Supergravity black holes, Love numbers, and harmonic coordinates

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A problem which has now come to a head in general relativity, primarily as a result of the many recent detections of gravitational waves from the coalescence of compact binaries, is that there are very few viable theories against which to "test" it. To perform realistic tests of theories of gravity, we need to be able to look beyond general relativity and evaluate the consistency of a parametrized, physically acceptable, family of black hole metric alternatives with observational data from, especially, gravitational wave detections using, for example, an agnostic Bayesian approach. In this paper we further examine properties of one class of such metrics, which in fact arise as solutions of ungauged supergravity. In particular, we examine the massless, neutral, minimally coupled scalar wave equation in a general stationary, axisymmetric background metric such as that of a charged rotating black hole, when the scalar field is either time independent or in the low-frequency, near-zone limit, with a view to calculating the Love numbers of tidal perturbations, and of obtaining harmonic coordinates for the background metric. For a four-parameter family of charged asymptotically flat rotating black hole solutions of ungauged supergravity theory known as STU black holes, which includes Kaluza-Klein black holes and the Kerr-Sen black hole as special cases, we find that all time-independent solutions, and hence the harmonic coordinates of the metrics, are identical to those of the Kerr solution. In the low-frequency limit we find the scalar fields exhibit the same $SL(2,R)$ symmetry as holds in the case of the Kerr solution. We point out extensions of our results to a wider class of metrics, which includes solutions of Einstein-Maxwell-dilaton theory.

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I. INTRODUCTION

The era of gravitational wave detection has arrived. So far, gravitational waves from the mergers of neutron stars and black holes have been observed, but it is also true that these have waveforms which we best know how to calculate. In practice, waveforms may be obtained by a variety of methods—analytical, numerical, and perturbative—often in some hybrid combination. Unless one looks back in the literature written more than half a century ago, one could easily forget how important was the study of stability of both black holes and neutron stars, in a variety of mass and spin configurations, for their acceptance in the domain of observable physics: If slightly perturbed compact objects could explode rather than radiate away relatively small quantities of gravitational wave energy, and settle down to another equilibrium configuration, observations of merging black holes or neutron stars would be dramatically different from what we are now able to detect. To date, most analyses of compact objects and their merging has taken place within the framework of general relativity. In this paper we shall look beyond general relativity, further examining a parametrized, physically acceptable, family of black hole metric alternatives against which it might be tested.

For now more than a century, tests of general relativity have typically attempted to show that predictions of the theory have been borne out by observation. Thus, the default assumption in interpreting such observations as gravitational waves by LIGO [1] or the proposed space-borne LISA mission [2], the apparent black hole shadow in M87 by EHT [3,4], the black hole at the center of the Milky Way [5], and x-ray emission from black hole accretion discs [6], is in terms of the Kerr metric. In the era to come there will be growing interest in attempting to show, or to exclude, the possibility that the results of observations may be better described by some alternative theory of gravity. Thus, in accessing the reliability of present-day interpretations, or in looking for physics beyond the standard model, it is important to compare with the predictions for alternative metrics; or, to adapt the terminology introduced in [7], with *Kerr metric foils*, that is, asymptotically flat, axisymmetric, stationary metrics that are regular outside a nonsingular event horizon.[1]

A foil is not meant to be a full-blown alternative theory of gravity. Rather it is intended to provide a parametrized, physically acceptable, family of black hole metric alternatives to the more extensively studied Kerr class of metrics, against which potential departures from general relativity might be reasonably tested. To be useful for such purposes, one might require at the least that:

(i) The Hamilton-Jacobi equation for null geodesics is Liouville integrable.
(ii) The energy momentum tensor of all fields other than the metric satisfies acceptable positive-energy conditions.
(iii) The metric is a solution of a well-defined set of field equations, having a well-posed initial value problem.
(iv) The propagation of time-dependent solutions is causal.
(v) The spacetime of the foil has positive energy.

Although not essential, it is also highly desirable that the equations of motion, which might reasonably be assumed to contain only massless scalar and vector fields in addition to the metric, may be derived from an action principle, thus admitting a Hamiltonian formulation and therefore a well-defined notion of total energy and angular momentum. Moreover, progress would be facilitated if, at least to some degree, separability of the equations for linear perturbations of the solutions holds, analogous to the case for the Kerr metric, and also that the analysis of magnetic fields around the foil solutions be tractable. A final desirable requirement is that the foil solutions have some degree of uniqueness, to provide reassurance that predictions made using them are robust.

Various parametrized classes of metrics have been employed previously as foils, in order to probe the extent to which observational predictions would change as the spacetime geometry is modified away from that of the standard Kerr geometry. In some of these modified metrics may not necessarily obey basic natural requirements such as being free of naked singularities, or being free of closed timelike curves outside the horizon. Later proposals have addressed some of these issues. (See [8–10] for some examples.) These classes of metrics commonly involve arbitrary functions of one or more of the spacetime coordinates in order to parametrize the departure from the Kerr metric.

Another proposed class of foils that have been extensively considered are the Kerr-Sen black holes of string theory [11]. These are charged rotating black holes that arise as solutions of the low-energy limit of string theory, which reduce to the Kerr black hole if the charge is set to zero. They have the merit of being solutions of a known theory with well-posed field equations, they satisfy positive-energy conditions, have no pathologies such as closed timelike curves, and the massless wave equations for scalar and higher-spin fields are separable, thus allowing relatively straightforward calculation of certain important physical properties. Their use as foils has been discussed, for example, in [3–5,12,13].

