Boundedness from below of $SU(n)$ potentials

Renato M. Fonseca [*]

High Energy Physics Group Departamento de Física Teórica y del Cosmos, Universidad de Granada, E–18071 Granada, Spain and Institute of Particle and Nuclear Physics Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 18000 Prague 8, Czech Republic

(Received 7 December 2021; accepted 22 March 2022; published 14 April 2022)

Vacuum stability requires that the scalar potential is bounded from below. Whether or not this is true depends only on the scalar quartic interactions, but even so the analysis is arduous and has only been carried out for a limited set of models. Complementing the existing literature, this work contains the necessary and sufficient conditions for two $SU(n)$ invariant potentials to be bounded from below. In particular, expressions are given for models with the fundamental and the 2-index (anti)symmetric representations of this group. A sufficient condition for vacuum stability is also provided for models with the fundamental and the adjoint representations. Finally, some considerations are made concerning the model with the gauge group $SU(2)$ and the scalar representations 1, 2, and 3; such a setup is particularly important for neutrino mass generation and lepton number violation.

DOI: 10.1103/PhysRevD.105.075014

I. INTRODUCTION

The study of scalar potentials can be a formidable task given that these are quartic functions of several variables. Despite the difficulty, their analysis is crucial as the scalar minima correspond to the possible vacuum configurations.

A given vacuum state cannot be absolutely stable if the scalar potential acquires lower values for some other choice of field values. Of particular concern are those cases where the potential is not bounded from below (BFB), meaning that it acquires arbitrarily large negative values. If this were to happen it would be for field values far from the origin, in which case quadratic and trilinear interactions can be neglected. Even so, deriving the BFB conditions quickly becomes a very complicated problem as the number of scalar fields increases, so much so that in the literature one can find the derivation of these conditions for just a few models. Among the cases which were considered is the two Higgs doublet model [1,2], the type-II seesaw potential with the Higgs doublet plus an $SU(2)$ triplet [3,4], special three Higgs doublet models [5–7], and also an $SU(3)$ invariant potential with three triplets [8]. Several other works have analyzed the vacuum stability of specific models or discussed general techniques for doing so [1,9–12].

Of particular relevance to the following discussion is the analysis in Ref. [4] on the BFB conditions for the Standard Model potential with the inclusion of a scalar triplet, which refined the results in [3]. This corresponds to an $SU(2)$ invariant potential with the scalar representations 2 and 3. Following up on that analysis, the aim of the present work is threefold:

(1) Generalize the results of [4] to $SU(n)$ invariant potentials with the fundamental representation plus a 2-index representation—the symmetric, the antisymmetric, or the adjoint. This last representation presents a unique difficulty, hence I will only derive a sufficient condition (which is not a necessary one) for the potential to be bounded from below.

(2) A crucial step in the derivation of the BFB conditions in [4]—namely the shape of Fig. 1—was not demonstrated explicitly up to now, as it was obtained via elaborate manipulations of expressions in a computer. In this work I provide a fully analytical understanding of these calculations.

(3) The Standard Model potential supplemented by a scalar singlet and a scalar triplet (a 1-2-3 $SU(2)$ potential, in reference to the sizes of the irreducible fields) is important in the context of neutrino mass generation, and also lepton number violation [13]. For such a complicated potential, instead of providing in full generality the BFB conditions which are both necessary and sufficient, I will derive them for an important special case where one of the quartic

[*]renatofonseca@ugr.es

couplings is neglected. Furthermore, a sufficient condition will be given for the general case.

It is worth pointing out that extending the results of [4] to $SU(n)$, with $n > 2$, is not a mere mathematical curiosity. Indeed, it is plausible that the fundamental laws of physics are symmetric under a group larger than the Standard Model one, such as $SU(3) \times SU(3) \times U(1)$ [14–17], $SU(4) \times SU(2) \times SU(2)$ [18], $SU(5)$ [19], and even bigger special unitary groups (see for instance [20] and the references contained therein). The viability of the associated models requires several irreducible scalar representations, in some cases coinciding with the ones analyzed in this work [21]. In other cases, such as the Georgi-Glashow $SU(5)$ model [19], the field content studied in this work is just part of the full scalar sector, and if so the conditions presented here are still applicable—they are necessary (but not sufficient) for the potential to be bounded from below.

The rest of this document is structured as follows. Section II introduces and analyzes the $SU(n)$ invariant scalar potential with a fundamental and a 2-index symmetric representation. The BFB conditions depend on two crucial parameters, α and β, which are considered in detail in Sec. III and Appendix A. With a thorough understanding of them, in Sec. IV I derive the BFB conditions for the potential mentioned in Sec. II with a 2-index symmetric representation. Some modifications are necessary in the case of a 2-index antisymmetric representation, as explained in Sec. V. One can also find there an analysis of the more complicated setup where the 2-index representation is the adjoint. The 1-2-3 model mentioned earlier is considered in Sec. VI. Finally, for the reader's convenience, a summary of the results can be found at the very end. Appendix B supplements the discussion in the main text.

II. AN $SU(n)$ INVARIANT POTENTIAL

Consider a scalar ϕ_i transforming under the fundamental representation of $SU(n)$ as well as a Δ_{ij} transforming under the 2-index symmetric representation of this group. These fields can be viewed as a vector and a matrix, which change under an $SU(n)$ transformation U as follows:

$$\phi \to U\phi, \tag{1}$$

$$\Delta \to U\Delta U^T. \tag{2}$$

There are five quartic terms allowed by the symmetry, which are

$$V^{(4)} = \frac{\lambda_\phi}{2}(\phi^\dagger\phi)^2 + \frac{\lambda_\Delta}{2}[\mathrm{Tr}(\Delta\Delta^*)]^2 + \frac{\lambda'_\Delta}{2}\mathrm{Tr}(\Delta\Delta^*\Delta\Delta^*) + \lambda_{\phi\Delta}(\phi^\dagger\phi)\mathrm{Tr}(\Delta\Delta^*) + \lambda'_{\phi\Delta}\phi^\dagger\Delta\Delta^*\phi. \tag{3}$$

The field Δ has $n(n+1)/2$ independent components, but it is always possible to cast Δ in a diagonal form $\mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$ with a gauge transformation. In this basis,[1] the quartic potential reads

$$V^{(4)} = \frac{\lambda_\phi}{2}\left(\sum\nolimits_i |\phi_i|^2\right)^2 + \frac{\lambda_\Delta}{2}\left(\sum\nolimits_i |\Delta_i|^2\right)^2 + \frac{\lambda'_\Delta}{2}\sum_i |\Delta_i|^4 + \lambda_{\phi\Delta}\left(\sum_i |\phi_i|^2\right)\left(\sum_i |\Delta_i|^2\right) + \lambda'_{\phi\Delta}\sum_i |\phi_i|^2|\Delta_i|^2. \tag{4}$$

The above expression depends only on the $2n$ non-negative variables $|\phi_i|^2$ and $|\Delta_i|^2$, and the dependence is quadratic. Hence one can in principle use the copositivity[2] conditions [9] for a $2n$-dimensional matrix to infer the values of the λ parameters for which $V^{(4)}$ is always positive. The problem is that these conditions become quite complicated for square matrices with 4 or more rows. I will therefore follow an approach in line with [4] which is more readily applicable to variable n's.

Note that with a rescaling

$$|\phi_i|^2 \to \frac{1}{\sqrt{\lambda_\phi}}|\tilde{\phi}_i|^2, \tag{5}$$

$$|\Delta_i|^2 \to \frac{1}{\sqrt{\lambda_\Delta + \lambda'_\Delta}}|\tilde{\Delta}_i|^2, \tag{6}$$

one can deduce that whether or not the potential is bounded from below must depend on the five λ's only through the three combinations

$$\kappa'_\Delta \equiv \frac{\lambda'_\Delta}{\lambda_\Delta + \lambda'_\Delta}, \qquad \kappa_{\phi\Delta} \equiv \frac{\lambda_{\phi\Delta}}{\sqrt{\lambda_\phi}\sqrt{\lambda_\Delta + \lambda'_\Delta}}, \qquad \kappa'_{\phi\Delta} \equiv \frac{\lambda'_{\phi\Delta}}{\sqrt{\lambda_\phi}\sqrt{\lambda_\Delta + \lambda'_\Delta}}, \tag{7}$$

plus the signs of λ_ϕ and $\lambda_\Delta + \lambda'_\Delta$, which need to be positive. Indeed, to check that this last statement is true it suffices to consider the specific field directions where only ϕ_1 is nonzero, and also the case when only Δ_1 is nonzero. Despite the allure of working with only three κ's, I will not use them in the following discussion.

