Analytic meronic black holes, gravitating solitons, and higher-spins in the Einstein $SU(N)$-Yang-Mills theory

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We construct meronic black holes and solitons in the Einstein $SU(N)$-Yang-Mills theory in $D = 4$ and $D = 5$ dimensions. These analytical solutions are found by combining the generalized hedgehog ansatz with the Euler parametrization of the $SU(N)$ group from which the Yang-Mills equations are automatically satisfied for all values of N while the Einstein equations can be solved analytically. We explicitly show the role that the color number N plays in the black hole thermodynamics as well as in the gravitational spin from isospin effect. Two remarkable results of our analysis are that, first, meronic black holes can be distinguished by colored black holes by looking at the spin from isospin effect (which is absent in the latter but present in the former). Second, using the theory of nonembedded ansatz for $SU(N)$ together with the spin from isospin effect, one can build fields of arbitrary high spin out of scalar fields charged under the gauge group. Hence, one can analyze interacting higher spin fields in asymptotically flat space-times without "introducing by hand" higher spin fields. Our analysis also discloses an interesting difference between the spin from isospin effect in $D = 4$ and in $D = 5$.

I. INTRODUCTION

Yang-Mills (YM) theory is one of the main ingredients of the standard model which up to now has been phenomenologically extremely successful. Since the main open problems in high energy physics such as color confinement are nonperturbative in nature, it is of great interest to analyze topologically nontrivial configurations of the YM theory which are believed to play a fundamental role in the nonperturbative phase of the theory (see [1–14] and references therein).

A very interesting class of configurations that play an important role in the nonperturbative phase of the YM theory are the so-called merons[1] introduced in [15]. One of the characteristics of merons is that they can always be brought in the form $A = \lambda \tilde{A}$, where $\tilde{A}$ is a pure gauge field. Since such an ansatz would be trivial in Abelian gauge theories, merons are genuine non-Abelian configurations. It is known that merons connect different topological sectors

of the theory and these are related to instantons [16–19]. Also, lattice studies show that, as far as confinement is concerned, merons play a very important role, as can be seen in [16–18]. The existence of merons can be traced back to the appearance of Gribov copies [20] as merons can be interpreted as tunneling events between different Gribov vacua [21].

However, most of the studies of merons up to now (with the exception of [22]) have been devoted to the $SU(2)$ symmetry group case. In the present case we will focus on the $SU(N)$-YM theory (for arbitrary values of N) minimally coupled to general relativity (GR). We will be interested in genuine $SU(N)$ configurations: namely, configurations that are not trivial embedding of $SU(2)$ into $SU(N)$. This technical detail will be especially relevant in the analysis of the physical effects of nonembedded gravitating merons.[2]

The great importance to carefully analyze the coupling of GR with YM theory arises (at the very least) from two considerations. First of all, there are situations of high

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[1]Although the name "meron" is generally used to describe Euclidean solutions, in this work we will call merons to configurations with $\lambda = 1/2$ in Lorentzian space-time, which we will show in the following sections.

[2]Here it is worth to emphasize that the term "nonembedded," which will be adopted here, is very common in the literature on the Skyrme model after the pioneering papers [23,24], where the authors constructed the first numerical examples of genuine $SU(3)$ configurations in the Skyrme model [which are not trivial embeddings of $SU(2)$ solutions into $SU(3)$].

physical interest (such as close to black holes and neutron stars or in cosmology) in which the coupling of YM theory with GR cannot be neglected. Moreover, the coupling of topologically nontrivial configurations in YM theory with GR can be even useful to regularize them. For instance, merons, which on flat space-times are singular, when coupled to GR can become regular (see, for instance, [25–28] and references therein).

Many of the results in Einstein-YM are numerical [29–33], and these solutions have been derived in the case of the $SU(2)$ gauge group. In the Einstein $SU(2)$-YM system rigorous results are also known [34] [in-depth analysis of the $SU(N)$ case can be found in Refs. [35–37]).

In the present paper we will construct explicit analytic examples of nonembedded gravitating merons in the Einstein $SU(N)$-YM theory for arbitrary values of N. However, the main result of the paper is not the construction of the analytic solutions in itself[3] but rather the nontrivial physical effects which can be made manifest only with a careful group-theoretical analysis. The solutions that we will construct below disclose peculiar characteristics of the $SU(N)$ gauge group [which are absent in the $SU(2)$ case] as well as the quite nontrivial differences between the cases in $D = 4$ and $D = 5$ dimensions. One of the interesting features will arise from the analysis of the *spin-from-isospin effect* [38–40], comparing the new configurations with $N > 2$ with the usual $N = 2$ case.

A similar question about "genuine $SU(N)$ configurations with $N \geq 3$" in the low energy limit of QCD (which is described by the Skyrme model [41]) was answered in the seminal works [23,24], and recently in [42–44]. In Refs. [23,24], the first *numerical* example of a *non-embedded* solution representing a *dibaryon* (a bound state of two baryons) was constructed in the $SU(3)$-Skyrme model [this numerical construction of nonembedded configurations was extended to the $SU(N)$-Skyrme model in [45]]. Time after, in [43], combining the Balachandran ansatz and the generalized hedgehog ansatz with some known results on the Euler angles for $SU(3)$ [46–48], the first analytical solutions with high topological charge that describe gravitating dibaryons as well as dibaryons in flat space-time at finite density were constructed in the Einstein $SU(3)$-Skyrme model [43]. These dibaryons are genuine $SU(3)$ features in the sense that they are not trivial embeddings of $SU(2)$ in $SU(3)$. Finally, very recently, the generalized hedgehog ansatz has been combined with the Euler parametrization of the $SU(N)$ group describing the so-called nuclear pasta phases at finite density in the $SU(N)$-Skyrme model [42,44]. These solutions are genuine $SU(N)$, due to the image of $SU(2)$ through the Euler ansatz

construction is just a submanifold but not a subgroup of $SU(N)$, as we will show below. In this sense the map is not an embedding of $SU(2)$ into $SU(N)$ but just of S_3 into $SU(N)$ [49].