Viewed as foils, the Kerr-Sen black holes have just one parameter of freedom that parametrizes deviations from the Kerr black hole, namely the electric charge. We have previously proposed that a convenient larger family of foils is provided by the rotating black holes of STU supergravity. In previous work we studied the electrodynamics [14]; the initial value problem [15]; the photon sphere and sonic horizons [16]; the equatorial timelike geodesics [17]; and the stability of massless, minimally-coupled scalar fields [18], for a six-parameter family of rotating black holes of STU supergravity. These solutions admit a separation of variables for the scalar wave equation [19,20]. In particular, this class of black hole spacetimes contains as special cases: (i) the Kerr-Newman metric, (ii) its analogue in Kaluza-Klein theory [21], and (iii) the Kerr-Sen analogue in string theory [11].

With this in mind, in this paper we shall study a class of rotating black hole solutions in STU supergravity as foils against which results for the Kerr black hole can be compared. STU supergravity is $\mathcal{N}=2$ supergravity coupled to three vector supermultiplets, and its bosonic sector therefore comprises gravity coupled to four Maxwell-type gauge fields and six scalar fields [22]. Our focus will be on the rotating black holes in which each of the four gauge fields carries an independent electric charge, so the solutions are characterized by six independent parameters (the mass, the angular momentum, and four electric charges). If the electric charges are turned off the solution reduces to the Kerr metric. We emphasize that we are not proposing these STU supergravity black holes as physically realistic solutions in the observed universe; rather, we are proposing that they constitute a useful and convenient multiparameter family of foils whose properties can be compared with those of the Kerr metric itself. Importantly, these metrics satisfy all of the desiderata that we highlighted above. They provide a wider class of foils than some that have been considered previously (such as the Kerr-Sen black hole that was employed as a foil in [3–5,12,13], and which is contained as a special case within the STU black holes that we are considering).

In this work, our main focus will again be on scalar fields in the STU supergravity black hole backgrounds, since they are easier to work with than the full gravitational perturbations while still carrying the expectation that they will be indicative of behaviors that would be manifested also in a more extensive analysis. One aim in the present paper is to extend our previous work to an examination of tidal effects

[1]The “foils” of [7] were wormholes; ours are black holes.

induced on rotating STU black holes by orbiting black holes or neutron stars. These effects are characterized by dimensionless numbers analogous to those introduced by Love in his purely Newtonian study of tidal distortions of the shape of the earth by the moon [23]. The definition of tidal Love numbers[2] for static $SO(d)$ invariant asymptotically flat vacuum black holes in $d+1$ dimensions is given in [24], where the reader will find an extensive list of references to earlier work. Using the equations for static gravitational perturbations of such metrics given in [25], the remarkable result was obtained that the Love numbers vanish if $d = 3$ but are nonvanishing for $d > 3$.

Recently [26–28], it was shown that the vanishing of the tidal Love numbers also holds for the Kerr solution. This may be seen [28] by taking an appropriate limit of the Teukolsky equation [29] that governs scalar, spinor, vector and tensor perturbations. This achievement has sparked off attempts to provide an underlying explanation for this remarkable phenomenon [30–33].

The paper is organized as follows. In Sec. II, we show by considering an arbitrary stationary metric written in the form of a time fibration over a spatial 3-dimensional base metric, that static (time independent) solutions of the scalar wave equation are governed by an equation that depends only on the spatial 3-metric. In particular, this means that not only for the charged rotating STU supergravity black holes, which we discuss explicitly, but also more generally in much broader classes of charged black holes, the static scalar wave functions obey the same equation in the charged metrics as they do in the uncharged ones. In Sec. III we show that this means the static Love numbers for the scalar field in the charged supergravity black holes are the same as those in an uncharged Kerr black hole. We also comment on the distinction between the static Love numbers, which therefore vanish, and the dynamical response that arises as a consequence of tidal interactions. In Sec. IV, we show that if one looks at the properties of the scalar wave equation for nonzero but low frequency fields, then just as occurs in the Kerr background, one finds an $SL(2,\mathbb{R})$ symmetry in the near-zone regime.

In Sec. V, we study the time-independent solutions of the massless scalar wave equation in two classes of static black hole backgrounds, showing how in each case the 3-dimensional spatial metric is conformal to the metric on 3-dimensional hyperbolic space. This provides a geometrical insight into the $SL(2,\mathbb{R})$ symmetries of the static wave functions in these cases. One of the classes of static black holes we study here are 4-charge black holes in STU supergravity. The other class comprises charged black holes in Einstein-Maxwell-dilaton (EMD) gravity. We consider these black holes for arbitrary values of the constant characterizing the coupling of the dilaton to the Maxwell field. Only in certain special discrete cases do these EMD black holes coincide with black holes in the STU supergravity class.

In Sec. VI, we apply some of our previous results to the construction of harmonic coordinates for the charged black hole metrics. These are of interest because the Einstein equations then become a semilinear symmetric hyperbolic system, which can be useful for studying certain mathematical properties of the solutions. They can also be used in order to define the energy and momentum in terms of the Landau-Lifshitz energy-momentum pseudotensor. The paper ends with conclusions in Sec. VII.

