Retrolensing by two photon spheres of a black-bounce spacetime

Naoki Tsukamoto^{®*}

Department of General Science and Education, National Institute of Technology, Hachinohe College, Aomori 039-1192, Japan

(Received 19 February 2022; accepted 28 March 2022; published 20 April 2022)

We investigate retrolensing by two photon spheres in a novel black-bounce spacetime suggested by Lobo *et al.* which can correspond to a Schwarzschild black hole, a regular black hole, and a traversable wormhole including an Ellis-Bronnikov wormhole. In a case, the wormhole has a throat which acts as a photon sphere and it has another photon sphere outside of the throat. With the sun as a light source, an observer, and the wormhole are lined up in this order, sunlight reflected slightly outside of the throat and barely outside and inside of the outer photon sphere can reach the observer. We show that the light rays reflected by the outer photon sphere are dominant in retrolensing light curves in the case.

DOI: 10.1103/PhysRevD.105.084036

I. INTRODUCTION

Compact objects such as black holes and wormholes, which have unstable circular light orbits [1,2], reflect light rays like a mirror due to their strong gravity [3–20]. This phenomena can be used as a complementary method to detect the compact objects such as the observations of gravitational waves [21], shadow [22], and x-ray echo [23]. The unstable circular light orbit and stable circular light orbit are called the photon sphere and antiphoton sphere,¹ respectively, and theoretical and observational aspects of the photon spheres and antiphoton spheres have been discussed in Refs. [28–34] and generalized and alternative surfaces of the photon spheres are also suggested [35].

In 2002, Holz and Wheeler considered that sunlight is reflected by the photon sphere of a black hole passing by the Solar System [36]. The gravitational lensing with the deflection angle of lights $\alpha \sim \pi$ in a configuration, that the light source, an observer, and a photon sphere as a lens are lined up in this order, is called retrolensing. Retrolensing by photon spheres not only in black hole spacetimes [37–42] but also in a wormhole spacetime [17,43] and in naked singularity spacetimes [44,45] were investigated. The effect of light rays with the deflection angle $\alpha \sim 3\pi$ on retrolensing light curves were studied [43].

A wormhole spacetime [46,47] described by general relativity has a structure with nontrivial topology. Wormholes can have photon spheres and an antiphoton sphere on and/or off their throat and some wormholes can be black hole mimickers [48–54]. Simpson and Visser suggested a spacetime with a metric

$$ds^{2} = -\left(1 - \frac{2m}{\sqrt{r^{2} + a^{2}}}\right)dt^{2} + \frac{dr^{2}}{1 - \frac{2m}{\sqrt{r^{2} + a^{2}}}} + (a^{2} + r^{2})(d\vartheta^{2} + \sin^{2}\vartheta d\varphi^{2}), \qquad (1.1)$$

where *a* and *m* are nonnegative constants, which describes a regular black hole metric and a wormhole metric with a photon sphere on one side of a region against a throat and they called it black-bounce spacetime [55]. Its metric is useful to compare black holes and wormholes. Its gravitational lensing [56–59], shadows and accretion disks [60–64], the motion of S-stars around SgrA* [65], epicyclic oscillatory motion [66], field sources [67], and a rotating counterpart [68–71] have been investigated. The alternatives of the black-bounce spacetime were also studied [72–78].

In general, a spacetime with a photon sphere and no antiphoton sphere has the common behavior of light rays near the photon sphere. Thus, gravitational lensing in the Simpson-Visser spacetime in a strong gravitational field has a similar nature to the one in the Schwarzschild spacetime if we consider a light source on the same side of the region against the wormhole throat. This is because only one photon sphere in the Simpson-Visser spacetime affects on gravitational lensing in the strong gravitational field under the assumed configuration. Recently, compact objects with two photon spheres have been suggested [18,79–86] and they can be distinguished from compact objects with one photon sphere by the observations in the strong gravitational field.

Lobo *et al.* [73] have suggested a wide class of blackbounce spacetimes with a metric

tsukamoto@rikkyo.ac.jp

¹An antiphoton sphere may cause instability of a compact object due to the slow decay of linear waves [24–26]. The deflection angle of a light scattered by the antiphoton sphere has been studied in Ref. [27].

$$ds^{2} = -A(r)dt^{2} + \frac{dr^{2}}{A(r)} + (a^{2} + r^{2})(d\vartheta^{2} + \sin^{2}\vartheta d\varphi^{2}),$$
(1.2)

where A(r) is given by

$$A(r) \equiv 1 - \frac{2mr^{K}}{(a^{2N} + r^{2N})^{\frac{K+1}{2N}}}.$$
 (1.3)

If K = 0 and N = 1 are chosen, it recovers the metric of the Simpson-Visser spacetime (1.1). Lobo *et al.* pointed out that the metric with K = 0 and $N \ge 2$ has a similar nature to the Simpson-Visser spacetime and they considered a metric function

$$A(r) = 1 - \frac{2mr^2}{(a^2 + r^2)^{3/2}}$$
(1.4)

by setting K = 2 and N = 1 to find a novel blackbounce spacetime with new properties [73]. It is a Schwarzschild metric for a = 0 and m > 0, a regular black hole metric for $0 < a/m \le 4\sqrt{3}/9$, a wormhole metric for $4\sqrt{3}/9 < a/m$, and the Ellis-Bronnikov wormhole metric for a > 0 and m = 0 [87,88]. Tsukamoto pointed out that the metric with K = 0and N = 2 has a photon sphere off a wormhole throat and an additional photon sphere on the throat for $4\sqrt{3}/9 < a/m < 2\sqrt{5}/5$ [89]. The novel black-bounce metric can be the simplest one with two photon spheres.² Guerrero *et al.* studied geometrically thin accretion disks in the black-bounce spacetime [90].

In this paper we investigate retrolensing in the novel black-bounce spacetime by using the deflection angle of light rays in strong deflection limits [12,17,19].³ It would be natural to pay attention to the effects of the throat to distinguish between black holes and wormholes since the wormholes are characterized by the throat. In the case of $4\sqrt{3}/9 < a/m < 2\sqrt{5}/5$, the novel black-bounce spacetime can be distinguished from the Simpson-Visser spacetime since light rays reflected slightly inside of the outer photon sphere affect retrolensing light curves in the novel black-bounce spacetime.

