

## Friedmann-Lemaître-Robertson-Walker cosmology through the lens of gravitoelectromagnetism

Valerio Faraoni<sup>Ⓛ</sup>,\* Geneviève Vachon,<sup>†</sup> Robert Vanderwee,<sup>‡</sup> and Sonia Jose<sup>§</sup>

*Department of Physics and Astronomy, Bishop's University,  
2600 College Street, Sherbrooke, Québec J1M 1Z7, Canada*

 (Received 22 January 2022; accepted 12 April 2022; published 26 April 2022)

Friedmann-Lemaître-Robertson-Walker cosmology is examined from the point of view of gravitoelectromagnetism, in the approximation of spacetime regions small in comparison with the Hubble radius. The usual Lorentz gauge is not appropriate for this situation, while the Painlevé-Gullstrand gauge is rather natural. Several nontrivial features and differences with respect to “standard” asymptotically flat gravitoelectromagnetism are discussed.

DOI: [10.1103/PhysRevD.105.083530](https://doi.org/10.1103/PhysRevD.105.083530)

### I. INTRODUCTION

The weak-field, slow-motion limit of general relativity (GR) produces Newtonian gravity while, by allowing for relativistic motions (but keeping the gravitational field weak), one obtains the linearized version of GR. It is well known (e.g., [1]) that linearized gravity can be recast formally as a Maxwell-like theory by introducing a gravitoelectric and a gravitomagnetic potential. Gravitoelectromagnetism has a long history and several applications (e.g., [2–16] and references therein) and it is universally recognized as a characteristic of GR. Certain geometries that are solutions of the Einstein equations are usually not contemplated from the point of view of gravitoelectromagnetism in their weak-field limit. Here we address the gravitoelectromagnetic limit of Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. The weak-field limit is obtained in the approximation of small regions of space around an observer’s worldline and small intervals of time centered around a particular time (for example, the present time of that observer). Comoving observers are commonly used in cosmology, but we will introduce also the point of view of radial freely falling observers and of Painlevé-Gullstrand observers of a de Sitter space osculating the FLRW universe. The view of FLRW cosmology through the lens of gravitoelectromagnetism is quite unconventional and exhibits several differences with respect to “standard” linearized GR in asymptotically flat spacetimes. In particular, in spite of certain similarities, cosmological gravitoelectromagnetism offers the chance to discuss gauges different from the usual Lorentz gauge, which are necessarily encountered in this context. As expected,

because of spatial isotropy the gravitomagnetic field vanishes identically, while the gravitoelectric field is purely radial. Overall, the contexts of standard linearized GR and of the local approximation of FLRW cosmology with a de Sitter space are quite different.

To recap, there are three motivations for this work. First, there is the curiosity to explore the paradigm of gravitoelectromagnetism in cosmology, a context in which (to the best of our knowledge) it has not been discussed thus far. Second, we are interested in finding physically meaningful contexts in which the usual Lorentz gauge does not apply and one needs to expand the box of existing tools in gravitoelectromagnetism (the only other gauge used in the literature is the Bakolopoulos-Kanti one discussed in Sec. III). Last but not least, everything we know about structure formation in the universe comes from  $N$ -body simulations in the early Universe. These simulations are Newtonian in spite of the fact that they are performed on a box with side equal to a few times the Hubble radius. The reason why this is not a problem and Newtonian simulations remain accurate has been discussed in [17–19]: essentially, it boils down to the fact that the peculiar velocities of dark matter particles are small compared to the Hubble flow at redshift  $z \simeq 100$  (when the simulations begin), but this statement is extrapolated from calculations in a less than transparent way and depends on the gauge adopted [17]. In any case, it sounds like stating that gravitomagnetic effects are negligible in comparison with gravitostatic ones (which are Newtonian), and it seems to beg for the point of view of gravitoelectromagnetism, which we therefore develop here for unperturbed and perturbed FLRW cosmology. The gauge-invariant approach of [19] to the problem of Newtonian cosmological perturbations forming early structures is based on splitting the dynamics of dark matter particles into a local (Newtonian) part and a cosmological part by introducing the fictitious potential

\*vfaraoni@ubishops.ca

†gvachon18@ubishops.ca

‡rvanderwee20@ubishops.ca

§sjose21@ubishops.ca

$\Phi = -GM_{\text{MSH}}/R = -Gm/R + H^2R^2/2$ , where  $M_{\text{MSH}}$  is the Misner-Sharp-Hernandez mass contained in a sphere of (physical) radius  $R$ ,  $m$  is the mass generating the local Newtonian perturbation, and  $H$  is the Hubble function. The splitting of  $\Phi$  comes from a splitting of the Misner-Sharp-Hernandez mass [19]. This procedure teases out the local dynamics from the cosmological expansion in a gauge-invariant way but, although it makes sense physically, it was based on guessing  $\Phi$  rather than deriving it rigorously. Here, applying gravitoelectromagnetism to perturbed FLRW universes, we show that  $\Phi$  is nothing but the gravitostatic potential (while the gravitomagnetic contributions are negligible).

We follow the notation and conventions of Ref. [1]: the metric signature is  $-+++$ ,  $G$  is Newton's constant, and units are used in which the speed of light  $c$  is unity. Round brackets around indices denote symmetrization.

## II. LINEARIZED GENERAL RELATIVITY AND GRAVITOELECTROMAGNETISM

In linearized GR [1] it is assumed that an asymptotically Cartesian coordinate system exists in which the spacetime metric assumes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2.1)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric and the perturbations  $h_{\mu\nu}$  are small,  $|h_{\mu\nu}| \ll 1$ . The metric perturbations are supposed to be of order  $\mathcal{O}(\epsilon)$ , where  $\epsilon$  is a small dimensionless parameter and, in linearized theory, the Einstein equations are written by discarding terms of an order higher than  $\mathcal{O}(\epsilon)$ . The first order Einstein tensor is [1]

$$G_{\mu\nu}^{(1)} = -\frac{1}{2}\partial^\alpha\partial_\alpha\bar{h}_{\mu\nu} + \partial^\alpha\partial_{(\nu}\bar{h}_{\mu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}. \quad (2.2)$$

It is convenient to use the quantity

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha, \quad (2.3)$$

where indices are raised and lowered with the unperturbed tensors  $\eta^{\alpha\beta}$  and  $\eta_{\alpha\beta}$ . The Lorentz gauge

$$\partial^\mu\bar{h}_{\mu\nu} = 0 \quad (2.4)$$

is then imposed in order to simplify the first order Einstein equations  $G_{\mu\nu}^{(1)} = 8\pi GT_{\mu\nu}$  to

$$\partial^\alpha\partial_\alpha\bar{h}_{\mu\nu} = -16\pi GT_{\mu\nu}. \quad (2.5)$$

The matter energy-momentum tensor is usually assumed to be of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu \quad (2.6)$$

describing a dust with energy density  $\rho$  and 4-velocity field  $u^\mu$ .

