


# Nonequatorial scalar rings supported by magnetized Schwarzschild-Melvin black holes

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 (Received 2 April 2022; accepted 11 April 2022; published 26 April 2022)

It has recently been demonstrated that magnetized black holes in composed Einstein-Maxwell-scalar-Gauss-Bonnet field theories with a nonminimal negative coupling of the scalar field to the Gauss-Bonnet curvature invariant may support spatially regular scalar hairy configurations. In particular, it has been revealed that, for Schwarzschild-Melvin black-hole spacetimes, the onset of the near-horizon spontaneous scalarization phenomenon is marked by the numerically computed dimensionless critical relation  $(BM)_{\text{crit}} \simeq 0.971$ , where  $\{M, B\}$  are respectively the mass and the magnetic field of the spacetime. In the present paper we prove, using analytical techniques, that the boundary between bald Schwarzschild-Melvin black-hole spacetimes and hairy (scalarized) black-hole solutions of the composed Einstein-Maxwell-scalar-Gauss-Bonnet theory is characterized by the exact dimensionless relation  $(BM)_{\text{crit}} = \sqrt{\frac{\sqrt{6}-2}{2\sqrt{6}}} + \sqrt{\frac{\sqrt{6}-1}{2}}$  for the critical magnetic strength. Intriguingly, we prove that the critical dimensionless magnetic parameter  $(BM)_{\text{crit}}$  corresponds to magnetized black holes that support a pair of linearized nonminimally coupled thin scalar rings that are characterized by the nonequatorial polar angular relation  $(\sin^2 \theta)_{\text{scalar-ring}} = \frac{690-72\sqrt{6}+4\sqrt{3258\sqrt{6}-7158}}{789} < 1$ . It is also proved that the classically allowed angular region for the negative-coupling near-horizon spontaneous scalarization phenomenon of magnetized Schwarzschild-Melvin spacetimes is restricted to the black-hole poles,  $\sin^2 \theta_{\text{scalar}} \rightarrow 0$ , in the asymptotic large-strength magnetic regime  $BM \gg 1$ .

DOI: [10.1103/PhysRevD.105.084056](https://doi.org/10.1103/PhysRevD.105.084056)

## I. INTRODUCTION

The celebrated no-hair conjecture in black-hole physics [1,2] has asserted that static scalar field configurations cannot be supported in black-hole spacetimes that contain spatially regular absorbing horizons. Early mathematical investigations of the Einstein-matter field equations [3–8] have revealed, in accord with the spirit of this influential conjecture, that black holes cannot support scalar fields which are minimally coupled to the Ricci scalar of the curved spacetime. Similar conclusions have been obtained for scalar fields with a nontrivial (nonminimal) coupling to the Ricci curvature scalar [7,8].

However, recent mathematical studies [9–22] of the coupled Einstein-matter field equations have revealed the physically intriguing fact that black holes with spatially regular horizons can support hairy matter configurations which are made of scalar fields with a direct nonminimal coupling to the Gauss-Bonnet invariant  $\mathcal{G}$  of the curved black-hole spacetime [23–29].

The critical boundary between bald black holes and hairy (scalarized) black-hole spacetimes in Einstein-Gauss-Bonnet

field theories is marked by the presence of “cloudy” configurations, linearized nonminimally coupled scalar fields which are supported by the familiar black-hole solutions of the Einstein field equations. In particular, depending on the matter content of the theory and the assumed symmetry of the spacetime, cloudy nonminimally coupled scalar field configurations have been studied in Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman black-hole spacetimes [9–22].

The spontaneous scalarization phenomenon of black holes in Einstein-Gauss-Bonnet field theories is closely related to the fact that the Klein-Gordon wave equation of the nonminimally coupled scalar field  $\phi$  contains an effective spatially dependent mass term. This mass term, which has the compact linearized form  $-\eta\phi\mathcal{G}$  [see Eq. (10) below] [30], may become *negative* in the vicinity of the black-hole horizon, implying that the effective potential of the composed black-hole-scalar-field system may behave as an attractive potential well that binds the scalar field to the near-horizon region of the central black hole.