This paper is organized as follows. In Secs. II and III, we review the deflection angle in strong deflection limits in the novel black-bounce spacetime and we investigate the percent errors of the deflection angle in strong deflection limits, respectively. We investigate the retrolensing in Sec. IV and we conclude our result in Sec. V. In this paper, we use units in which the light speed and Newton's constant are unity.

II. DEFLECTION ANGLE IN STRONG DEFLECTION LIMITS

In this section, we review briefly the deflection angle of light rays in strong deflection limits [12,17,19] in the novel black-bounce spacetime with the function (1.4) suggested by Lobo et al. [73]. The spacetime is a black hole spacetime with a primary photon sphere and an event horizon for $0 \le a/m \le 4\sqrt{3}/9$; it is a wormhole spacetime with the primary photon sphere off a throat and a secondary photon sphere on the throat for $4\sqrt{3}/9 < a/m < 2\sqrt{5}/5$, and it is a wormhole spacetime with the primary photon sphere on the throat for $2\sqrt{5}/5 < a/m$. Here and hereinafter, we call the photon spheres in order from the largest: We call the outer (inner) photon sphere primary (secondary) photon sphere for $4\sqrt{3}/9 < a/m < 2\sqrt{5}/5$ and we call only one photon sphere primary photon sphere in other cases.

We consider that light rays come from spatial infinity, they are reflected by a compact object, and they go into spatial infinity. The deflection angle of the light ray with an impact parameter b is given by [89]

$$\alpha = 2 \int_{r_0}^{\infty} \frac{dr}{\sqrt{a^2 + r^2} \sqrt{\frac{a^2 + r^2}{b^2} - A}} - \pi, \qquad (2.1)$$

where r_0 is the closest distance of the light ray. The deflection angle of the light ray reflected little outside of the primary photon sphere in a strong deflection limit $b \rightarrow b_m + 0$, where b_m is a critical impact parameter, is expressed by

$$\alpha = -\bar{a}\log\left(\frac{b}{b_{\rm m}} - 1\right) + \bar{b} + O\left(\left(\frac{b}{b_{\rm m}} - 1\right)\log\left(\frac{b}{b_{\rm m}} - 1\right)\right),\tag{2.2}$$

where \bar{a} and \bar{b} are parameters determined by a and m [12,18,89,98].⁴ On the other hand, the deflection angle of light rays reflected slightly inside the primary photon sphere in a strong deflection limit $b \rightarrow b_{\rm m} - 0$ is given by

$$\alpha = -\bar{c}\log\left(\frac{b_{\rm m}}{b} - 1\right) + \bar{d} + O\left(\left(\frac{b_{\rm m}}{b} - 1\right)\log\left(\frac{b_{\rm m}}{b} - 1\right)\right),\tag{2.3}$$

²Note that it has another photon sphere in the other region against the throat. Thus, it has three photon spheres in total.

³References [18,20,40–43,91–102] show the details of strong deflection limits and their applications.

⁴The order of the vanishing term is estimated as in Ref. [12] as $O(b/b_{\rm m}-1)$ while it should be read as $O((b/b_{\rm m}-1) \times \log (b/b_{\rm m}-1))$ as shown in Refs. [94,96,98].

where \bar{c} and \bar{d} are obtained by *a* and *m* [19,89].⁵ The details of the following calculations of the deflection angle in the strong deflection limits are shown in Ref. [89].

A. For $a/m \le 4\sqrt{3}/9$

For $a/m \le 4\sqrt{3}/9$, light rays with the impact parameter $b \to b_{\rm m} + 0$, where the critical impact parameter $b_{\rm m}$ is given by

$$b_{\rm m} = \frac{(r_{\rm m}^2 + a^2)^{\frac{5}{4}}}{(r_{\rm m}^2 - 2a^2)^{\frac{1}{2}}m^{\frac{1}{2}}},$$
(2.7)

are reflected slightly outside of the photon sphere at $r = r_{\rm m}$, where $r_{\rm m}$ is the largest solution of an equation

$$(2a2 - 3r2)m + (a2 + r2)3/2 = 0.$$
 (2.8)

The parameters \bar{a} and \bar{b} in the deflection angle (2.2) are obtained as

$$\bar{a} = \frac{(a^2 + r_{\rm m}^2)^{\frac{5}{4}}}{r_{\rm m}\sqrt{3(r_{\rm m}^2 - 4a^2)m}}$$
(2.9)

and

$$\bar{b} = \bar{a} \log \frac{6r_{\rm m}^4(r_{\rm m}^2 - 4a^2)}{(a^2 + r_{\rm m}^2)^2(r_{\rm m}^2 - 2a^2)} + I_{\rm R} - \pi, \qquad (2.10)$$

respectively, where $I_{\rm R}$ is given by

$$I_{\rm R} = \int_0^1 g(z) dz,$$
 (2.11)

⁵Notice that the deflection angle in the strong deflection limit $b \rightarrow b_{\rm m} - 0$ in Refs. [18,45] is given by

$$\alpha = -\bar{c}\log\left(\frac{b_{\rm m}^2}{b^2} - 1\right) + \bar{d}' + O\left(\left(\frac{b_{\rm m}}{b} - 1\right)\log\left(\frac{b_{\rm m}}{b} - 1\right)\right).$$
(2.4)

In this paper, however, we use Eq. (2.3) according to Ref. [89]. From an approximation

$$\left(\frac{b_{\rm m}^2}{b^2} - 1\right) \sim 2\left(\frac{b_{\rm m}}{b} - 1\right),\tag{2.5}$$

we obtain a relation

$$\bar{d} = -\bar{c}\log 2 + \bar{d}'. \tag{2.6}$$

The difference between \bar{d} and \bar{d}' causes apparent differences between the formulas of retrolensing in the following section and the ones in Refs. [18,45].

and q(z) is defined as

$$g(z) = \frac{2r_{\rm m}}{\sqrt{r_{\rm m}^2 + a^2(1-z)^2}} \left[\frac{r_{\rm m}^2 + a^2(1-z)^2}{b_{\rm m}^2} - (1-z)^2 + \frac{2mr_{\rm m}^2(1-z)^3}{[r_{\rm m}^2 + a^2(1-z)^2]^{\frac{3}{2}}} \right]^{-\frac{1}{2}} - \frac{2(a^2 + r_{\rm m}^2)^{\frac{5}{4}}}{r_{\rm m}\sqrt{3m(r_{\rm m}^2 - 4a^2)}|z|}.$$

$$(2.12)$$

B. For
$$4\sqrt{3}/9 < a/m < 2\sqrt{5}/5$$

In the case of $4\sqrt{3}/9 < a/m \le 2\sqrt{5}/5$, the wormhole has a primary photon sphere off a throat at $r = r_{\rm m}$ which is the larger solution of Eq. (2.8) and the secondary photon sphere on the throat at $r = r_{\rm sc} = 0$.