Gravitoelectromagnetism is introduced by noting that the linearized Einstein equations in the Lorentz gauge (2.4) assume the form of Maxwell equations and that the geodesic equation resembles the equation for the Lorentz force acting on a particle of unit charge [1–15] [there are, however, subtleties in the Lorentz force equation when  $\phi_{(g)}$  and  $\vec{A}_{(g)}$  are time-dependent [20]]. The line element is written as

$$ds^2 = -(1 - 2\phi_{(g)})dt^2 + 2\vec{A}_{(g)} \cdot d\vec{x}dt + (1 + 2\phi_{(g)})\delta_{ij}dx^i dx^j, \quad (2.7)$$

from which one reads off the gravitoelectromagnetic potentials  $\phi_{(g)}$  and  $\vec{A}_{(g)}$  [2–15].

The three-dimensional projection of the timelike geodesic equation for a massive particle of 3-velocity  $\vec{v}$  assumes the form analogous to the Lorentz force equation [1]

$$\vec{a} = -\vec{E}_{(g)} - 4\vec{v} \times \vec{B}_{(g)}. \quad (2.8)$$

In the following, we develop gravitoelectromagnetism for FLRW cosmology and we compare it with the “standard” version summarized in this section.

## III. GRAVITOELECTROMAGNETISM IN FLRW SPACETIME

Let us consider now the FLRW metric in comoving coordinates  $(t, x, y, z)$

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (3.1)$$

$$= -dt^2 + a^2(t)(dr^2 + r^2 d\Omega_{(2)}^2), \quad (3.2)$$

where the last line uses polar comoving coordinates  $(t, r, \vartheta, \varphi)$  and  $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$  is the line element on the unit 2-sphere. The areal radius is

$$R(t, r) = a(t)r = \sqrt{X^2 + Y^2 + Z^2}, \quad (3.3)$$

where  $X^i \equiv a(t)x^i$  ( $i = 1, 2, 3$ ) are (oriented) physical lengths along the  $x^i$  axes, while the comoving coordinates  $x^i$  instead follow the expansion of the cosmic fluid. More precisely, two points located on the  $x^i$  axis and separated by the comoving infinitesimal distance  $dx^i$  have physical separation  $a(t)dx^i$  at time  $t$ . Two such points at finite comoving distance  $x^i$  have physical separation  $X^i = a(t)x^i$  [however,  $dX^i$  does not coincide with the physical

infinitesimal separation  $a(t)dx^i$  unless  $a(t)$  is approximated with its value  $a(t_0)$  at the time  $t_0$ .

In order to write the FLRW metric as a formal Minkowski metric plus small perturbations, it is convenient to switch to the use of coordinates  $X^i$  instead of  $x^i$ , and of the areal radius  $R$  as the radial coordinate instead of the comoving  $r$ . We have

$$dx^i = \frac{dX^i - HX^i dt}{a}, \quad dr = \frac{dR - HRdt}{a}, \quad (3.4)$$

where  $H \equiv \dot{a}/a$  is the Hubble function and an overdot denotes differentiation with respect to the comoving time  $t$ . Substituting into the FLRW line element (3.2), one obtains [21]

$$\begin{aligned} ds^2 &= -(1 - H^2 R^2)dt^2 - 2HX^i dt dX^i + dX^2 + dY^2 + dZ^2, \\ &= -(1 - H^2 R^2)dt^2 - 2HRdt dR + dR^2 + R^2 d\Omega_{(2)}^2, \\ &= (\eta_{\mu\nu} + h_{\mu\nu})dX^\mu dX^\nu, \end{aligned} \quad (3.5)$$

where, in the last line,<sup>1</sup> the metric is formally the Minkowski metric  $\eta_{\mu\nu}$  plus a deviation  $h_{\mu\nu}$  from it that, at this stage, is not yet required to be small. Explicitly, we have

$$h_{00} = H^2 R^2, \quad h_{0i} = -HX^i, \quad h_{ij} = 0. \quad (3.6)$$

This form of the metric resembles linearized gravitational theory where the  $h_{\mu\nu}$ s are small. To establish a parallel with linearized GR, we now assume that the corrections to the formal Minkowski metric appearing in Eq. (3.5) are small. There is a conceptual difference with respect to ‘‘standard’’ linearized GR. While usually one assumes the existence of an asymptotically Cartesian coordinate system in which the metric splits as  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  [1], in cosmology we have the opposite situation. Spacetime is asymptotically (indeed, exactly) FLRW and one obtains  $|h_{\mu\nu}| \ll 1$  only by restricting to spacetime regions small with respect to the Hubble radius  $H^{-1}$ , which implies

$$H|X^i| \leq HR \ll 1. \quad (3.7)$$

The physical meaning of this approximation is that spacetime is locally flat and the effects of the cosmological expansion can only be felt by systems of size non-negligible with respect to the radius of curvature of spacetime, in this case the Hubble radius  $H^{-1}$ . However, Eqs. (3.5) and (3.6) are exact, no expansion is required for

<sup>1</sup>We stress that, in the line element (3.5),  $t$  is still the comoving time and the only difference with respect to Eq. (3.2) is the coordinate switch  $x^i \rightarrow X^i$ : we are now considering observers using a Schwarzschild-like radius and moving radially with respect to the comoving observers.

their validity, and the  $h_{\mu\nu}$  are not *a priori* small. It is only when one wants the  $h_{\mu\nu}$  to be small in order to mirror linearized gravity, and to introduce gravitoelectromagnetism (which is our goal here), that one restricts oneself to regions much smaller than  $H^{-1}$  and uses  $\epsilon \equiv HR$  as a smallness parameter.

In practice, when one studies cosmological physics in the neighborhood of a certain instant of time  $t_0$ , for example the present time in the history of the Universe in cosmography, one expands the Hubble function  $H(t)$  around the present time  $t_0$ . If one allows  $|t - t_0|$  to be arbitrary, then light signals reaching the observer at time  $t_0$  can arrive from distant regions of the Universe, breaking the assumption that only regions with  $HR \ll 1$  are considered. Therefore, as done in cosmography, we replace the Hubble function  $H(t)$  with its value  $H_0 \equiv H(t_0)$  and we consider only time intervals such that  $H_0|t - t_0| \ll 1$ , in addition to restricting to regions with  $H_0 R \ll 1$ . The local deviations of the spacetime metric from the Minkowski one then read

$$h_{00} = H_0^2 R^2, \quad h_{0i} = -H_0 X^i, \quad h_{ij} = 0 \quad (3.8)$$

in coordinates  $(t, X^i)$  or, with equivalent terminology, in the gauge in which the line element assumes the form

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})dX^\mu dX^\nu, \\ &= -(1 - H_0^2 R^2)dt^2 - 2H_0 X^i dt dX^i + \delta_{ij} dX^i dX^j. \end{aligned} \quad (3.9)$$

The approximation  $H(t) \simeq H_0 = \text{const}$  is equivalent to replacing the exact FLRW manifold with a de Sitter spacetime with Hubble constant equal to the value  $H_0 \equiv H(t_0)$  of the Hubble function of the real FLRW spacetime.