Intriguingly, it has recently been demonstrated in the physically important work [31] that, in Einstein-Gauss-Bonnet

field theories, the phenomenon of black-hole spontaneous scalarization can be triggered by the presence of magnetic fields. In particular, it has been demonstrated numerically in [31] that, for Schwarzschild-Melvin black-hole spacetimes, the spontaneous scalarization phenomenon is magnetically-induced in the sense that only black holes in the dimensionless strong magnetic regime

$$BM > (BM)_{\text{crit}} \simeq 0.971 \quad (1)$$

can support scalar field configurations with a nonminimal *negative* coupling to the Gauss-Bonnet invariant of the curved magnetized spacetime (Here  $M$  and  $B$  are respectively the mass and the magnetic field of the black-hole spacetime).

The main goal of the present compact paper is to explore, using analytical techniques, the onset of the spontaneous scalarization phenomenon in magnetized Schwarzschild-Melvin black-hole spacetimes of the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory with negative values of the nonminimal coupling parameter  $\eta$ .

In particular, below we shall derive a remarkably compact analytical expression for the critical magnetic strength  $(BM)_{\text{crit}}$  which marks the boundary between scalarless Schwarzschild-Melvin black holes and scalarized (hairy) magnetized black-hole configurations. In addition, we shall reveal the physically intriguing fact that the black-hole solutions of the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory can support pairs of infinitesimally thin static scalar rings which are located above and below the equator of the central supporting black hole.

## II. DESCRIPTION OF THE SYSTEM

We shall study, using analytical techniques, the onset of the negative-coupling spontaneous scalarization phenomenon of magnetized Schwarzschild-Melvin black holes in the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory whose action is given by the integral expression [31]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + f(\phi) \mathcal{G} \right], \quad (2)$$

for the magnetically-dependent Gauss-Bonnet invariant of the Schwarzschild-Melvin black hole.

The action (2) of the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory yields the characteristic Klein-Gordon differential equation [31]

where  $F_{\mu\nu}$  is the Maxwell electromagnetic tensor of the spacetime. The term  $f(\phi)\mathcal{G}$ , which is essential for the existence of spontaneously scalarized black-hole solutions, represents the direct (nonminimal) coupling between the massless scalar field  $\phi$  and the Gauss-Bonnet invariant

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \quad (3)$$

of the black-hole spacetime.

The magnetized Schwarzschild-Melvin black-hole solution of the composed field theory (2) is characterized by the curved line element [31–35]

$$ds^2 = \Lambda^2 [-f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\theta^2] + \Lambda^{-2} r^2 \sin^2 \theta d\varphi^2, \quad (4)$$

where

$$\Lambda = \Lambda(r, \theta; B) = 1 + \frac{1}{4} B^2 r^2 \sin^2 \theta. \quad (5)$$

The metric function

$$f(r) = 1 - \frac{2M}{r}, \quad (6)$$

whose simple root

$$r_{\text{H}} = 2M \quad (7)$$

determines the radius of the black-hole horizon, is expressed in terms of the mass  $M$  of the spacetime.

From Eqs. (3), (4), (5), and (6) one obtains the (rather cumbersome) near-horizon dimensionless functional expression [31,36]

$$M^4 \mathcal{G}(r \rightarrow r_{\text{H}}^+, \theta; M, B) = [4(1 + B^2 M^2 \sin^2 \theta)^8]^{-1} \times \{ 3[B^8 M^8 \sin^8 \theta - 2B^4 M^4 \sin^4 \theta + [1 - B^4 M^4 \sin^2(2\theta)]^2] \\ + 24\cos^2 \theta (B^8 M^8 \sin^6 \theta - B^6 M^6 \sin^4 \theta + B^2 M^2) + 16B^4 M^4 \cos^4 \theta (1 - 6B^2 M^2 \sin^2 \theta) \} \quad (8)$$

$$\nabla^\nu \nabla_\nu \phi = \mu_{\text{eff}}^2 \phi \quad (9)$$

for the scalar field, where the presence of the magnetically-dependent effective mass term in (9),

$$\mu_{\text{eff}}^2(\theta; M, B) = -\eta \cdot \mathcal{G}(\theta; M, B), \quad (10)$$

is a direct outcome of the nonminimal coupling between the scalar field of the theory and the Gauss-Bonnet invariant of the spacetime. The physical parameter  $\eta$  in (10) plays the role of an expansion coefficient in the weak-field functional behavior [31]

$$f(\phi) = \frac{1}{2}\eta\phi^2 \quad (11)$$

of the scalar coupling function. This physical parameter [37] determines the strength of the direct nonminimal coupling between the massless scalar field and the Gauss-Bonnet invariant (8) of the magnetized Schwarzschild-Melvin black hole.