1. Rays reflected barely outside of the primary photon sphere

Light rays reflected slightly outside of the primary photon sphere are calculated by using parameters \bar{a} , \bar{b} in the deflection angle (2.2), and the critical impact parameter $b_{\rm m}$ given by Eqs. (2.9), (2.10), and (2.7), respectively.

2. Rays reflected little inside the primary photon sphere

We obtain parameters \bar{c} and \bar{d} in the deflection angle (2.3) of rays reflected slightly inside of the primary photon sphere as

$$\bar{c} = \frac{2(a^2 + r_{\rm m}^2)^{\frac{5}{4}}}{r_{\rm m}\sqrt{3(r_{\rm m}^2 - 4a^2)m}}$$
(2.13)

and

$$\bar{d} = \bar{c} \log \left(\frac{6r_{\rm m}^4(r_{\rm m}^2 - 4a^2)}{(r_{\rm m}^2 + a^2)^2(r_{\rm m}^2 - 2a^2)} \left(\frac{r_{\rm m}}{r_{\rm c}} - 1 \right) \right) + I_{\rm r} - \pi,$$
(2.14)

where $I_{\rm r}$ is defined by

$$I_{\rm r} \equiv \int_{1-\frac{r_{\rm m}}{r_{\rm c}}}^{1} g(z)dz, \qquad (2.15)$$

where r_c is the smaller positive solution of V(r) = 0. Here, V(r) is the effective potential of the critical impact parameter $b = b_m$ defined by

$$V(r) \equiv \left\{ \left[\frac{1}{a^2 + r^2} - \frac{2mr^2}{(a^2 + r^2)^{\frac{5}{2}}} \right] b^2 - 1 \right\} E^2, \quad (2.16)$$

where E is the conserved energy of the light ray, and g(z) and b_m are given by Eqs. (2.12) and (2.7), respectively.

3. Rays reflected barely outside of the secondary photon sphere

The deflection angle of light rays reflected little outside of the secondary photon sphere at the throat $r = r_{sc} = 0$ in a strong deflection limit $b \rightarrow b_{sc} + 0$, where $b_{sc} = a$ is the critical impact parameter for the secondary photon sphere, is written in

$$\alpha = -\bar{a}\log\left(\frac{b}{b_{sc}} - 1\right) + \bar{b} + O\left(\left(\frac{b}{b_{sc}} - 1\right)\log\left(\frac{b}{b_{sc}} - 1\right)\right),$$
(2.17)

where \bar{a} and \bar{b} are are obtained as

$$\bar{a} = \sqrt{\frac{a}{a+2m}} \tag{2.18}$$

and

$$\bar{b} = \bar{a}\log\frac{4(a+2m)}{a} + I_{\mathcal{R}} - \pi,$$
 (2.19)

respectively, where $I_{\mathcal{R}}$ is defined by

$$I_{\mathcal{R}} \equiv \int_{0}^{1} \left[\frac{2}{z(2-z)} \sqrt{\frac{a}{a+2m(1-z)^{3}}} - \frac{1}{z} \sqrt{\frac{a}{a+2m}} \right] dz.$$
(2.20)



C. For $a/m > 2\sqrt{5}/5$

For $a/m > 2\sqrt{5}/5$, rays with $b \to b_m + 0$, where the critical impact parameter is $b_m = a$, are reflected slightly outside of the primary photon sphere on the throat $r = r_m = 0$, with \bar{a} and \bar{b} in the deflection angle (2.2) given by Eqs. (2.18) and (2.19), respectively.

III. PERCENT ERRORS OF THE DEFLECTION ANGLE IN THE STRONG DEFLECTION LIMITS

We define the percent errors of the deflection angles α of Eqs. (2.2), (2.3), and (2.17) in the strong deflection limits against the deflection angle of Eq. (2.1) as

$$\frac{\alpha \text{ of Eq.}(2.2) - \alpha \text{ of Eq.}(2.1)}{\alpha \text{ of Eq.}(2.1)} \times 100, \qquad (3.1)$$

$$\frac{\alpha \text{ of Eq.}(2.3) - \alpha \text{ of Eq.}(2.1)}{\alpha \text{ of Eq.}(2.1)} \times 100, \qquad (3.2)$$

and

$$\frac{\alpha \text{ of Eq.}(2.17) - \alpha \text{ of Eq.}(2.1)}{\alpha \text{ of Eq.}(2.1)} \times 100, \qquad (3.3)$$

respectively. We plot the percent errors (3.1), (3.2), and (3.3) against the deflection angle of Eq. (2.1) in Figs. 1, 2, and 3, respectively. Figure 1 shows that the absolute value of the percent error of retrolensing with $\alpha \sim \pi$ by the Schwarzschild black hole is about 2% and that it is larger



FIG. 1. The percent error (3.1) of the deflection angle α of a light ray reflected nearly outside of the primary photon sphere against α of Eq. (2.1). Thick solid (red), thick dashed (green), thick dotted (magenta), thick dot-dashed (blue), thin solid (black), and thin dashed (orange) curves denote the percent error (3.1) for a/m = 0, 0.5, 0.83, 0.86, 0.9, and 3.0, respectively.

FIG. 2. The percent error (3.2) of the deflection angle α of a light ray reflected slightly inside of the primary photon sphere against α of Eq. (2.1). Solid (red), dashed (green), dotted (magenta), and dot-dashed (blue), curves denote the percent error (3.2) for a/m = 0.77, 0.8, 0.83, and 0.86, respectively.