As in linearized gravity [1], one can introduce  $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha$ , which has the only nonvanishing components

$$\bar{h}_{00} = \frac{H_0^2 R^2}{2}, \quad (3.10)$$

$$\bar{h}_{0i} = \bar{h}_{i0} = -H_0 X^i, \quad (3.11)$$

$$\bar{h}_{ij} = \frac{H_0^2 R^2}{2} \delta_{ij}, \quad (3.12)$$

in coordinates  $(t, X^i)$ , in which the line element (3.9) can be written as

$$ds^2 = -(1 - 2\Phi)dt^2 + 2A_i dX^i dt + \delta_{ij} dX^i dX^j. \quad (3.13)$$

Here

$$\Phi = \frac{H_0^2 R^2}{2}, \quad \vec{A} = -H_0 \vec{X} \quad (3.14)$$

can be regarded as the gravitoelectric and gravitomagnetic potentials, respectively. There is, however, something very unconventional about this identification: usually [1], the linearization of the Einstein equations and the formulation of gravitoelectromagnetism are performed by imposing the Lorentz gauge  $\partial^\mu \bar{h}_{\mu\nu} = 0$  in which the linearized Einstein equations simplify and the resulting line element assumes the form (2.7). Here, instead, the line element appears in the different form (3.9). Our gauge (3.10)–(3.12) is incompatible with the Lorentz gauge because  $\partial^\mu \bar{h}_{\mu 0} = -3H_0 \neq 0$ . Is this a problem? *A priori*, it is not: the gravitoelectromagnetic potentials are gauge dependent and the gravitoelectric and gravitomagnetic fields are gauge independent, as expected [10,20]. Clearly, the metric looks different in the two gauges and physical interpretations based on such gauges will be different.

There is, however, a more substantial conceptual and gauge-independent difference between standard linearized gravity and the linearized version of cosmology. In the former, the matter stress-energy tensor  $T_{\mu\nu}$  is assumed to describe a dust [Eq. (2.6)]. In the cosmological context, instead, we have replaced the exact FLRW space with its de Sitter approximation at time  $t_0$ , which means that  $T_{\mu\nu}$  has necessarily the form of the effective energy-momentum tensor of a cosmological constant  $\Lambda = 3H_0^2$ ,

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} = -3H_0^2(\eta_{\mu\nu} + h_{\mu\nu}). \quad (3.15)$$

Contrary to a dust, this effective stress-energy tensor has nonvanishing pressure

$$P_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G} = -\frac{3H_0^2}{8\pi G}; \quad (3.16)$$

it depends in an essential way from the metric perturbations  $h_{\mu\nu}$ . In particular, in the gauge (3.10)–(3.12) adopted, the nondiagonal components

$$T_{0i} = -3H_0^2 h_{0i} = 3H_0^3 X^i \quad (3.17)$$

describe an energy current which is generated by the transformation from the comoving coordinates  $x^i$  (which expand with the cosmic substratum) to the (oriented) physical lengths  $X^i$  along the spatial axes. This means that the observers at rest in coordinates  $(t, X^i)$  (which we call “Schwarzschild-like observers” because they use the Schwarzschild-like areal radius as the radial coordinate) move radially with respect to the comoving observers and see a spatial current of radially moving matter, while comoving observers see the cosmic fluid at rest. This current is due to the use of spatial coordinates not adapted to the spatial symmetries. The de Sitter approximation to the FLRW metric satisfies the Einstein-Friedmann equations (here listed in comoving coordinates)

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \quad (3.18)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \quad (3.19)$$

$$\dot{\rho} + 3H(P + \rho) = 0, \quad (3.20)$$

with  $K = 0$  in the approximation

$$H(t) \simeq H_0, \quad \rho(t) \simeq \rho(t_0) = \frac{3H_0^2}{8\pi G} \simeq -P(t) \quad (3.21)$$

and

$$a(t) = a_0 e^{\int H(t) dt} \simeq e^{H_0 t}, \quad (3.22)$$

where we set  $a_0 = 1$  for convenience.

There is another important difference between standard linearized gravity and the local approximation to cosmology: usually, one assumes that  $h_{\mu\nu} = \mathcal{O}(\epsilon)$ , where  $\epsilon$  is a smallness parameter, and keeps only terms of order  $\epsilon$  in the Einstein equations while discarding higher order terms.<sup>2</sup> In our expansion of the FLRW metric, we have metric components with different orders of magnitude in the dimensionless expansion parameter  $\epsilon = H_0 R$ :

$$h_{00} = \mathcal{O}(\epsilon^2), \quad h_{0i} = \mathcal{O}(\epsilon), \quad (3.23)$$

while the  $h_{ij}$  are *exactly* zero. As a consequence, our context is not the usual first order GR and the comparison of results is necessarily limited. In particular, we should not expect a one-to-one correspondence between these two contexts. With this *caveat*, let us proceed.

As expected from the spherical symmetry about every spatial point, the gravitoelectric field  $\vec{E}_{(g)} = -\vec{\nabla}\Phi$  is purely radial. The gravitomagnetic potential  $\vec{A} = -H_0 \vec{X}$  is also purely radial and the gravitomagnetic field then vanishes,

$$\vec{B}_{(g)} = \vec{\nabla} \times \vec{A} = 0. \quad (3.24)$$

The spatial acceleration of a test particle of unit mass is

$$\vec{a} = \vec{E}_{(g)} = -\vec{\nabla}\Phi = -H_0^2 R \vec{e}_R, \quad (3.25)$$

where  $\vec{e}_R$  is the spatial unit vector in the radial direction in coordinates  $(t, \vec{X})$ . Moreover, in the approximation made  $H(t) \simeq H(t_0) \equiv H_0$ , the gravitoelectric and gravitomagnetic potentials are time independent,

<sup>2</sup>We are not concerned here with expansions in inverse powers of the speed of light, which one finds in standard linearized GR.

$$\frac{\partial \Phi}{\partial t} = \frac{\partial A_i}{\partial t} = 0, \quad (3.26)$$

which removes certain unpleasant terms in the Lorentz force equation associated with the time dependence and reported, e.g., in Ref. [22].