### III. ONSET OF NEGATIVE-COUPLING NEAR-HORIZON SPONTANEOUS SCALARIZATION IN MAGNETIZED SCHWARZSCHILD-MELVIN BLACK-HOLE SPACETIMES

In the present section we shall study the onset of the negative-coupling spontaneous scalarization phenomenon in the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory (2). In particular, we shall reveal the interesting fact that, at the onset of the magnetically-induced spontaneous scalarization phenomenon, Schwarzschild-Melvin black-hole spacetimes can support nonequatorial thin rings which are made of massless scalar fields with a negative coupling to the Gauss-Bonnet invariant of the magnetized spacetime.

The presence of an effective *negative* mass term (a binding potential well) in the Klein-Gordon wave equation (9) of the nonminimally coupled scalar field provides a necessary condition for the existence of spatially regular supported field configurations (bound-state scalar clouds) in the black-hole spacetime [11,13,17,18,22]. Intriguingly, from Eq. (8) one learns that, depending on the values of the dimensionless magnetic parameter  $BM$  of the spacetime and the polar angle  $\theta$ , the effective mass term (10) in the

Klein-Gordon differential equation (9) may become negative in the vicinity of the horizon of the Schwarzschild-Melvin black hole.

In particular, the *onset* of the near-horizon spontaneous scalarization phenomenon in the magnetized black-hole spacetime (4) is marked by the critical functional relation [17,22,38]

$$\min\{\mu_{\text{eff}}^2(\theta; M, B)\} \rightarrow 0^-. \quad (12)$$

For negative values of the nonminimal coupling parameter  $\eta$  of the composed Einstein-Maxwell-scalar-Gauss-Bonnet field theory (2), the characteristic relation (12) yields the critical functional relation

$$\min\{G(\theta; M, B)\} \rightarrow 0^- \quad (13)$$

at the onset of the spontaneous scalarization phenomenon.

Interestingly, and most importantly for our analysis, as we shall now show explicitly, the set of coupled equations [see Eq. (13)]

$$G(\theta; M, B) = 0 \quad (14)$$

with

$$\frac{dG(\theta; M, B)}{d\theta} = 0, \quad (15)$$

which determine the onset of the near-horizon spontaneous scalarization phenomenon of the magnetized black-hole spacetime (4), can be solved *analytically*.

To see this, it is convenient to define the dimensionless variables

$$\beta \equiv (BM)^2; \quad x \equiv \sin^2\theta, \quad (16)$$

in terms of which the near-horizon Gauss-Bonnet invariant (8) can be written in the dimensionless form [39]

$$M^4\mathcal{G}(r \rightarrow r_{\text{H}}^+, x; \beta) = [4(1 + \beta x)^8]^{-1} \times \{3[\beta^4 x^4 - 2\beta^2 x^2 + [1 - 4\beta^2 x(1 - x)]^2] + 24(1 - x)(\beta^4 x^3 - \beta^3 x^2 + \beta) + 16\beta^2(1 - x)^2(1 - 6\beta x)\}. \quad (17)$$

Substituting (17) into Eqs. (14) and (15), one obtains the coupled equations

$$(16\beta^2 + 24\beta + 3) - 8\beta(12\beta^2 + 7\beta + 3)x + 2\beta^2(24\beta^2 + 84\beta + 17)x^2 - 72\beta^3(\beta + 1)x^3 + 27\beta^4 x^4 = 0 \quad (18)$$

and [40]

$$-2(12\beta^2 + 7\beta + 3) + \beta(24\beta^2 + 84\beta + 17)x - 54\beta^2(\beta + 1)x^2 + 27\beta^3 x^3 = 0 \quad (19)$$

for the critical dimensionless variables  $\{x_{\text{crit}}, \beta_{\text{crit}}\}$ .

Interestingly, the coupled polynomial equations (18) and (19), which determine the onset of the near-horizon spontaneous scalarization phenomenon of the magnetized Schwarzschild-Melvin black hole (4), can be solved

analytically to yield the remarkably compact critical solution [41]

$$\beta_{\text{crit}} = \frac{1}{2} - \frac{1}{\sqrt{6}} + \sqrt{\frac{\sqrt{6}-1}{2}} \quad (20)$$

with

$$x_{\text{crit}} = \frac{690 - 72\sqrt{6} + 4\sqrt{3258\sqrt{6} - 7158}}{789}. \quad (21)$$

From the analytically derived expression (21) one deduces that the onset of the spontaneous scalarization phenomenon at the critical magnetic strength (20) is characterized by the presence of a pair of infinitesimally thin nonequatorial scalar rings which are supported by the magnetized black hole at the unique polar angles  $\theta_{\text{ring}}^- = 63.177^\circ$  and  $\theta_{\text{ring}}^+ = 116.823^\circ$  [see Eqs. (16) and (21)] [42].

