## Unitarity and the Finkelstein-Kajantie problem in diffractive hadron production

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The Froissart-Martin bound states that two-particle diffraction cross section cannot rise with s faster than  $\ln^2 s$ , where s is the Mandelstam variable. However, the diffraction production of many hadron showers separated by large rapidity gaps, when calculated within the standard Pomeron approach, leads to cross sections rising much faster than the Froissart-Martin bound. This is the point of Finkelstein-Kajantie problem. We consider the unitarization procedure based on Dyson-Schwinger equations with input froissaron propagators and a three-froissaron vertex depending on angular momenta of froissarons in it. The developed diffraction production model allows to resolve Finkelstein-Kajantie problem.

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#### I. INTRODUCTION

A problem of the unitarity violation in the Pomeron models has been well known since 1960s [1–3]. It is more related to a three-Pomeron (3P) interaction vertex rather than to a possible large intercept  $[\alpha(0) > 1]$  of a bare Pomeron. If the Pomeron trajectory is linear, so  $\alpha(t) =$  $\alpha(0) + \alpha't$  with  $\alpha(0) = 1$ , and the 3P-vertex,  $r_{3P}$ , is constant, then the total hadron cross section does not depend asymptotically on hadron energy,  $\sigma_t(s) \propto$  $(s/s_0)^{\alpha(0)-1} = \text{const} (s_0 \sim 1 \text{ GeV}^2)$ . At the same time the contribution of diffraction production of high-effective mass hadron showers, separated by the large enough rapidity gaps (Fig. 1),  $\sigma_t(s)$  rise vigorously with energy demonstrating inconsistency of this simplified scheme (see [4–6] and references therein).

This process is pictured by the loop diagrams shown in Fig. 2. The equivalence of left and right diagrams is a consequence of the generalized optical theorem.

The total cross section of the *n* showers production is determined by the imaginary part of the loop diagram which has in  $j = \omega + 1$ , *t*-representation of the following form

$$\phi(\omega, t=0) \propto (r_{3P}^2)^{n-1} \frac{(\ln \omega/\omega)^{n-1}}{\omega}.$$
 (1)

<sup>\*</sup>martynov@bitp.kiev.ua <sup>†</sup>gters.hep@gmail.com One can obtain from this equation that at  $s \to \infty$ 

$$\sigma_{\text{diifr}}^{n}(s) \propto (r_{3P}^{2} \ln(s/s_{0}) \ln \ln(s/s_{0}))^{n-1}, \qquad s_{0} \sim 1 \text{ GeV}^{2},$$
(2)

which severely contradicts the Froissart-Martin bound [7–9]. This is the essence of the Finkelstein-Kajantie problem [2,10,11].

In the 1970s, Cardy proposed considering additional Pomeron rescatterings which had to screen large rapidity gaps [12]. It was believed that in a black disc limit such a screening can put the cross section back to Froissart-Martin unitarity bound [13,14]. Unfortunately, any eikonal-type screening appears to be not enough [15,16], at least for the simplest input contribution to the amplitudes of single diffraction dissociation (SDD), central diffraction production (CDP), or double diffraction dissociation (DDD). In Ref. [17] the differential cross sections have been written in a general form taking into account enhanced Reggeon diagrams. However, it is most likely, that the explicit final result for integrated (over rapidity and impact parameter variables) cross section cannot be obtained in analytical



FIG. 1. Diffraction production of n hadron showers.

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FIG. 2. Cross section of a diffraction production of n hadron showers as (n - 1)-loop diagram.



FIG. 3. Graphic definition of three-froissaron vertex.

form. Another way to fix the problem, namely assumption that the 3P-vertex depends on t,  $r_{3P}(t) \propto t$  at  $t \rightarrow 0$  [6,18] is not supported by the data at high energies.