II. SCALAR WAVE EQUATION IN CHARGED ROTATING BLACK HOLE BACKGROUNDS

A. Charged from uncharged black holes

It was observed in [34] that the charged rotating black holes of four-dimensional ungauged STU supergravity can be conveniently constructed by starting from the neutral Kerr black hole as a seed solution,[3] written in the time-fibered Kaluza-Klein form

$$ds_4^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2U}(dt + \mathcal{A}_i dx^i)^2 + e^{-2U}\gamma_{ij}dx^i dx^j, \tag{2.1}$$

where the base metric γ_{ij}, the Kaluza-Klein vector $\mathcal{A}_i$ and the scalar U depend only on the three spatial coordinates x^i. By performing a Kaluza-Klein reduction of the STU supergravity theory on the time coordinate, using a general ansatz of the form (2.1) for the metric and

$$\hat{A}_\mu dx^\mu = A_i dx^i + \chi dt, \tag{2.2}$$

for each of the four STU supergravity gauge potentials, the resulting three-dimensional theory has an $O(4,4)$ global symmetry that can be used to introduce up to eight charges (independent electric and magnetic charges for the four gauge fields), after lifting back to four dimensions again.

An important point about this procedure of starting with a neutral seed solution and introducing charges by using the global symmetries of the three-dimensional reduced theory, is that the charge parameters enter the final four-dimensional metric only via the scalar function U and the Kaluza-Klein vector $\mathcal{A}_i$ in Eq. (2.1). This is because the metric γ_{ij} in the three-dimensional reduced theory is invariant under the $O(4,4)$ global symmetry. In other

[2]There has also been introduced a notion of dissipative Love numbers, about which we will say more below.

[3]To be more precise, the seed solution should be taken to be the Kerr-NUT metric in general, with the seed NUT parameter eventually enabling the cancellation of a further NUT charge that can arise when certain combinations of sufficiently many charges are introduced using the generating procedure.

words, in the final four-dimensional charged solution, taking the form (2.1), the three-dimensional base metric γ_{ij} is unchanged from the form it took in the original seed solution.

It is worth emphasizing that this procedure for generating charged solutions from an uncharged seed solution can be applied much more generally than in the specific case of the charged STU supergravity black holes we are considering in this paper. For any theory whose Kaluza-Klein reduction to three dimensions yields a theory with global symmetry group G, one can (i) reduce from a stationary solution using the metric ansatz (2.1); (ii) act with the symmetry G; and then (iii) lift it back to four dimensions again. Quite generally, since the three-dimensional metric is invariant under G, the charge parameters in the lifted solution enter only via the scalar function U and the Kaluza-Klein vector $\mathcal{A}_i$. Extensive discussions of the three-dimensional global symmetries for rather general higher-dimensional starting points can be found, for example, in [35,36].

As can easily be seen from Eq. (2.1), the determinant of the four-dimensional metric $g_{\mu\nu}$ is related to that of the three-dimensional metric γ_{ij} by $\sqrt{-g} = e^{-2U}\sqrt{\gamma}$, and so

$$\sqrt{-g}\left(\frac{\partial}{\partial s_4}\right)^2 = \sqrt{-g}g^{\mu\nu}\partial_\mu \otimes \partial_\nu = -\sqrt{\gamma}e^{-4U}\partial_t^2 + \sqrt{\gamma}\gamma^{ij}(\partial_i - \mathcal{A}_i\partial_t)(\partial_j - \mathcal{A}_j\partial_t). \tag{2.3}$$

This means that the four-dimensional D'Alembertian wave operator $\Box$ on scalar functions is given by

$$\Box\Psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\mu\Psi) = -e^{-2U}\partial_t^2\Psi + e^{2U}\gamma^{ij}(\nabla_i - \mathcal{A}_i\partial_t)(\nabla_j - \mathcal{A}_j\partial_t)\Psi, \tag{2.4}$$

where ∇_i is the covariant derivative in the three-dimensional base metric γ_{ij}. In particular, this means that if the wave function[4] $\Psi(t, \mathbf{x})$ is taken to be independent of time, $\Psi(t, \mathbf{x}) = \Psi(\mathbf{x})$, then

$$\Box\Psi(\mathbf{x}) = e^{2U}\gamma^{ij}\nabla_i\nabla_j\Psi(\mathbf{x}) = \frac{e^{2U}}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j\Psi(\mathbf{x})). \tag{2.5}$$

Thus a time-independent solution of the massless wave equation obeys

$$\gamma^{ij}\nabla_i\nabla_j\Psi(\mathbf{x}) = 0, \tag{2.6}$$

and this depends only on the metric γ_{ij} of the three-dimensional base. As already observed, this is identical in the original neutral seed solution and in the charged solution.

B. Charged rotating black holes in STU supergravity

Written in the Kaluza-Klein form (2.1), the neutral Kerr solution is given by the three-dimensional base metric

$$\gamma_{ij}dx^i dx^j = (\rho^2 - 2Mr)\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \Delta\sin^2\theta d\phi^2, \tag{2.7a}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr, \tag{2.7b}$$

together with the vector $\mathcal{A}_i$ and scalar U:

$$\mathcal{A}_i dx^i = \frac{2Mar\sin^2\theta}{\rho^2 - 2Mr}d\phi, \qquad e^{2U} = 1 - \frac{2Mr}{\rho^2}. \tag{2.8}$$

(Note that we can write $\mathcal{A}_i dx^i = a(e^{-2U} - 1)\sin^2\theta d\phi$ in the seed solution.)

