Inhomogeneous states in the two-dimensional linear sigma model at large N

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In this paper we consider inhomogeneous solutions of two-dimensional linear sigma model in the large N limit. These solutions are similar to the ones found recently in the two-dimensional CP^N sigma model. The solution exists only for some range of coupling constant. We calculate the energy of the solutions as a function of the model parameters and show that it is negative. We analyze the zero modes of the soliton and argue that they can be interpreted as rotational excitations. The case of the nonlinear model at finite temperature is also discussed. The free energy of the inhomogeneous solution is shown to change sign at some critical temperature indicating possible phase transition.

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I. INTRODUCTION

The two-dimensional linear sigma model is a theory of N real scalar fields with a quartic $O(N)$ symmetric interaction. The model has two dimensionful parameters; particle mass and coupling constant. In the limit of infinite coupling one can obtain the nonlinear $O(N)$ sigma model. For that reason, the linear model can be thought of as a generalization of the nonlinear one. These models can be solved in the large N limit, see [1] for a nonlinear model review. In turn, the $O(N)$ sigma model is quite similar to the CP^N sigma model.

Recently the large N CP^N sigma model was considered on a finite interval with various boundary conditions [2–8] and on circle [9–12]. It was shown [4] that in some cases the ground state field configuration must be inhomogeneous. Namely, the expectation values of the fields must depend on the spatial coordinate. This observation stimulated the search for inhomogeneous solutions of the model on the whole plane without boundaries. Such solutions were originally constructed in [13] by analogy with the Gross-Neveu model [14–16] and studied in [17–20]. It was shown in [19] that the energy of the solution is negative. In other words, such an inhomogeneous configuration has lower energy than the homogeneous state. However, these solutions break the internal $O(N)$ symmetry of the theory so zero modes appear and the solution may suffer from infrared (IR) divergences. This IR physics probably prevents the system on the whole plane from collapsing into an inhomogeneous phase, despite energy consideration. Therefore, the interpretation of the solutions remains unclear.

The purpose of this paper is to expand the analysis of inhomogeneous states to the case of the linear sigma model. In the linear model context, similar solutions were considered in [21,22]. Some of our solutions were considered a long time ago in [22]. We revisit these solutions and explore their properties in greater detail. In particular, we carefully compute the energy of these solutions and find that for some values of parameters the energy is negative (similarly to [19]). These solutions are plagued by the zero-mode problem; quantum effective action seems to diverge due to the presence of zero frequency fluctuations. We use an approach similar to [23–25] to examine this problem. We consider canonical quantization and identify creation operators associated with zero modes. We find that zero modes can correspond to soliton rotations in internal space and thus can be safely treated by moduli space formalism. Note that the zero-mode question also arises in the so-called gray soliton models [26,27].

This paper is organized as follows. In Sec. II we review the actions of the considered models and discuss their properties in the large N limit. We introduce the gap equation for the models and determine their spectrum. In Sec. III we discuss inhomogeneous solutions of the gap equations and in Sec. IV we calculate the soliton energy. In Sec. V we turn to the zero-mode problem and consider it in detail. First we review the method of [23] for the kink in the ϕ^4 model with broken Z_2 symmetry and then generalize the method to the sigma model case. In Sec. VI we consider the finite temperature case. We find that at high temperature the soliton energy becomes positive.

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II. THE MODELS

The two-dimensional linear sigma model in Euclidean space is defined by the action

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 + \frac{g}{2N}(n_a^2 - r)^2 \right). \quad (1)$$

The index $a = 1 \ldots N$ enumerates the field components and we always assume that N is large. The constant g is the coupling with the dimension of mass squared. When the coupling strength is large, the potential imposes the constraint $n^2 = r$ at the classical level and the theory reduces to the nonlinear sigma model. In the general case, r corresponds to the classical value of n^2. However, due to the Coleman theorem, continuous symmetry cannot be spontaneously broken in two dimensions and quantum fluctuations should restore the value $n^2 = 0$.