The Pomeron with  $\alpha(t) = 1 + \varepsilon + \alpha' t$  as input in the eikonal ([19] and references to earlier papers therein, [20]), quasieikonal [21], *U*-matrix [22] unitarization and their generalization [23] lead to the elastic scattering amplitude which does not violate the Froissart-Martin limit for total cross section. Namely, unitarization leads to  $\sigma_t \approx 8\pi\varepsilon\alpha' \ln^2(s/s_0)$ . Such an amplitude (in a simplified form at  $s \to \infty$ ) can be represented in impact parameter representation  $H(s,b) = g\Theta(R(s) - b)$ , where  $R(s) \propto \ln(s/s_0)$  and  $g \le 1$ . In the  $\omega$ -representation this amplitude is not a pole—it looks like a pair of two complex branch points colliding at t = 0,

$$\phi(\omega, t) \propto (\omega^2 + a^2 \vec{q}^2)^{-3/2}, \quad \omega = j - 1, \quad \vec{q}^2 = -t.$$
 (3)

Such a Pomeron was called a froissaron [24,25] because it saturates the Froissart-Martin bound (in a functional form), i.e., provides the maximality of strong interactions.

By introducing the proper three-froissaron (3F) vertex shown in Fig. 3, one can hope to extend an unitarization to the shower production.

Anyway, there are two possibilities: either we start from the single pomeron pole with the intercept  $\alpha(0) =$  $1 + \varepsilon > 1$  (and then apply to it a not well-defined unitarization procedure) or we consider from the very beginning a more complicated Pomeron singularity [for instance, in the form (3)]. The question may be asked: does froissaron satisfy the Dyson-Schwinger equation (DSE), provided that the three-froissaron vertex is chosen appropriately.

The DSE for propagator is given in the Fig. 4.

Analytically the DSE for the propagator has the form,

$$G(\omega, \vec{q}^2) = G_0(\omega, \vec{q}^2) + G_0(\omega, \vec{q}^2) \Sigma(\omega, \vec{q}^2) G(\omega, \vec{q}^2), \quad (4)$$

where

$$\Sigma(\omega, \vec{q}^2) = -\frac{1}{2!} \int_{\uparrow} \frac{d\omega'}{2\pi i} \int \frac{d^2 q'}{\pi} \Gamma_0(\{\omega\}, \{q\}) G(\omega', \vec{q}'^2)$$
$$\times G(\omega - \omega', (\vec{q} - \vec{q}')^2) \Gamma(\{\omega\}, \{q\}), \tag{5}$$

 $\omega \equiv j - 1$ , j is the Pomeron angle momentum, and  $\vec{q}$  is the transverse component of the Pomeron momentum,  $\vec{q}^2 \approx -t$ .

The equation for the three-Pomeron vertex  $\Gamma(\omega, \omega_1, \omega_2; \vec{q}, \vec{q}_1, \vec{q}_2)$  is given by the Fig. 5.

In the leading order the analytical form of the DSE for the 3P-vertex (if only 3P-vertices are considered) is

$$\Gamma(\{\omega\};\{q\})$$

$$=\Gamma_{0}(\{\omega\};\{q\}) - \int \frac{d\omega'}{2\pi i} \int \frac{d^{2}q'}{\pi} \Gamma(\{\omega\};\{q\}) G(\omega',\vec{q}'^{2})$$

$$\times \Gamma(\{\omega\};\{q\}) G(\omega'-\omega_{1},(\vec{q}'-\vec{q}_{1})^{2})) \Gamma(\{\omega\};\{q\})$$

$$\times G(\omega-\omega',(\vec{q}-\vec{q}')^{2})) + (\omega_{1} \leftrightarrow \omega_{2},\vec{q}_{1} \leftrightarrow \vec{q}_{2}), \quad (6)$$

where the notations  $\{\omega\} = \omega, \omega', \omega - \omega', \{q\} = \vec{q}^2, \vec{q}'^2, (\vec{q} - \vec{q}')^2$  are used.



FIG. 4. DSE for the Pomeron propagator. The wavy line is the full propagator and the dashed line is the bare propagator. The black point is the bare or input 3P-vertex, the gray circle is the dressed or output vertex.



FIG. 5. Dyson-Schwinger equation for the 3P-vertex.