Charged rotating STU black holes carrying just four charges in total were obtained originally in [22], and were obtained in the Kaluza-Klein formulation in [34]; for these solutions, the metric is given by (2.1) with the base 3-metric again given by (2.7), and the 1-form $\mathcal{A}_{(1)}$ and scalar U now given by

$$\mathcal{A}_i dx^i = \frac{2Ma[r\Pi_c - (r - 2M)\Pi_s]\sin^2\theta}{\rho^2 - 2Mr}d\phi, \qquad e^{2U} = \frac{\rho^2}{W}\left(1 - \frac{2Mr}{\rho^2}\right), \tag{2.9}$$

where

$$W^2 = r_1r_2r_3r_4 + 2a^2\left[r^2 + Mr\sum_i s_i^2 + 4M^2\Pi_s(\Pi_c - \Pi_s) - 2M^2\sum_{i<j<k}s_i^2s_j^2s_k^2\right]\cos^2\theta + a^4\cos^4\theta,$$
$$r_i = r + 2Ms_i^2, \qquad s_i = \sinh\delta_i, \qquad c_i = \cosh\delta_i,$$
$$\Pi_s = s_1s_2s_3s_4, \qquad \Pi_c = c_1c_2c_3c_4. \tag{2.10}$$

As already remarked, the charge parameters (the boost parameters δ_i) enter only in the expressions for $\mathcal{A}_i$ and U, and thus the scalar wave operator for time-independent wave functions, given by (2.5), remains unchanged from that for the original Kerr metric.[5]

[4]Note that in this paper we are using the term "wave function" in a purely classical sense.

[5]Note that for the particular combination of four charges that are introduced in this example, it suffices to take just the Kerr metric as the seed solution, since no NUT charge is generated in this case.

More generally, the full eight-charge rotating black hole family of solutions was obtained using these methods in [37].[6] As can be seen from the expressions given there, the charge parameters again enter only in the Kaluza-Klein vector $\mathcal{A}_i$ and the scalar U in the metric (2.1), and so again the massless wave equation for the case of time-independent wave functions is independent of the charge parameters.

III. TIME-INDEPENDENT SCALAR WAVE FUNCTIONS AND LOVE NUMBERS

Consider a time-independent solution $\Psi(r,\theta,\phi)$ of the massless scalar wave equation in the background of a charged rotating STU supergravity black hole. From Eqs. (2.5) and (2.7), it will obey

$$\partial_r(\Delta\partial_r\Psi)+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\Psi)-\frac{(\rho^2-2Mr)m^2}{\Delta\sin^2\theta}\Psi=0. \quad (3.1)$$

From the expressions for ρ and Δ in Eqs. (2.7) we have

$$\frac{\rho^2-2Mr}{\Delta\sin^2\theta}=\frac{1}{\sin^2\theta}-\frac{a^2}{\Delta}, \quad (3.2)$$

and so for factorized solutions with $\Psi(r,\theta,\phi)=R(r)S(\theta)e^{im\phi}$, the massless scalar wave equation separates, giving

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S)+\left[\lambda-\frac{m^2}{\sin^2\theta}\right]S=0, \quad (3.3)$$

implying that $S(\theta)$ is just the associated Legendre function $P_\ell^m(\cos\theta)$, and the separation constant λ is

$$\lambda=\ell(\ell+1), \qquad \ell=0,1,2,\ldots, \quad (3.4)$$

and therefore the radial equation is

$$\partial_r(\Delta\partial_r R)+\frac{a^2m^2}{\Delta}R-\ell(\ell+1)R=0. \quad (3.5)$$

A. Love numbers

In nonrelativistic gravity the Newtonian potential U of a tidally distorted body of mass M and mean radius R is given in spherical coordinates by [26,39]

$$U=\frac{M}{r}-\sum_{\ell,m}\frac{(\ell-2)!}{\ell!}E_{\ell m}r^\ell\left[1+2k_\ell\left(\frac{R}{r}\right)^{2\ell+1}\right]Y_{\ell m}(\theta,\phi), \quad (3.6)$$

where $Y_{\ell m}$ are spherical harmonics, $E_{\ell m}$ are a measure of the moments of an external tidal field and $k_\ell E_{\ell m}R^{2\ell+1}$ is a measure of the deformation of the gravitational field of the body. The quantity R is included on dimensional grounds, and renders the coefficients k_ℓ, known as tidal Love numbers, or TLNs, dimensionless. The name Love refers to the elastician Augustus Edward Hough Love who introduced the k_ℓ coefficients in [23]. The Love numbers provide a characterization of the elasticity or rigidity of the body, with larger Love numbers corresponding to greater elasticity. However, as Love noted [23], tidal forces are often dynamical, and (3.6) represents only the static part of an elastic body's response to tidal deformation.