To examine the model in the large N limit we start by rewriting the action via an auxiliary field λ

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 + \frac{\lambda}{2}((n_a)^2 - r) - \frac{N\lambda^2}{8g} \right). \quad (2)$$

Note that the minus sign of the λ^2 term corresponds to the positive potential. Now the action is quadratic in the n field. We fix one component $n^N = n$ to examine the inhomogeneous solutions and integrate out all other components n^a, $a = 1, \ldots, N-1$. Thus, we obtain effective actions as a function of fields n and λ,

$$S_{\text{eff}} = \frac{N}{2}\operatorname{tr}\log(-\partial^2 + \lambda) + \int d^2x \left(\frac{1}{2}(\partial n)^2 + \frac{\lambda}{2}(n^2 - r) - \frac{N\lambda^2}{8g} \right). \quad (3)$$

Let us first consider homogeneous solutions with $n = 0$. The vacuum expectation value $\lambda = m^2 = \text{const}$ gives the mass to the n field quanta. Differentiating Eq. (3) with respect to λ we obtain the gap equation

$$\frac{N}{8\pi}\log\frac{M^2}{m^2} - \frac{1}{2}r - \frac{Nm^2}{4g} = 0. \quad (4)$$

The first term comes from differentiating the trace of the logarithm term. This trace can be evaluated as follows:

$$\frac{\partial}{\partial\lambda}\operatorname{tr}\log(-\partial^2 + \lambda) = \operatorname{tr}\frac{1}{-\partial^2 + \lambda} = \int \frac{d^2k}{k^2 + m^2} = \frac{1}{4\pi}\log\frac{M^2}{m^2}.$$

To regularize the trace we used a UV cutoff M^2.

We introduce coupling constant renormalization

$$r = \frac{N}{4\pi}\log\frac{M^2}{\Lambda^2}, \quad (5)$$

where Λ is a dynamically generated scale. We choose this renormalization procedure to make it similar to the nonlinear sigma model case. The coupling constant g is not renormalized, which is consistent with the diagrammatic expansion in two dimensions. Now we can explain what we mean by a large coupling; a strong coupling limit corresponds to the case $|g| \gg \Lambda^2$.

The final version of homogeneous gap equation reads

$$m^2 = \frac{g}{2\pi}\log\frac{\Lambda^2}{m^2}. \quad (6)$$

This equation defines the mass of the particles as a function of the coupling and the mass scale. In the strong coupling limit $g \gg \Lambda^2$ the solution is $m = \Lambda$. In the case of the positive coupling constant ($g > 0$) this equation always has a unique solution; in the small coupling limit we obtain

$$m^2 \approx \frac{g}{2\pi}\log\frac{2\pi\Lambda^2}{g}. \quad (7)$$

Thus, mass increases with the coupling constant g and can be made arbitrarily small.

We also consider the model with quartic interaction and positive mass and coupling constant. Its Euclidean action is

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 + \frac{1}{2}m_0^2 n_a^2 + \frac{g}{4N}(n_a^2)^2 \right). \quad (8)$$

Similarly to the previous case, we can rewrite the action via an auxiliary field λ

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 + \frac{1}{2}\lambda n_a^2 - \frac{N}{4g}(\lambda - m_0^2)^2 \right). \quad (9)$$

Now we can integrate out the scalar fields and obtain the effective action for λ and $n = n_N$

$$S_{\text{eff}} = \frac{N}{2}\operatorname{tr}\log(-\partial^2 + \lambda) + \int d^2x \left(\frac{1}{2}(\partial n)^2 + \frac{1}{2}\lambda n^2 - \frac{N}{4g}(\lambda - m_0^2)^2 \right). \quad (10)$$

The homogeneous gap equation reads ($n = 0$, $\lambda = m^2$, and M is the UV cutoff),

$$m^2 = m_0^2 + \frac{g}{4\pi}\log\frac{M^2}{m^2}. \quad (11)$$

The solution for the physical mass of the particles, m^2, is always unique. Note that if the coupling constant is small we can substitute m_0 instead of m in the logarithmic term and thus obtain the usual one-loop mass renormalization. However, the large N limit allows us to consider the coupling of arbitrary strength.

III. INHOMOGENEOUS SOLUTIONS

Now we turn to the inhomogeneous solutions; namely, the stationary points of effective action (3) or (10) for which both fields $\lambda(x)$ and $n(x)$ can depend on spacial coordinates, but are constant in (Euclidean) time.