General properties of the DSE in the framework of Reggeon field theory (with a single Pomeron input pole) were discussed in details by Gribov [4]. Two regimes—the weak coupling and the strong coupling ones—were analyzed. The weak coupling regime was noted as preferable; however, it is not supported by available experimental data.

The first attempt to discuss the froissaron (3) as an input in the DSE was made by Ball [26]. He considered the model in which the output propagator and 3F-vertex are proportional to input ones

$$G(w, q^{2}) = gG_{0}(\omega, q^{2}),$$

$$\Gamma(\omega, \omega', q, q') = \gamma \Gamma_{0}(\omega, \omega', q, q'),$$

$$G_{0}(\omega, q^{2}) = 2\pi a^{2}(\omega^{2} + a^{2}q^{2})^{-3/2},$$

$$\Gamma_{0}(\omega, \omega', q, q') = \gamma_{0}(\omega^{2} + a^{2}q^{2})^{3/2}/(2\pi a^{2}),$$
(7)

and have obtained two algebraic equations for couplings g and  $\gamma$ .

However, one can see that in this simple scheme the integrated DSE cross section,  $\sigma_{SDE}(s) = 0$ . We think that the problem can be fixed only if the input 3F-vertex is changed for a more complicated form.

Developing this idea is one of the goals of our work. In Sec. II we define the main ingredients in our scheme and estimate corrections to the input propagator and 3F-vertex in the DS equations. Differential and integrated cross sections of diffraction production processes are estimated at asymptotic energy in Sec. III.

We would like to emphasize here that our approach is based on two main assumptions: First, the input Reggeon in DSE is a froissaron [Eq. (3)], which for t = 0 is located in the  $\omega$ -plane at the point  $\omega = 0$  not violating the Froissart-Martin bound. Second, we assume that the 3F-vertex depends on the angular and spatial momenta of the froissarons in it; therefore, the factorization of propagators and vertices takes place only in the  $(\omega, t)$ -representation, but does not hold in the (s, t) one, which is valid for the input Pomeron in the form of a simple pole.

In Sec. III, cross sections of diffraction processes are calculated at  $s \to \infty$ . The limits for free parameters at which the diffraction cross sections do not exceed Froissart-Martin bound are obtained. The results are summarized in the conclusion.

## II. PROPAGATOR AND 3F VERTEX: RESTRICTIONS ON THE VERTEX PARAMETERS

In accordance with a general form of the partial amplitude of elastic scattering at low  $\omega$  and  $q_i^2$  we consider the universal propagator for the froissaron

$$G_0(\omega, q) = \frac{E(\omega, q)}{(\omega^2 + a^2 q^2)^{3/2}},$$
(8)

where the function  $E(\omega, q)$  is a finite function at any  $\omega, \vec{q}$ , providing the main contributions in the integrals over  $\omega, \vec{q}$ in the region where  $\omega^2 \approx aq^2 \approx 0$ . We recall that our main interest is concentrated in the limit of high-energy and lowtransferred momenta, which correspond to the above mentioned  $\omega \to 0$ .

Now we suppose that in accordance with the structure of the froissaron singularity in  $G_0(\omega, q)$  at  $\omega^2 + \omega_0^2 = 0$ , the function  $E(\omega, q)$  depends on  $\omega$  through the variable  $\kappa = (\omega^2 + \omega_0^2)^{1/2}$  and it can be expanded in powers of  $\kappa$ ,

$$G_0(\omega, q) = \frac{E_0(q) + \kappa E_1(q) + \kappa^2 E_2(q)}{\kappa^3} = \sum_{k=0}^2 G_0^{(k)}(\omega, q),$$
(9)

where k corresponds (at q = 0) to the contribution of the triple pole (k = 0), double pole (k = 1) and single pole (k = 2). The DSE diagrams for propagator and vertex with arbitrary values of k are shown in Figs. 6 and 7. Thus the froissaron propagator can be written in the form with main and subasymptotic (SA) corrections terms

$$G_{0}(\omega,q) = \frac{E_{0}(q)}{(\omega^{2} + \omega_{0}^{2})^{3/2}} + \frac{E_{1}(q)}{\omega^{2} + \omega_{0}^{2}} + \frac{E_{2}(q)}{(\omega^{2} + \omega_{0}^{2})^{1/2}},$$
  

$$\omega_{0} = aq.$$
(10)