In general relativity the situation is obviously more complicated and, for black holes, even more so. In particular, if the Love numbers are zero this does not indicate, as has sometimes been supposed, that a black hole has no response to an external tidal field. In fact, early studies of perturbations of black holes found that their surrounding spacetime would respond with quasinormal ringing [40,41], and York [42] showed that these actually resulted in changes in the area of the event horizon of a black hole. In numerical relativity it was later demonstrated that during black hole mergers, cross sections of the event horizon could become singular [43]. Thus, the static Love numbers being zero for black holes represents only part of the full story.

In examining the Love numbers of the Kerr solution, the approach usually taken has been to focus on the Teukolsky equations [29], which govern gauge invariant parts of the Weyl tensor arising from gravitational perturbations of the metric, or some equivalent formulation. These equations are necessarily rather complicated, and in any case so far their generalizations are not available for all of the rotating black holes we wish to consider. However, on the basis of our results, some insight may be gained from the behavior of static, i.e., time-independent, solutions of the wave equation $\Box f=0$ where $\Box$ is the covariant D'Alembertian of any metric of the form (2.1).

In [31], static Love numbers, which determine the response to time-independent external fields, were found to vanish in four-dimensional Einstein theory both for spherical and spinning black holes (see also [24,28,30,32,44–46]). Although this might appear to be at variance with [26,27], it is clear from [28] that dynamics is at the root of this apparent discrepancy. Specifically, while the static Love number is indeed zero for the Kerr black hole, there is a dissipative response (in the Weyl tensor) proportional to a superradiance factor, namely

[6]Because of the existence of an $SL(2,\mathbb{R})^3$ global symmetry of the STU theory in four dimensions, it actually suffices from the point of view of generality to construct a five-charge solution, since the remaining three charges can then be introduced in four dimensions by acting with the $U(1)^3$ maximal compact subgroup of $SL(2,\mathbb{R})^3$. This technique was employed for the case of the static STU black holes in [38].

$$-\mathrm{i}(m\Omega_H - \omega)\left[\frac{2Mr_+}{r_+ - r_-}\right]\nu_{\ell m}^{\mathrm{Kerr}}$$

which, except for the axisymmetric $m = 0$ mode, does not vanish in the zero-frequency limit (nor, indeed, in Schwarzschild with nonzero frequency). Here r_- and r_+ label the inner and outer horizons, Ω_H is the angular momentum of the outer horizon, and the $\nu_{\ell m}$ are new, *dissipative*, Love numbers. Note that the sign of this term changes in the superradiant regime. This discrepancy is specifically addressed in [27], and in its discussion of the papers [30,47].

As implied by our earlier discussion, the radial equation arising from the separation of variables for time-independent solutions of the massless scalar wave equation in the background of any charged rotating STU supergravity black hole is given by (3.5). Since this is identical in form to the corresponding radial equation in the Kerr black hole background, the same analysis given in [31,33] carries over identically to the charged supergravity black hole cases.

IV. NEAR-ZONE $SL(2,\mathbb{R})$ SYMMETRY

It was shown in [48] that the massless scalar wave equation for low-frequency wave functions in the Schwarzschild geometry exhibits a "hidden" $SL(2,\mathbb{R})$ symmetry. This observation was subsequently extended to more general black hole backgrounds, including the Kerr black hole in [31]. In this section, we show that the hidden $SL(2,\mathbb{R})$ symmetry is present also for the low-frequency massless wave equation in the background of the 4-charge rotating STU supergravity black holes. This is noteworthy because, unlike the zero-frequency results that we discussed earlier, which were insensitive to any of the details of the charge parameters, here the low-frequency massless wave equation does involve dependence on the charge parameters.

Considering now time-dependent massless scalar wave functions $\Phi = R(r)S(\theta)e^{-\mathrm{i}\omega t + \mathrm{i}m\phi}$, one can see from the expressions in Eqs. (2.7), (2.9), and (2.10) that the scalar wave operator (2.4) separates, giving the angular equation

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta S) + \left[\lambda + a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right]S = 0, \tag{4.1}$$

and the radial equation

$$\partial_r(\Delta\partial_r R) + (V_0 + V_1)R = \lambda R, \tag{4.2}$$

where

$$V_0 = \frac{4M^2(\Pi_c r_+ + \Pi_s r_-)^2}{\Delta}\left[(\omega - m\Omega)^2 - 4m\omega\Omega\frac{r - r_+}{r_+ - r_-}\right], \tag{4.3}$$

$$V_1 = \frac{2am\omega M(\Pi_c + \Pi_s)}{\kappa(r - r_-)(\Pi_c r_+ + \Pi_s r_-)} + \frac{8\omega^2 M^3(\Pi_c^2 - \Pi_s^2)}{r - r_-} + \omega^2(b_0 + b_1 r + r^2), \tag{4.4}$$

$$b_0 = 4M^2\left(1 + \sum_i s_i^2 + \sum_{i<j} s_i^2 s_j^2\right), \quad b_1 = 2M\left(1 + \sum_i s_i^2\right). \tag{4.5}$$

Here κ, the surface gravity of the outer horizon, and Ω, the angular velocity of the outer horizon, are given by

$$\kappa = \frac{r_+ - r_-}{4M(\Pi_c r_+ + \Pi_s r_-)}, \quad \Omega = \frac{a}{2M(\Pi_c r_+ + \Pi_s r_-)}. \tag{4.6}$$