[1] One can also make all ϕ_i—or all Δ_i—real and non-negative. I will nevertheless abstain from making this further simplification.

[2] A matrix M is copositive if for every vector $x \neq 0$ with real and non-negative entries it is true that $x^T M x > 0$ (sometimes the sign $\geq$ is considered instead). The fact that the entries of the vector cannot be negative is crucial. While this might seem a concept which is too specific to be useful in generic calculations, its importance and usefulness in the assessment of the stability of scalar potentials is well established.

Let us now introduce the variables[3]

$$\alpha \equiv \frac{\sum_i |\Delta_i|^4}{(\sum_i |\Delta_i|^2)^2} \quad \text{and} \quad \beta \equiv \frac{\sum_i |\Delta_i|^2 |\phi_i|^2}{(\sum_i |\Delta_i|^2)(\sum_i |\phi_i|^2)}, \tag{8}$$

so that

$$V^{(4)} = \frac{1}{2} \begin{pmatrix} \sum_i |\phi_i|^2 \\ \sum_i |\Delta_i|^2 \end{pmatrix}^T \begin{pmatrix} \lambda_\phi & \lambda_{\phi\Delta} + \beta\lambda'_{\phi\Delta} \\ \lambda_{\phi\Delta} + \beta\lambda'_{\phi\Delta} & \lambda_\Delta + \alpha\lambda'_\Delta \end{pmatrix} \times \begin{pmatrix} \sum_i |\phi_i|^2 \\ \sum_i |\Delta_i|^2 \end{pmatrix}. \tag{9}$$

This expression is positive if and only if for all values of α and β the 2×2 matrix above is copositive.[4] In turn, that is true if and only if

$$\lambda_\phi > 0 \quad \text{and} \quad \lambda_\Delta + \alpha\lambda'_\Delta > 0 \quad \text{and}$$
$$\lambda_{\phi\Delta} + \beta\lambda'_{\phi\Delta} + \sqrt{\lambda_\phi(\lambda_\Delta + \alpha\lambda'_\Delta)} > 0, \tag{10}$$

for all values of α and β. With rather straightforward steps, we have reduced the initial problem, with $n + n(n+1)/2$ field directions, first down to $2n$ variables (the $|\phi_i|^2$ and the $|\Delta_i|^2$) and eventually down to just two (α and β). However, to get rid of these remaining field-dependent quantities, we must first understand what is the range of values they can take.

III. THE ALLOWED VALUES OF α AND β

The price to pay for reducing the $2n$ non-negative field quantities $|\phi_i|^2$ and $|\Delta_i|^2$ to just α and β is that the range of the new variable is not obvious. It is rather easy to see that $\max(\alpha) = 1$ when just one $|\Delta_i|^2$ is different from zero, while on the other hand $\min(\alpha) = 1/n$ is reached when all $|\Delta_i|^2$ have a constant value. As for β, if just a single ϕ_i is different from zero, and the same is true for the corresponding Δ_i ($\Delta_{j\neq i} = 0$) then we reach a maximum β value of 1. If on the other hand a single ϕ_i is different from zero and only one $\Delta_{j\neq i}$ is non-null, then β reaches a minimum of 0.

So $\alpha \in [1/n, 1]$ and $\beta \in [0, 1]$. Nevertheless, the allowed region for (α, β) is not a rectangle. For example, when α is minimal ($= 1/n$), all the $|\Delta_i|^2$ must have the same value c which means that β is forced to be $c(\sum_i |\phi_i|^2)/[(\sum_i |\phi_i|^2)nc] = 1/n$ as well.

The border of the allowed area for (α, β) can be found following a generic method proposed long ago in [22,23].

[3]I am assuming that at least one $|\phi_i|$ and at least one $|\Delta_i|$ is nonzero. If $\sum_i |\Delta_i|^2 = 0$ then $V^{(4)}$ is positive if and only if $\lambda_\phi > 0$ (a condition which has already been mentioned), while if $\sum_i |\phi_i|^2 = 0$ it is required (and sufficient) that $\lambda_\Delta + \lambda'_\Delta > 0$ and also $\lambda_\Delta + \lambda'_\Delta/n > 0$. This last condition has not been mentioned in the text yet, but it will appear eventually, so there is no loss of generality in considering that $\sum_i |\phi_i|^2, \sum_i |\Delta_i|^2 \neq 0$.

[4]The case where all $|\phi_i|^2$ and all $|\Delta_i|^2$ are simultaneously null is known to lead to $V^{(4)} = 0$, therefore it deserves no further attention.

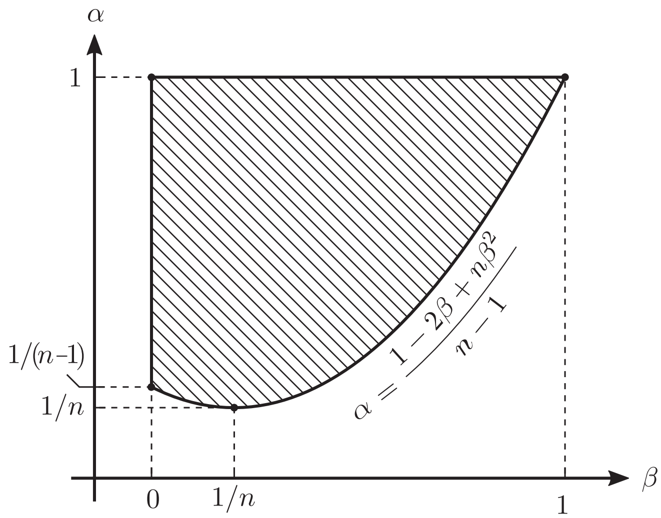

FIG. 1. Allowed values of the important parameters α and β, defined in Eq. (8), when Δ is symmetric.

These two quantities can be seen as functions of the variables $|\phi_i|^2$ plus the $|\Delta_i|^2$, and at the border the vectors $(\partial\alpha/\partial|\phi_i|^2, \partial\beta/\partial|\phi_i|^2)^T$ and $(\partial\alpha/\partial|\Delta_j|^2, \partial\beta/\partial|\Delta_j|^2)^T$ for all i and j must be proportional to each other [the null vector $(0,0)^T$ is allowed as well]. That is because at the border of the allowed area for (α, β) it should not be possible to move in two independent directions in the (α, β) plane by making small variations of the $|\phi_i|^2$ and the $|\Delta_i|^2$. The only caveat is that these last variables cannot be negative; hence, for $|\phi_i|^2 = 0$ and for $|\Delta_i|^2 = 0$ the previous restriction does not apply. Such nuance can be taken into account by saying that the $2n$ vectors

$$|\phi_j|^2(\partial\alpha/\partial|\phi_j|^2, \partial\beta/\partial|\phi_j|^2)^T = x_j(0, y_j - \beta)^T, \tag{11}$$

and

$$|\Delta_k|^2(\partial\alpha/\partial|\Delta_k|^2, \partial\beta/\partial|\Delta_k|^2)^T = y_k(2(y_k - \alpha), x_k - \beta)^T, \tag{12}$$

must either be null or proportional to some constant vector. The notation $x_j \equiv |\phi_j|^2/\sum_i |\phi_i|^2$ and $y_j \equiv |\Delta_j|^2/\sum_i |\Delta_i|^2$ was used to reduce the complexity of the expressions (note that by definition $\sum_i x_i = \sum_i y_i = 1$). It is straightforward but tedious to carefully go through all cases in which the above vectors are all aligned with each other, or null. Therefore, a description of the various possibilities is relegated to Appendix A.

The conclusion of the discussion contained therein is that the allowed values of (α, β) correspond to the shaded area in Fig. 1, including the border lines. Note that—as expected—this shape grows with n since the $SU(n)$-invariant potential can be seen as a special case of the $SU(n+1)$ invariant where some field components are set to zero (Appendix B discusses this topic in more detail). As a consequence, the BFB conditions on the λ's become more stringent as n increases. Of particular relevance is the lower part of this shape, which is defined by the quadratic relation

$$\alpha = \frac{1-2\beta+n\beta^2}{n-1}. \quad (13)$$

The Fig. 1 shown without proof in [4] corresponds to the special situation where $n=2$, in which case the allowed region for (α,β) is symmetric under reflection around the vertical axis $\beta=1/2$; for $n>2$ there is a qualitative change as the point $(\beta,\alpha)=(0,1/(n-1))$ becomes distinct from (0,1).