In the present paper, the ansatz proposed in [42] for the $SU(N)$-Skyrme model will be adapted to the Einstein $SU(N)$-YM case in order to construct analytical solutions describing nonembedded meronic black holes (BHs). It is important to highlight that, recently, this ansatz [considering $\lambda = \lambda(r)$] has allowed the construction of analytical solutions describing inhomogeneous condensates in the Yang-Mills-Higgs theory in $(2 + 1)$ dimensions [50] as well as in $(3 + 1)$ dimensions [51].

The present analysis has three quite nontrivial outcomes. First, one can tell apart merons BHs from colored BHs using the spin from isospin effect: while an asymptotically flat meron BH changes the spin of a scalar test field, a colored black hole does not. This is a very intriguing way to distinguish a colored BH from a meron BH. Second, using the technology of nonembedded ansatz in $SU(N)$, *one can generate test fields with arbitrary high spin*. This is a really powerful result since it allows us to study the dynamics of higher spin fields without introducing any explicit higher spin field but, actually, just analyzing the dynamics of a self-interacting scalar field (charged under the gauge group) living in asymptotically flat $SU(N)$ nonembedded meron BHs (with large enough N). It is worthwhile to remind the reader here of the severe technical problems which are encountered when analyzing the interactions of higher spin fields related to the Coleman-Mandula theorem and its generalizations (see [52–56]) "preventing" a nontrivial interacting S matrix in a flat space for particles with high enough spins.[4] The present approach provides with a valid and sound alternative to the analysis of higher spin interactions in (asymptotically) flat space-times: one can just consider a four-dimensional renormalizable scalar field theory for a Higgs field (which, consequently, has quartic vertices) charged under the $SU(N)$ gauge group and living in the background of a nonembedded $SU(N)$ (gravitating) meron. In the asymptotic region, due to the presence of the nonembedded meron BH, the scalar field becomes a higher spin field. Hence, the present construction allows us to study interacting higher spin fields in asymptotically flat space-times. A further byproduct of our framework is that the structure of the spin from isospin effect in $D = 4$ is slightly different from the one in $D = 5$ dimensions. The reasons behind this difference will also be discussed.

The paper is organized as follows: in Sec. II we give a brief review of the Einstein $SU(N)$-YM theory and we present the ansatz that allows us to construct analytical solutions. In Sec. III we construct BH solutions in $D = 4$,

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[3]Although in a different form and with different ansatz, spherical black holes in Einstein $SU(N)$-YM theory have been already discussed in the literature (see [37] and references therein).

[4]We will mention the relations of the present approach with recent developments in higher spin field theory in the next sections.

and we study the spin from isospin effect and how higher spin fields can be generated. In Sec. IV we construct BH solutions in $D = 5$ and we compare its characteristics with those of the $D = 4$ case. In Sec. V, using a similar ansatz, we found an analytic gravitating soliton solution. Section VI is devoted to the conclusions and perspectives.

II. THE EINSTEIN $SU(N)$-YANG-MILLS THEORY

In this section we make a brief review of the Einstein $SU(N)$-Yang-Mills theory and also we introduce the general ansatz that allows us to construct analytical solutions.

A. Field equations

The action of Einstein $SU(N)$-Yang-Mills theory is given by

$$I = \int d^D x \sqrt{-g} \left(\frac{R - 2\Lambda}{\kappa} - \frac{1}{2e^2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \right), \quad (2.1)$$

where R is the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength of the gauge field A_μ, κ is the Newton's coupling constant, Λ the cosmological constant and e is the YM coupling.

Here we use the convention $c = \hbar = 1$, Greek indices $\{\mu, \nu, \rho, \ldots\}$ run over the D-dimensional space-time with mostly plus signature and Latin indices $\{a, b, c, \ldots\}$ are reserved for those of the internal space (in the present paper we will consider the cases $D = 4$ and $D = 5$).

The YM field equations are

$$\nabla_\nu F^{\mu\nu} + i[A_\nu, F^{\mu\nu}] = 0, \quad (2.2)$$

where ∇_μ is the Levi-Civita covariant derivative.

The Einstein equations, on the other hand, are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2.3)$$

with

$$T_{\mu\nu} = \frac{2}{e^2} \mathrm{Tr}(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}), \quad (2.4)$$

the energy-momentum tensor of the YM field.

B. General ansatz

We consider a meronlike ansatz for the YM field

$$A_\mu = -i\lambda(x^\mu)(U^{-1}\partial_\mu U), \quad (2.5)$$

where $U(x)$ is in a subgroup of $SU(N)$. It is well known that there are many ways of embedding $SU(2)$ into $SU(N)$. It was Dynkin the first to consider the classification of such embeddings [49] (see [57] for details and applications in

gauge theory). We choose what is sometimes called the "maximal" embedding, which is the only one which gives rise to a irreducible representation of $SU(2)$ of spin $j = (N - 1)/2$ (in agreement with the nomenclature in the Skyrme literature, we will call these configurations "nonembedded"). We may parametrize it in terms of the generalized Euler angles as follows

$$U = e^{-iF_1(x^\mu)T_3} e^{-iF_2(x^\mu)T_2} e^{-iF_3(x^\mu)T_3}, \quad (2.6)$$

where the matrices T_a are explicitly given by

$$T_1 = \frac{1}{2} \sum_{j=2}^{N} \sqrt{(j-1)(N-j+1)}(E_{j-1,j} + E_{j,j-1}), \quad (2.7)$$

$$T_2 = \frac{i}{2} \sum_{j=2}^{N} \sqrt{(j-1)(N-j+1)}(E_{j-1,j} - E_{j,j-1}), \quad (2.8)$$

$$T_3 = -\sum_{j=1}^{N} \left(\frac{N+1}{2} - j \right) E_{j,j}, \quad (2.9)$$

with

$$(E_{i,j})_{mn} = \delta_{im}\delta_{jn}. \quad (2.10)$$

They are chosen so that the following relations are satisfied:

$$[T_a, T_b] = i\epsilon_{abc} T_c, \quad \mathrm{Tr}(T_a T_b) = \frac{N(N^2 - 1)}{12} \delta_{ab}. \quad (2.11)$$