### A. Bakopoulos-Kanti gauge

A gauge similar to the one used in this section is reported in linearized GR by Bakopoulos and Kanti [20,22]. This is the only instance that we are aware of in which gravitoelectromagnetism is discussed in a gauge different from the Lorentz gauge (2.4). Specifically, in the context of the linearized theory summarized in Sec. II, the Bakopoulos-Kanti gauge is [20,22]

$$\bar{h}_{00} = \phi_{(g)}, \quad \bar{h}_{0i} = -A_i^{(g)}, \quad \bar{h}_{ij} = \phi_{(g)} \delta_{ij} \quad (3.27)$$

or, equivalently,

$$h_{00} = 2\phi, \quad h_{0i} = -A_i^{(g)}, \quad h_{ij} = 0; \quad (3.28)$$

these authors derive the result that this gauge choice is only possible *in vacuo*,  $T_{\mu\nu} = 0$ . At first sight, this result seems to conflict with the gauge that we obtained in FLRW space, but this conclusion would be incorrect. In fact, the two contexts are quite different: first, Bakopoulos and Kanti [20,22] assume the stress-energy tensor of a dust, while we assume that of a cosmological constant (3.15). Second, in standard linearized theory the metric perturbations are all of the same (first) order  $h_{\mu\nu} = \mathcal{O}(\epsilon)$ , while this is not true in the de Sitter space approximating a FLRW universe. Indeed, by denoting loosely with  $R$  the radius of curvature of spacetime, the standard linearized Einstein equations (2.5) give, in order of magnitude,  $\epsilon/R^2 \sim \rho$ , where all terms of order higher than  $\mathcal{O}(\epsilon)$  are discarded. In the cosmological case, the stress-energy tensor (3.15) proportional to  $H_0$  gives, instead, an equation of the form  $h \simeq H_0^2 R^2 = \mathcal{O}(\epsilon^2)$ , where the right hand side is of *second order* in the smallness parameter  $\epsilon = H_0 R$ . Therefore, this right-hand side would be dropped from the linearized field equations in “standard” theory and one would conclude that this gauge only applies to vacuum, but the cosmological context is quite different from the usual linearized theory (moreover, vacuum cosmology without  $\Lambda$  is meaningless). The procedure that we followed, and the standard results on Painlevé-Gullstrand coordinates for static spherical spacetimes that we discuss in the next section and that agree with the previous procedure, are legitimate and do not contradict Ref. [20] because of the different assumptions.

In the light of the fact that FLRW spacetimes are spherically symmetric, we can think of the Bakopoulos-Kanti gauge in such situations. By virtue of the Jebsen-Birkhoff theorem [1], if a linearized geometry is expressed

in the Bakopoulos-Kanti gauge *and* is spherical, then it must be the linearization of the Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 + \frac{dr^2}{1 - 2Gm/r} + r^2 d\Omega_{(2)}^2 \quad (3.29)$$

because it is a vacuum, spherical, and asymptotically flat solution of the Einstein equations (this conclusion applies also to the spacetime outside spherical black holes in most scalar-tensor theories of gravity in “reasonable” situations, see [23–26]). Indeed, it is not even necessary to linearize the Schwarzschild metric to recast it in the Painlevé-Gullstrand gauge [27–29]

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dT^2 + 2\sqrt{\frac{2Gm}{r}} dT dr + dr^2 + r^2 d\Omega_{(2)}^2, \quad (3.30)$$

where

$$T = t + 4m \left( \sqrt{\frac{r}{2Gm}} + \frac{1}{2} \ln \left| \frac{\sqrt{\frac{r}{2Gm}} - 1}{\sqrt{\frac{r}{2Gm}} + 1} \right| \right) \quad (3.31)$$

is the Painlevé-Gullstrand time [27–29]. This gauge coincides with the Bakopoulos-Kanti gauge without the need to assume  $|h_{\mu\nu}| \ll 1$ . This situation is well known and we conclude that the Bakopoulos-Kanti gauge is most interesting in nonspherical situations.

## IV. RELATION WITH PAINLEVÉ-GULLSTRAND OBSERVERS

The Schwarzschild-like observers used in the previous section to discuss gravitoelectromagnetism in FLRW cosmology employ the comoving time  $t$  but differ from comoving observers, as they move radially. In the first part of this section we recall known material from a variety of sources in the literature with the purpose of elucidating the physical meaning of these observers (which we do at the end of this section).

First, let us recall the transformation from comoving to Schwarzschild-like coordinates for spatially flat FLRW universes and, in particular, for the special de Sitter case that we use to approximate a FLRW universe. Beginning from the spatially flat FLRW metric in comoving coordinates (3.2) and using the areal radius  $R \equiv a(t)r$ , we have obtained the line element

$$ds^2 = -(1 - H^2 R^2) dt^2 - 2HX^i dt dX^i + \delta_{ij} dX^i dX^j. \quad (4.1)$$

The cross term in  $dt dR$  can now be eliminated by introducing the new time  $T$  defined by

$$dT = \frac{1}{F} (dt + \beta dR), \quad (4.2)$$

where  $F(t, R)$  is an integrating factor satisfying

$$\frac{\partial}{\partial R} \left( \frac{1}{F} \right) = \frac{\partial}{\partial t} \left( \frac{\beta}{F} \right) \quad (4.3)$$

to guarantee that  $dT$  is a locally exact differential, while  $\beta(t, r)$  is, for the moment, an unknown function [30]. Substituting  $dt = FdT - \beta dR$  into the line element yields

$$\begin{aligned} ds^2 = & -(1 - H^2 R^2) F^2 dT^2 \\ & + 2F[(1 - H^2 R^2)\beta - HR] dT dR \\ & + [1 - (1 - H^2 R^2)\beta^2 + 2\beta HR] dR^2 + R^2 d\Omega_{(2)}^2. \end{aligned} \quad (4.4)$$

Setting

$$\beta(t, R) = \frac{HR}{1 - H^2 R^2} \quad (4.5)$$

reduces the FLRW line element to its Schwarzschild-like form

$$ds^2 = -(1 - H^2 R^2) F^2 dT^2 + \frac{dR^2}{1 - H^2 R^2} + R^2 d\Omega_{(2)}^2. \quad (4.6)$$

In the special case of de Sitter space the Hubble function  $H$  is constant and  $F = 1$  satisfies Eq. (4.3), which transforms (4.6) into the de Sitter line element in static coordinates. As done in the previous section, we approximate the spatially flat FLRW space with a de Sitter space by replacing  $H(t)$  with  $H_0 \equiv H(t_0)$  around a fixed time  $t_0$ . The result is

$$ds^2 \simeq -(1 - H_0^2 R^2) dT^2 + \frac{dR^2}{1 - H_0^2 R^2} + R^2 d\Omega_{(2)}^2 \quad (4.7)$$

for  $H_0 R \ll 1$ .

Let us review now the Painlevé-Gullstrand coordinates for de Sitter space, which are a special case of the more general Martel-Poisson family [29] derived in [21] for de Sitter space.