FIG. 3. The percent error (3.3) of the deflection angle α of a light ray reflected nearly outside of the secondary photon sphere against α of Eq. (2.1). Solid (red), dashed (green), dotted (magenta), and dot-dashed (blue), curves denote the percent error (3.3) for a/m = 0.77, 0.8, 0.83, and 0.86, respectively.

than gravitational lensing by the photon sphere of the black hole in usual lens configurations with $\alpha \sim 2\pi$. From Figs. 1 and 3, we notice that the absolute value of the percent errors of light rays reflected slightly outside of the primary and secondary photon spheres on the wormhole throat can be small even if we consider retrolensing. From Fig. 2, the absolute value of the percent error of the light rays reflected barely inside of the primary photon sphere is relatively large.

The absolute value of the percent error is large in an almost marginally unstable photon sphere case $a/m = 2\sqrt{5}/5 + \epsilon$, where $0 < \epsilon \ll 1$, while it is small in a case with $a/m = 2\sqrt{5}/5 - \epsilon$ as shown in Figs. 1–3. This is caused by the degeneracy of the outer photon sphere and an antiphoton sphere into a marginally unstable photon sphere in a limit $a/m \rightarrow 2\sqrt{5}/5 - 0$. The deflection angle of the light rays reflected by the marginally unstable photon sphere would diverge nonlogarithmically in the strong deflection limits.

IV. RETROLENSING

In this section, we investigate retrolensing by the photon spheres.

A. Lens equation

A light ray emitted by the sun S is reflected by the photon sphere L with the deflection angle α , reaches to an observer O, and the observer sees it as an image I as shown in Fig. 4. We use the Ohanian lens equation [7,41,103] expressed by

$$\beta = \pi - \bar{\alpha}(\theta) + \theta + \bar{\theta}, \qquad (4.1)$$



FIG. 4. Lens configuration. A light ray emitted by the sun S with a source angle $\beta \equiv \angle OLS$ is reflected with a deflection angle α by a lens object L and it is observed by an observer O as an image I with an image angle θ . The effective deflection angle $\bar{\alpha}$ is given by $\bar{\alpha} = \alpha - 2\pi n$, where *n* is the winding number of the ray, $\bar{\theta}$ is an angle between the ray and the line LS, and D_{ol} , D_{ls} , and D_{os} are distances between O and L, between L and S, and between O and S, respectively.

where $\beta \equiv \angle OLS$ is a source angle,

$$\bar{\alpha} \equiv \alpha \pmod{2\pi} \tag{4.2}$$

is an effective deflection angle, $\theta \equiv \angle \text{IOL}$ is an image angle, and $\overline{\theta}$ is an angle between the ray and a line LS.

We assume that L, O, and S are almost aligned in this order. Under the assumption, we obtain $\beta \sim 0$, $\bar{\alpha} \sim \pi$, $\alpha \sim \pi + 2\pi n$, and $D_{\rm ls} = D_{\rm ol} + D_{\rm os}$, where *n* is the winding number of the light, and $D_{\rm ls}$, $D_{\rm ol}$, and $D_{\rm os}$ are distances between L and S, between O and L, and between O and S, respectively. We also assume that $\theta = b_{\rm m}/D_{\rm ol} \ll 1$ and $\bar{\theta} = b_{\rm m}/D_{\rm ls} \ll 1$ and we neglect the terms of the small angles θ and $\bar{\theta}$ in the Ohanian lens equation. We regard the sun as a uniform-luminous disk with a finite size on the observer's sky.

B. Image separations and magnifications

1. Light rays reflected slightly outside of the primary and secondary photon spheres

We consider the image separations and magnifications of rays reflected little outside of the photon spheres in the retrolensing configuration. First, we concentrate on the primary photon sphere. Note that we use \bar{a} (2.9) and \bar{b} (2.10) in this case. By using Eq. (2.2), $\alpha = \bar{\alpha} + 2\pi n$, and $b = \theta D_{\text{ol}}$, and the assumptions, we obtain the positive solution of the lens equation (4.1) as

$$\theta = \theta_n^{\text{out}}(\beta) \equiv (1 + e^{[b - (1 + 2n)\pi + \beta]/\bar{a}})\theta_{\text{m}}, \qquad (4.3)$$

where $\theta_{\rm m}$ is the image angle of the primary photon sphere given by $\theta_{\rm m} \equiv b_{\rm m}/D_{\rm ol}$, and it is plotted in Fig. 5. Its magnification is obtained as



FIG. 5. Image separations $2\theta_0^{\text{out}}$ (top), $2\theta_0^{\text{in}}$ (middle), and $2\theta_0^{\text{sc}}$ (bottom) with $D_{\text{ol}} = 0.01 \text{ pc}$, $\beta = 0$, and n = 0 as functions of a/m. Solid (red), dashed (green), and dotted (cyan) curves denote the image separations for $m = 60 M_{\odot}$, $30 M_{\odot}$, and $10 M_{\odot}$, respectively.

$$\mu_n^{\text{out}}(\beta) = -\frac{D_{\text{os}}^2}{D_{\text{ls}}^2} s(\beta) \theta_n^{\text{out}} \frac{d\theta_n^{\text{out}}}{d\beta} = -\frac{D_{\text{os}}^2}{D_{\text{ls}}^2} \frac{\theta_m^2 e^{[b-(1+2n)\pi]/\bar{a}}}{\bar{a}} \times (1 + e^{[\bar{b}-(1+2n)\pi]/\bar{a}}) s(\beta), \qquad (4.4)$$

where $s(\beta)$ is an integral over the disk on a source plane [104–106] given by

$$s(\beta) = \frac{1}{\pi \beta_{\rm s}^2} \int_{\rm disk} d\beta' d\phi, \qquad (4.5)$$

where $\beta_s \equiv R_s/D_{ls}$ is the dimensionless radius of the sun, where R_s is the radius of the sun, β' is a dimensionless radial coordinate normalized by D_{ls} on the source plane, and ϕ is an azimuthal coordinate around an origin which is an intersection point between an axis $\beta = 0$ and the source plane.⁶ The function $s(\beta)$ is expressed by