It is worthwhile to emphasize that the above generators are an irreducible representation of $SU(2)$, which is not true for all embbedings [46–48]. In the case of $SU(3)$, for instance, one may take one half of the first three Gell-Mann matrices as generators of $SU(2)$, which form a spin $1/2$ representation of $SU(2)$. However, it is not irreducible, because its three 3×3 matrices have zeros everywhere except for their 2×2 first blocks, where the spin matrices are embedded. The above T_a matrices, on the contrary, form the spin-j irreducible representation of $SU(2)$, with $j = (N - 1)/2$. This may be seen directly from the diagonal element (2.9), or by noting that

$$(\vec{T})^2 = \sum_{a=1}^{3} T_a T_a = \sigma(N)\mathbf{1}, \quad (2.12)$$

$$\sigma(N) = \frac{(N^2 - 1)}{4} = j(j + 1). \quad (2.13)$$

Picking the irreducible representation of $SU(2)$ for all values of N implies that for every N we are using a representation with different spin. This means that $(\vec{T})^2$ (which will play an important role to define the "square of

the total angular momentum operator") depends on N. One can see that $\sigma(N)$ grows with N^2 so that, for the irreducible embedding ansatz presented here, the total angular momentum will also grow with N (as it will be discussed in the next sections).

C. A short review on merons

Classic results on gravitating merons and their physical applications in the case of Einstein-YM theory with the $SU(2)$ gauge group are in [58–68].[5]

A meron can always be brought in the following form:

$$A_\mu = -i\lambda(U^{-1}\partial_\mu U), \qquad \lambda \neq 0, 1, \qquad (2.14)$$

which is proportional to a pure gauge term without being, of course, a pure gauge configuration. Therefore the existence of merons is an *intrinsically non-Abelian feature*. The first example on flat space-time was constructed by de Alfaro, Fubini and Furlan in Ref. [15], and it has $\lambda = 1/2$. Although, in principle, λ could take any value different from zero and one, here we will show that even in the case of the $SU(N)$ gravitating meron $\lambda = 1/2$ is indeed a special value.

The field strength $F_{\mu\nu}$ of the meron in Eq. (2.14) is proportional to the commutator,

$$F_{\mu\nu} = -i\lambda(\lambda - 1)[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]. \qquad (2.15)$$

Recently[6] in [25–28], it has been possible to analyze explicitly the physical effects generated by $SU(2)$ meron BHs. In particular, it has been shown that the asymptotically flat case is a very interesting arena to implement the usual spin from isospin effect without worrying about the singularities associated to the meron (which are hidden behind the BH horizon). In the present paper, we will ask the following questions:

(1) Is the Einstein $SU(N)$-YM case physically different from the already known $SU(2)$ case?
(2) Are there genuine $SU(N)$ configurations which are absent in the $SU(M)$ case with $M < N$?
(3) Which are the physical effects associated to these genuine $SU(N)$ configurations?

The above interesting questions can be answered in a very elegant way combining the group theoretical tools developed in Refs. [46–48], both with the idea of non-embedded ansatz developed in [23,24], as well as with the recent results in [42,43].

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[5]It is interesting to note that in [58] the authors constructed the first example of a $SU(2)$ meron black hole. However, the concept of meron was invented after such black hole was constructed. That is why the authors of [58] do not mention the connection with merons.

[6]Using a strategy developed originally to analyze the Skyrme model (see [44,69–77]).

III. BLACK HOLES IN $D = 4$

In this section we construct meron BHs in the Einstein $SU(N)$-YM theory in $D = 4$.

A. Analytic meron black hole solutions

We impose spherical symmetry considering the metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (3.1)$$

The meron in Eq. (2.5) that satisfies identically the complete set of YM equations in Eq. (2.2) is given by

$$F_1(x^\mu) = -\phi, \qquad F_2(x^\mu) = 2\theta, \qquad F_3(x^\mu) = \phi, \quad (3.2)$$

together with the particular value of λ mentioned above,

$$\lambda = \frac{1}{2}. \qquad (3.3)$$

From the Einstein equations in Eq. (2.3), we obtain for the metric function $f(r)$ the following expression

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 + \frac{8\lambda^2(\lambda - 1)^2\kappa}{e^2 r^2}\frac{(N-1)N(N+1)}{6}$$

$$= 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 + \frac{\kappa}{2e^2 r^2}T_N, \qquad (3.4)$$

with $T_N = \frac{(N-1)N(N+1)}{6}$ as the tetrahedral numbers for $N = 2, 3, \ldots$.