#### IV. THE CLASSICALLY ALLOWED POLAR ANGULAR REGION FOR THE SPONTANEOUS SCALARIZATION PHENOMENON OF MAGNETIZED SCHWARZSCHILD-MELVIN BLACK HOLES

In the present section we shall reveal the physically interesting fact that the negative-coupling near-horizon spontaneous scalarization phenomenon of Schwarzschild-Melvin black holes is characterized by two critical values of the dimensionless magnetic strength parameter  $BM$ . These critical magnetic strengths mark the boundaries between three qualitatively different spatial behaviors of the composed magnetized-black-hole-nonminimally-coupled-scalar-field configurations.

We shall now discuss the three qualitatively different regimes of the magnetic parameter  $BM$ :

- (1) Case I: In the dimensionless weak magnetic regime

$$\beta < \beta_{\text{crit}}^- \equiv \frac{1}{2} - \frac{1}{\sqrt{6}} + \sqrt{\frac{\sqrt{6}-1}{2}}, \quad (22)$$

the near-horizon Gauss-Bonnet invariant (8) of the Schwarzschild-Melvin black hole is positive definite in the entire polar angular range  $\theta \in [0, \pi]$ . Thus, Schwarzschild-Melvin black holes in the dimensionless magnetic regime (22) cannot support negatively coupled massless scalar fields in their near-horizon region.

- (2) Case II: In the intermediate-strength magnetic regime

$$\frac{1}{2} - \frac{1}{\sqrt{6}} + \sqrt{\frac{\sqrt{6}-1}{2}} \equiv \beta_{\text{crit}}^- \leq \beta \leq \beta_{\text{crit}}^+ \equiv 1, \quad (23)$$

the near-horizon Gauss-Bonnet invariant (8) of the Schwarzschild-Melvin black hole becomes negative in the polar angular range

$$\frac{1}{3} \left[ 2 + \frac{2 - 2\sqrt{\frac{2}{3}}}{\beta} - \frac{\sqrt{12 - \frac{12-4\sqrt{6}}{\beta} - \frac{8\sqrt{6}-11}{\beta^2}}}{\sqrt{3}} \right] \leq \sin^2\theta \leq \frac{1}{3} \left[ 2 + \frac{2 - 2\sqrt{\frac{2}{3}}}{\beta} + \frac{\sqrt{12 - \frac{12-4\sqrt{6}}{\beta} - \frac{8\sqrt{6}-11}{\beta^2}}}{\sqrt{3}} \right]. \quad (24)$$

The polar range (24) defines, in the dimensionless magnetic regime (23), the classically allowed angular region for the spontaneous scalarization phenomenon of negatively-coupled scalar fields in the near-horizon region of the magnetized Schwarzschild-Melvin black hole.

Interestingly, from the analytically derived relation (24) one learns that the classically allowed angular width for the near-horizon spontaneous scalarization phenomenon is infinitesimally thin in the near-critical magnetic regime  $\beta/\beta_{\text{crit}}^- \rightarrow 1^+$ . In particular, defining the near-critical relation

$$\beta = \beta_{\text{crit}}^- \cdot (1 + \epsilon), \quad 0 \leq \epsilon \ll 1, \quad (25)$$

one finds from (24) the nontrivial (nonlinear) critical functional behavior

$$\Delta(\sin^2\theta) = \frac{\sqrt{\frac{32\alpha}{3}(3\sqrt{2} - 2\sqrt{3} + 6\alpha)}}{2 + 3\sqrt{2}\alpha - \alpha^2} \cdot \sqrt{\epsilon}, \quad (26)$$

$$\alpha \equiv \sqrt{\sqrt{6}-1}$$

for the infinitesimally thin classically allowed angular interval.