The decomposition of the potential in the radial equation as the sum of the two terms V_0 and V_1 is motivated by the analysis in [31] for the case of the Kerr metric. The term V_0 contains all the terms that contribute at leading order in the near-zone region defined by

$$\omega(r - r_+) \ll 1, \tag{4.7}$$

while the term V_1 is of subleading order in a near-zone expansion.[7]

Following [31], and making the appropriate changes for our case, we now define $SL(2,\mathbb{R})$ generators as follows:

$$L_0 = \kappa^{-1}\partial_t,$$
$$L_{\pm 1} = e^{\pm\kappa t}\left(\mp\Delta^{1/2}\partial_r + \kappa^{-1}\partial_r(\Delta^{1/2})\partial_t + \frac{a}{\Delta^{1/2}}\partial_\phi\right). \tag{4.8}$$

These obey the $SL(2,\mathbb{R})$ algebra

$$[L_m, L_n] = (m - n)L_{m+n}, \quad -1 \le m \le 1, \quad -1 \le n \le 1. \tag{4.9}$$

Defining the quadratic $SL(2,\mathbb{R})$ Casimir operator $\mathcal{C}_2 \equiv L_0^2 - \frac{1}{2}(L_{-1}L_1 + L_1 L_{-1})$, the scalar wave equation in the near zone can (i.e., with the V_1 term suppressed) be written as

$$\mathcal{C}_2\Phi = \lambda\Phi, \tag{4.10}$$

with the radial function R satisfying the near-zone equation

[7]As noted in [31], there is some arbitrariness in the choice of the decomposition into the terms V_0 and V_1. The important point is that the terms in V_0 with $(r - r_+)$ in the denominator are the dominant ones in the near zone. The last term in the expression for V_0 in (4.3) (for which the $(r - r_+)$ factor coming from $\Delta = (r - r_+)(r - r_-)$ in the denominator, is canceled by the factor in the numerator), could just as well be assigned to V_1. It is included in V_0 in order to give a precise formulation of the near-zone equation that exhibits the $SL(2,\mathbb{R})$ symmetry, in Eq. (4.11) as seen below.

$$\partial_r(\Delta\partial_r R) + V_0 R = \lambda R, \tag{4.11}$$

Thus, the near-zone massless scalar wave equation exhibits an $SL(2,\mathbb{R})$ symmetry in the 4-charge rotating black hole background, extending the previous findings for the Schwarzschild [48] and Kerr [31] black holes.

V. TIME-INDEPENDENT SCALAR FIELDS IN STATIC BLACK HOLE BACKGROUNDS

It was observed in [32] that in the case of the Schwarzschild black hole background, the symmetries of the time-independent scalar wave functions, reflected in the tower of ladder operators that related the solutions with different values of ℓ in the decomposition in spherical harmonics $Y_{\ell m}(\theta,\varphi)$, were related to the fact that the massless time-independent wave operator was conformally related to that on three-dimensional hyperbolic space. In this section, we demonstrate that this feature extends to two broad classes of static black hole solutions, namely those in a general Einstein-Maxwell-dilaton (EMD) theory, and to the static 4-charge black holes of STU supergravity.

A. Static Einstein-Maxwell-dilaton black holes

These black holes are solutions of the theory described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}[R - 2(\partial\phi)^2 - e^{-2a\phi}F^2], \tag{5.1}$$

where the dimensionless constant, a, can be arbitrary. The static black hole solutions were obtained in [49]. In a convenient parametrization described in [16], the metrics are given in terms of an isotropic radial coordinate ρ by

$$ds^2 = -e^{2U}dt^2 + \Phi^4(d\rho^2 + \rho^2 d\Omega^2), \tag{5.2}$$

where

$$e^{2U} = V^2 W^{\frac{2(1-a^2)}{1+a^2}}(CD)^{-\frac{2}{1+a^2}}, \qquad \Phi^2 = (CD)^{\frac{1}{1+a^2}}W^{\frac{2a^2}{1+a^2}},$$
$$C = 1 + \frac{u^2}{\rho}, \qquad D = 1 + \frac{v^2}{\rho}, \qquad W = 1 + \frac{uv}{\rho},$$
$$V = 1 - \frac{uv}{\rho}, \tag{5.3}$$

where u and v are constants parametrizing the mass and the charge.

It can be seen that

$$\sqrt{-g}g^{\rho\rho} = \rho^2 VW\sin\theta, \qquad \sqrt{-g}g^{\theta\theta} = VW\sin\theta,$$
$$\sqrt{-g}g^{\varphi\varphi} = \frac{VW}{\sin\theta}, \tag{5.4}$$

and therefore a time-independent solution Ψ of the massless scalar wave equation obeys

$$\partial_\rho(\rho^2 VW\partial_\rho\Psi) + VW\nabla^2_{(\theta,\varphi)}\Psi = 0, \tag{5.5}$$

where $\nabla^2_{(\theta,\varphi)} = \csc\theta\partial_\theta(\sin\theta\partial_\theta) + \csc^2\theta\partial^2_\varphi$ is the scalar Laplacian on the unit sphere. In terms of the standard radial coordinate r defined by

$$r = \rho CD = \rho + u^2 + v^2 + \frac{u^2 v^2}{\rho}, \tag{5.6}$$

the outer and inner horizons are located at $r_\pm = (u \pm v)^2$, and defining $\bar{\Delta} = (r - r_+)(r - r_-)$ we have