Let us notice that the first terms in the Eq. (10) has a pair of branch points colliding at  $\omega_0 = 0(q = 0)$  and generating a triple pole. The numerators  $E_k(\omega, q, k = 0, 1, 2)$  can be chosen for simplicity in exponential form either  $e^{-B_k q}$  or  $e^{-B_k q^2}$ , although it can be more sophisticated. The amplitudes with such terms can be calculated in (s, b)-representation, or at least can be estimated at



FIG. 6. General 3-Reggeon (3R) vertex depending on the kinds of legs.

 $b \ll \xi = \ln(s/s_0)$  and  $b \gg \xi = \ln(s/s_0)$ . The details for k = 0 are given in the Appendix A.

We impose certain requirements to the vertex function, from which the intervals for the vertex parameters can be set.

- (1) The vertex could not have singularities in  $\omega_i$ ,  $q_i \sim 0$  which lead to its infinity and could not cancel a singularity of the propagator (while it can make it a more soft). It means that vertex can (must) have zeros at some of these variables.
- (2) The vertex must be symmetric in  $\omega_i$ ,  $\vec{q_i}$ , i = 1, 2, at least for  $k_1 = k_2$ .
- (3) Diffractive differential and integrated cross sections could not violate the unitarity restriction.
- (4) In accordance with the experimental data differential cross section the single diffraction dissociation does not vanish at t = 0.
- (5) Most likely, the experiments show that diffractive integrated cross sections rise with energy slower then total and elastic integrated cross sections.
- (6) Corrections to propagator and vertex in the DSEs would be small at small  $\omega_i$  and  $q_i^2$ .

We consider here the 3F-vertex function in a factorized form. Each of the three factors at the vertex (Fig. 6) depends on the type of Reggeon corresponding to it in accordance with Eq. (10), and has the same functional form. Generally,

 $k_1$   $k_{01}$   $k_2$   $k_{02}$ 

FIG. 7. General form of the  $\Sigma^{(k_0;k_1,k_2)}(\omega_0, \vec{q_0})$  diagram.

the factor corresponding to the input froissaron (with  $\omega_0, \vec{q_0}$ ) can differ from other, output ones. For instance, it can have free parameters which are different from those at other factors shown in the next equations. To avoid a nonprincipal complexity we do not consider such a possibility here.

$$\Gamma_{0}^{(k,k_{1},k_{2})}(\omega,\omega_{1},\omega_{2},\vec{q},\vec{q}_{1},\vec{q}_{2}) = \gamma_{k}(\kappa,q)\gamma_{k_{1}}(\kappa_{1},q_{1})\gamma_{k_{2}}(\kappa_{2},q_{2}),$$

$$\kappa_{i} = (\omega_{i}^{2} + \omega_{0i}^{2})^{1/2} = (\omega_{i}^{2} + a^{2}q_{i}^{2})^{1/2}, \quad \omega_{i} = \omega,\omega_{1}.\omega_{2},$$
(11)

where

$$\gamma_i(\kappa_i, q_i) = \gamma_i(0, q_i) \kappa_i^{\mu(k_i)}, \qquad (12)$$

and

$$\mu(k) = \mu_1(k) + k,$$
 (13)

with  $\mu_1(k)$  coming from the 3F-vertex while the second term *k* takes into account the kind of Reggeon from the expansion (10) of the f-propagator.

To satisfy point 1 from the list of requirements to 3F vertex we must impose

$$0 < \mu_1(k) < 3$$
 at any  $k = 0, 1, 2.$  (14)

It will be shown in Sec. III A that  $\mu_1(k=0) \equiv \mu_0 \neq 0$ . Otherwise this term leads to the integrated diffraction cross section rising with energy  $\propto \ln^5(s/s_0)$  (F-K problem).

