}\rangle$ as a consequence of discontinuities in a planar background field, extending the results in Refs. [3,7]. Our computations show that their functional dependence on the background field is at most quadratic, while they are local. These considerations have been confirmed by an alternative WKB-type approach, proving that they are not relevant in the computation of Casimir

forces. For the mathematically oriented reader, we have also dedicated a section regarding their formal interpretation in terms of distribution.

It is important to mention that, contrary to the situation when other renormalization prescriptions are employed as in Ref. [17], we do not obtain a so-called pressure anomaly. Moreover, we do not find the analog of the van der Waals anomaly discussed in Ref. [10], which in our scalar model would consist in a violation of the semiclassical conservation equation for the energy-stress tensor.

In spite of the obtained results, there are still many open questions. The first one is related to the intrinsic character of the background field in a given body and its surroundings and how they are affected by a displacement of the body. Another interesting issue is whether our results regarding the surface divergencies can be extended to nonplanar geometries, either by considering the perturbative or the WKB-type approach. These lines are currently being studied.

ACKNOWLEDGMENTS

APPENDIX A: FIRST-ORDER STRESS-ENERGY TENSOR AS A DISTRIBUTION

We have seen in Sec. VI D that, when we consider the first-order SE tensor for a barrier, we obtain a formally divergent expression. However, that expression is well defined as a distribution. As explained in Ref. [42] (see also Ref. [43] for an introductory course), we can interpret Eq. (64) as the Fourier transform of a distribution,

$$\int dq_1 e^{-iq_1 x}(e^{iaq_1}-e^{ibq_1})q_1\log\left(\frac{q_1^2+i\epsilon}{\mu^2}\right)$$
$$=2\mathcal{F}[\alpha(q_1)](a-x)-2\mathcal{F}[\alpha(q_1)](b-x), \quad (A1)$$

where

$$\alpha(q):=q\log\left(\frac{|q|}{\mu}\right). \quad (A2)$$

To be more explicit, we may recast this expression in the notation of Ref. [42] as

$$\alpha(q)=q_+\log(q_+)-q_-\log(q_-)-q\log(\mu), \quad (A3)$$

defining the distributions

$$x_\pm^\lambda = \Theta(\pm x)|x|^\lambda. \tag{A4}$$

Employing several identities that can be found in Ref. [42], we end up with

$$\mathcal{F}_{\text{Shilov}}[\alpha(x)] = -i\pi(-2(\log(\mu) + \gamma - 1)\delta'(x) \\ + \hat{x}^{-2} - 2x_-^{-2}), \tag{A5}$$

where both x^{-2} and x_-^{-2} are themselves distributions that are defined by their actions on test functions ϕ:

$$(x^{-2}, \phi) = \int_0^\infty dx\, x^{-2}[\phi(x) + \phi(-x) - 2\phi(0)], \tag{A6}$$

$$(x_-^{-2}, \phi) = \int_0^\infty dx\, x^{-2}[\phi(-x) - \phi(0) + x\phi'(0)\Theta(1 - x)]. \tag{A7}$$

The SE tensor can be readily obtained combining the previous equations:

$$\langle T_{\mu\nu}\rangle_{\text{ren}}^{(1)} = \frac{\lambda(\delta^3{}_\mu\delta^3{}_\nu + \eta_{\mu\nu})}{48(2\pi)^2}\{2(-1 + \gamma + \log(\mu)) \\ \times (\delta'(a - z) - \delta'(b - z)) \\ - (z - a)^{-2} + 2(z - a)_-^{-2} \\ + (z - b)^{-2} - 2(z - b)_-^{-2}\}. \tag{A8}$$

If one is interested just in mean values of $\langle T_{\mu\nu}\rangle_{\text{ren}}^{(1)}$ over a finite region, then using (A8), one obtains a finite number.

APPENDIX B: SECOND-ORDER CONTRIBUTIONS IN $\langle T^{\mu\nu}\rangle_{\text{ren}}$ FOR A BARRIER AND DISTRIBUTIONS

In general, Eq. (57) involves regularized quadratic forms as given by the definition of F_3 in (56). The mathematical theory has been extensively studied in Ref. [42]; in this Appendix, we will follow a physicist approach, performing a change of variables that converts the quadratic forms into linear ones. We can define the transformation

$$(p_1, p_2) := (q_1 q_2, q_1(q_1 - q_2)), \tag{B1}$$

as well as its inverse, defining the functions $h^\pm(\cdot, \cdot)$:

$$(q_1, q_2) = \pm\left(\sqrt{p_1 + p_2}, \frac{p_1}{\sqrt{p_1 + p_2}}\right) \\ =: h^\pm(p_1, p_2). \tag{B2}$$

Taking into account the map of the domains, we get

$$H := \int_{-\infty}^\infty dq_1 dq_2 \frac{f(q_1, q_2)}{[q_1(q_1 - q_2) - i\epsilon_1](q_1 q_2 + i\epsilon_2)} \\ = \int_{-\infty}^\infty \int_{-p_1}^\infty dp_2 dp_1 \frac{\bar{f}(p_1, p_2)}{[p_2 - i\epsilon_1](p_1 + i\epsilon_2)} \tag{B3}$$

for an arbitrary well-behaved function $f(\cdot, \cdot)$, if we use the additional definition

$$\bar{f}(p_1, p_2) := \frac{1}{2(p_1 + p_2)}[f(h^+(p_1, p_2)) + f(h^-(p_1, p_2))]. \tag{B4}$$