$$\bar{\Delta} = \rho^2 V^2 W^2. \tag{5.7}$$

The time-independent wave equation (5.5) becomes

$$\partial_r(\bar{\Delta}\partial_r\Psi) + \nabla^2_{(\theta,\varphi)}\Psi = 0, \tag{5.8}$$

and this can be seen to be the Laplace equation in the 3-metric

$$ds_3^2 = dr^2 + \bar{\Delta}d\Omega^2. \tag{5.9}$$

Following the same steps as were described in [32] in the case of the Schwarzschild metric, we find here for the static EMD black holes that the conformally rescaled metric $d\tilde{s}_3^2 = \Omega^2 ds_3^2$, with $\Omega = L^2/\bar{\Delta}$, is given in terms of the isotropic radial coordinate ρ by

$$d\tilde{s}_3^2 = \frac{L^4}{(\rho^2 - u^2 v^2)^2}(d\rho^2 + \rho^2 d\Omega^2). \tag{5.10}$$

This can be recognized as the homogeneous metric on three-dimensional hyperbolic space.

B. Static 4-charge STU supergravity black holes

These can be conveniently written, in terms of an isotropic radial coordinate, in the form [16]

$$ds^2 = -\Pi^{-1/2}f_+^2 f_-^2 dt^2 + \Pi^{1/2}(d\rho^2 + \rho^2 d\Omega^2),$$
$$\Pi = \prod_{I=1}^{4} C_I D_I, \qquad f_\pm = 1 \pm \frac{m}{2\rho}, \qquad C_I = 1 + \frac{m e^{2\delta_I}}{2\rho},$$
$$D_I = 1 + \frac{m e^{-2\delta_I}}{2\rho}, \tag{5.11}$$

where the constants m and δ_I parametrize the mass and the four charges. The massless wave equation for a time-independent function Ψ is therefore given by

$$\partial_\rho(\rho^2 f_+ f_- \partial_\rho \Psi) + f_+ f_- \nabla^2_{(\theta,\varphi)}\Psi = 0. \tag{5.12}$$

In terms of the radial coordinate r defined by

$$r = \rho + m + \frac{m^2}{4\rho}, \tag{5.13}$$

and defining $\bar\Delta \equiv r(r-2m) = \rho^2 f_+^2 f_-^2$, Eq. (5.12) becomes

$$\partial_r(\bar\Delta \partial_r \Psi) + \nabla^2_{(\theta,\varphi)}\Psi = 0. \tag{5.14}$$

This is the Laplace equation in the metric $ds_3^2 = dr^2 + \bar\Delta d\Omega^2$. The conformally rescaled metric $d\tilde s_3^2 = \Omega^2 ds_3^2$, with $\Omega = L^2/\bar\Delta$, is recognizable as the metric

$$d\tilde s_3^2 = \frac{L^4}{(\rho^2 - \frac{m^2}{4})^2}(d\rho^2 + \rho^2 d\Omega^2), \tag{5.15}$$

on three-dimensional hyperbolic space, when written in terms of the isotropic coordinate ρ.

In summary, we have seen that both in the case of the static EMD black holes and the static 4-charge STU black holes, the massless time-independent scalar wave equation reduces to the same form as was found in [32] in the case of the Schwarzschild metric. Thus the construction of the time-independent wave functions in terms of ladder operators proceeds in the same way as was seen in [32], with the underlying conformally related hyperbolic metric providing a geometric understanding for the ladder structure.

In all the static backgrounds considered, the time-independent solutions of the massless scalar wave equation obey the Laplace equation

$$g^{ij}\nabla_i\nabla_j\Psi \equiv \nabla^2\Psi = 0, \tag{5.16}$$

in a 3-metric of the form (5.9) with $\bar\Delta$ given by

$$\bar\Delta = (r - r_+)(r - r_-). \tag{5.17}$$

In the conformally rescaled metric $d\tilde s_3^2 = \Omega^2 ds_3^2$ with $\Omega = L^2/\bar\Delta$, the conformally invariant three-dimensional scalar operator $(\tilde\nabla^2 - \frac{1}{8}\tilde R)$ is related to the corresponding operator in the untilded metric by

$$\left(\tilde\nabla^2 - \frac{1}{8}\tilde R\right)\tilde\Psi = \Omega^{-5/2}\left(\nabla^2 - \frac{1}{8}R\right)\Psi, \tag{5.18}$$

where

$$\tilde\Psi = \Omega^{-1/2}\Psi. \tag{5.19}$$

As can be easily verified, when $\bar\Delta$ is given by (5.17) the Ricci scalar in the untilded metric (5.9) is given by

$$R = \frac{(r_+ - r_-)^2}{2\bar\Delta^2}, \tag{5.20}$$

while the Ricci scalar of the tilded metric $d\tilde s_3^2 = \Omega^2 ds_3^2 = (L^4/\bar\Delta^2)ds_3^2$ is given by

$$\tilde R = -\frac{3(r_+ - r_-)^2}{2L^4}. \tag{5.21}$$

Thus it follows that $-\frac{1}{8}\tilde R + \frac{1}{8}\Omega^{-2}R = -\frac{1}{6}\tilde R$, and so from Eq. (5.18) we see that if Ψ obeys $\nabla^2\Psi = 0$ then $\tilde\Psi$ obeys