IV. THE CONDITIONS FOR THE λ'S

We may now return to the inequalities in (10). Since they must hold for all α and β, substituting α in $\lambda_\Delta+\alpha\lambda'_\Delta>0$ by the smallest $(1/n)$ and the largest (1) values this variable can take, we conclude that this last inequality is equivalent to

$$n\lambda_\Delta+\lambda'_\Delta>0 \quad \text{and} \quad \lambda_\Delta+\lambda'_\Delta>0. \quad (14)$$

As observed already in [4], the left-hand side of $\lambda_{\phi\Delta}+\beta\lambda'_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\alpha\lambda'_\Delta)}>0$ is a monotonic function of both α and β, hence it is enough that this condition holds on the border of the allowed $\alpha\beta$-region, which is convex. In turn this is true if the inequality holds for the points $(\beta,\alpha)=(0,1/(n-1))$, (0,1), (1,1) and the parabolic lower part of the shaded region in Fig. 1. From the points we get the constraints

$$\begin{aligned}&\lambda_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{n-1}\right)}>0 \text{ and}\\ &\lambda_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\lambda'_\Delta)}>0 \quad \text{and}\\ &\lambda_{\phi\Delta}+\lambda'_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\lambda'_\Delta)}>0. \end{aligned} \quad (15)$$

Five inequalities have so far been derived for the λ's. The second condition in Eq. (10) must also hold for the parabolic lower part of the border, and that constitutes the last problem to be dwelt with. In practice, we must find the constraints on the quartic scalar couplings which make

$$f(\beta)\equiv\lambda_{\phi\Delta}+\beta\lambda'_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{1-2\beta+n\beta^2}{n-1}\lambda'_\Delta\right)} \quad (16)$$

positive for all $\beta\in[0,1]$. The sign of the second derivative of this function does not change and in fact it is the same as the one of λ'_Δ,

$$\mathrm{sign}[f''(\beta)]=\mathrm{sign}(\lambda'_\Delta), \quad (17)$$

so f has a single stationary point (where $f'(\beta)=0$) and it is an absolute minimum if $\lambda'_\Delta>0$. Note that if $\lambda'_\Delta\le 0$ the value of $\lambda_{\phi\Delta}+\beta\lambda'_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\alpha\lambda'_\Delta)}$ is minimized instead for $(\beta,\alpha)=(0,1)$ or (1,1), and both of these cases were already taken into account above.

The final condition is then

$$f'(0)>0 \quad \text{or} \quad f'(1)<0 \quad \text{or} \quad \min[f(\beta)]>0, \quad (18)$$

where $\min[f(\beta)]$ can be found by requiring that $f'(\beta)=0$ without caring if the value of β is between 0 and 1. In fact, the first two inequalities in the expression above are necessary because if $f'(0)$ is positive or $f'(1)$ is negative the derivative of $f(\beta)$ is null outside the interval $\beta\in[0,1]$.[5] It is then rather simple to resolve the logical condition (18) is terms of λ's.

In summary, the necessary and sufficient BFB condition for the $SU(n)$ invariant potential (3) which have been derived over the previous paragraphs is the following:

$$\begin{aligned}&\lambda_\phi>0 \quad \text{and} \quad n\lambda_\Delta+\lambda'_\Delta>0 \quad \text{and} \quad \lambda_\Delta+\lambda'_\Delta>0 \quad \text{and}\\ &\lambda_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{n-1}\right)}>0 \quad \text{and} \quad \lambda_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\lambda'_\Delta)}>0 \quad \text{and}\\ &\lambda_{\phi\Delta}+\lambda'_{\phi\Delta}+\sqrt{\lambda_\phi(\lambda_\Delta+\lambda'_\Delta)}>0 \quad \text{and} \quad \left[\lambda'_{\phi\Delta}-\frac{1}{n-1}\frac{\lambda'_\Delta\sqrt{\lambda_\phi}}{\sqrt{\lambda_\Delta+\frac{\lambda'_\Delta}{n-1}}}>0 \text{ or}\right.\\ &\left.\lambda'_{\phi\Delta}+\frac{\lambda'_\Delta\sqrt{\lambda_\phi}}{\sqrt{\lambda_\Delta+\lambda'_\Delta}}<0 \text{ or } n\lambda_{\phi\Delta}+\lambda'_{\phi\Delta}+\sqrt{\left(n\frac{\lambda_\Delta}{\lambda'_\Delta}+1\right)[n\lambda'_\Delta\lambda_\phi-(n-1)\lambda'^2_{\phi\Delta}]}>0\right]. \end{aligned} \quad (19)$$

[5] In that case, the minimum of $f(\beta)$ in the [0, 1] interval is at one of the end-points ($\beta=0$ or 1). This corresponds to the points $(\beta,\alpha)=(0,1/(n-1))$ and (1,1), which were already considered previously.

This set of inequalities generalizes to any $SU(n)$ the somewhat more compact formulas given in [4] for $n=2$. The expression inside the square brackets corresponds to condition (18); the first two square roots appearing in it must be positive due to the other constraints [in particular (14)]. On the other hand, if the first two conditions in the above OR expression are false, then the argument of the last square root will always be positive hence the full expression always makes sense.

V. OTHER SCALARS

A. The 2-index antisymmetric representation

Let us now consider what happens if Δ transforms as the 2-index antisymmetric representation. The gauge transformation is the same as in Eq. (1), hence the relevant potential is the one given in Eq. (3), but now Δ is to be viewed as a generic $n\times n$ antisymmetric matrix. This feature makes it impossible to diagonalize Δ with a gauge transformation. One can however block diagonalize it into the form

$$\Delta = \mathrm{diag}\left[\begin{pmatrix} 0 & \Delta_1 \\ -\Delta_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Delta_2 \\ -\Delta_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \Delta_{\lfloor n/2 \rfloor} \\ -\Delta_{\lfloor n/2 \rfloor} & 0 \end{pmatrix}, (0)_{\text{if } n=\text{odd}}\right], \tag{20}$$

where $\lfloor n/2 \rfloor$ stands for the greatest integer lesser than or equal to $n/2$. If n is odd, there must be an extra diagonal entry equal to 0. Nevertheless, the potential (3) is only sensitive to the matrix combination $\Delta^*\Delta$ which can be diagonalized,

$$\Delta^*\Delta = -\mathrm{diag}(|\Delta_1|^2, |\Delta_1|^2, |\Delta_2|^2, |\Delta_2|^2, \ldots, |\Delta_{\lfloor n/2 \rfloor}|^2, |\Delta_{\lfloor n/2 \rfloor}|^2, 0_{\text{if } n=\text{odd}}). \tag{21}$$

Two differences with the symmetric Δ can promptly be discerned:

(1) There is an overall minus sign in $\Delta^*\Delta$. This can be taken into account by swapping $\lambda_{\phi\Delta}$ and $\lambda'_{\phi\Delta}$ by $-\lambda_{\phi\Delta}$ and $-\lambda'_{\phi\Delta}$ in the BFB conditions. I will tacitly assume that this change has been done from now on.

(2) The eigenvalues of $\Delta^*\Delta$ appear repeated, except a zero when n is odd.

Let us then consider first the case when n is even. Using the notation $n \equiv 2n'$ and $|\Phi_i|^2 \equiv |\phi_{2i-1}|^2 + |\phi_{2i}|^2$ we may write

$$\alpha = \frac{\sum_i^{n'} 2|\Delta_i|^4}{(\sum_i^{n'} 2|\Delta_i|^2)^2} = \frac{1}{2}\frac{\sum_i^{n'} |\Delta_i|^4}{(\sum_i^{n'} |\Delta_i|^2)^2}, \tag{22}$$

$$\beta = \frac{\sum_i^{n'} |\Delta_i|^2(|\phi_{2i-1}|^2 + |\phi_{2i}|^2)}{(\sum_i^{n'} 2|\Delta_i|^2)[\sum_i^{n'}(|\phi_{2i-1}|^2 + |\phi_{2i}|^2)]} \equiv \frac{1}{2}\frac{\sum_i^{n'} |\Delta_i|^2|\Phi_i|^2}{(\sum_i^{n'} |\Delta_i|^2)(\sum_i^{n'} |\Phi_i|^2)}. \tag{23}$$

Apart from the $1/2$ factors, these expressions are exactly what one would have if Δ was a symmetric matrix with dimension n'. Hence, the allowed $\alpha\beta$-region is as depicted in Fig. 1, but shrunk by a factor of two in both axis, and using $n' = n/2$ instead of n. That means that for $SU(n)$ the border of the figure goes through the points $(0, 1/(n-2))$, $(1/n, 1/n)$, $(0, 1/2)$, and $(1/2, 1/2)$. Based on these comments, it is rather straightforward to make the necessary changes to the conditions (19) in order to obtain the BFB conditions when Δ is antisymmetric and n is even (these are given explicitly below).