Similarly to the [19] we use the ansatz

$$\lambda = m^2\left(1 - \frac{2}{\cosh^2 mx}\right), \qquad n \sim \frac{1}{\cosh mx}. \tag{12}$$

Varying the action with respect to n we obtain equation

$$(-\partial_x^2 + \lambda(x))n(x) = 0, \tag{13}$$

which is satisfied by (12) automatically. Variation with respect to λ leads to the gap equation

$$\frac{N}{4\pi}\sum_n \frac{|f_n(x)|^2}{2E_n} + \frac{1}{2}(n^2 - r) - \frac{N}{4g}\lambda = 0. \tag{14}$$

Here the summation is over the eigenfunction of the differential operator $-\partial_x^2 + \lambda(x)$,

$$(-\partial_x^2 + \lambda(x))f_n(x) = E_n^2 f_n(x). \tag{15}$$

The eigenfunctions for the field configuration (12) are

$$f_k(x) = \frac{-ik + m\tanh mx}{\sqrt{k^2+m^2}}e^{ikx}, \qquad E_n^2 = k^2 + m^2. \tag{16}$$

The operator $-\partial_x^2 + \lambda$ has a zero mode $f_0 \sim 1/(\cosh mx)$. Later we will argue that it corresponds to the rotations of the solution in the internal space. Partition function integration over the zero modes yields the volume of the moduli space of the solution that is the volume of $(N-1)$-dimensional sphere. Therefore, for now we explicitly exclude this mode from the summation in (14).

After substitution (16) in the gap equation (14) we find that coordinate independent terms cancel due to (6) and the inhomogeneous part gives the amplitude of the n field,

$$n^2 = \frac{N\lambda}{2g} - N\int\frac{dk}{4\pi}\left(\frac{1}{\sqrt{k^2+m^2}} - \frac{1}{\sqrt{k^2+\Lambda^2}}\right) + \frac{N}{4\pi}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+m^2}}\frac{m^2}{k^2+m^2}\frac{1}{\cosh^2 mx}. \tag{17}$$

After straightforward integration we obtain

$$n^2 = \frac{N}{2\pi}\left(1 - \frac{2\pi m^2}{g}\right)\frac{1}{\cosh^2 mx}. \tag{18}$$

The solution exists if $n^2 \geq 0$ or

$$\frac{g}{2\pi m^2} > 1. \tag{19}$$

Thus, we have found a solution if the coupling is strong enough.

Now we turn to the model with action (10). The ansatz (12) is the same as previously. The gap equation reads as

$$\frac{N}{4\pi}\sum_n \frac{|f_n(x)|^2}{2E_n} + \frac{1}{2}n^2 - \frac{N}{2g}(\lambda - m_0^2) = 0. \tag{20}$$

Therefore, we can calculate n^2; the result is given by Eq. (18). Again the solution exists only when condition (19) is satisfied. However, one should remember that in this case the meaning of the coupling constant g is different.

IV. ENERGY OF THE SOLUTIONS

Now we are going to calculate the energy of the solution found in the previous section. We introduce a large but finite time cutoff β and calculate the regularized Euclidean effective action S_{reg} of this solution. To deal with divergences we use the Pauli-Villars regularization and subtract the action for the homogeneous solution. The energy is $E = S_{\mathrm{reg}}/\beta$.

The Pauli-Villars regularized effective action for the model (2) is

$$S_{\mathrm{reg}} = \frac{N}{2}\sum_i C_i \mathrm{Tr}\log(-\partial^2 + \lambda + M_i^2) + \int d^2x\left(\frac{1}{2}(\partial n)^2 + \frac{\lambda}{2}(n^2 - r) - \frac{N\lambda^2}{8g}\right). \tag{21}$$

The summation is over regulator fields, M_i, $i = 0, 1, 2$, are regulator masses and C_i are coefficients satisfying

$$\sum_i C_i = 0; \qquad \sum_i C_i M_i^2 = 0; \qquad C_0 = 1, \qquad M_0 = 0.$$

The coupling constant r can be expressed in terms of the regulators' masses as

$$r = -\frac{N}{4\pi}\sum_{i=1,2}\log\frac{M_i^2}{\Lambda^2}. \tag{22}$$

Subtracting from (21) (similar to the expression for homogeneous configuration) we obtain energy

$$E = \frac{N}{2}\int\frac{d\omega}{2\pi}\sum_i C_i\log(\omega^2 + M_i^2) + \frac{N}{2}\int\frac{d\omega}{2\pi}\sum_n\sum_i C_i\log\frac{\omega^2 + E_n^2 + M_i^2}{\omega^2 + E_{n0}^2 + M_i^2} + \int dx\left(\frac{1}{2}(\partial_x n)^2 + \frac{1}{2}\lambda n^2 - \frac{r}{2}(\lambda - m^2) - \frac{N}{8g}(\lambda^2 - m^4)\right). \tag{23}$$