The classically allowed angular region (24) for the near-horizon spontaneous scalarization phenomenon of the magnetized Schwarzschild-Melvin black holes in the intermediate magnetic regime (23) is a monotonically increasing function of the magnetic strength parameter  $\beta$ . In particular, it is characterized by the limiting property

$$\frac{15 - 4\sqrt{6}}{9} \leq \sin^2\theta \leq 1 \quad \text{for } \beta = \beta_{\text{crit}}^+ \quad (27)$$

for the critical magnetic strength  $\beta = \beta_{\text{crit}}^+$ .

It is interesting to point out that the black-hole configuration with the critical magnetic strength  $\beta_{\text{crit}}^+ = 1$  is unique in the sense that the classically

allowed polar angular region for the negative-coupling near-horizon spontaneous scalarization phenomenon extends in this case all the way to the equator of the magnetized black hole.

- (3) Case III: In the dimensionless strong magnetic regime

$$1 \equiv \beta_{\text{crit}}^+ < \beta, \quad (28)$$

the near-horizon Gauss-Bonnet invariant (8) of the Schwarzschild-Melvin black holes becomes negative in the polar angular range

$$\frac{1}{3} \left[ 2 + \frac{2 - 2\sqrt{\frac{2}{3}}}{\beta} - \frac{\sqrt{12 - \frac{12 - 4\sqrt{6}}{\beta} - \frac{8\sqrt{6} - 11}{\beta^2}}}{\sqrt{3}} \right] \leq \sin^2\theta \leq \frac{1}{3} \left[ 2 + \frac{2 + 2\sqrt{\frac{2}{3}}}{\beta} - \frac{\sqrt{12 - \frac{12 + 4\sqrt{6}}{\beta} + \frac{8\sqrt{6} + 11}{\beta^2}}}{\sqrt{3}} \right]. \quad (29)$$

The polar range (29) defines the classically allowed angular region for the negative-coupling near-horizon spontaneous scalarization phenomenon of strongly magnetized [see Eq. (28)] Schwarzschild-Melvin black holes.

Interestingly, one finds that, in the strong magnetic regime (28), the two polar boundaries of the classically allowed region (29) are monotonically decreasing functions of the dimensionless magnetic parameter  $\beta$ . In particular, one finds that the classically allowed angular interval (29) gradually shrinks to the infinitesimally thin polar region [43]

$$\frac{1 - \sqrt{\frac{2}{3}}}{\beta} \leq \sin^2\theta \leq \frac{1 + \sqrt{\frac{2}{3}}}{\beta} \quad \text{for } \beta \rightarrow \infty \quad (30)$$

in the asymptotically large magnetic regime. One therefore concludes that the classically allowed polar region for the negative-coupling near-horizon spontaneous scalarization phenomenon of the magnetized Schwarzschild-Melvin spacetime is restricted to the vicinity of the black-hole poles in the asymptotic large-strength magnetic regime  $\beta \gg 1$ .

## V. SUMMARY AND DISCUSSION

It has recently been shown [31] that magnetized black holes may support nonminimally coupled scalar hairy configurations. In particular, the recently published important work [31] has revealed the physically interesting fact that, in composed Einstein-Maxwell-scalar-Gauss-Bonnet field theories with negative values of the nonminimal

coupling parameter  $\eta$ , there exists a critical magnetic strength [see Eq. (1)],

$$(BM)_{\text{crit}} \simeq 0.971, \quad (31)$$

above which Schwarzschild-Melvin black holes can support near-horizon massless scalar field configurations with a nonminimal direct coupling to the magnetically-dependent Gauss-Bonnet curvature invariant (8).

In the present paper we have studied, using *analytical* techniques, the onset of the negative-coupling near-horizon spontaneous scalarization phenomenon in magnetized Schwarzschild-Melvin black-hole spacetimes. The main results derived in this paper and their physical implications are as follows:

- (1) We have proved that the critical black-hole magnetic strength  $(BM)_{\text{crit}}$ , which marks the boundary between bald Schwarzschild-Melvin black holes and hairy (scalarized) black holes in the Einstein-Maxwell-scalar-Gauss-Bonnet field theory (2) with a negative nonminimal Gauss-Bonnet-scalar-field coupling, is given by the compact dimensionless analytical expression [see Eqs. (16) and (20)]

$$(BM)_{\text{crit}} = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{6}}} + \sqrt{\frac{\sqrt{6} - 1}{2}}. \quad (32)$$

It is worth emphasizing the fact that the analytically derived critical black-hole magnetic strength (32) agrees remarkably well with the corresponding numerical value (31) of the critical magnetic field as originally presented in the physically interesting work [31].