When n is odd, $\Delta^*\Delta$ contains an unpaired null eigenvalue, which is an important feature. If we were to define $n \equiv 2n' + 1$, then α is as given in Eq. (22). However, the denominator of β now depends on $|\phi_n|$ while the numerator does not,

$$\beta = \frac{1}{2}\frac{\sum_i^{n'} |\Delta_i|^2|\Phi_i|^2}{(\sum_i^{n'} |\Delta_i|^2)(\sum_i^{n'} |\Phi_i|^2 + |\phi_n|^2)}. \tag{24}$$

This is a decreasing function of $|\phi_n|$, reaching a maximum given by Eq. (23) (when $|\phi_n| = 0$) and a minimum of 0 when $|\phi_n| \to \infty$. Therefore, compared to Fig. 1, the allowed $\alpha\beta$-region shrinks by a factor of two in both axes and n' replaces n. Furthermore, for all values of α ($1/(2n')$ to $1/2$) β can be null, which means that in (β, α) coordinates, a straight line connecting $(0, 1/2n')$ to $(1/2n', 1/2n')$ forms part of the border of the allowed space. Figure 2 shows some examples.

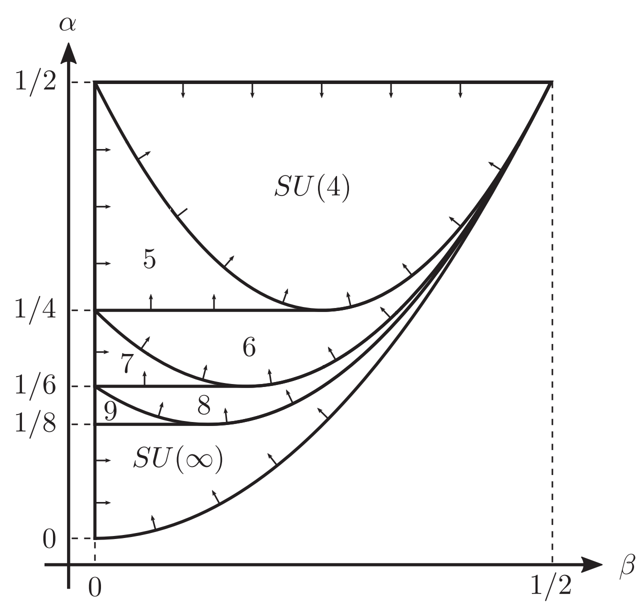

FIG. 2. The allowed region for the parameters α and β, as defined in Eqs. (22), (23), and (24), when Δ is antisymmetric. The numbers shown refer to the $SU(n)$ group under consideration. The shape of the allowed region is markedly different for odd n's when compared to even n's; nevertheless, the area always increases with n.

Note that the cases $n=2,3$ are exceptional, since n' is 1 and α has a fixed value of $1/2$. In other words $\mathrm{Tr}(\Delta^*\Delta\Delta^*\Delta)=[\mathrm{Tr}(\Delta^*\Delta)]^2/2$ and therefore $V^{(4)}$ contains only four independent coupling (it depends on λ_Δ and λ'_Δ only through the combination $\lambda_\Delta+\lambda'_\Delta/2$). For $n=2$, β also has the fixed value $1/2$, while for $n=3$ it can be any number between 0 and $1/2$.

Taking into account the above considerations, the BFB condition in (19) for the symmetric representation is modified to the following form, which is valid for all values of n, regardless of its parity. First define $\tilde{n}$ to be the largest even integer smaller or equal to n: $\tilde{n}=n$ if n is even, otherwise $\tilde{n}=n-1$. Then for $n>3$ the BFB conditions are the following:

$$\begin{aligned}
&\lambda_\phi>0 \quad \text{and} \quad \tilde{n}\lambda_\Delta+\lambda'_\Delta>0 \quad \text{and} \quad 2\lambda_\Delta+\lambda'_\Delta>0 \quad \text{and}\\
&-\lambda_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{2n-\tilde{n}-2}\right)}>0 \quad \text{and} \quad -\lambda_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{2}\right)}>0 \quad \text{and}\\
&-\lambda_{\phi\Delta}-\frac{\lambda'_{\phi\Delta}}{2}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{2}\right)}>0 \quad \text{and} \quad \left[-\lambda'_{\phi\Delta}-\frac{1}{\tilde{n}-2}\frac{2\lambda'_\Delta\sqrt{\lambda_\phi}}{\sqrt{\lambda_\Delta+\frac{\lambda'_\Delta}{\tilde{n}-2}}}>0 \text{ or}\right.\\
&\left.-\lambda'_{\phi\Delta}+\frac{\lambda'_\Delta\sqrt{\lambda_\phi}}{\sqrt{\lambda_\Delta+\frac{\lambda'_\Delta}{2}}}<0 \text{ or } -\tilde{n}\lambda_{\phi\Delta}-\lambda'_{\phi\Delta}+\sqrt{\left(\tilde{n}\frac{\lambda_\Delta}{\lambda'_\Delta}+1\right)\left[\tilde{n}\lambda'_\Delta\lambda_\phi-\left(\frac{\tilde{n}}{2}-1\right)\lambda'^2_{\phi\Delta}\right]}>0\right].
\end{aligned} \tag{25}$$

For $n=2$ (Δ is an $SU(2)$ singlet) the conditions are

$$\begin{aligned}
&\lambda_\phi>0, \quad \text{and} \quad 2\lambda_\Delta+\lambda'_\Delta>0 \quad \text{and}\\
&-\lambda_{\phi\Delta}-\frac{\lambda'_{\phi\Delta}}{2}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{2}\right)}>0,
\end{aligned} \tag{26}$$

while for $n=3$ [Δ^* is an $SU(3)$ triplet] it is additionally necessary that

$$-\lambda_{\phi\Delta}+\sqrt{\lambda_\phi\left(\lambda_\Delta+\frac{\lambda'_\Delta}{2}\right)}>0. \tag{27}$$

B. The adjoint representation

We may move on to the significantly more elaborate case where Δ transforms as an adjoint representation Δ^i_j,

$$\Delta\to U\Delta U^\dagger. \tag{28}$$

This Δ can be viewed as a traceless Hermitian matrix with n^2-1 real degrees of freedom. Reusing the same names for the λ quartic couplings, the most general $SU(n)$ invariant potential can be written as

$$\begin{aligned}
V^{(4)}=&\frac{\lambda_\phi}{2}(\phi^\dagger\phi)^2+\frac{\lambda_\Delta}{2}[\mathrm{Tr}(\Delta^2)]^2+\frac{\lambda'_\Delta}{2}\mathrm{Tr}(\Delta^4)\\
&+\lambda_{\phi\Delta}(\phi^\dagger\phi)\mathrm{Tr}(\Delta^2)+\lambda'_{\phi\Delta}\phi^\dagger\Delta\Delta\phi,
\end{aligned} \tag{29}$$

which is an expression somewhat similar to the one in Eq. (3). With a gauge transformation it is always possible to diagonalize Δ, however unlike when Δ was symmetric, the matrix must remain traceless,[6]

$$\Delta=\mathrm{diag}(\Delta_1,\Delta_2,\ldots,\Delta_{n-1},-\Delta_1-\Delta_2\cdots-\Delta_{n-1}). \tag{30}$$

This leads to nontrivial complications in the analysis of $V^{(4)}$, as was pointed out in [22]; Ref. [1] also considered this setup, arriving at necessary and sufficient conditions for the potential to be bounded from below when both λ'_Δ and $\lambda'_{\phi\Delta}$ are negative. We may define α and β as before [see Eq. (8)], with the understanding that $\Delta_n=-\Delta_1-\Delta_2\cdots-\Delta_{n-1}$, and try to find the allowed values of these two variables. The authors of [22] conjectured that the configurations associated to the border of the valid $\alpha\beta$-space are those of the form

$$\phi=(0,0,\ldots,0,1)^T, \tag{31}$$

[6]The reader might be puzzled by the fact that in the case of $SU(2)$, the adjoint and the 2-index symmetric representations are the same. Yet the text implies that if we treat Δ as a symmetric matrix (let us call it Δ_S), the best that can be done with the gauge symmetry is to cast it in a diagonal form (two real degrees of freedom), while Δ seen as a traceless Hermitian matrix (Δ_H) can be reduced to a real traceless diagonal matrix, with only one real degree of freedom. The reason behind this apparent contradiction is that Δ_S may represent a complex triplet, while Δ_H must stand for a real triplet, with half of the degrees of freedom to start with. Even if we take Δ_S to be a real matrix, the two cases would still be inequivalent due to a different choice of basis (as can be seen from the fact that $\Delta_S\epsilon$ is not Hermitian, with ϵ being the Levi-Civita matrix).