It turns out that the meron in this case is just the Wu-Yang monopole, whose singularity is dressed under the BH horizon. In fact,

$$A^i = -\frac{1}{r^2}\epsilon^{ija}x_j T_a, \qquad (3.5)$$

where, $(x_1, x_2, x_3) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The above solution has exactly the same form as the one in Minkowski space-time, but be aware that the x^i are only asymptotically the Cartesian coordinates of flat space. It is a straightforward computation to check that twice the Wu-Yang monopole field in Eq. (3.5) gives vanishing field strength, that is, as pure gauge as expected for a meron with $\lambda = 1/2$. Now, if one performs a gauge transformation using a group element of the form (2.6), with

$$F_1(x^\mu) = -\phi, \qquad F_2(x^\mu) = -\theta, \qquad F_3(x^\mu) = \phi, \quad (3.6)$$

then the YM potential transforms to the "Dirac gauge"

$$A = (1 - \cos\theta)d\phi T_3. \qquad (3.7)$$

This potential has a Dirac string singularity at $\theta = 0$, and the field strength is given by

$$F_{\mu\nu} = f_{\mu\nu}T_3,$$

where $f_{\mu\nu}$ is the field of the Dirac monopole, with $f_{\theta\phi} = \sin\theta$, the only nonvanishing component. The field is effectively Abelian, and its contribution to the action in Eq. (2.1) is

$$\frac{1}{2e^2}\int d^4x\sqrt{-g}f_{\mu\nu}f^{\mu\nu}\mathrm{Tr}[T_3^2] = \frac{N(N^2-1)}{24e^2}\int d^4x\sqrt{-g}f_{\mu\nu}f^{\mu\nu},$$

where we have used Eq. (2.11). This means that, in fact, the effective coupling constant Q is given by

$$Q^2 = \frac{12e^2}{N(N^2-1)}. \qquad (3.8)$$

The resulting metric is precisely the Reissner-Nordström metric in (anti–)de Sitter with unitary magnetic charge,

$$g = \frac{1}{Q} = \frac{N(N^2-1)}{12e^2}. \qquad (3.9)$$

Note that the monopole in the Dirac gauge is not of the form (2.14). Its double is not pure gauge. Actually, it may be multiplied by any constant to get a monopole solution with any magnetic charge. However, if the magnetic charge is not unitary, then we will not be able to perform a gauge transformation that takes it to the Wu-Yang form, that is, it will not be a meron anymore. Indeed, the gauge transformation from the meronic configuration to the Abelian Dirac monopole is singular at the origin (see the discussion on pages 13 and 14 of [78]). Since two gauge potentials are gauge equivalent if and only if there is a *proper gauge transformation* (namely, a *smooth gauge transformation which is also well behaved at infinity*[7]) from one configuration to the other, one can conclude that the present meronic configuration and the Dirac monopole are not gauge equivalent. Note also that if one would not define gauge equivalence using proper gauge transformations one would arrive at absurd conclusions such as that the (anti–)de Sitter space-time in $(2+1)$ dimensions is the same as the Bañados-Teitelboim-Zanelli black hole (as these two configurations are connected by an improper gauge transformation).

Even though the above solution is well known, there is an interesting feature arising from the dependence of the effective charge g with N as seen in Eq. (3.9). If the cosmological constant Λ is positive, then for a horizon to

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[7]Well behaved at infinity means that the group-valued element U which generates such gauge transformation must approach the center of the gauge group at spatial infinity: see the discussion in [79]. Note that the group element of the form (2.6) [with $F_1(x^\mu) = -\phi$, $F_2(x^\mu) = -\theta$, $F_3(x^\mu) = \phi$] not only is singular at the origin but also does not approach the center of $SU(2)$ (which is $\pm \mathbf{1}_{2\times2}$) at spatial infinity.

exist the magnetic (or electric) charge must satisfy $g^2 < (4\Lambda)^{-1}$. Therefore, these merons cease to exist for big enough N. There are also bound for the mass. If the cosmological constant vanishes, for instance, then for a horizon to dress the singularity the mass must be such that $M^2 > g^2$. Therefore, as N grows, the mass of the merons are forced to grow as well.

Obviously, spherically symmetric BHs in the Einstein $SU(N)$-YM theory have been already discussed in depth in the literature (see, for instance, [35–37,80–82] and references therein). In fact, the idea of the present construction (using an explicit "nonembedded" ansatz for the meronic field) is that it discloses in a very neat way the fact that the spin from isospin effect depends actually on "the N" of the gauge group $SU(N)$, so that the interactions of test scalar fields [charged under $SU(N)$] with the gravitating merons discussed here can generate fields of arbitrary high spin (if N is large enough). This fact has not been noticed before (to the best of our knowledge) and is a novel outcome of our technique.

B. About colored black holes

It is well known that the Einstein-YM theory admits spherically symmetric BHs solutions with a non-Abelian hair (see [83–85] and references therein) in which the non-Abelian electric and magnetic fields decay too fast to give rise to charges. Despite their instability [86,87], the very important role of such non-Abelian hairy BHs (especially in the application of holography) cannot be underestimated [88,89]. Here we want just to emphasize that these BHs can be written very easily using the present approach. We will consider the following metric

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \qquad (3.10)$$

together with a radial profile for the YM field, namely

$$A_\mu = -i\lambda(r)(U^{-1}\partial_\mu U),$$

and

$$F_1(x^\mu) = -\phi, \qquad F_2(x^\mu) = 2\theta, \qquad F_3(x^\mu) = \phi. \qquad (3.11)$$

Of course, $\lambda = 1/2$, would give the meron BH, while hairy colored BHs must be found numerically. The YM equations are reduced to the following equation for the profile

$$\lambda'' + \frac{(fh)'}{2fh}\lambda' - \frac{2\lambda(\lambda-1)(2\lambda-1)}{r^2h} = 0. \qquad (3.12)$$