In the next section we consider three specified choices of the function  $\mu_1(k)$ 

(a) 
$$\mu_1(k) = \mu_0,$$
  
(b)  $\mu_1(k) = \mu_0(1 + \lambda k),$   
(c)  $\mu_1(k) = \frac{\mu_0}{1 + \lambda k}.$  (15)

Equation (a) chooses the universal form of 3F-vertex independent on the corrections to propagators. Equations (b) and (c) describe an increasing and decreasing k with power of  $\kappa$  in the vertex.

Now let us consider a small, main, correction to the input propagator  $G_0(\omega, t)$  and vertex  $\Gamma_0(\{\omega\}, \{\vec{q}\})$ .

#### A. DS-corrections: All Reggeons are froissarons

Let us consider here the main corrections to the propagator and vertex in DSE. In the equations for propagator (9), (10), and vertex (11) only the main term (with k = 0) is taken into account. In this section we ignore the subasymptotic, coming from propagator corrections [Eq. (9)] which are small in the region of  $\omega$ , q considered here.



FIG. 8. Corrections to three-Reggeon vertex.

$$G_0^{(0)}(\omega,q) = \frac{E_{0,+}(q)}{(\omega^2 + \omega_0^2)^{3/2}} \quad \omega_0 = \sqrt{a}q \qquad (16)$$

$$\Gamma_0^{(0)}(\omega, \omega_1, \omega_2, \vec{q}, \vec{q}_1, \vec{q}_2) = \gamma_1(\kappa, q)\gamma_2(\kappa_1, q_1)\gamma_3(\kappa_2, q_2), 
\kappa_{i+} = (\omega_i^2 + a_+^2 q_i^2)^{1/2}, \qquad \omega_i = \omega, \omega_1.\omega_2$$
(17)

where

$$\gamma_i(\kappa_i, q) = \gamma_i(0, q_i) \kappa_i^{\mu(k_i=0)}.$$
(18)

It will be shown in Sec. III A that  $\mu(k = 0) \equiv \mu_0 \neq 0$ . Otherwise this term leads to the integrated diffraction cross section rising with energy  $\propto \ln^5(s/s_0)$ . (This is a F-K problem or paradox).

## 1. Smallness of the DS-corrections

*Propagator:* 

First, let us consider the "simplest" case with one parameter  $\mu_0$  in the vertex  $\Gamma$ , all  $k_i = 0$ . The input froissaron propagator  $G_0^{(0)}$  and input vertex  $\Gamma_0^{(0)}$  are defined by Eqs. (16) and (17), correspondingly.

In this case we have for the first corrections to the  $G_0^{(0)}$  propagator (all propagators and vertices in the first approximation are the input ones  $G_0^{(0)}, \Gamma_0^{(0)}$ ),

$$G_{1}(\omega, q) = G_{0}^{(0)}(\omega, q^{2}) + \Delta G,$$
  
$$\Delta G = G_{0}^{(0)}(\omega, q) \Sigma_{0}^{(0)}(\omega, q) G_{0}^{(0)}(\omega, q), \quad (19)$$

$$\Sigma_{0}^{(0)}(\omega,q) = \int_{C} \frac{d\omega'}{2\pi i} \int \frac{d^{2}q'}{\pi} \Gamma_{0}^{(0)}(\{\omega\},\{\vec{q}\}) G_{0}^{(0)}(\omega,q_{1}) \\ \times G_{0}^{(0)}(\omega,q_{2}) \Gamma_{0}^{(0)}(\{\omega\},\{\vec{q}\}).$$
(20)

Now let us estimate the corrections to  $G_0$  at  $s \to \infty$ . In Eq. (20) the essential region of integration is  $\omega(\omega') \sim aq(aq') \sim 1/\xi$ . Therefore, from Eqs. (19) and (20) we have

$$\Delta G \propto G_0^{(0)}(\omega^2)^{3/2}(\omega^2)^{(-3/2)}(\omega^2)^{3\mu_0}(\omega^2)^{-3} = G_0^{(0)}(\omega^2)^{3\mu_0-3},$$
(21)

i.e.,

$$G_1 = G_0^{(0)} [1 + \mathcal{O}(\omega^{6(\mu_0 - 1)})].$$
(22)