Begin from the de Sitter line element in Schwarzschild-like coordinates and define a new time coordinate  $\bar{T}$  by

$$d\bar{T} = dT + \frac{\sqrt{1 - pf}}{f} dR, \quad (4.8)$$

where  $f \equiv 1 - H_0^2 R^2$  and  $p$  is a parameter labeling different charts (it is straightforward to check that the differential  $d\bar{T}$  is exact). The physical meaning of  $p$  is obtained by writing the equation of outgoing ( $\dot{R} > 0$ ) radial timelike geodesics [21,31]

$$\frac{ds^2}{d\tau^2} = -f \left( \frac{dT}{d\tau} \right)^2 + \frac{1}{f} \left( \frac{dR}{d\tau} \right)^2 = -1, \quad (4.9)$$

where  $\tau$  is the proper time along timelike geodesics. Because of the presence of the timelike Killing vector  $T^a = (\partial/\partial T)^a$  in the de Sitter metric approximating the FLRW universe, the energy is conserved along these radial timelike geodesics and, denoting with  $p^c = mu^c$  the four-momentum of a particle of mass  $m$  and 4-velocity  $u^c$ ,  $p_a T^a = -E$  is constant along the geodesic. If  $\bar{E} \equiv E/m$  denotes the particle energy per unit mass, then  $u^0 = dT/d\tau = \bar{E}/f$ ,

$$\left( \frac{dR}{d\tau} \right)^2 = \bar{E}^2 - f, \quad (4.10)$$

and

$$\frac{dR}{d\tau} = \pm \sqrt{\bar{E}^2 - f}, \quad (4.11)$$

where the upper sign refers to outgoing and the lower sign to ingoing geodesics. Introducing  $p \equiv 1/\bar{E}^2$ , the radial component of the 4-velocity reads [21]

$$\frac{dR}{d\tau} = \frac{dR dt}{dt d\tau} = \pm \gamma(v) v = \pm \frac{v}{\sqrt{1 - v^2}} = \pm \sqrt{\bar{E}^2 - f}, \quad (4.12)$$

where  $\gamma(v)$  is the Lorentz factor and  $v = |\vec{v}|$  is the magnitude of the coordinate 3-velocity.

At the origin it is

$$\left| \frac{dR}{d\tau} \right|_{R=0} = \frac{v_0}{\sqrt{1 - v_0^2}} = \sqrt{\bar{E}^2 - 1}, \quad (4.13)$$

$$p \equiv \frac{1}{\bar{E}^2} = 1 - v_0^2, \quad (4.14)$$

and the parameter  $p$  spans the range  $0 < p \leq 1$  (this is similar to the case of Martel-Poisson coordinates in Schwarzschild space [29]).

The outgoing ‘‘Martel-Poisson’’ observer freely falling from rest from the origin  $R = 0$  perceives the geometry

$$ds^2 = -f d\bar{T}^2 + 2\sqrt{1 - pf} d\bar{T} dR + p dR^2 + R^2 d\Omega_{(2)}^2 \quad (4.15)$$

where the time coordinate  $\bar{T}$  is given explicitly by [21]

$$\begin{aligned} \bar{T} = & T + \sqrt{1 - p} \int dR \frac{\sqrt{1 + \frac{p}{1-p} H_0^2 R^2}}{1 - H_0^2 R^2}, \\ = & T + \frac{\sqrt{p}}{H_0} \left[ \frac{1}{p} \tanh^{-1} \left( \frac{1}{\sqrt{p(1-p)}} \frac{H_0 R}{\sqrt{1 + \frac{p}{1-p} H_0^2 R^2}} \right) \right. \\ & \left. - \sinh^{-1} \left( \sqrt{\frac{p}{1-p}} H_0 R \right) \right] + \text{const.} \end{aligned} \quad (4.16)$$

The special parameter value  $p = 1$  gives Painlevé-Gullstrand coordinates (see [31] for a discussion of different radial geodesic observers in FLRW cosmology) and it is now clear that it corresponds to vanishing initial velocity  $v_0 = 0$  of the freely falling observer at the origin. With  $p = 1$ , the de Sitter line element (4.15) assumes the Painlevé-Gullstrand form [21]

$$ds^2 = -f d\bar{T}^2 + 2H_0 R d\bar{T} dR + dR^2 + R^2 d\Omega_{(2)}^2. \quad (4.17)$$

The time slices are flat and the Painlevé-Gullstrand time is simply [21]

$$\bar{T} = T - \frac{1}{2H_0} \ln |1 - H_0^2 R^2| + \text{const}, \quad (4.18)$$

which was used in previous literature [32].

The Schwarzschild-like observers seeing the geometry (3.9) and using comoving time  $t$  and coordinates  $X^i = a(t)x^i$  are not Painlevé-Gullstrand observers, although the line element (3.9) has the Painlevé-Gullstrand form with flat spatial sections. The reason is that all freely falling observers are related by a Lorentz boost and do not accelerate with respect to each other (indeed, in a general spacetime freely falling observers, which do not accelerate with respect to each other, are determined up to a Lorentz transformation [33]). The line element (3.9) is Lorentz invariant and has the same form for all these observers boosted with respect to Painlevé-Gullstrand ones. However, the special initial condition  $v_0 = 0$  at  $R = 0$  is satisfied only by Painlevé-Gullstrand observers (using the time  $\bar{T}$ ) and not by all those Lorentz boosted with respect to them.

### A. Geodesic observers in FLRW and de Sitter

In a FLRW universe sourced by a perfect fluid, the comoving observers are not, in general, geodesic because they are subject to the pressure gradient  $\nabla^\mu P$  and they accelerate. Because of spatial isotropy,  $P = P(t)$  and  $\nabla^\mu P$  points in the direction of comoving time. In de Sitter space the pressure  $P = -\frac{\Lambda}{8\pi G}$  is constant,  $\nabla^\mu P$  vanishes identically and the comoving observers of the effective fluid in de Sitter space are geodesic. Therefore, freely falling and comoving observers in de Sitter space differ only by a Lorentz boost, which agrees with what we have already found with different considerations. Painlevé-Gullstrand observers are special radial geodesic observers, as shown above.