$$s(\beta) = \frac{2}{\pi\beta_{\rm s}^2} \left[\pi(\beta_{\rm s} - \beta) + \int_{-\beta + \beta_{\rm s}}^{\beta + \beta_{\rm s}} \arccos \frac{\beta^2 + \beta'^2 - \beta_{\rm s}^2}{2\beta\beta'} d\beta' \right]$$

$$(4.6)$$

and

$$s(\beta) = \frac{2}{\pi\beta_s^2} \int_{\beta-\beta_s}^{\beta+\beta_s} \arccos\frac{\beta^2 + \beta'^2 - \beta_s^2}{2\beta\beta'} d\beta' \quad (4.7)$$

for $\beta \leq \beta_s$ and $\beta_s < \beta$, respectively. Note that we get $s(0) = 2/\beta_s$ for the perfectly aligned case. There are negative solutions $\theta \sim -\theta_n^{\text{out}}(\beta)$ for each winding number *n* and their magnifications are obtained as $-\mu_n^{\text{out}}$ approximately. The total magnification of the couple of images for all *n* is given by

$$\mu_{\text{tot}}^{\text{out}}(\beta) \equiv 2 \sum_{n=0}^{\infty} |\mu_n^{\text{out}}(\beta)|$$

= $2 \frac{D_{\text{os}}^2}{D_{\text{ls}}^2} \frac{\theta_m^2}{\bar{a}} |s(\beta)| \left[\frac{e^{(\bar{b}-\pi)/\bar{a}}}{1 - e^{-2\pi/\bar{a}}} + \frac{e^{2(\bar{b}-\pi)/\bar{a}}}{1 - e^{-4\pi/\bar{a}}} \right]$ (4.8)

and, in the perfectly aligned case, it becomes

$$\mu_{\text{tot}}^{\text{out}}(0) = 4 \frac{D_{\text{os}}^2}{D_{\text{ls}}^2} \frac{\theta_{\text{m}}^2}{\bar{\alpha}\beta_{\text{s}}} \left[\frac{e^{(\bar{b}-\pi)/\bar{\alpha}}}{1 - e^{-2\pi/\bar{\alpha}}} + \frac{e^{2(\bar{b}-\pi)/\bar{\alpha}}}{1 - e^{-4\pi/\bar{\alpha}}} \right].$$
(4.9)

Second, we comment on the secondary photon sphere. In this case. if we read $\theta_{\rm m}$ as $\theta^{\rm sc} \equiv b_{\rm sc}/D_{\rm ol}$ and if we use \bar{a} and \bar{b} given by Eqs. (2.18) and (2.19), respectively, the above formulas for the images of the rays reflected little outside of the primary photon sphere can be used as the ones of the secondary photon sphere.

2. Light rays reflected barely inside of the primary photon sphere

By using the deflection angle (2.3), the image angle of rays reflected slightly inside of the primary photon sphere is obtained as the positive solution of the lens equation:

$$\theta = \theta_n^{\rm in}(\beta) \equiv \frac{\theta_{\rm m}}{1 + e^{[\bar{d} - (1+2n)\pi + \beta]/\bar{c}}}.$$
 (4.10)

Its magnification for each n is obtained as

$$u_n^{\rm in}(\beta) = \frac{D_{\rm os}^2}{D_{\rm ls}^2} \frac{\theta_{\rm m}^2 e^{[\bar{d} - (1+2n)\pi]/\bar{c}}}{\bar{c}(1 + e^{[\bar{d} - (1+2n)\pi]/\bar{c}})^3} s(\beta).$$
(4.11)

⁶Note that $s(\beta) = 1/\beta$ for a point source.

1



FIG. 6. The apparent magnification with $\beta = 0$ and $D_{\rm ol} = 0.01$ pc for given a/m. Solid (red), dashed (green), and dotted (cyan) curves denote the apparent magnification of the retrolensing caused by the photon spheres with the mass $m = 60 M_{\odot}$, $30 M_{\odot}$, and $10 M_{\odot}$, respectively.



FIG. 7. Motion of the sun S with a radius R_s on a source plane which is orthogonal to an optical axis $\beta = 0$. We assume that an observer O and a lens L are at rest and the sun moves with the orbital velocity v = 30 km/s on the source plane. The smallest source angle is denoted by β_{\min} .

Due to a negative solution $\theta \sim -\theta_n^{in}(\beta)$ for each winding number *n*, the total magnification of the couple of images for all the winding number *n* is given by



FIG. 8. Light curves with $D_{\rm ol} = 0.01$ pc. Top: solid (red), dashed (green), dotted (cyan), long-dashed (magenta), and long-dashed-short-dashed (black) curves denote light curves with a/m = 0, 0.5, 0.8, 1, and 3, respectively, and $m = 30 M_{\odot}$ and $\beta_{\rm min} = 0$. Middle: solid (red), dashed (green), dotted (cyan), and long-dashed (magenta) curves denote light curves with $\beta_{\rm min} = 0$, $0.5\beta_{\rm s}$, $\beta_{\rm s}$, and $1.5\beta_{\rm s}$, respectively, and m = $30 M_{\odot}$ and a/m = 0.8. Bottom: solid (red), dashed (green), and dotted (cyan) curves denote light curves with $m = 60 M_{\odot}$, $30 M_{\odot}$, and $10 M_{\odot}$, respectively, and $\beta_{\rm min} = 0$ and a/m = 0.8.

$$\mu_{\text{tot}}^{\text{in}}(\beta) = 2 \sum_{n=0}^{\infty} |\mu_n^{\text{in}}(\beta)|$$

= $2 \sum_{n=0}^{\infty} \frac{D_{\text{os}}^2}{D_{\text{ls}}^2} \frac{\theta_m^2 e^{[\bar{d} - (1+2n)\pi]/\bar{c}}}{\bar{c}(1 + e^{[\bar{d} - (1+2n)\pi]/\bar{c}})^3} |s(\beta)| \quad (4.12)$

and it is, in the perfect-aligned case,

$$\mu_{\text{tot}}^{\text{in}}(0) = 4 \sum_{n=0}^{\infty} \frac{D_{\text{os}}^2}{D_{\text{ls}}^2} \frac{\theta_m^2 e^{[\bar{d} - (1+2n)\pi]/\bar{c}}}{\bar{c}(1 + e^{[\bar{d} - (1+2n)\pi]/\bar{c}})^3 \beta_{\text{s}}}.$$
 (4.13)

The apparent magnification of the retrolensing in the perfect-aligned case is shown in Fig. 6. If there are light rays reflected barely inside of the primary photon sphere, they are dominant. On the other hand, if there are rays reflected slightly outside of the secondary photon sphere, their effect on light curves can be ignored since they are fainter than the others.