The first term is the zero-mode contribution, the second term comes from the continuous spectrum and the last term is from the classical part of the action. To calculate the second term one should remember that the eigenvalues in the continuous spectra of homogeneous and inhomogeneous configurations are the same but eigenvalue densities are different. The difference as function of momentum k can be expressed in terms of phase shifts $\delta(k)$ of the eigenmodes,

$$\rho(k) = \frac{1}{\pi}\frac{\mathrm{d}\delta}{\mathrm{d}k} = -\frac{2m}{\pi(k^2+m^2)}.$$

Otherwise calculation is straightforward. The final expression for energy is

$$E = -\frac{Nm}{\pi}\left(1+\log\frac{\Lambda}{m} - \frac{\pi m^2}{3g}\right) = -\frac{Nm}{\pi}\left(1+\frac{2\pi m^2}{3g}\right). \tag{24}$$

Here the first term is the conformal anomaly contribution, the second is due to the renormalization of the coupling constant r, and the third is a contribution of the λ^2 term. In the last transformation the homogeneous gap equation was used; this expression is negative when $g > 0$.

For model (8) the calculation can be performed in similar way. The result is

$$E = -\frac{Nm}{\pi} - \frac{4m^3}{3g}N. \tag{25}$$

Clearly the energy is always negative.

V. ZERO MODES

While solving the gap equation (14) we explicitly excluded the zero mode from the summation. Otherwise, the presence of this mode invalidates the whole calculation due to the term with $E_n = 0$ in the denominator. To justify this exclusion, we argue that this zero mode describes the soliton rotations in the internal space of the theory. Therefore, the zero mode corresponds to the orientational moduli and can be treated by the collective-coordinates formalism.

Our argument is based on canonical quantization of the theory. In this section we work in the Minkowski space. We introduce a displacement operator connecting the homogeneous state with the one-soliton sector of the theory. We investigate how this operator commutes with generators of internal rotations and show that in the kink background zero-mode operators transform exactly into the rotation generators. To make the logic more clear, we first review how the standard kink in the $\lambda\psi^4$ theory translational module is described in this language. Our approach is very close to the series of works [23–25]. First we review their results for the kink. Then we apply similar reasoning to the sigma model.

A. Kink in $\lambda\phi^4$ model

In this section we review the analysis of the kink in model with Lagrangian

$$\mathcal{L} = \frac{1}{2}((\partial_t\phi)^2 - (\partial_x\phi)^2) + \frac{\lambda}{4}(\phi - v)^2(\phi+v)^2. \tag{26}$$

The corresponding normal-ordered Hamiltonian is

$$H = \int dx\mathcal{H}(x);$$
$$\mathcal{H}(x) = \frac{1}{2}:\pi(x)^2: + \frac{1}{2}:(\partial_x\phi(x))^2: + \frac{\lambda}{4}:(\phi^2(x)-v^2)^2, \tag{27}$$

and the canonical momentum is $\pi(x) = \partial_t\varphi(x)$. The theory has two possible (classical) ground states with $\phi = \pm v$, particle mass is $m = \sqrt{2\lambda}v$. The theory ground state with $\phi = -v$ is $|-\rangle$ and $\langle -|\phi|-\rangle = -v$.

Classical theory with action (26) enjoys the kink solution

$$f(x) = \frac{m}{\sqrt{2\lambda}}\tanh\left(\frac{mx}{2}\right). \tag{28}$$

To construct a Hilbert space state, which corresponds to the kink we introduce the shift operator

$$D_f = \exp\left(-i\int dx f_1(x)\pi(x)\right), \qquad f_1(x) = f(x) + v.$$

From the canonical commutation relation we obtain

$$[D_f, \phi(x)] = -f_1(x)D_f.$$

Therefore, for the state $|f\rangle = D_f|-\rangle$ we have

$$\langle -|D_f^\dagger\phi(x)D_f|-\rangle = f(x),$$

and this state has the correct form factor describing the kink. The true solitonic state can be obtained from $|f\rangle$ perturbatively by some operator O_1 constructed from the creation and annihilation operator,

$$|K\rangle = D_f O_1|-\rangle.$$

Next we expand the field and corresponding momenta in terms of the modes around the classical solution,

$$\phi(x) = g_B(x)\phi_0 + \int\frac{dk}{2\pi}\frac{B_k^\dagger + B_{-k}}{\sqrt{2\omega_k}}g_k(x); \tag{29}$$

$$\pi(x) = g_B\pi_0 + i\int \frac{dk}{2\pi}\sqrt{\frac{\omega_k}{2}}(B_k^\dagger - B_{-k})g_k. \qquad (30)$$