### B. FLRW gravitoelectromagnetism and quasilocal mass

It is well known [29,34,35] that the line element of a spherically symmetric (possibly time-dependent) spacetime can be recast in the Painlevé-Gullstrand form

$$ds^2 = -\left(1 - \frac{2GM_{\text{MSH}}(\bar{t}, R)}{R}\right) d\bar{t}^2 \pm 2\sqrt{\frac{2GM_{\text{MSH}}(\bar{t}, R)}{R}} d\bar{t} dR + dR^2 + R^2 d\Omega_{(2)}^2, \quad (4.19)$$

where  $R$  is the areal radius,  $M_{\text{MSH}}(\bar{t}, R)$  is the Misner-Sharp-Hernandez mass of a sphere of radius  $R$ , and one can choose either sign in front of the time-radius cross term (see the discussion in [21]). The expression (4.19) holds when  $M_{\text{MSH}}$  is non-negative. The Misner-Sharp-Hernandez mass is defined by<sup>3</sup> [36,37]

$$1 - \frac{2GM_{\text{MSH}}}{R} \equiv \nabla^c R \nabla_c R. \quad (4.20)$$

This definition is expressed by a scalar equation, therefore  $M_{\text{MSH}}$  is coordinate invariant. The Hawking-Hayward quasilocal mass [38,39] reduces to the Misner-Sharp-Hernandez mass in spherical symmetry [40] and, in this case, it is the Noether charge associated with the conservation of the Kodama current and with spherical symmetry [40]. In general, however, Painlevé-Gullstrand observers with zero initial velocity cannot be used in nonstatic (spherical) spacetimes because their introduction makes use of energy conservation along radial timelike geodesics [21,29]. Before approximating  $H(t)$  with  $H(t_0)$  in the spatially flat FLRW universe, one can introduce the coordinates  $(t, X^i)$  which turn the FLRW line element into what looks like the Painlevé-Gullstrand form with flat spatial sections. However, these coordinates are not those associated with freely falling radial observers with zero initial velocity until the approximation  $H(t) \simeq H_0$  is made: Painlevé-Gullstrand observers can be introduced in the Sitter space, but not in general (nonstatic) FLRW universes [21].

As noted, Painlevé-Gullstrand coordinates are not defined in spherical spacetimes or spacetime regions in which the Misner-Sharp-Hernandez mass becomes negative. This is the case, e.g., of anti-de Sitter space with the physical interpretation that the repulsion of the negative cosmological constant prohibits a freely falling observer with zero initial velocity from leaving the origin  $R = 0$  [21]. When  $M \geq 0$  and Painlevé-Gullstrand coordinates are defined, their characterizing feature is that spatial sections are flat.

In a spatially flat FLRW universe, the Misner-Sharp-Hernandez mass defined by Eq. (4.20) reads

$$M_{\text{MSH}} = \frac{H^2 R^3}{2G} = \frac{4\pi R^3}{3} \rho, \quad (4.21)$$

where, in the last equality, we used the Friedmann equation (3.18) in a spatially flat universe. This is

<sup>3</sup>Since the areal radius  $R$  is defined only in spherical symmetry, so is the Misner-Sharp-Hernandez mass [36,37].

consistent with the expression of  $M_{\text{MSH}}$  obtained by comparing the line element (3.9) with the form (4.19) for general spherical geometries.

By comparing the forms (3.13) and (4.19) of the line element, one can express the gravitoelectric and gravitomagnetic potentials as functions of the Misner-Sharp-Hernandez mass,

$$\Phi = \frac{GM_{\text{MSH}}}{R}, \quad (4.22)$$

$$\vec{A} = \sqrt{\frac{2GM_{\text{MSH}}}{R}} \vec{e}_R = \sqrt{2\Phi} \vec{e}_R \quad (4.23)$$

to first order in the perturbation.

### V. PERTURBED FLRW UNIVERSE

We now discuss a toy model of a perturbed FLRW universe, in which there is a single, spherically symmetric, scalar perturbation described by the post-Newtonian potential  $\phi$ . The line element in the Newtonian gauge is

$$ds^2 = -(1+2\phi)dt^2 + a^2(t)(1-2\phi)(dr^2 + r^2 d\Omega_{(2)}^2), \quad (5.1)$$

where, for the moment, we allow the spherically symmetric post-Newtonian potential to depend on time,  $\phi = \phi(t, r)$  with  $|\phi| \ll 1$ . Consistently with the fact that the peculiar velocities of scalar perturbations (both primordial dark matter perturbations and well-developed galaxies) are usually small in comparison with the Hubble flow, the vector perturbations are neglected, which leads to the absence of the gravitomagnetic potential  $\vec{A}$  in this gauge. This fact is consistent with gravitoelectromagnetism when terms of higher order in  $v/c$  (where  $v$  is a typical velocity) are neglected.

The areal radius is

$$R(t, r) = a(t)r\sqrt{1-2\phi(t, r)} \quad (5.2)$$

and its gradient

$$\nabla_\mu R = \frac{\dot{a}r(1-2\phi) - ar\dot{\phi}}{\sqrt{1-2\phi}} \delta_{0\mu} + \frac{a(1-2\phi - r\phi')}{\sqrt{1-2\phi}} \delta_{1\mu} \quad (5.3)$$

(where a prime denotes differentiation with respect to the comoving radius  $r$  of the FLRW background) gives

$$\nabla^c R \nabla_c R = 1 - H^2 R^2 (1 - 2\phi) + 2HR^2 \dot{\phi} - 2r\phi' \quad (5.4)$$

to first order. Equation (4.20) then gives the Misner-Sharp-Hernandez mass

$$M_{\text{MSH}}(t, r) = \frac{H^2 R^3}{2G} + \frac{rR\phi'}{G} - \frac{HR^3}{G} (H\phi + \dot{\phi}). \quad (5.5)$$

The first contribution to the right-hand side has a cosmological nature (in a spatially flat universe, this is the mass of the cosmic fluid enclosed by the sphere of radius  $R$ ); the second contribution is purely local, while the third contribution is mixed. Thus far, we have performed an expansion in powers of  $\phi$ , keeping only linear terms. We now restrict to regions much smaller than the Hubble radius, obtaining two expansions with smallness orders  $\mathcal{O}(\phi) = \mathcal{O}(r\phi')$  and  $HR$ . The mixed term  $-\frac{HR^3}{G}(H\phi + \dot{\phi})$  is of higher order than the two previous terms and is usually discarded unless tiny relativistic effects are searched for in cosmology [41–45].