On the other hand, the components of the energy-momentum tensor are

$$T_{tt} = 4T_N \times \frac{f}{e^2 r^4}(2\lambda^2 - 4\lambda^3 + 2\lambda^4 + r^2 h\lambda'^2),$$

$$T_{rr} = 4T_N \times -\frac{1}{e^2 h r^4}(2\lambda^2 - 4\lambda^3 + 2\lambda^4 - r^2 h\lambda'^2),$$

$$T_{\theta\theta} = 4T_N \times \frac{2}{e^2 r^2}(\lambda - 1)^2 \lambda^2,$$

$$T_{\phi\phi} = \sin^2\theta T_{\theta\theta},$$

while the components of the Einstein tensor (with cosmological constant) are given by

$$G_{tt} + \Lambda g_{tt} = \frac{f}{r^2}(1 - h - rh') - \Lambda f,$$

$$G_{rr} + \Lambda g_{rr} = \frac{1}{r^2 fh}(fh - f + rhf') + \Lambda\frac{1}{h},$$

$$\begin{aligned}G_{\theta\theta} + \Lambda g_{\theta\theta} = &\frac{r}{4f^2}(f[rf'h' + 2h(f' + rf'')] \\ &+ 2f^2 h' - rhf'^2) + \Lambda r^2,\end{aligned}$$

$$G_{\phi\phi} + \Lambda g_{\phi\phi} = \sin^2\theta(G_{\theta\theta} + \Lambda g_{\theta\theta}).$$

This equations system (where N only enters as an overall factor in the energy-momentum tensor) has been already analyzed, so that the known numerical solutions of the references mentioned above can be adapted to the present case.

Here we only want to mention that the key difference between meron BHs and colored BHs appears in the Klein-Gordon equation

$$(\Box - m^2)\Phi = 0, \qquad \Box = D_\mu D^\mu, \qquad (3.13)$$

for a scalar field Φ charged under the gauge group. In the asymptotically flat case, the terms that should give rise to the spin from isospin effect [which are $g^{\mu\nu}A_\mu A_\nu \Phi$ and $g^{\mu\nu}(A_\mu)\nabla_\nu\Phi$] decay faster than in the case of the meron BH, so that, in the asymptotic region of the colored BHs, such terms are unable to form the contribution "$(\vec{J})^2/r^2$" (which will be discussed in the next section) needed to transform bosons into fermions (and vice versa).

C. Gravitational spin from isospin effect in $SU(N)$

In general, the presence of a background field breaks the natural symmetries of a theory. For instance, the $SU(N)$ Klein-Gordon or Dirac equations, in which the Yang-Mills field is explicitly given, will break rotational invariance (unless the given field is spherically symmetric). However, there are situations in which the field is indeed symmetric, but the corresponding gauge potential, which appears in the equations, is not. In that case, the orbital angular momentum $\vec{l}$ will not be a symmetry generator. However, it is possible to compensate the lack of invariance of the potential under spatial rotations with an appropriate gauge

rotation. For example, the potential in Eq. (3.5) is not invariant under rotations. However, if one performs the same $SU(2)$ gauge rotation to both space-time indices and internal indices, then the symmetry is recovered. The operator that generates such a transformation is

$$\vec{J} = \vec{l} + \vec{T}, \qquad (3.14)$$

where the vector $\vec{T}$ is formed by the generators of the nonembedded subgroup of $SU(N)$ defined in Eqs. (2.7)–(2.10), while $\vec{l}$ is the usual orbital angular momentum operator. Hence, $\vec{J}$ should be considered as the total angular momentum of the system.

It is precisely this *spherical symmetric up to an internal rotation* which gives rise to the *Jackiw-Rebbi-Hasenfratz-'t Hooft* mechanism, or "spin form isospin" effect [38,39], according to which the excitations of a Bosonic field charged under $SU(2)$ around a background gauge field with the above characteristics behave as Fermions.[8] We are interested here in the case of $SU(N)$, in which the meron solution discussed in the previous section will do the same trick. A quick way to derive the spin from isospin phenomena is to analyze the Klein-Gordon equation in Eq. (3.13) for a scalar field Φ (which will be assumed to belong to the fundamental representation) charged under $SU(N)$, being in this case ∇_μ the Levi-Civita covariant derivative corresponding to the metric in Eqs. (3.1) and (3.4), and A_μ is the $SU(N)$ meron gauge potential in Eqs. (2.5), (2.6), and (3.2). For the present purpose, it is enough to restrict us to the static case, set $\Lambda = 0$ and to explore the asymptotic region, where the metric is Minkowski. We also set $m = 0$, so that Eq. (3.13) becomes

$$\begin{aligned}(\nabla_i + iA_i)&(\nabla^i + iA^i)\Phi \\ &= (\nabla^2 + 2iA_i\nabla^i + i(\nabla^i A_i) - A_i A^i)\Phi. \qquad (3.15)\end{aligned}$$

The first term in Eq. (3.15) is the Laplacian,

$$\nabla^2\Phi = \frac{1}{r^2}[\partial_r(r^2\partial_r\Phi) - \vec{L}^2\Phi],$$

where

$$\vec{L} = -i\vec{r} \times \vec{\nabla}$$

is the orbital angular momentum operator. Using Eq. (3.5), the second term in Eq. (3.15) is

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[8]An effect which is very similar to the *Jackiw-Rebbi-Hasenfratz-'t Hooft* mechanism occurs for Skyrmions [41] (for a detailed review, see [3]). Indeed, the excitations around the Skyrme soliton with winding number equal to one can behave as fermions.