C. Retrolensing light curves

We assume that the lens L is at rest against the observer O as shown in Fig. 7. The retrolensing light curves by the photon spheres are shown in Fig. 8.

V. CONCLUSION

We have investigated retrolensing in a novel blackbounce spacetime [73], which is a black hole spacetime with a photon sphere or a wormhole spacetime with one or two photon spheres on one side of a region against a throat. If the wormhole has two photon spheres there, light rays reflected by the inner photon sphere on a throat can be ignored since the magnifications are dimmer than the ones by the outer photon sphere. However, the retrolensing by the wormhole with two photon spheres is brighter than the black hole with one photon sphere since rays reflected slightly not only outside but also inside of the outer photon sphere of the wormhole reach an observer due to absence of an event horizon. We can distinguish the retrolensing from other variable phenomena since the retrolensing light curve has characteristic shapes as shown in Fig. 8; it can be observed on the ecliptic, and it has precise solar spectra [36].

The light rays reflected near the outer photon sphere with the winding number n = 1 reach the observer a few milliseconds later than ones with n = 0. However, the rays with $n \ge 1$ are fainter than the lights with n = 0. Therefore, we can ignore the effect of the rays with $n \ge 1$ on the retrolensing light curves.

We have shown the percent errors of the deflection angles of retrolensing in strong deflection limits. We have found that the percent errors depend on the parameters of the metric and the cases of reflections: (i) a reflection nearly outside of the photon sphere on the throat, (ii) a reflection barely outside of the photon sphere off the throat, (iii) a reflection slightly inside of the photon sphere off the throat. We have shown that the case (iii) has larger error than the cases (i) and (ii). Thus, we should improve approximations in the strong deflection limit for the retrolensing of the rays reflected nearly inside of the photon sphere. One may use an improved approximation (2.4) which was applied in an usual lens configuration [19,89] and in the retrolensing configuration [89]. We will investigate the details of the improved approximation in a forthcoming paper [107].

We do not consider the case of $a/m = 2\sqrt{5}/5$ because the deflection angles of the light rays reflected by the outer photon do not diverge logarithmically in the forms of Eqs. (2.2) and (2.3). Note that the observables of the outer photon sphere with the vanishing winding number n = 0 obtained by using Eqs. (2.2) and (2.3) in the strong deflection limits have relativity large error for $a/m \sim 2\sqrt{5}/5 - \epsilon$. In the case of $a/m = 2\sqrt{5}/5$, the outer photon sphere and an antiphoton sphere degenerate into a marginally unstable photon sphere and the deflection angles of the light rays reflected by it would diverge nonlogarithmically in the strong deflection limits. We can obtain the deflection angle of the light rays reflected slightly outside of the marginally unstable photon sphere in the strong deflection limit as well as Ref. [20]. On the other hand, the case of the rays reflected slightly inside of the marginally unstable photon sphere and retrolensing by the marginally unstable photon sphere are left as future works.

ACKNOWLEDGMENTS

The author thanks an anonymous referee for his or her useful comments.

- [1] V. Perlick, Living Rev. Relativity 7, 9 (2004).
- [2] V. Perlick and O. Y. Tsupko, Phys. Rep. 947, 1 (2022).
- [3] Y. Hagihara, Jpn. J. Astron. Geophys. 8, 67 (1931), https:// adsabs.harvard.edu/full/1930JaJAG...8...67H.
- [4] C. Darwin, Proc. R. Soc. A 249, 180 (1959).
- [5] R. d' E. Atkinson, Astron. J. 70, 517 (1965).
- [6] J.-P. Luminet, Astron. Astrophys. 75, 228 (1979), https:// adsabs.harvard.edu/full/1979A%26A....75..228L.
- [7] H. C. Ohanian, Am. J. Phys. 55, 428 (1987).
- [8] R. J. Nemiroff, Am. J. Phys. 61, 619 (1993).
- [9] S. Frittelli, T. P. Kling, and E. T. Newman, Phys. Rev. D 61, 064021 (2000).