As in any spherically symmetric spacetime, the line element can be written<sup>4</sup> in the Painlevé-Gullstrand form (4.19) [34,35], which becomes

$$ds^2 = -[1 - H^2 R^2 - 2r\phi' + 2HR^2(H\phi + \dot{\phi})]d\bar{t}^2 \pm 2\sqrt{H^2 R^2 + 2r\phi' - 2HR^2(H\phi + \dot{\phi})}d\bar{t}dR + dR^2 + R^2 d\Omega_{(2)}^2. \quad (5.6)$$

At this stage, we do not yet have gravitoelectromagnetism, which requires the metric to be Minkowskian with small corrections. By neglecting the time dependence of  $H(t)$  and  $\phi(t, r)$ , one makes the now familiar approximations

$$H(t) \simeq H_0, \quad H_0 R \ll 1, \quad \dot{\phi} \simeq 0, \quad HR\phi \simeq 0, \quad (5.7)$$

obtaining

$$ds^2 \simeq ds_{(0)}^2 = -(1 - H_0^2 R^2 - 2r\phi')d\bar{t}^2 \pm 2\sqrt{H_0^2 R^2 + 2r\phi'}d\bar{t}dR + dR^2 + R^2 d\Omega_{(2)}^2. \quad (5.8)$$

The usual identification of the gravitoelectromagnetic potentials follows:

$$\Phi = \frac{H_0^2 R^2}{2} + r\phi', \quad (5.9)$$

$$\vec{A} = \pm \sqrt{H_0^2 + \frac{2r\phi'}{R^2}} \vec{X}. \quad (5.10)$$

Again, the gravitomagnetic potential is purely radial, giving gravitomagnetic field  $B_{(g)} = \vec{\nabla} \times \vec{A} = 0$ . If we assume that the FLRW perturbation is due to a single (constant) point mass  $m$ , then  $\phi = -Gm/r$  and  $\Phi \simeq \frac{H_0^2 R^2}{2} + \frac{Gm}{R}$ .

<sup>4</sup>Since the FLRW geometry is not static, Martel-Poisson and Painlevé-Gullstrand observers cannot be introduced, but the spherical metric can always be cast in the form (4.19).



The decomposition (5.5) of the Misner-Sharp-Hernandez mass in three contributions was performed in Ref. [19] in the context of the potential problem that  $N$ -body simulations of large scale structures are Newtonian, even though they span volumes larger than the Hubble volume at the redshift of structure formation [17,18,41–45]. There, a “potential”  $\sim \frac{H^2 R^2}{2} + \frac{Gm}{R}$  was introduced *ad hoc* to quantify the degree of “non-Newtonianity” of dark matter perturbations (the result was that the Newtonian simulations of large scale structures are adequate) [19]. It was not realized, however, that this fictitious potential appears in the gravitoelectromagnetic description of cosmology in the approximation in which the FLRW background is replaced with a de Sitter one.

## VI. CONCLUSIONS

We have examined FLRW cosmology from the perspective of the most well-known version of gravitoelectromagnetism in linearized GR. The alternative formulation of gravitoelectromagnetism using electric and magnetic parts of the Weyl tensor with respect to a given observer (e.g., [9]) does not apply to FLRW universes, in which the Weyl tensor vanishes identically [1].

In retrospect, even the “standard” picture of gravitoelectromagnetism is not so standard when applied to FLRW universes. In fact, one must replace the exact FLRW manifold with its instantaneous de Sitter approximation, which implies that the matter stress-energy tensor must necessarily be the effective one associated with a cosmological constant  $\Lambda = 3H_0^2$ , and not that of a dust. Moreover, in order for the spacetime metric to be the Minkowski one plus small perturbations, one must restrict oneself to spacetime regions small in comparison with the Hubble radius  $H^{-1}$ , instead of large regions far away from localized energy distributions.

A freely falling (geodesic) observer will always see the spacetime metric as the flat one plus small perturbations in a local expansion [46,47]. Freely falling observers are determined up to a Lorentz boost (e.g., [33]). In FLRW universes, it is natural to consider freely falling radial observers, to which are associated special coordinates in cosmology [21,31,32,48–50]. Since the FLRW universe is approximated locally with an osculating de Sitter space, which is locally static, one can introduce Martel-Poisson observers and their special subclass, the Painlevé-Gullstrand observers [21]. It is rather natural to formulate gravitoelectromagnetism in the Painlevé-Gullstrand gauge.

This is different from the usual Lorentz gauge and is more similar to the Bakopoulos-Kanti gauge [20,22]. In asymptotically flat linearized GR, the Bakopoulos-Kanti gauge is valid only *in vacuo* but the situation is different in cosmology, in which the metric components have two different orders of smallness.

As expected from spatial isotropy, the gravitoelectric field is purely radial and the gravitomagnetic field vanishes identically as a consequence of the gravitomagnetic potential  $\vec{A}$  being radial. Due to the spherical symmetry of FLRW spaces about every spatial point, one can introduce the Misner-Sharp-Hernandez quasilocal mass [36,37] and we have expressed the gravitoelectromagnetic potentials  $\Phi, \vec{A}$  in terms of it.

It is also interesting to consider perturbed FLRW universes from the perspective of gravitoelectromagnetism. For simplicity, we have considered the situation of a single spherically symmetric metric perturbation. The analysis of Ref. [19] of the physics of  $N$ -body simulations, which are Newtonian even though the box used is a few Hubble scales in size, was based on the splitting of the Misner-Sharp-Hernandez mass into local and cosmological perturbations, discarding a much smaller contribution [19]. Here the fictitious potential used in [19] has been shown to coincide with the gravitoelectrostatic potential of FLRW universes, making more meaningful the discussion of [19]. One could generalize the discussion to arbitrary (small) cosmological perturbations, in which case the Misner-Sharp-Hernandez mass (defined only in spherical symmetry) cannot be used. However, one can use its Hawking-Hayward quasilocal generalization [38–40], as done in Ref. [19]. We do not repeat the discussion of [19] here, the conclusion being the rather obvious generalization of the gravitoelectromagnetic potentials to the nonspherical case.

To conclude, even though gravitoelectromagnetism in FLRW cosmology could be expected to be rather trivial, it is not: we have uncovered several nontrivial aspects and many differences with respect to the usual discussion of linearized GR in asymptotically flat spaces.

## ACKNOWLEDGMENTS

This work is supported, in part, by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2016-03803 to V.F.) and by a Bishop’s University Graduate Entrance Scholarship to S. J.