$$2iA_i\nabla^i = \frac{2i}{r^2}T_a\epsilon^{aji}x_j\nabla_i = -2T_aL_a.$$

The third term vanishes because $\nabla^i A_i = 0$, as one may verify directly. Finally, for the last term,

$$-A_iA^i = -\frac{1}{r^4}(r^2\delta^{ab} - x^ax^b)T_aT_b = -\frac{1}{r^2}[\vec{T}^2 - (\hat{r}\cdot\vec{T})^2],$$

where $\hat{r}\cdot\vec{T} = x_aT_a/r$ is the projection of $\vec{T}$ along the direction of $\vec{r}$. Putting all together, Eq. (3.15) turns out to be

$$0 = \frac{1}{r^2}\partial_r(r^2\partial_r\boldsymbol{\Phi}) + \frac{1}{r^2}(-\vec{L}^2 - 2T_aL_a - \vec{T}^2 + (\hat{r}\cdot\vec{T})^2)\boldsymbol{\Phi},$$

$$= \frac{1}{r^2}\partial_r(r^2\partial_r\boldsymbol{\Phi}) - \frac{1}{r^2}(\vec{J}^2 - (\hat{r}\cdot\vec{T})^2)\boldsymbol{\Phi}. \tag{3.16}$$

Here $\vec{J}$ is the total angular momentum in Eq. (3.14). We see that it forms in the Klein-Gordon equation, supplementing the orbital part as it should. Therefore, one can generate higher spin fields in asymptotically flat space-times using test scalar fields (charged under the gauge group) living in the $SU(N)$ meron BHs constructed in the previous subsections.

D. Higher spin fields from nonembedded ansatz in $D = 4$

The classic results in [52–56], showed that, under "normal" circumstances, in flat space-times one cannot formulate a consistent quantum field theory with massless particles with spins greater than two. The same approach also suggests similar negative results in asymptotically flat space-times. Soon after these original references, some positive partial results on how to define consistent (cubic) interactions between higher spin fields were obtained in [90–92]. However, the problem to define consistent renormalizable interactions between higher spin fields on (asymptotically) flat space-times remained. A situation with negative cosmological constants (due to its role as an effective infrared cutoff) was disclosed in [93,94] (an in-depth analysis of the current situation can be found in [95–101] and references therein). To the best of authors' knowledge, the only well-established case (so far) in which it is possible to define a consistent interaction in four-dimensional (asymptotically) flat space-times is the cubic vertex (see, for a modern perspective, [102–104] and references therein). In particular, in those references, a complete classification of the possible cubic vertices has been performed. It is worth to emphasize that, within their approach, the spectrum is reducible and consist of propagating massless particles with spin s, $s - 2$, $s - 4$, … and so on. Consequently, this modern formulation is different from [90], in which case the field equations describe a single massless degree of freedom of a particile with spin s.

In this sense, the spin from isospin effect corresponding to the nonembedded gravitating merons constructed in the previous sections is more similar to [90] rather than to the modern references mentioned above. The reason is that with the choice of the generators in Eqs. (2.7), (2.9), and (2.10) one gets an irreducible representation of $SO(3)$ of spin $j = (N - 1)/2$. Hence, due to the *conversion of isospin into spin* (see [105,106]) a scalar field charged under the gauge group $SU(N)$ becomes a field of spin $j = (N - 1)/2$. One way to see this (which has been already discussed in the previous sections) is that the ansatz for the gauge field in Eqs. (2.5)–(2.7), (2.9), and (2.10) is not spherically symmetric, but the lack of spherical symmetry can be compensated by an internal rotation of spin $j = (N - 1)/2$ so that the "true" angular momentum operator acting on such a scalar field corresponds to a spin-j field.

Now, if one wants to consider interactions one can analyze the well-known (renormalizable in $D = 4$) scalar field Lagrangian for the Higgs field charge under the $SU(N)$ gauge group with a quartic Higgs potential whose field equations and Lagrangian read, respectively,

$$g^{\mu\nu}(\nabla_\mu + iA_\mu)(\nabla_\nu + iA_\nu)\boldsymbol{\Phi} = -\gamma(v^2 - |\boldsymbol{\Phi}|^2)\boldsymbol{\Phi}, \tag{3.17}$$

$$I[\boldsymbol{\Phi}] = \frac{1}{4}\int d^4x\sqrt{-g}(\text{Tr}[D_\mu\boldsymbol{\Phi}D^\mu\boldsymbol{\Phi}] - \gamma(v^2 - |\boldsymbol{\Phi}|^2)^2),$$

$$\boldsymbol{\Phi}^2 = -\frac{1}{2}\text{Tr}[\boldsymbol{\Phi}\boldsymbol{\Phi}]. \tag{3.18}$$

The above theory is renormalizable in $D = 4$ and the corresponding Feynman rules in coordinates space can be defined in the usual way (taking care of the nontrivial background). In order to display the interplay between the vertices and the spin of $\boldsymbol{\Phi}$, one can expand explicitly in terms of eigenfunctions $\boldsymbol{\Phi}$ of $\vec{J}^2$ and $\hat{r}\cdot\vec{T}$. Clearly, being the original theory $I[\boldsymbol{\Phi}]$ well defined in $D = 4$, the interaction vertices will be well defined as well, and, since the field $\boldsymbol{\Phi}$ acquires a spin $j = (N - 1)/2$ due to the background, one can interpret the usual Feynman rules as Feynman rules for spin $(N - 1)/2$ fields. The original no-go theorems [52–56] are avoided since the presence of the gravitating meron breaks the symmetry of the vacuum and changes the topology of space-time. Thus, as long as the backreaction of $\boldsymbol{\Phi}$ on the background can be neglected, in principle this construction works. Of course, there are severe technical complications to implement this program in practice due to the fact that the nontrivial background prevents one from finding easily the propagators in Fourier space. We hope to return to this interesting issue in a future work.

IV. BLACK HOLES IN $D = 5$

In this section we construct meron BHs in the Einstein $SU(N)$-YM theory in $D = 5$.