Using the Sokhotski-Plemelj theorem to rewrite the $(p_i \pm i\epsilon)^{(-1)}$ factors, we may further simplify this expression. In particular, if the contributions corresponding to integrals of Dirac δ vanish (as is the case for the barrier), then we get

$$H \to H_{\text{PV,PV}} = \int_0^\infty \frac{dp_1}{p_1} \int_0^{p_1} \frac{dp_2}{p_2}\{\bar{f}(p_1, p_2) \\ - \bar{f}(p_1, -p_2) + \bar{f}(p_2, p_1) - \bar{f}(-p_2, p_1)\}. \tag{B5}$$

For a barrier, it proves convenient to perform the rescalings $p_2 \to p_1 p_2$ and afterward $p_1 \to (1 \pm p_2)p_1$, depending on whether the argument in $\bar{f}$ is $\pm p_2$. The integral in p_1 can then be performed, and the remaining integral in p_2 is convergent for $x \neq a, b$. The contributions that are divergent at the surface of the barrier can be isolated. To sketch the kind of computations involved for a barrier, consider the following example:

$$\int dq_1 dq_2 \tilde{\sigma}_b^2(q_1 - q_2)\tilde{\sigma}_b^2(q_2)e^{-iq_1 x} \frac{F_1}{F_3}\bigg|_{\text{div}} = 8\int_0^1 \frac{dp_2}{p_2^2(p_2^2 - 1)}[p_2^4(\log(1 - p_2) - 2\log(p_2) + \log(p_2 + 1)) \\ - 2p_2^3(\log(1 - p_2) - \log(p_2 + 1)) \\ + 2p_2(\log(1 - p_2) - \log(p_2 + 1)) - \log(1 - p_2) - \log(p_2 + 1)] \\ \times \log\left(\frac{|a - x||b - x|}{|b - x + (x - a)p_2||a - x + (x - b)p_2|}\right). \tag{B6}$$

The divergence therefore arises either directly from a $\log(|x - y|)$ term or from a term that renders the integrand singular when $x = y$. Both of them can be easily handled, and the sum over all the form factors results in Eq. (68).

APPENDIX C: ADIABATIC COEFFICIENTS

If we try to solve Eq. (84) by using Eq. (86) together with the *ansatz*

$$W(x) = \sum_{j=0}^{\infty} W_j(x), \qquad \text{(C1)}$$

where W_j is a term of adiabatic order j, then we find that the coefficients with odd j vanish. The first ones for even j are given by

$$W_0(x) = \omega(x),$$

$$W_2(x) = -\frac{3\omega'(x)^2}{8\omega(x)^3} + \frac{\omega''(x)}{4\omega(x)^2},$$

$$W_4(x) = \frac{\omega^{(4)}(x)}{16\omega(x)^4} - \frac{13\omega''(x)^2}{32\omega(x)^5} - \frac{297\omega'(x)^4}{128\omega(x)^7}$$
$$- \frac{5\omega^{(3)}(x)\omega'(x)}{8\omega(x)^5} + \frac{99\omega'(x)^2\omega''(x)}{32\omega(x)^6}. \qquad \text{(C2)}$$

APPENDIX D: GREEN'S FUNCTION FOR PLANAR GEOMETRIES

In this Appendix, we review the results of Ref. [8]. If we have a discontinuous background field, we can obtain the solutions to the inhomogeneous Eq. (77) by gluing together the solutions to the inhomogeneous problem in each slab, which we call $e_{\pm,i}$. As is customary, there will exist two solutions; we will call them $e_\pm$, according to whether they decay fast enough at $\pm\infty$. If we ask $e_\pm$ and their first derivatives to be continuous at $z = a$, b, the expansion reads as

$$e_+(z) := \begin{cases} e_{3,+}, & z > b \\ A_+ e_{2,+} + B_+ e_{2,-}, & a < z < b, \\ C_+ e_{1,+} + D_+ e_{1,-}, & z < a \end{cases} \qquad \text{(D1)}$$

$$e_-(z) := \begin{cases} C_- e_{3,+} + D_- e_{3,-}, & z > b \\ A_- e_{2,+} + B_- e_{2,-}, & a < z < b, \\ e_{1,-}, & z < a \end{cases} \qquad \text{(D2)}$$

in terms of the coefficients

$$A_+ = \frac{[e_{3,+}, e_{2,-}](b)}{[e_{2,+}, e_{2,-}](b)}, \qquad A_- = -\frac{[e_{2,-}, e_{1,-}](a)}{[e_{2,+}, e_{2,-}](a)}, \qquad B_+ = -\frac{[e_{3,+}, e_{2,+}](b)}{[e_{2,+}, e_{2,-}](b)}, \qquad B_- = \frac{[e_{2,+}, e_{1,-}](a)}{[e_{2,+}, e_{2,-}](a)},$$

$$C_+ = \frac{A_+[e_{2,+}, e_{1,-}](a) - B_+[e_{1,-}, e_{2,-}](a)}{[e_{1,+}, e_{1,-}](a)}, \qquad C_- = -\frac{A_-[e_{2,+}, e_{3,-}](b) + B_-[e_{2,-}, e_{3,-}](b)}{[e_{3,+}, e_{3,-}](b)},$$

$$D_+ = \frac{B_+[e_{1,+}, e_{2,-}](a) - A_+[e_{2,+}, e_{1,+}](a)}{[e_{1,+}, e_{1,-}](a)}, \qquad D_- = -\frac{B_-[e_{3,+}, e_{2,-}](b) + A_-[e_{3,+}, e_{2,+}](b)}{[e_{3,+}, e_{3,-}](b)} \qquad \text{(D3)}$$

and the Wronskians $[f, g](x) := f(x)g'(x) - f'(x)g(x)$. The main difference with the results in Ref. [8] resides in the fact that our Wronskians are the usual ones.

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