- [10] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D 62, 084003 (2000).
- [11] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, Gen. Relativ. Gravit. **33**, 1535 (2001).
- [12] V. Bozza, Phys. Rev. D 66, 103001 (2002).
- [13] V. Perlick, Phys. Rev. D 69, 064017 (2004).
- [14] K. K. Nandi, Y. Z. Zhang, and A. V. Zakharov, Phys. Rev. D 74, 024020 (2006).
- [15] K. S. Virbhadra, Phys. Rev. D 79, 083004 (2009).
- [16] V. Bozza, Gen. Relativ. Gravit. 42, 2269 (2010).
- [17] N. Tsukamoto and T. Harada, Phys. Rev. D 95, 024030 (2017).
- [18] R. Shaikh, P. Banerjee, S. Paul, and T. Sarkar, J. Cosmol. Astropart. Phys. 07 (2019) 028.
- [19] R. Shaikh, P. Banerjee, S. Paul, and T. Sarkar, Phys. Rev. D 99, 104040 (2019).
- [20] N. Tsukamoto, Phys. Rev. D 102, 104029 (2020).
- [21] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. **116**, 061102 (2016).
- [22] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), Astrophys. J. 875, L1 (2019).
- [23] D. R. Wilkins, L. C. Gallo, E. Costantini, W. N. Brandt, and R. D. Blandford, Nature (London) 595, 657 (2021).
- [24] J. Keir, Classical Quantum Gravity **33**, 135009 (2016).
- [25] V. Cardoso, L. C. B. Crispino, C. F. B. Macedo, H. Okawa, and P. Pani, Phys. Rev. D 90, 044069 (2014).
- [26] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, Phys. Rev. Lett. **119**, 251102 (2017).
- [27] R. Kudo and H. Asada, Phys. Rev. D 105, 084014 (2022).
- [28] S. Hod, Phys. Lett. B 727, 345 (2013); 776, 1 (2018).
- [29] N. G. Sanchez, Phys. Rev. D 18, 1030 (1978); Y. Decanini,
 A. Folacci, and B. Raffaelli, Phys. Rev. D 81, 104039 (2010); S. W. Wei, Y. X. Liu, and H. Guo, Phys. Rev. D 84, 041501 (2011).
- [30] W. H. Press, Astrophys. J. **170**, L105 (1971); C. J. Goebel, Astrophys. J. **172**, L95 (1972); I. Z. Stefanov, S. S. Yazadjiev, and G. G. Gyulchev, Phys. Rev. Lett. **104**, 251103 (2010); B. Raffaelli, Gen. Relativ. Gravit. **48**, 16 (2016).
- [31] M. A. Abramowicz, Mon. Not. R. Astr. Soc. 245, 733 (1990), https://adsabs.harvard.edu/full/1990MNRAS.245.
 .733A; W. Hasse and V. Perlick, Gen. Relativ. Gravit. 34, 415 (2002).
- [32] P. Mach, E. Malec, and J. Karkowski, Phys. Rev. D 88, 084056 (2013); E. Chaverra and O. Sarbach, Classical Quantum Gravity 32, 155006 (2015); M. Cvetic, G. W. Gibbons, and C. N. Pope, Phys. Rev. D 94, 106005 (2016); Y. Koga and T. Harada, Phys. Rev. D 94, 044053 (2016).
- [33] C. Barcelo and M. Visser, Nucl. Phys. B584, 415 (2000);
 Y. Koga, Phys. Rev. D 101, 104022 (2020).
- [34] W. L. Ames and K. S. Thorne, Astrophys. J. 151, 659 (1968); J. L. Synge, Mon. Not. R. Astron. Soc. 131, 463 (1966); H. Yoshino, K. Takahashi, and K. i. Nakao, Phys. Rev. D 100, 084062 (2019).
- [35] C. M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, J. Math. Phys. (N.Y.) 42, 818 (2001); G. W. Gibbons and C. M. Warnick, Phys. Lett. B 763, 169 (2016); H. Yoshino, K. Izumi, T. Shiromizu, and Y. Tomikawa, Prog. Theor. Exp. Phys. 2017, 063E01 (2017); D. V. Gal'tsov and K. V. Kobialko, Phys. Rev. D 100, 104005 (2019); M. Siino, Classical Quantum Gravity 38, 025005 (2021); K. Izumi,

Y. Tomikawa, T. Shiromizu, and H. Yoshino, Prog. Theor. Exp. Phys. **2021**, 083E02 (2021).

- [36] D. E. Holz and J. A. Wheeler, Astrophys. J. 578, 330 (2002).
- [37] F. De Paolis, G. Ingrosso, A. Geralico, and A. A. Nucita, Astron. Astrophys. 409, 809 (2003).
- [38] F. De Paolis, A. Geralico, G. Ingrosso, A. A. Nucita, and A. Qadir, Astron. Astrophys. **415**, 1 (2004).
- [39] A. Abdujabbarov, B. Ahmedov, N. Dadhich, and F. Atamurotov, Phys. Rev. D 96, 084017 (2017).
- [40] E. F. Eiroa and D. F. Torres, Phys. Rev. D 69, 063004 (2004).
- [41] V. Bozza and L. Mancini, Astrophys. J. 611, 1045 (2004).
- [42] N. Tsukamoto and Y. Gong, Phys. Rev. D 95, 064034 (2017).
- [43] N. Tsukamoto, Phys. Rev. D 95, 084021 (2017).
- [44] G. Zaman Babar, F. Atamurotov, and A. Zaman Babar, arXiv:2104.01340.
- [45] N. Tsukamoto, Phys. Rev. D 105, 024009 (2022).
- [46] M. Visser, Lorentzian Wormholes: From Einstein to Hawking (American Institute of Physics, Woodbury, NY, 1995).
- [47] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).
- [48] T. Ohgami and N. Sakai, Phys. Rev. D 91, 124020 (2015).
- [49] S. Paul, R. Shaikh, P. Banerjee, and T. Sarkar, J. Cosmol. Astropart. Phys. 03 (2020) 055.
- [50] S. Kasuya and M. Kobayashi, Phys. Rev. D 103, 104050 (2021).
- [51] Y. Nambu, S. Noda, and Y. Sakai, Phys. Rev. D 100, 064037 (2019).
- [52] K. A. Bronnikov and J. C. Fabris, Phys. Rev. Lett. 96, 251101 (2006).
- [53] K. A. Bronnikov, V. N. Melnikov, and H. Dehnen, Gen. Relativ. Gravit. 39, 973 (2007).
- [54] T. Damour and S. N. Solodukhin, Phys. Rev. D 76, 024016 (2007).
- [55] A. Simpson and M. Visser, J. Cosmol. Astropart. Phys. 02 (2019) 042.
- [56] J. R. Nascimento, A. Y. Petrov, P. J. Porfirio, and A. R. Soares, Phys. Rev. D 102, 044021 (2020).
- [57] A. Övgün, Turk. J. Phys. 44, 465 (2020).
- [58] N. Tsukamoto, Phys. Rev. D 103, 024033 (2021).
- [59] X. T. Cheng and Y. Xie, Phys. Rev. D 103, 064040 (2021).
- [60] H. C. D. Lima, Jr., L. C. B. Crispino, Pedrp V. P. Cunha, and C. A. R. Herdeiro, Phys. Rev. D 103, 084040 (2021).
- [61] K. A. Bronnikov, R. A. Konoplya, and T. D. Pappas, Phys. Rev. D 103, 124062 (2021).
- [62] M. Guerrero, G. J. Olmo, D. Rubiera-Garcia, and D. S. C. Gómez, J. Cosmol. Astropart. Phys. 08 (2021) 036.
- [63] P. Bambhaniya, S. K, K. Jusufi, and P. S. Joshi, Phys. Rev. D 105, 023021 (2022).
- [64] J. Schee and Z. Stuchlík, J. Cosmol. Astropart. Phys. 01 (2022) 054.
- [65] R. Della Monica and I. de Martino, J. Cosmol. Astropart. Phys. 03 (2022) 007.
- [66] Z. Stuchlík and J. Vrba, Universe 7, 279 (2021).
- [67] K. A. Bronnikov and R. K. Walia, Phys. Rev. D 105, 044039 (2022).
- [68] J. Mazza, E. Franzin, and S. Liberati, J. Cosmol. Astropart. Phys. 04 (2021) 082.