A. Analytic meronic black hole solutions

We consider a five-dimensional, spherically symmetric, space-time ansatz:

$$ds^2 = -f(r)^2 dt^2 + \frac{1}{f(r)^2} dr^2$$
$$+ \frac{r^2}{4}(d\gamma^2 + d\theta^2 + d\phi^2 + 2\cos\theta d\gamma d\phi), \quad (4.1)$$

together with the YM field given by Eqs. (2.5) and (2.6), with

$$F_1(x^\mu) = -\phi, \qquad F_2(x^\mu) = -\theta, \qquad F_3(x^\mu) = -\gamma, \quad (4.2)$$

$$\lambda = \frac{1}{2}. \quad (4.3)$$

These fields satisfy the YM equations. They correspond to a $D = 5$ meron, an analog of the $D = 4$ case described in the previous section. The Einstein equations may be explicitly solved:

$$f(r)^2 = 1 - \frac{2m}{r^2} - \frac{\Lambda}{6} r^2$$
$$- \frac{2}{3} \times 24(\lambda-1)^2 \lambda^2 \frac{\kappa \log(r)}{e^2 r^2} \frac{(N-1)N(N+1)}{6},$$
$$= 1 - \frac{2m}{r^2} - \frac{\Lambda}{6} r^2 - \frac{\kappa \log(r)}{e^2 r^2} T_{\text{N}}. \quad (4.4)$$

Here $T_{\text{N}} = \frac{(N-1)N(N+1)}{6}$ are the tetrahedral numbers. The constant λ has been left arbitrary so one can see that when the YM field is pure gauge, $\lambda = 1$, the metric reduces to Schwarzschild-(anti–)de Sitter in $D = 5$.

B. Gravitational spin from isospin effect in $SU(N)$

As in the previous section, in order to study the spin from isospin effect, we will analyze the Klein-Gordon equation in Eq. (3.13) in $D = 5$ for a scalar field $\mathbf{\Phi}$ charged under $SU(N)$, with ∇_μ the Levi-Civita covariant derivative corresponding this time to the metric in Eqs. (4.1) and (4.4). Here A_μ is the $SU(N)$ meron gauge potential in Eqs. (2.5), (2.6), and (4.2).

In $D = 5$ the orbital angular momentum is given, in Cartesian coordinates, by

$$\mathcal{L}_{AB} = -i(x_A \partial_B - x_B \partial_A),$$

where x^A, $A = 1, \ldots, 4$ are the spatial indices. They satisfy the $SO(4)$ algebra. Because $SO(4) = SO(3) \times SO(3)$, the above generators may be divided into two sets, each satisfying the $SO(3)$ algebra. Explicitly,

$$L_a^\pm = \epsilon_a{}^{bc} \mathcal{L}^{bc} \pm \mathcal{L}_{4a}, \qquad [L_a^\pm, L_b^\pm] = i\epsilon^c{}_{ab} L_c^\pm,$$
$$[L_a^+, L_b^-] = 0, \quad (4.5)$$

where a, $b = 1$, 2, 3. We call L_a^+, L_b^- the right and left angular momentum, respectively. It is useful to write these generators in the spherical coordinates of the 3-sphere defined in the metric (4.1). For example, the right angular momentum is given by

$$L_1^+ = i\left(\cos\gamma \cot\theta \partial_\gamma + \sin\gamma \partial_\theta - \frac{\cos\gamma}{\sin\theta} \partial_\phi\right), \quad (4.6)$$

$$L_2^+ = i\left(\sin\gamma \cot\theta \partial_\gamma - \cos\gamma \partial_\theta - \frac{\sin\gamma}{\sin\theta} \partial_\phi\right), \quad (4.7)$$

$$L_3^+ = -i\partial_\gamma. \quad (4.8)$$

In this form, the generators are well defined not only in Minkowski space but also in the BH geometry in Eq. (4.1). In terms of these, the D'Alambert operator is

$$\Box = -\frac{1}{f^2} \partial_t^2 + \frac{1}{r^3} \partial_r(r^3 f^2 \partial_r) - \frac{1}{r^2} \frac{1}{2} \mathcal{L}_{AB} \mathcal{L}^{AB},$$
$$= -\frac{1}{f^2} \partial_t^2 + \frac{1}{r^3} \partial_r(r^3 f^2 \partial_r) - \frac{2}{r^2}[(\vec{L}^+)^2 + (\vec{L}^-)^2].$$

As in the $D = 4$ case, we now consider the Klein-Gordon equation for a scalar field $\mathbf{\Phi}$ in the background of the right-handed meron,

$$(\Box + i\nabla_\mu A^\mu + 2iA^\mu \nabla_\mu - A^\mu A_\mu - m^2)\mathbf{\Phi} = 0. \quad (4.9)$$

Substituting the explicit expressions for A_μ and $g_{\mu\nu}$ given by Eqs. (2.5), (2.6), (4.1)–(4.4), and (4.9) takes the form

$$\left(-\frac{1}{f^2} \partial_t^2 + \frac{1}{r^3} \partial_r(r^3 f^2 \partial_r)\right.$$
$$\left. - \frac{1}{r^2}(2(\vec{J}^+)^2 + 2(\vec{J}^-)^2 - \sigma(N)\mathbf{1}) - m^2\right)\mathbf{\Phi} = 0, \quad (4.10)$$

where **1** is the $N \times N$ identity matrix, $\sigma(N)$ is given in Eq. (2.13) and

$$J_a^+ = L_a^+ + T_a, \qquad J_a^- = L_a^-.$$

From this equation we see that the angular momentum is given by the pair J_a^+, J_a^- which, besides the orbital part L_a^+, L_a^-, has a contribution from the generators of $SU(N)$. In this case, only the right angular momentum J_a^+ gets shifted. Of course, there is nothing special about the right angular momentum. A second solution of the Yang-Mills-Einstein

system exists which shifts the left angular momentum J_a^- instead. It is obtained by replacing the group element U of the above solution by by U^{-1}. The metric (3.4) and the meron form (2.5) of the gauge field are the same.