Here g_B is zero mode and g_k includes both the bound state and the continuous spectrum. The Hamiltonian up to second order in field fluctuation is

$$H = M_1 + \frac{\pi_0^2}{2} + \int \frac{dk}{2\pi}\omega_k B_k^\dagger B_k.$$

M_1 is the kink mass with a one-loop correction.

Operators $B_k^\dagger$ create excited states in the kink sector. Interpretation of the π_0 operator is less trivial. This operator should correspond to translational zero mode, which describes the motion of the kink as a whole. To provide this connection let us consider the physical momentum of the field operator

$$P = -\int dx\,\pi(x)\partial_x\phi(x).$$

In its rest frame the kink does not move and we should have

$$P|K\rangle = 0.$$

The transformed momentum operator is

$$D_f^\dagger P D_f = P' = -\int dx\,\pi(x)\partial_x(\phi(x) + f(x))$$
$$= P - \int dx\,\pi(x)\partial_x f(x).$$

The derivative of the kink profile connected with the zero mode is

$$\partial_x f(x) = \sqrt{M_0}g_B(x).$$

Integration in the formula for momentum separates the π_0 component [$\int dx\pi(x)g_B(x) = \pi_0$],

$$P' = P - \sqrt{M_0}\pi_0. \qquad (31)$$

The kink state is paramtrized as $|K\rangle = D_f O_1|-\rangle$; therefore we should have

$$P'O_1|-\rangle = 0, \qquad (P - \sqrt{M_0}\pi_0)O_1|-\rangle = 0.$$

Therefore, in the kink sector we have the relation

$$\pi_0 = \frac{P}{\sqrt{M_0}}, \qquad (32)$$

and the corresponding term in the Hamiltonian

$$\frac{\pi_0^2}{2} = \frac{P^2}{2M_0}$$

is nothing but the standard formula for kinetic energy in the nonrelativistic approximation. Further terms in perturbation theory should provide relativistic corrections to this expression. Thus, we have established the connection between the zero mode component of the canonical momentum and the motion of the kink.

B. Sigma model

Now we perform a similar calculation for the sigma model. For simplicity we consider case $g \to \infty$—the general case can be studied in the same way. The Hamiltonian of the model is

$$H = \int dx\frac{1}{2}((\pi_a(x))^2 + (\partial_x n_a(x))^2 + \lambda(n_a(x)^2 - r)). \qquad (33)$$

The ground state of this model in the canonical quantization formalism was described in [28].

Now we wish to consider an inhomogeneous state which resembles the kink state in the $\lambda\phi^4$ model. The fields in this state are

$$\lambda(x) = m^2\left(1 - \frac{2}{\cosh^2 mx}\right), \qquad n^a(x) = Al^a f_0.$$

Here A is a normalization factor that was determined from the gap equation previously, f_0 is the zero model, l^a is a unit vector that describes the soliton direction in the internal state. Note that these fields are not solutions of any classical equations of motion, but were determined from effective quantum action in previous sections. However, we can proceed in a manner similar to the construction of the kink quantum state.

Let us define the field shift operator as

$$D_f = \exp\left(-i\int dx\,\pi_a(x)l_a A f_0(x)\right). \qquad (34)$$

This operator transforms the vacuum state $|0\rangle$ to a state with the correct n form factor,

$$\langle 0|D_f^\dagger n_a(x)D_f|0\rangle = Al^a f_0.$$

Note that we introduce the appropriate value of the λ field by hand because it does not have an appropriate canonical momentum.