- [1] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).
- [2] A. Matte, Sur des nouvelles solutions oscillatoires des equations de la gravitation, *Can. J. Math.* **5**, 1 (1953).
- [3] V. B. Braginsky, C. M. Caves, and K. S. Thorne, Laboratory experiments to test relativistic gravity, *Phys. Rev. D* **15**, 2047 (1977).
- [4] P. Teyssandier, Rotating nonspherical masses and their effects on the precession of a gyroscope, *Phys. Rev. D* **16**, 946 (1977).
- [5] P. Teyssandier, Rotating stratified ellipsoids of revolution and their effects on the dragging of inertial frames, *Phys. Rev. D* **18**, 1037 (1978).
- [6] R. Jantzen, P. Carini, and D. Bini, The many faces of gravitoelectromagnetism, *Ann. Phys. (N.Y.)* **215**, 1 (1992).
- [7] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton, 1995).
- [8] M. A. G. Bonilla and J. M. M. Senovilla, Very Simple Proof of the Causal Propagation of Gravity in Vacuum, *Phys. Rev. Lett.* **78**, 783 (1997).
- [9] R. Maartens and B. A. Bassett, Gravitoelectromagnetism, *Classical Quantum Gravity* **15**, 705 (1998).
- [10] S. J. Clark and R. W. Tucker, Gauge symmetry and gravitoelectromagnetism, *Classical Quantum Gravity* **17**, 4125 (2000).
- [11] J. M. M. Senovilla, Super-energy tensors, *Classical Quantum Gravity* **17**, 2799 (2000).
- [12] M. L. Ruggiero and A. Tartaglia, Gravitomagnetic effects, *Nuovo Cimento B* **117**, 743 (2002).
- [13] B. Mashhoon, Gravitoelectromagnetism: A brief review, [arXiv:gr-qc/0311030](https://arxiv.org/abs/gr-qc/0311030).
- [14] M. L. Ruggiero, A note on the gravitoelectromagnetic analogy, *Universe* **7**, 451 (2021).
- [15] M. L. Ruggiero and A. Ortolan, Gravitomagnetic resonance in the field of a gravitational wave, *Phys. Rev. D* **102**, 101501(R) (2020).
- [16] M. L. Ruggiero, A. Ortolan, and C. C. Speake, Galactic dynamics in general relativity: The role of gravitomagnetism, [arXiv:2112.08290](https://arxiv.org/abs/2112.08290).
- [17] N. E. Chisari and M. Zaldarriaga, Connection between Newtonian simulations and general relativity, *Phys. Rev. D* **83**, 123505 (2011); **84**, 089901(E) (2011).
- [18] S. R. Green and R. M. Wald, Newtonian and relativistic cosmologies, *Phys. Rev. D* **85**, 063512 (2012).
- [19] V. Faraoni, M. Lapierre-Leonard, and A. Prain, Do Newtonian large-scale structure simulations fail to include relativistic effects?, *Phys. Rev. D* **92**, 023511 (2015).
- [20] A. Bakopoulos and P. Kanti, From GEM to electromagnetism, *Gen. Relativ. Gravit.* **46**, 1742 (2014).
- [21] V. Faraoni and G. Vachon, When Painlevé–Gullstrand coordinates fail, *Eur. Phys. J. C* **80**, 771 (2020).
- [22] A. Bakopoulos and P. Kanti, Novel ansatzes and scalar quantities in gravito-electromagnetism, *Gen. Relativ. Gravit.* **49**, 44 (2017).
- [23] S. W. Hawking, Black holes in the Brans–Dicke theory of gravitation, *Commun. Math. Phys.* **25**, 167 (1972).
- [24] J. D. Bekenstein, Black hole hair: 25 years after, [arXiv:gr-qc/9605059](https://arxiv.org/abs/gr-qc/9605059).
- [25] T. P. Sotiriou and V. Faraoni, Black Holes in Scalar-Tensor Gravity, *Phys. Rev. Lett.* **108**, 081103 (2012).
- [26] S. Bhattacharya, K. F. Dialektopoulos, A. E. Romano, and T. N. Tomaras, Brans–Dicke Theory with  $\Lambda > 0$ : Black Holes and Large Scale Structures, *Phys. Rev. Lett.* **115**, 181104 (2015).
- [27] P. Painlevé, La mécanique classique et la théorie de la relativité, *C. R. Acad. Sci. (Paris)* **173**, 677 (1921).
- [28] A. Gullstrand, Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie, *Ark. Mat. Astron. Fys.* **16**, 1 (1912).
- [29] K. Martel and E. Poisson, Regular coordinate systems for Schwarzschild and other spherical space-times, *Am. J. Phys.* **69**, 476 (2001).
- [30] V. Faraoni, *Cosmological and Black Hole Apparent Horizons*, Lecture Notes in Physics (Springer, New York, 2015), Vol. 907.
- [31] G. Vachon, R. Vanderwee, and V. Faraoni, Revisiting geodesic observers in cosmology, *Eur. Phys. J. C* **81**, 820 (2021).
- [32] M. K. Parikh, New coordinates for de Sitter space and de Sitter radiation, *Phys. Lett. B* **546**, 189 (2002).
- [33] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), p. 72.
- [34] G. Abreu and M. Visser, Kodama time: Geometrically preferred foliations of spherically symmetric spacetimes, *Phys. Rev. D* **82**, 044027 (2010).
- [35] A. B. Nielsen and M. Visser, Production and decay of evolving horizons, *Classical Quantum Gravity* **23**, 4637 (2006).
- [36] C. W. Misner and D. H. Sharp, Relativistic equations for adiabatic, spherically symmetric gravitational collapse, *Phys. Rev.* **136**, B571 (1964).
- [37] W. C. Hernandez and C. W. Misner, Observer time as a coordinate in relativistic spherical hydrodynamics, *Astrophys. J.* **143**, 452 (1966).
- [38] S. W. Hawking, Gravitational radiation in an expanding universe, *J. Math. Phys. (N.Y.)* **9**, 598 (1968).
- [39] S. A. Hayward, Quasilocal gravitational energy, *Phys. Rev. D* **49**, 831 (1994).
- [40] S. A. Hayward, Gravitational energy in spherical symmetry, *Phys. Rev. D* **53**, 1938 (1996).
- [41] J. Adamek, D. Daverio, R. Durrer, and M. Kunz, General relativistic  $N$ -body simulations in the weak field limit, *Phys. Rev. D* **88**, 103527 (2013).
- [42] J. Adamek, R. Durrer, and M. Kunz,  $N$ -body methods for relativistic cosmology, *Classical Quantum Gravity* **31**, 234006 (2014).
- [43] J. Adamek, D. Daverio, R. Durrer, and M. Kunz, General relativity and cosmic structure formation, *Nat. Phys.* **12**, 346 (2016).
- [44] E. Bentivegna and M. Bruni, Effects of Nonlinear Inhomogeneity on the Cosmic Expansion with Numerical Relativity, *Phys. Rev. Lett.* **116**, 251302 (2016).
- [45] J. T. Giblin, J. B. Mertens, and G. D. Starkman, Departures from the Friedmann–Lemaître–Robertson–Walker Cosmological Model in an Inhomogeneous Universe: A Numerical Examination, *Phys. Rev. Lett.* **116**, 251301 (2016).

- 
- [46] C. V. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [47] K. P. Marzlin, Fermi coordinates for weak gravitational fields, *Phys. Rev. D* **50**, 888 (1994).
- [48] R. Gautreau, Geodesic coordinates in the de Sitter universe, *Phys. Rev. D* **27**, 764 (1983).
- [49] R. Gautreau, Curvature coordinates in cosmology, *Phys. Rev. D* **29**, 186 (1984).
- [50] A. A. Grib and Y. V. Pavlov, Some effects of different coordinate systems in cosmology, *Eur. Phys. J. Plus* **136**, 318 (2021).