Note also that, in addition to the angular momentum, the expression multiplying r^{-2} in Eq. (4.10) contains a term proportional to the identity. This is much simpler than the $D = 4$ case, where the extra term is $(\hat{r} \cdot \vec{T})^2$, as seen in Eq. (3.16). The reason behind this reduction (similar to what happens for the BH in [25]), lies in the term $A_\mu A^\mu$ in Eq. (4.9), which in the $D = 5$ case, turns out to be proportional to $(\vec{T})^2$. Then, the spin of the particles becomes exactly σ_N according to Eq. (2.12).

There is another important difference between the black hole solutions in $D = 4$ and $D = 5$ presented above, namely, the first has vanishing topological charge while the latter has a finite one. In fact, consider the following standard definition of the topological charge,

$$ B = \frac{1}{24\pi^2} \int_\Sigma \rho_B, \qquad \rho_B = \epsilon^{ijk} \text{Tr}[\mathcal{L}_i \mathcal{L}_j \mathcal{L}_k], $$

where Σ is any three-dimensional spatial surface defined by $t = \text{const}$ and $r = \text{const}$ while

$$ \mathcal{L}_\mu = U^{-1} \partial_\mu U = \Omega_\mu^a T_a \qquad (4.11) $$

are the Maurer-Cartan form components, Ω_μ^a are the left-invariant 1-forms components of an element $U(x) \in SU(N)$ parametrized as in Eq. (2.6). Note that a necessary (but not sufficient) condition for having a nonzero topological charge is that the functions F_i in Eq. (2.6) must be independent. This effectively occurs in the case of the BH in $D = 5$ considered above, where each function depends linearly on a different coordinate of the 3-sphere [see Eq. (4.2)], and it is possible to verify that $B \neq 0$ on Σ by integrating into the ranges of the coordinates in Eq. (4.1). On the other hand, in the case of the meron BH in $D = 4$, the functions F_i are not independent [see Eq. (3.2)], and it is direct to check that $\rho_B = 0$ identically.

V. GRAVITATING SOLITON

In this section we present an analytic self-gravitating soliton solution in $D = 4$. Although this configuration has compact spatial sections (and, consequently, no spin from isospin effect) it possesses interesting features which are worth mentioning.[9]

We consider a static space-time metric that is a product of $\mathbb{R} \times S^3$ with a constant scale factor ρ_0, namely

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[9]See [107] for the construction of gravitating merons in D-dimensional massive Yang-Mills theory and the Skyrme model.

$$ ds^2 = -dt^2 + \frac{\rho_0^2}{4}((d\gamma + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2), $$

$$ (5.1) $$

together with the following ansatz for the gauge field

$$ F_1(x^\mu) = \gamma, \qquad F_2(x^\mu) = \theta, \qquad F_3(x^\mu) = \phi, \quad (5.2) $$

and

$$ \lambda = \frac{1}{2}. \qquad (5.3) $$

With the above ansatz the $SU(N)$-YM equations are identically satisfied, while the Einstein equations provide the following constraints between the coupling constants

$$ \rho_0^2 = \frac{\kappa T_N}{e^2}, \qquad \Lambda = \frac{3}{2} \frac{e^2}{\kappa T_N}. \qquad (5.4) $$

The energy density of the soliton is then

$$ T_{00} = \frac{3}{2} \frac{T_N}{e^2 \rho_0^4} = \frac{\Lambda}{\kappa} = \frac{3}{2} \frac{e^2}{\kappa^2 T_N}. \qquad (5.5) $$

One can see that, if one requires having a static gravitating configuration, then the cosmological constant must scale as $1/N$, so that it must be small and positive when N is large.

One can also consider a time-dependent scale factor, $\rho = \rho(t)$, in which case the field equations read

$$ \ddot{\rho} - \frac{1}{3}\Lambda\rho + \frac{\kappa T_N}{2e^2\rho^3} = 0, \qquad (5.6) $$

$$ \dot{\rho}^2 - \frac{1}{3}\Lambda\rho^2 + 1 - \frac{\kappa T_N}{2e^2\rho^2} = 0. \qquad (5.7) $$

The above equations system represents a cosmological space-time whose source is the energy-momentum tensor of a nonembedded $SU(N)$ meron, because still in this dynamical case the YM equations are identically satisfied for $\lambda = 1/2$. We hope to come back on the analysis of these cosmological space-time in a future publication.

VI. CONCLUSIONS AND PERSPECTIVES

In this paper we have constructed meron BHs and self-gravitating soliton solutions in the Einstein $SU(N)$-YM theory in $D = 4$ and $D = 5$ dimensions for all values of N. These analytic configurations have been found by combining the generalized hedgehog ansatz with the Euler parametrization of the $SU(N)$ group from which the YM equations are automatically satisfied for all values of N, while the Einstein equations can be solved analytically.

One of the main results of this work is that we explicitly show the role that the color number N plays in the

gravitational spin from isospin effect. In fact, meron BHs can be distinguished by colored BHs by looking at the spin from isospin effect, because this effect is present only in the meron BHs constructed here.

In order to compute the spin generated from the isospin we have considered a Bosonic field charged under $SU(N)$ around the background gauge field of the BH solutions, showing that this mechanism works differently for the BHs in $D = 4$ and $D = 5$. This difference lies in the presence of a nonzero topological charge for the ansatz of the $D = 5$ case.

Also, using the theory of nonembedded ansatz for $SU(N)$ together with the spin from isospin effect, one can build fields of arbitrary high spin out of scalar fields

charged under the gauge group. Hence, one can analyze interacting higher spin fields in asymptotically flat spacetimes without introducing by hand higher spin fields.

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