$$\left(\tilde\nabla^2 - \frac{1}{6}\tilde R\right)\tilde\Psi = 0. \tag{5.22}$$

In other words, in all the static metrics we are considering here the time-independent solutions of the massless scalar wave equation in the four-dimensional static black hole background are conformally related to solutions of Eq. (5.22) in three-dimensional hyperbolic space. Note that, as was observed in [32] in the case of the Schwarzschild metric, the coefficient of the Ricci scalar in this equation is the one associated with the conformally-invariant scalar wave equation of four dimensions, not three. As was discussed in [32], the fact that the metric ds_3^2 whose Laplacian gives the time-independent scalar wave functions is conformal to the $d\tilde s_3^2$ hyperbolic metric, which has $SO(3,1)$ as its symmetry group, provides one way to understand the $SL(2,R)$ symmetry and ladder structure of the wave functions. In the conformal frame of the hyperbolic metric the conformally rescaled wave functions $\tilde\Psi$ are eigenfunctions of the Laplace operator $\tilde\nabla^2$ shifted by the constant $-\frac{1}{6}\tilde R$.

VI. HARMONIC COORDINATES

We turn, in this section, to an application of our previous results, enabling the construction of harmonic coordinates, also known as wave coordinates. Let

$$\mathfrak{g}^{\mu\nu} = \sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu}. \tag{6.1}$$

Then, by definition, a harmonic coordinate chart, $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i)$, is one in which the De Donder gauge condition

$$\partial_\mu \mathfrak{g}^{\mu\nu} = 0, \tag{6.2}$$

holds. It follows that in harmonic coordinates the D'Alembertian of any scalar function f is given by

$$\Box f \stackrel{*}{=} g^{\mu\nu}\partial_\mu\partial_\nu f \tag{6.3}$$

where $\stackrel{*}{=}$ means "equal in harmonic coordinates." In particular, harmonic coordinates satisfy

$$\Box x^\alpha = 0. \tag{6.4}$$

The main interest of harmonic coordinates is that in these coordinates the Einstein equations become a semilinear symmetric hyperbolic system, since the highest derivative term has the same form as the right-hand side of (6.3), with f representing each component of the metric $g_{\alpha\beta}$ [50]. They also permit a reformulation of the Einstein equations and the definition of total energy and momentum in terms of the Landau-Lifshitz energy momentum pseudo-tensor [39].

It is a standard result that for the Schwarzschild metric, a set of harmonic coordinates is provided by [39,51]

$$x^\alpha = (t, (r-M)\sin\theta\cos\phi, (r-M)\sin\theta\sin\phi, (r-M)\cos\theta). \tag{6.5}$$

This result has been extended to the case of the Reissner-Nordström black hole in [52], the Kerr black hole in [53], and the Kerr-Newman black hole in [54]. In view of the universal form of the scalar D'Alembertian (2.4) for the STU charged black holes we are considering in this paper, it is evident that the procedure found in [53] for constructing harmonic coordinates in the Kerr geometry can be immediately carried over to the general case of the charged rotating STU black holes. Thus we define a new azimuthal coordinate $\tilde{\phi}$ by setting

$$\tilde{\phi} = \phi + \int^r \frac{a dr'}{\Delta(r')}, \tag{6.6}$$

and then it can be seen that the coordinates x^α defined by

$$\begin{aligned} x^\alpha = (t, &[(r-M)\cos\tilde{\phi} - a\sin\tilde{\phi}]\sin\theta, \\ &[(r-M)\sin\tilde{\phi} + a\cos\tilde{\phi}]\sin\theta, (r-M)\cos\theta) \end{aligned} \tag{6.7}$$

are harmonic in the charged black hole metric (2.1), where the base 3-metric γ_{ij} is given by that of the Kerr seed metric in Eq. (2.7). In any stationary, axisymmetric metric for which all metric components are independent of both ϕ and t, and such that $g^{t\theta}$, g^{tr}, $g^{\phi\theta}$, and $g^{\phi r}$ all vanish, so that $g^{\phi t} = g^{t\phi}$ gives the only nonvanishing cross term, then t (and, in fact, ϕ too) is a harmonic function, and we can seek the spatial harmonic coordinates by looking only at the time independent part of (6.4). Thus, irrespective of $\mathcal{A}_i$ and U, the coordinates x^α given in Eq. (6.7) satisfy $\Box x^\alpha = 0$. It follows that, in complete generality, Eq. (6.7) provides a set of harmonic coordinates for all charged rotating black holes in the class (2.1) with the Kerr seed 3-metric.

VII. CONCLUSIONS

We have studied the charged, asymptotically flat, rotating black hole solutions of ungauged STU supergravity, which include Kaluza-Klein black holes and Kerr-Sen black holes as special cases, and we find that the time-independent solutions of the massless scalar wave equation are identical to those of the Kerr solution. This implies that the Love numbers for scalar perturbations, which had previously been shown to vanish in the Kerr background, vanish also for all these charged black holes. The harmonic coordinates for the charged black holes also coincide with those for the Kerr metric. In the low-frequency limit, we find the scalar fields exhibit the same $SL(2,R)$ symmetry as holds in the case of the Kerr solution. We have pointed out extensions of our results to a wider class of black-hole metrics, including solutions of Einstein-Maxwell-dilaton theory.

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