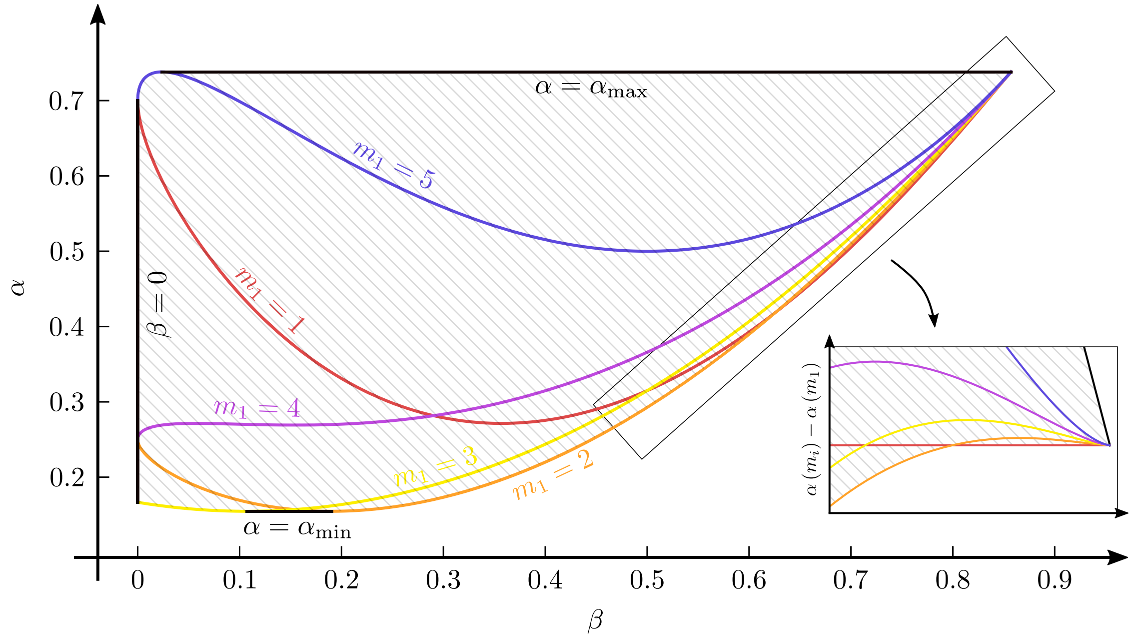

FIG. 3. Demarcation lines of the allowed $\alpha\beta$-space (shaded area) for an $SU(7)$ invariant potential with a fundamental and an adjoint representation. The curved lines (in color) follow Eq. (33), while the straight ones (in black) are described by the expressions (37)–(39), obtained in Ref. [22]. The inlet clarifies what is happening on the right side of the plot, with the $m_1 = 1, 2, 3$ curves all being important for the demarcation of the bottom border line.

$$\Delta = \mathrm{diag}(\underbrace{a, \ldots, a}_{m_1}, \underbrace{b, \ldots, b}_{m_2}, -am_1 - bm_2), \tag{32}$$

plus some lesser important cases to be discussed later.[7] Note that $n = m_1 + m_2 + 1$, so for a fixed n only one of the integers $m_{1,2}$ can be picked freely (for definiteness I take m_1 as the independent variable). We get the following relation between α and β for this particular VEV configuration, with a and b eliminated,

$$\alpha = \beta^2(1 + A + 2B + C) - 2\beta(A + B) + A \pm \frac{4(m_1 - m_2)}{(m_1 + m_2)^3}\sqrt{\frac{\beta}{m_1 m_2}}[m_1 + m_2 - (1 + m_1 + m_2)\beta]^{3/2}, \tag{33}$$

with

$$A \equiv \frac{m_1^2 - m_1 m_2 + m_2^2}{m_1 m_2 (m_1 + m_2)}, \qquad B \equiv \frac{m_1^2 - 4m_1 m_2 + m_2^2}{m_1 m_2 (m_1 + m_2)^2}, \qquad C \equiv \frac{m_1^2 - 6m_1 m_2 + m_2^2}{m_1 m_2 (m_1 + m_2)^3}. \tag{34}$$

There are two choices for each choice of m_1, depending on the sign selected for the last term in the α expression, but it is sufficient to always pick the plus sign, as the minus sign can be replicated by swapping m_1 and m_2 ($m_1 \to n - 1 - m_1$). Unlike when Δ was symmetric (or skew-symmetric), the border of the $\alpha\beta$-space is no longer composed exclusively of straight lines and a parabola; now the relation between α and β is significantly more complicated and furthermore one should consider more than a single curve, since m_1 can take values from 1 to $n - 2$. One might have hoped that a single m_1 is relevant for the demarcation of the border line, but this is not the case; several of them contribute, each for some specific range of β.

Figure 3 illustrates what happens for $SU(7)$ (that is $n = 7$). One can see there that the border line is also made-up of horizontal and vertical straight lines (see [22]); nevertheless, they are irrelevant for the stability of the vacuum.[8] Noting that $\beta \in [0, (n-1)/n]$ and $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ with

$$\alpha_{\min} = \begin{cases} \frac{1}{n} & n \text{ even} \\ \frac{n^2+3}{n(n^2-1)} & n \text{ odd} \end{cases}, \tag{35}$$

[7]Numerical scans suggest that this conjecture is true.

[8]The reason is as follows. We need to find the minimum of the expressions appearing in the inequalities (10) however, since these expressions are monotonous functions of α and β, one can disregard straight portions of the $\alpha\beta$-border line (it is enough to consider their endpoints where the expressions will always reach a minimum).

$$\alpha_{\max} = \frac{(n-1)^3+1}{(n-1)n^2}, \tag{36}$$

there are the following straight lines,

$$\alpha = \alpha_{\min} \quad \text{and} \quad \beta \in \left[\frac{n-1}{n(n+1)}, \frac{n+1}{n(n-1)}\right]$$
$$\text{(line exists only for even } n\text{)}, \tag{37}$$

$$\alpha = \alpha_{\max} \quad \text{and} \quad \beta \in \left[\frac{1}{n(n-1)}, \frac{n-1}{n}\right], \tag{38}$$

$$\beta = 0 \quad \text{and} \quad \alpha \in \begin{cases} \left[\frac{n^2-2n+4}{n^3-3n^2+2n}, \frac{n^2-5n+7}{(n-2)(n-1)}\right] & n \text{ even} \\ \left[\frac{1}{n-1}, \frac{n^2-5n+7}{(n-2)(n-1)}\right] & n \text{ odd} \end{cases}. \tag{39}$$

Since the shape of the $\alpha\beta$-space is quite elaborate, we may focus instead on the rectangle containing it and derive the following simple but potentially useful BFB condition —which is sufficient but not necessary for vacuum stability. It consists on demanding that all the following expressions are positive,

$$\lambda_\phi, \qquad \lambda_\Delta + \alpha_{\min/\max}\lambda'_\Delta,$$
$$\lambda_{\phi\Delta} + \beta_{\min/\max}\lambda'_{\phi\Delta} + \sqrt{\lambda_\phi(\lambda_\Delta + \alpha_{\min/\max}\lambda'_\Delta)}. \tag{40}$$

One should take every combination of α and β at their minimum and maximum values (see Eqs. (35), (36), and the text immediately preceding them), hence there is a total of $1+2+4=7$ quantities to be checked.