- (2) We have revealed the fact that *nonequatorial* thin scalar rings can be supported in magnetized black-hole spacetimes. In particular, one finds that, in the dimensionless critical limit

$$BM \rightarrow [(BM)_{\text{crit}}]^+, \quad (33)$$

the effective mass term (10) of the composed Schwarzschild-Melvin-black-hole-nonminimally-coupled-massless-scalar-field system becomes *negative* in a pair of narrow nonequatorial polar rings which are characterized by the analytically derived angular relation [see Eqs. (16), (21), (25) and (26)]

$$\sin^2\theta_{\text{ring}} = \frac{18 + 8(3\sqrt{2} - 2\sqrt{3})\alpha}{(2 + 3\sqrt{2}\alpha - \alpha^2)^2}, \quad \alpha \equiv \sqrt{\sqrt{6} - 1}. \quad (34)$$

This physically interesting property of the Schwarzschild-Melvin curved spacetime implies

that magnetized black holes can support, in the critical limit (33), cloudy configurations of the nonminimally coupled massless scalar fields in the form of two infinitesimally thin nonequatorial scalar rings which are characterized by the polar angles [see Eq. (34)]

$$\theta_{\text{ring}}^- = 63.177^\circ; \quad \theta_{\text{ring}}^+ = 116.823^\circ. \quad (35)$$

- (3) It has been revealed that the Schwarzschild-Melvin black hole with the critical magnetic strength  $BM = 1$  is unique in the sense that the near-horizon angular region for which the Gauss-Bonnet invariant becomes nonpositive (thus allowing for the onset of the negative-coupling spontaneous scalarization phenomenon) extends in this case all the way to the equator of the magnetized black hole [see Eq. (24)].
- (4) We have revealed the fact that, for strong magnetic fields in the  $BM \gg 1$  regime, the classically allowed angular region for the negative-coupling spontaneous scalarization phenomenon of magnetized

Schwarzschild-Melvin spacetimes gradually shrinks as the value of the dimensionless magnetic strength  $BM$  increases. In particular, we have proved that the near-horizon spontaneous scalarization phenomenon is restricted to the narrow near-polar angular interval [see Eqs. (16) and (30)] [44]

$$\frac{\sqrt{1 - \sqrt{\frac{2}{3}}}}{BM} \leq \theta_{\text{scalar}} \leq \frac{\sqrt{1 + \sqrt{\frac{2}{3}}}}{BM} \quad \text{for } BM \gg 1 \quad (36)$$

in the asymptotic large-strength magnetic regime  $BM \gg 1$ .

### ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

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- [34] Here  $(t, r, \theta, \phi)$  are the familiar Schwarzschild spacetime coordinates.
- [35] We shall use natural units in which  $G = c = 1$ .
- [36] Due to the asymptotic properties of the Melvin Universe [31–33] it cannot describe the full spacetime of astrophysically realistic black holes. However, as nicely emphasized in [31], the magnetized Schwarzschild-Melvin solution is expected to capture, approximately, the essential physical properties of the composed black-hole-scalar-field system in the vicinity of the black-hole horizon.
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- [39] Here we have used the identities  $\sin^2(2\theta) = 4\sin^2\theta\cos^2\theta = 4\sin^2\theta(1 - \sin^2\theta)$ .
- [40] Note that we have used the relation (18) in Eq. (19).
- [41] It is important to point out that the numerical value of Eq. (20) is approximately 0.943 and its square root is approximately 0.971, thus in excellent agreement with the numerical value (1) obtained in [31].
- [42] Note that, due to the azimuthal symmetry of the magnetized Schwarzschild-Melvin black-hole spacetime [see the curved line element (4)], the critical angular relation (21) describes two nonequatorial scalar rings with  $\phi \in [0, 2\pi]$ .
- [43] It is worth noting that, as nicely emphasized in [31], the horizon geometry of a magnetized Schwarzschild-Melvin black hole is deformed in the same spatial direction as the external magnetic field of the spacetime, and it therefore has the curved form of a prolate spheroid.
- [44] The second classically allowed region for the near-horizon spontaneous scalarization phenomenon in the large-magnetic  $BM \gg 1$  regime is characterized by the narrow near-polar angular interval  $\frac{\sqrt{1-\sqrt{\frac{5}{3}}}}{BM} \leq \pi - \theta_{\text{scalar}} \leq \frac{\sqrt{1+\sqrt{\frac{5}{3}}}}{BM}$ .