We can expand the fields and corresponding canonical momenta in the eigenfunctions of the inhomogeneous Hamiltonian. The expansion is similar to the case of the kink,

$$n(x)_a = f_0(x)n_{0a} + \int \frac{dk}{2\pi}\frac{B_k^{a\dagger} + B_{-k}^a}{\sqrt{2\omega_k}} f_k(x); \qquad (35)$$

$$\pi(x)_a = f_0\pi_0^a + i\int \frac{dk}{2\pi}\sqrt{\frac{\omega_k}{2}}(B_k^{a\dagger} - B_{-k}^a)f_k. \qquad (36)$$

Here $f_k(x)$ are continuum spectrum modes. The main difference from the previous model is the internal space index a. The Hamiltonian is again

$$H = E_1 + \frac{(\pi^a)^2}{2} + \int \frac{dk}{2\pi}\omega_k B_k^{a\dagger}B_k^a.$$

Here E_1 is quantum one-loop energy of the soliton, which is negative according to our previous calculation.

The theory has a $O(N)$ rotational symmetry. The corresponding generators are

$$J^{ab} = \int dx(\pi^a(x)\phi^b(x) - \pi^b(x)\phi^a(x)). \qquad (37)$$

In analogy with the kink consideration we can define shifted operators

$$J_1^{ab} = D_f^\dagger J^{ab} D_f = J^{ab} + \int dx(\pi^a(x)l^b - \pi^b(x)l^a)f(x). \qquad (38)$$

Therefore, integration yields the zero-mode components of the canonical momentum with some coefficient,

$$J_1^{ab} = J^{ab} + A(\pi_0^a l^b - \pi_0^b l^a).$$

Thus, we have a clear relation between the zero-mode components and the rotational operator. We can use the same argument as in the kink state to show that the soliton state should be annihilated by J_1^{ab} operator.

Therefore, we can express the soliton rotation operators as

$$J^{ab} = -A(\pi_0^a l^b - \pi_0^b l^a).$$

The square of this operator is

$$J^{ab}J^{ab} \equiv J^2 = 2A^2(\delta^{ab} - l^a l^b)\pi_0^a\pi_0^b.$$

Let us define additional operator

$$R = l^a\pi_0^a.$$

Therefore, the momentum squared is

$$(\pi_0^a)^2 = \frac{J^2}{2A^2} + R^2$$

and the π_0^2 term in the Hamiltonian is expressed as a sum of rotational energy and an additional term, R^2. Therefore, we established a connection between rotational energy and the zero-mode operators.

VI. FINITE TEMPERATURE

In this section we discuss the model with the action (2) at finite temperature. For simplicity, we restrict the analysis only to the case of large g; namely, we consider only the case of the nonlinear sigma model. The effective action can be calculated in the same way as in zero-temperature case. The only difference is that the trace should be taken over periodic fields with period $\beta = 1/T$ in Euclidean time. Therefore, instead of integration over all frequencies we calculate the sum over the Matsubara frequencies $\Omega_n = 2\pi Tn$.

First, we consider homogeneous saddle points of the action. The gap equation yields

$$NT\int \frac{dk}{2\pi}\sum_{n\in\mathbb{Z}}\frac{1}{k^2 + m^2 + \Omega_n^2} - r = 0, \qquad (39)$$

or after the summation over the frequencies

$$\frac{N}{4\pi}\int_0^\infty \mathrm{d}k\left(\frac{\coth\left(\frac{1}{2T}\sqrt{k^2+m^2}\right)}{\sqrt{k^2+m^2}} - \frac{1}{\sqrt{k^2+\Lambda^2}}\right) = 0. \qquad (40)$$

The second term here is the integral representation of the coupling constant r. From this equation we can determine mass as a function of temperature. In the low-temperature limit $T \ll \Lambda$ we have $m \approx \Lambda$, and for $T \gg \Lambda$ the solution is

$$m = \frac{\pi T}{\log(\kappa T/\Lambda)}, \qquad \kappa \approx 7.08. \qquad (41)$$

Now consider the soliton solution (12). From the gap equation we obtain

$$n^2 = \frac{N}{4\pi}\int \frac{m^2\mathrm{d}k}{(k^2+m^2)^{3/2}}\coth\left(\frac{\sqrt{k^2+m^2}}{2T}\right)\frac{1}{\cosh^2 mx} = \frac{NA}{\cosh^2 mx}. \qquad (42)$$

The last equation is a definition of the amplitude A of the condensate. At high temperatures we have

$$A \approx \frac{T}{4m}\left(1 + \frac{m^2}{6T^2}\right). \qquad (43)$$

Now we turn to the calculation of the free energy of the soliton. The free energy is connected with the regularized action (21) as $E = TS_{\mathrm{reg}}$. The only subtlety of the calculation is that we have to take care of the zero mode in the determinant term of the effective action. If we treat the zero mode in the Gaussian approximation as all other modes, it will lead to an infrared divergence. This is because the n field in solution (12) breaks the internal $O(N)$ symmetry in the model which results in orientational zero modes. As is typical for calculations with solitons, we have to integrate over moduli space rather than use Gaussian approximation.