VI. THE 1-2-3 $SU(2)$ POTENTIAL

Neutrino masses can be generated at tree level by introducing in the Standard Model a scalar Δ with the $SU(2)_L \times U(1)_Y$ quantum numbers (**3**, 1). Via the seesaw type-II mechanism, neutrinos acquire a mass $m_\nu = Y_\nu \mu \langle\phi\rangle^2/m^2_{\Delta^0}$ where

(i) Y_ν is the Yukawa coupling matrix regulating the interaction $L_i^T C L_j \Delta$ between left-handed leptons and Δ;
(ii) m_{Δ^0} stands for the mass of the neutral component Δ;
(iii) μ is a mass which controls the strength of the trilinear interaction $\phi^\dagger\Delta\phi^*$ between Δ and the Higgs doublet ϕ.

Note that lepton number is restored in the limit where μ vanishes, so this symmetry protects μ from big radiative corrections, and that is why the smallness of m_ν is usually attributed to the tiny value of this mass parameter.

As an alternative, lepton number might be spontaneously violated. To that end one can introduce a scalar singlet with no hypercharge and two units of lepton number [13], so that

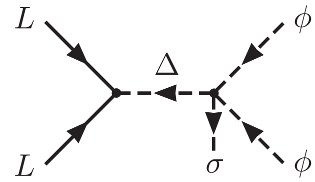

FIG. 4. Neutrino mass diagram in the 1-2-3 model. When σ acquires a nonzero vacuum expectation value, the $LL\phi\phi$ Weinberg operator [25] is generated (L and ϕ represent the left-handed leptons and the Higgs doublet).

an interaction $\frac{\lambda_{\sigma\phi\Delta}}{2}\sigma\phi^\dagger\Delta\phi^* + \text{H.c.}$ is allowed by all symmetries; once this scalar acquires a vacuum expectation value, an effective μ equal to $\lambda_{\sigma\phi\Delta}\langle\sigma\rangle$ is generated (see Fig. 4). With a singlet σ (**1**), a doublet ϕ (**2**), and a triplet Δ (**3**), this setup is sometimes called the 1-2-3 model. The full scalar potential reads

$$\begin{aligned} V^{(4)}(\phi,\Delta,\sigma) = V^{(4)}(\phi,\Delta) + \frac{\lambda_\sigma}{2}|\sigma|^4 \\ + \lambda_{\sigma\phi}|\sigma|^2\phi^\dagger\phi + \lambda_{\sigma\Delta}|\sigma|^2\mathrm{Tr}(\Delta\Delta^*) \\ + \left(\frac{\lambda_{\sigma\phi\Delta}}{2}\sigma\phi^\dagger\Delta\phi^* + \text{H.c.}\right), \end{aligned} \tag{41}$$

where $V^{(4)}(\phi,\Delta)$ contains only terms with ϕ and Δ and was given previously in Eq. (3). Once again a gauge transformation can be used to diagonalize Δ [$\to \mathrm{diag}(\Delta_1,\Delta_2)$], in which case we may make the replacements $\phi^\dagger\phi \to |\phi_1|^2 + |\phi_2|^2$, $\mathrm{Tr}(\Delta\Delta^*) \to |\Delta_1|^2 + |\Delta_2|^2$ and $\phi^\dagger\Delta\phi^* \to (\phi_1^*)^2\Delta_1 + (\phi_2^*)^2\Delta_2$. This last expression is the only one sensitive to the phases of the fields, so the potential above is minimal when

$$\frac{\lambda_{\sigma\phi\Delta}}{2}\sigma\phi^\dagger\Delta\phi^* + \text{H.c.} = -|\lambda_{\sigma\phi\Delta}||\sigma|(|\phi_1|^2|\Delta_1| + |\phi_2|^2|\Delta_2|). \tag{42}$$

We have seen that $V^{(4)}(\phi,\Delta)$ depends only on four field components—$|\phi_{1,2}|$ and $|\Delta_{1,2}|$—or equivalently $|\phi_1|^2 + |\phi_2|^2$, $|\Delta_1|^2 + |\Delta_2|^2$, α and β [see Eq. (8)]. With the introduction of σ, the minimum of the potential will depend only on one extra field $|\sigma|$,[9] nevertheless the potential itself becomes significantly more complicated, with four new λ's. In fact, to find the BFB conditions of the 1-2-3 potential it would be necessary to minimize a

[9]In analogy with α and β, we may define the variable

$$\gamma \equiv \frac{|\phi_1|^2|\Delta_1| + |\phi_2|^2|\Delta_2|}{(|\phi_1|^2 + |\phi_2|^2)\sqrt{|\Delta_1|^2 + |\Delta_2|^2}}.$$

Nevertheless, γ can be written as a function of α and β so it does not constitute an independent degree of freedom.

polynomial with a quadratic dependence on the $|\phi_i|^2$ and, crucially, a quartic dependence on the variables $|\sigma|$ and $|\Delta_i|$. The results on the copositivity of quadratic functions cannot be used here, and one can appreciate from [10,24] that handling multivariable quartic functions is very complicated, hence it seems unwise to try to find the necessary and sufficient BFB of the potential in Eq. (41).[10] However, for the study of neutrino masses in the 1-2-3 model it might be good enough to find some acceptable values of the λ's (not necessarily all of them).

One important case is when the coupling $\lambda_{\sigma\phi\Delta}$ is too small to be relevant for the stability of the vacuum. The neutrino mass matrix is given by the formula $Y_\nu \lambda^*_{\sigma\phi\Delta}\langle\sigma\rangle\langle\phi\rangle^2/m^2_{\Delta^0}$ with m_{Δ^0} often taken to be quite low—of the TeV order[11]—in which case the product $Y_\nu \lambda^*_{\sigma\phi\Delta}\langle\sigma\rangle$ would need to be tiny. Furthermore, without the coupling $\lambda_{\sigma\phi\Delta}$ the model becomes invariant under and extra $U(1)$ symmetry which acts only on σ, hence radiative corrections to this parameter became vanishingly small when $\lambda_{\sigma\phi\Delta} \approx 0$; this is therefore an important and well-motivated approximation.

In the absence of this coupling, the 1-2-3 potential becomes a quadratic function of the non-negative variables $|\phi_{1,2}|^2$, $|\Delta_{1,2}|^2$, and $|\sigma|^2$, hence the potential is bounded from below if and only if the symmetric matrix

$$\begin{pmatrix} \lambda_\phi & \lambda_\phi & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta} & \lambda_{\phi\Delta} & \lambda_{\sigma\phi} \\ \cdot & \lambda_\phi & \lambda_{\phi\Delta} & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta} & \lambda_{\sigma\phi} \\ \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \cdot & \lambda_\sigma \end{pmatrix} \tag{43}$$

is copositive. It is straightforward to obtain the explicit set of inequalities which the λ's must obey (for example with the method described in [26]; see also [9]), however I will not reproduce the expressions here since they are long and not very instructive.

If $|\lambda_{\sigma\phi\Delta}|$ is sizable one might consider the following strategy. For any scalar field configuration, it is either true that $|\sigma| \geq \sqrt{|\Delta_1|^2+|\Delta_2|^2}$ or the opposite, hence

$$\begin{aligned} &-|\sigma|(|\phi_1|^2|\Delta_1|+|\phi_2|^2|\Delta_2|) \leq -|\sigma|^2(|\phi_1|^2+|\phi_2|^2) \quad \text{or} \\ &-(|\Delta_1|^2+|\Delta_2|^2)(|\phi_1|^2+|\phi_2|^2). \end{aligned} \tag{44}$$

By replacing in the potential $V^{(4)}(\phi,\Delta,\sigma)$ the left term with the terms on the right, we get two potentials, both of which depend only on $|\phi_{1,2}|^2$, $|\Delta_{1,2}|^2$, and $|\sigma|^2$. Therefore, the 1-2-3 potential is bounded from below if both the following symmetric matrices are copositive,

$$\begin{pmatrix} \lambda_\phi & \lambda_\phi & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta} & \lambda_{\phi\Delta} & \lambda_{\sigma\phi}-|\lambda_{\sigma\phi\Delta}| \\ \cdot & \lambda_\phi & \lambda_{\phi\Delta} & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta} & \lambda_{\sigma\phi}-|\lambda_{\sigma\phi\Delta}| \\ \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \cdot & \lambda_\sigma \end{pmatrix}, \tag{45}$$

$$\begin{pmatrix} \lambda_\phi & \lambda_\phi & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta}-|\lambda_{\sigma\phi\Delta}| & \lambda_{\phi\Delta}-|\lambda_{\sigma\phi\Delta}| & \lambda_{\sigma\phi} \\ \cdot & \lambda_\phi & \lambda_{\phi\Delta}-|\lambda_{\sigma\phi\Delta}| & \lambda_{\phi\Delta}+\lambda'_{\phi\Delta}-|\lambda_{\sigma\phi\Delta}| & \lambda_{\sigma\phi} \\ \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \lambda_\Delta+\lambda'_\Delta & \lambda_{\sigma\Delta} \\ \cdot & \cdot & \cdot & \cdot & \lambda_\sigma \end{pmatrix}. \tag{46}$$

Note however that this is not a necessary condition; the 1-2-3 potential might be bounded from below even if it fails to pass this test.