- [69] R. Shaikh, K. Pal, K. Pal, and T. Sarkar, Mon. Not. R. Astron. Soc. **506**, 1229 (2021).
- [70] S. U. Islam, J. Kumar, and S. G. Ghosh, J. Cosmol. Astropart. Phys. 10 (2021) 013.
- [71] X. Jiang, P. Wang, H. Yang, and H. Wu, Eur. Phys. J. C 81, 1043 (2021).
- [72] H. Huang and J. Yang, Phys. Rev. D 100, 124063 (2019).
- [73] F. S. N. Lobo, M. E. Rodrigues, M. V. d. S. Silva, A. Simpson, and M. Visser, Phys. Rev. D 103, 084052 (2021).
- [74] E. Franzin, S. Liberati, J. Mazza, A. Simpson, and M. Visser, J. Cosmol. Astropart. Phys. 07 (2021) 036.
- [75] Z. Xu and M. Tang, Eur. Phys. J. C 81, 863 (2021).
- [76] N. Chatzifotis, E. Papantonopoulos, and C. Vlachos, Phys. Rev. D 105, 064025 (2022).
- [77] Y. Guo and Y. G. Miao, arXiv:2112.01747.
- [78] J. Barrientos, A. Cisterna, N. Mora, and A. Viganò, arXiv: 2202.06706.
- [79] R. Shaikh, P. Banerjee, S. Paul, and T. Sarkar, Phys. Lett. B 789, 270 (2019); 791, 422(E) (2019).
- [80] N. Godani and G. C. Samanta, Ann. Phys. (Amsterdam) 429, 168460 (2021).
- [81] Q. Gan, P. Wang, H. Wu, and H. Yang, Phys. Rev. D 104, 024003 (2021).
- [82] Q. Gan, P. Wang, H. Wu, and H. Yang, Phys. Rev. D 104, 044049 (2021).
- [83] J. Peng, M. Guo, and X. H. Feng, Phys. Rev. D 104, 124010 (2021).
- [84] M. Guerrero, G.J. Olmo, and D. Rubiera-Garcia, J. Cosmol. Astropart. Phys. 04 (2021) 066.
- [85] X. Wang, P. C. Li, C. Y. Zhang, and M. Guo, Phys. Lett. B 811, 135930 (2020).
- [86] M. Wielgus, J. Horak, F. Vincent, and M. Abramowicz, Phys. Rev. D 102, 084044 (2020).
- [87] H. G. Ellis, J. Math. Phys. (N.Y.) 14, 104 (1973); K. A. Bronnikov, Acta Phys. Pol. B 4, 251 (1973), https://www.actaphys.uj.edu.pl/R/4/3/251; C. Martinez and M. Nozawa, Phys. Rev. D 103, 024003 (2021).
- [88] L. Chetouani and G. Clément, Gen. Relativ. Gravit. 16, 111 (1984); K. Nakajima and H. Asada, Phys. Rev. D 85,

107501 (2012); N. Tsukamoto, T. Harada, and K. Yajima, Phys. Rev. D **86**, 104062 (2012); F. Abe, Astrophys. J. **725**, 787 (2010); Y. Toki, T. Kitamura, H. Asada, and F. Abe, Astrophys. J. **740**, 121 (2011); N. Tsukamoto and T. Harada, Phys. Rev. D **87**, 024024 (2013); R. Takahashi and H. Asada, Astrophys. J. **768**, L16 (2013); N. Tsukamoto and Y. Gong, Phys. Rev. D **97**, 084051 (2018).

- [89] N. Tsukamoto, Phys. Rev. D 104, 064022 (2021).
- [90] M. Guerrero, G. J. Olmo, D. Rubiera-Garcia, and D. S. C. Gómez, arXiv:2202.03809.
- [91] E. F. Eiroa, G. E. Romero, and D. F. Torres, Phys. Rev. D 66, 024010 (2002).
- [92] V. Bozza, F. De Luca, G. Scarpetta, and M. Sereno, Phys. Rev. D 72, 083003 (2005).
- [93] V. Bozza, F. De Luca, and G. Scarpetta, Phys. Rev. D 74, 063001 (2006).
- [94] S. V. Iyer and A. O. Petters, Gen. Relativ. Gravit. 39, 1563 (2007).
- [95] V. Bozza and G. Scarpetta, Phys. Rev. D 76, 083008 (2007).
- [96] N. Tsukamoto, Phys. Rev. D 94, 124001 (2016).
- [97] A. Ishihara, Y. Suzuki, T. Ono, and H. Asada, Phys. Rev. D 95, 044017 (2017).
- [98] N. Tsukamoto, Phys. Rev. D 95, 064035 (2017).
- [99] G. F. Aldi and V. Bozza, J. Cosmol. Astropart. Phys. 02 (2017) 033.
- [100] T. Hsieh, D.S. Lee, and C.Y. Lin, Phys. Rev. D 103, 104063 (2021).
- [101] K. Takizawa and H. Asada, Phys. Rev. D 103, 104039 (2021).
- [102] G. S. Bisnovatyi-Kogan and O. Y. Tsupko, Phys. Rev. D 105, 064040 (2022).
- [103] V. Bozza, Phys. Rev. D 78, 103005 (2008).
- [104] H. J. Witt and S. Mao, Astrophys. J. 430, 505 (1994).
- [105] R. J. Nemiroff and W. A. D. T. Wickramasinghe, Astrophys. J. 424, L21 (1994).
- [106] C. Alcock *et al.* (MACHO and GMAN Collaborations), Astrophys. J. **491**, 436 (1997).
- [107] N. Tsukamoto (to be published).