The moduli space of the soliton is the $(N-1)$-dimensional sphere S^{N-1} (the translational mode does not contribute to the effective action in the leading order of the $1/N$ expansion). Thus moduli space dynamics can be represented by a time-dependent unit vector $l^a(t)$ and the soliton configuration is

$$n^a(x,t) = n(x) l^a(t).$$

Here $n(x)$ is a solitonic solution. The orresponding effective action for l^a is

$$S_1 = \frac{1}{2}\int \mathrm{d}x\, n^2(x) \int \mathrm{d}t\, \dot{l}^a(t)^2 = \frac{M}{2}\int \mathrm{d}t\, \dot{l}^a(t)^2; \quad M = \frac{2NA}{m}. \tag{44}$$

This action is formally the same as for a nonrelativistic particle of mass M on a $(N-1)$-dimensional sphere with a unit radius. For this system the separation of energy levels is of order $1/M \sim m/N$. Therefore, the partition function can be calculated classically if the temperature is not too small ($T \gg m/N$). This assumption is reasonable in the large N limit. The classical partition function is

$$Z_1 = \frac{1}{(2\pi)^{(N-1)/2}} S^{N-1} (2\pi M T)^{(N-1)/2} \approx \left(\frac{2eAT}{m}\right)^{N/2}. \tag{45}$$

Here S^{N-1} is the area of the sphere. Thus, the zero-mode contribution to the free energy is $-T \log Z_1$.

After a straightforward computation we find the free energy of the soliton

$$\begin{aligned} E = &-\frac{Nm}{\pi} - \frac{Nm}{\pi}\log\frac{\Lambda}{m} - \frac{2NT}{\pi}\int_0^\infty \frac{m dk}{m^2+k^2} \\ &\times \log\left(1-\exp\left(-\frac{\sqrt{k^2+m^2}}{T}\right)\right) \\ &-\frac{NT}{2}\log\frac{2eAT}{m} \approx mN\left(\frac{1-\log 2}{2\pi}\log\frac{T}{\Lambda}\right. \\ &\left. + C - \frac{1}{4\log(\kappa T/\Lambda)}\right); \qquad C \approx 0.0945. \end{aligned} \tag{46}$$

Here, the first term is due to the zero-temperature fluctuations, the second term comes from the coupling-constant renormalization, and the third term is the contribution of thermal excitations. The last term comes from the zero mode.

The free energy is negative at small temperatures and increases with T. On the other hand, in the high-temperature limit, free energy becomes positive due to the thermal excitations. At the point $T \approx 1.044\Lambda$, which can be found numerically, the energy changes sign. This observation suggests that the model might undergo a phase transition of some kind.

VII. CONCLUSIONS

To sum up, we constructed inhomogeneous solutions in the linear sigma model. Similar to [19] we find that its energy is negative. These solutions are similar to those discussed in [22]. However, there are some important differences. First, in [22] only the dynamics of the λ field was considered and the symmetry-breaking scalar condensate n was not introduced. Therefore, our saddle-point equations are rather different from [22]. Second, in [22] the anomalous contribution to the energy density was not taken into account. Therefore, the energy is found to be positive and the solitons were interpreted as some excited (metastable) states of the theory.

The interpretation of zero modes was discussed. We have provided evidence that they are nothing but soliton rotations. In our solution the n^a classical field explicitly breaks $O(N)$ symmetry. Therefore, there is a family of solutions that transform into each other by the $O(N)$ rotations. These solutions can be parametrized with the unit vector l^a. Instead of Gaussian integration over zero-frequency fluctuations, we should integrate over all directions of l^a and obtain finite results.

The nonlinear sigma model at finite temperature was also discussed. We found that the soliton is present at all temperatures, but its energy becomes positive when the temperature is high enough. This might indicate a phase transition. The role of the solutions at zero temperature is still unclear. Negative energy suggests that they cannot be interpreted as an excited state of the theory. Instead, they might imply some nontrivial phase structure of the theory. We postpone these questions to future work.

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