VII. CONCLUSIONS

Scalar potentials are quartic functions of several field components, hence their analysis can be quite complicated. That is why it is only possible to write down the necessary and sufficient conditions for these functions to be bounded from below in simple cases, when the number of scalar representations is small. In this work, I have derived these constraints for $SU(n)$ invariants potentials with two fields; one transforming under the fundamental representation and the other as a 2-index representation (the symmetric or the anti-symmetric one). The case where the 2-index representation is the adjoint is substantially more complicated; hence, I have only provided in a closed form a sufficient condition for the potential to be stable.

The combination of fields above mentioned appears in several models extending the Standard Model gauge group. The special case where $n = 2$ and the scalars are a doublet and a triplet is particularly important because these fields participate in the seesaw type-II mechanism which might be responsible for neutrino mass generation. The BFB conditions for this scenario were already presented in [4], although a crucial step necessary to derive this result was not shown explicitly, as the relevant calculations were performed with a computer algebra system. In this work

[10]Neglecting the special case when $\sigma = 0$ (which was already addressed), one can make the variable substitution $|\Delta_i| \to |\sigma||\Delta'_i|$, turning the potential into a quadratic function of $|\phi_1|^2$, $|\phi_2|^2$ and $|\sigma|^2$, hence the known copositivity results can be applied to these three variables. The result is a complicated system of inequalities involving $|\Delta'_1|$ and $|\Delta'_2|$ which would still need to be resolved for all values of these variables. Nevertheless, for a numerical check of whether or not a specific potential is bounded from below, those inequalities might be of some use since for each set of λ's one only has to sample a 2-dimensional field space rather the original 12-dimensional one.

[11]This is a possibility rather than a requirement. Indeed the mass of Δ might as well be several orders of magnitude above the electroweak symmetry breaking scale.

I have provided a fully analytical proof of this result, which is valid for any $SU(n)$ group.

One can also add a scalar singlet to the Standard Model on top of the triplet used in the type-II seesaw mechanism. With the introduction of the singlet, lepton number can be broken spontaneously rather than explicitly, leading to important phenomenological consequences. Yet the scalar potential of this so-called 1-2-3 model contains nine quartic couplings, making it hard to derive necessary and sufficient BFB conditions in full generality. Therefore, I considered the physically well-motivated approximation where one of the interactions is negligible, in which case one can use well-known results on the copositivity of matrices to derive the relevant conditions. For those cases where all quartic couplings are relevant, we also derived a sufficient (but not necessary) condition which can be used to pick acceptable coupling constants.

ACKNOWLEDGMENTS

I acknowledge the financial support from MCIN/AEI (10.13039/501100011033) through Grant No. PID2019–106087GB-C22, from the Junta de Andalucía through Grant No. P18-FR-4314 (FEDER), from the Grant Agency of the Czech Republic (GAČR) through Contract No. 20-17490S, and also from the Charles University Research Center UNCE/SCI/013.

APPENDIX A: THE BORDER OF THE VALID $\alpha\beta$-REGION

As discussed in the main text, the $2n$ vectors in Eqs. (11) and (12) can be used to identify the allowed values of the α and β variables defined in Eq. (8). In particular, at the border of the valid $\alpha\beta$-region, these vectors must either be null or point in a single direction. By carefully considering the right-hand side of the expressions (11) and (12), one of the following possibilities must be true.

(1) For all i such that $x_i \neq 0$ (there must be at least one such case since $\sum_i x_i = 1$) we have $y_i = \beta$. We can further divide this possibility in three cases.

  (a) $\beta = 0$. This means that $x_i \neq 0$ implies $y_i = 0$. So we can have at most $n-1$ nonzero y_i which in turn means that $\alpha \in [1/(n-1), 1]$.

  (b) $\beta = \alpha \neq 0$. In this scenario, the vectors (11) and (12) are aligned only if the value of all nonzero x_i or all nonzero y_i is $\beta(=\alpha)$. So, we conclude that $\alpha = \beta = 1/m$ where m is the number of x_i or $y_i \neq 0$.

  (c) $\beta \neq 0, \alpha$. This is undoubtedly the most important case. By assumption, if $x_i \neq 0$ then $y_i = \beta$, and for all such cases the vectors $(2(\beta-\alpha), x_i - \beta)^T$ are proportional to each other only if the x_i take a constant value. In other words, there are m nonzero x_i and they all have the same value $1/m$ (because $\sum_i x_i = 1$), plus the corresponding y_i are equal to β. Any additional nonzero y_i must be paired with a null x_i, and in all such cases the alignment of the vectors $(2(y_i - \alpha), \beta)^T$ and $(2(\beta-\alpha), 1/m - \beta)^T$ requires that those y_i also have a constant value given by the expression $y_i = (m\beta^2 - \alpha)/(m\beta - 1) \equiv \omega$. Let us assume that there are m' such occurrences; the overall picture is this; there are m cases $(x_i, y_i) = (1/m, \beta)$, m' occurrences of $(x_i, y_i) = (0, \omega)$ and all other (x_i, y_i) are equal to (0,0). From the relation $1 = \sum_i y_i = m\beta + m'\omega$ we conclude that

$$\alpha = \frac{1 - 2m\beta + (m^2 + mm')\beta^2}{m'}. \quad \text{(A1)}$$

  The non-negative integers m and m' can take any values as long as $m \geq 1$ and $m + m' \leq n$. However, note that the case $m = 1$ and $m' = n - 1$ leads to the smallest value of α (for any fixed β). This important setup corresponds to the quadratic dependence of α on β shown in Eq. (13) which defines a line of utmost importance for the extraction of the boundedness from below condition of the scalar potential given in (3).

(2) There is at least one i such that $x_i \neq 0$ and the corresponding $y_i \neq \beta$. If this is the case, then all y_i must either be 0 or α in order for the vectors $y_i(y_i - \alpha, x_i - \beta)^T$ to be collinear with $(0,1)^T$. If we were to call m to the number of y_i different from zero (this must be an integer between 1 and $n-1$[12]), then we conclude from $\sum_i y_i = 1$ that $\alpha = 1/m$. The x_i are unconstrained in this scenario, so it follows that β can be anywhere in the range $[0, \alpha]$ (the value $\beta = 0$ is reached for example when a single $x_i = 1$ is paired with a null y_i; on the other hand when all null y_i have an associated $x_i = 0$ then $\beta = \alpha$).

The four cases above (1a, 1b, 1c, and 2) correspond only to potential fragments of the border of the $\alpha\beta$-region. In fact, some of them are in the interior of this space. Figure 1 depicts the actual border: the vertical line with $\beta = 0$ and $\alpha \in [1/(n-1), 1]$ (case 1.a), the horizontal line $\alpha = 1$ and $\beta \in [0, 1]$ (case 2 with $m = 1$) and the parabola (13) with $\beta \in [0, 1]$ (case 1c).

Note that for a fixed value of α, if we manage to find two valid values of β then all values in between them are equally achievable.[13] Using this fact, we conclude that

[12] At least one y_i must be null. Otherwise, if $m = n$ then all the y_i would have the value α and it would follow that $\beta = \sum_i x_i y_i = \alpha$, in contradiction with the assumption that some $y_i \neq \beta$.

all the space inside the border (shaded area in Fig. 1) is allowed as well.

APPENDIX B: ON THE n-DEPENDENCE OF THE BFB CONDITIONS

The scalar potentials discussed in this work are bounded from below only in part of the parameter space, with the volume of this space becoming smaller for larger $SU(n)$ groups. This is unlike other models, such as the one with two scalars in the fundamental representations of $SU(n)$.

In order to simplify the discussion of this feature, let us henceforth consider the case where there is a fundamental representation ϕ and a 2-index symmetric one Δ, leading to the quartic potential shown in Eq. (3); I will call it V_n if it is invariant under $SU(n)$. First, one should note that—for the same λ couplings—the potential V_{n+1} takes the same value as V_n when the $(n+1)$th entries of ϕ and Δ are null. As such, if Λ_n is the λ-space where V_n is bounded from below, it is quite obvious that $\Lambda_{n+1} \subseteq \Lambda_n$. To exclude the possibility that these two spaces are one and the same, it suffices to consider the special setup where $\lambda_\phi = \lambda_{\phi\Delta} = \lambda'_{\phi\Delta} = 0$[14] in which case

$$\frac{V_n}{(\sum_i |\Delta_i|^2)^2} = \frac{\lambda_\Delta}{2} + \frac{\lambda'_\Delta}{2} \underbrace{\frac{\sum_i |\Delta_i|^4}{(\sum_i |\Delta_i|^2)^2}}_{\equiv \alpha}. \tag{B1}$$

This expression depends only on α, whose range $[1/n, 1]$ increases with n. Therefore one concludes that Λ_{n+1} is not the same as Λ_n, i.e., $\Lambda_{n+1} \subset \Lambda_n$.

As mentioned earlier, the decrease with n of the stable parameter space of the $\phi\Delta$-model contrasts with the situation in other setups. For example, take two scalars ϕ_1 and ϕ_2 transforming under the fundamental representations of $SU(n)$ and, for simplicity, that they have some common charge under an extra $U(1)$ symmetry. The potential depends on $\phi_1^\dagger \phi_1$, $\phi_1^\dagger \phi_2$, and $\phi_2^\dagger \phi_2$ but crucially the range of values which these quantities can have is the same for any $n \geq 2$. Therefore, the BFB conditions do not depend on the $SU(n)$ group under consideration. Nevertheless, this is an exceptional case, as we will now see by considering the generic scenario where there are m scalars $\phi_{1,\ldots,m}$ transforming under the fundamental representation of $SU(n)$. If there is an extra $U(1)$ acting equally on all fields, the potential can only depend on the scalars through the inner products

$$X_{ij} \equiv \phi_i^\dagger \phi_j. \tag{B2}$$

With the help of an orthogonal basis e_i for the irreducible n-dimensional vector space where $SU(n)$ acts on $(e_i^\dagger e_j = \delta_{ij})$, we may write $\phi_i \equiv \sum_j B_{ij} e_j$ so that B can be any $m \times n$ dimensional matrix. It follows that $X = B^* B^T$. From linear algebra (and the Cholesky decomposition in particular) we conclude that any $m \times m$ Hermitian matrix X can be formed from a suitably chosen matrix B with a rank equal or larger than m; this is possible if and only if $n \geq m$. As such, for a fixed m the set of all possible values for X stays unchanged once n is made equal or larger than m, and therefore the analysis of the stability of a model with m scalars ϕ is the same for all $n \geq m$. For instance, with $m = 2$ scalars the stability conditions are the same for all $SU(n \geq 2)$, while for $m = 3$ scalars the BFB conditions depend on whether we consider the group $SU(2)$ or $SU(n \geq 3)$.

[13] One can see that it is so with the following reasoning. By definition $\alpha \equiv \sum_i (y_i)^2 = y^T y$ and $\beta = \sum_i x_i y_i = x^T y$ with the restriction that $\sum_i x_i = \sum_i y_i = 1$ so we may replace x with $x' = ty + (1-t)x$ where t is some number between 0 and 1. This replacement preserves α but changes β to $\beta' = \beta + t(\alpha - \beta)$, which means that if a point $\{\alpha, \beta\}$ is valid, so is any point $\{\alpha', \beta'\}$ in the line with $\alpha' = \alpha$ and $\beta' \in [\alpha, \beta]$ (or $[\beta, \alpha]$ if $\alpha > \beta$). As a consequence, if two valid points have the same α and distinct β's, then all points in between them are equally allowed.

[14] Strictly speaking, the parameter λ_ϕ must be positive in order for the quartic potential to be positive as well. We may then consider λ_ϕ to be very small, rather than exactly zero.

[1] K. G. Klimenko, Conditions for certain Higgs potentials to be bounded below, Theor. Math. Phys. **62**, 58 (1985).
[2] I. P. Ivanov, Minkowski space structure of the Higgs potential in 2HDM, Phys. Rev. D **75**, 035001 (2007); **76**, 039902(E) (2007).
[3] A. Arhrib, R. Benbrik, M. Chabab, G. Moultaka, M. C. Peyranère, L. Rahili, and J. Ramadan, The Higgs potential in the type II seesaw model, Phys. Rev. D **84**, 095005 (2011).
[4] C. Bonilla, R. M. Fonseca, and J. W. F. Valle, Consistency of the triplet seesaw model revisited, Phys. Rev. D **92**, 075028 (2015).
[5] N. Buskin and I. P. Ivanov, Bounded-from-below conditions for A_4-symmetric 3HDM, J. Phys. A **54**, 325401 (2021).
[6] F. S. Faro and I. P. Ivanov, Boundedness from below in the $U(1) \times U(1)$ three-Higgs-doublet model, Phys. Rev. D **100**, 035038 (2019).

[7] I. P. Ivanov and F. Vazão, Yet another lesson on the stability conditions in multi-Higgs potentials, J. High Energy Phys. 11 (2020) 104.

[8] A. Costantini, M. Ghezzi, and G. M. Pruna, Theoretical constraints on the Higgs potential of the general 331 model, Phys. Lett. B **808**, 135638 (2020).

[9] K. Kannike, Vacuum stability conditions from copositivity criteria, Eur. Phys. J. C **72**, 2093 (2012).

[10] K. Kannike, Vacuum stability of a general scalar potential of a few fields, Eur. Phys. J. C **76**, 324 (2016); **78**, 355(E) (2018).

[11] I. P. Ivanov, M. Köpke, and M. Mühlleitner, Algorithmic boundedness-from-below conditions for generic scalar potentials, Eur. Phys. J. C **78**, 413 (2018).

[12] G. Moultaka and M. C. Peyranère, Vacuum stability conditions for Higgs potentials with $SU(2)_L$ triplets, Phys. Rev. D **103**, 115006 (2021).

[13] J. Schechter and J. W. F. Valle, Neutrino decay and spontaneous violation of lepton number, Phys. Rev. D **25**, 774 (1982).

[14] M. Singer, J. W. F. Valle, and J. Schechter, Canonical neutral current predictions from the weak electromagnetic gauge group $SU(3) \times U(1)$, Phys. Rev. D **22**, 738 (1980).

[15] F. Pisano and V. Pleitez, An $SU(3) \times U(1)$ model for electroweak interactions, Phys. Rev. D **46**, 410 (1992).

[16] P. H. Frampton, Chiral Dilepton Model and the Flavor Question, Phys. Rev. Lett. **69**, 2889 (1992).

[17] R. M. Fonseca and M. Hirsch, A flipped 331 model, J. High Energy Phys. 08 (2016) 003.

[18] J. C. Pati and A. Salam, Lepton number as the fourth color, Phys. Rev. D **10**, 275 (1974); **11**, 703(E) (1975).

[19] H. Georgi and S. L. Glashow, Unity of All Elementary Particle Forces, Phys. Rev. Lett. **32**, 438 (1974).

[20] R. M. Fonseca, On the chirality of the SM and the fermion content of GUTs, Nucl. Phys. **B897**, 757 (2015).

[21] D. Harries, M. Malinský, and M. Zdráhal, Witten's loop in the minimal flipped $SU(5)$ unification revisited, Phys. Rev. D **98**, 095015 (2018).

[22] S. C. Frautschi and J. Kim, SU(5) Higgs problem with adjoint + vector representations, Nucl. Phys. **B196**, 301 (1982).

[23] C. J. Cummins and R. C. King, Absolute minima of the Higgs potential for the 75 of $SU(5)$, J. Phys. A **19**, 161 (1986).

[24] Y. Song and L. Qi, Analytical expressions of copositivity for 4th order symmetric tensors and applications, Anal. Appl. **19**, 779 (2021).

[25] S. Weinberg, Baryon and Lepton Nonconserving Processes, Phys. Rev. Lett. **43**, 1566 (1979).

[26] R. Cottle, G. Habetler, and C. Lemke, On classes of copositive matrices, Linear Algebra Appl. **3**, 295 (1970).