The first correction is small at  $\omega \sim q \sim 0$  if  $\mu_0 > 1$ . It follows from this inequality and (14) that

$$1 < \mu_0 < 3.$$
 (23)

Vertex:

The first correction  $\Delta_1\Gamma$ ,  $\Delta_2\Gamma$ , (Fig. 8) are the following

$$\Gamma_{1}(\omega_{0}, \omega_{1}, \omega_{2}, \overrightarrow{q_{0}}, \overrightarrow{q_{1}}, \overrightarrow{q_{2}}) \equiv \Gamma_{1} = \Gamma_{0}^{(0)} + \Delta_{1}\Gamma + \Delta_{2}\Gamma,$$

$$\Delta_{1}\Gamma = \int \frac{d\omega' d^{2} \overrightarrow{q'}}{2\pi i} \Gamma_{0}^{(0)}(\omega_{0}, \omega_{3}, \omega_{4}, \overrightarrow{q_{0}}, \overrightarrow{q_{3}}, \overrightarrow{q_{4}})$$

$$\times \Gamma_{0}^{(0)}(\omega_{1}, \omega_{3}, \omega_{5}, \overrightarrow{q_{1}}, \overrightarrow{q_{3}}, \overrightarrow{q_{5}})$$

$$\times \Gamma_{0}^{(0)}(\omega_{3}, \omega_{5}, \omega_{2}, \overrightarrow{q_{3}}, \overrightarrow{q_{5}}, \overrightarrow{q_{2}})G_{0}^{(0)}(\omega_{3}, q_{3}^{2})$$

$$\times G_{0}^{(0)}(\omega_{4}, q_{4}^{2})G_{0}^{(0)}(\omega_{5}, q_{5}^{2}),$$

$$\Delta_{2}\Gamma = \Delta_{1}\Gamma(\omega_{1} \Leftrightarrow \omega_{2}),$$
(24)



FIG. 9. SDD process (left) and the corresponding diagram (right) from generalized optical theorem,  $\xi = \xi_X + \xi_1$ .



FIG. 10. CDP process (left) and the corresponding diagram (right) from the generalized optical theorem,  $\xi = \xi_X + \xi_1 + \xi_2$ .

Similarly to the propagator case, consideration of the correction to  $\Gamma_0^{(0)}$  leads to

$$\Delta_1 \Gamma \propto \Gamma_0^{(0)} (\omega^2)^{3/2} (\omega^2)^{6 \cdot \mu_0/2} (\omega^2)^{-9/2} = \Gamma_0^{(0)} (\omega^2)^{3(\mu_0 - 1)}.$$
(25)

Because  $\Delta_1 \Gamma \approx \Delta_2 \Gamma$  in the considered limit, we have

$$\Gamma_1 = \Gamma_0^{(0)} [1 + \mathcal{O}(\omega^{6(\mu_0 - 1)})].$$
(26)

Again, the correction is small at  $\omega \sim q \sim 0$  if  $\mu_0 > 1$ .

Moreover, let us evaluate the factor coming from one additional Reggeon line between any two Reggeon lines. We have three new propagators, two new 3F-vertices, and one integral over the new loop. The additional factor has at  $\omega_i \sim$ ,  $q_i \ll 1$  the following behavior

$$(\omega^2)^{3/2} (\omega^2)^{-9/2} (\omega^2)^{(6\mu_0)/2} = \omega^{6(\mu_0 - 1)}.$$
 (27)

Inserting one Reggeon loop into the Reggeon line we have the same additional factor,  $\omega^{12(\mu_0-1)}$ . And finally, inserting one Reggeon line with one loop (six new Reggeons, four



FIG. 11. DDD process (left) and the corresponding diagram (right) from generalized optical theorem,  $\xi = \xi_{X1} + \xi_{X2} + \xi_1$ .

new vertices, and two new loops), we again obtain the same factor.

This means that an increase in the number of Reggeons and vertices only increases the number of ever-smaller corrections in DS-equations if  $\mu_0 > 1$ .

#### **B. DS-corrections: General case**

Here we consider the general term defined in the propagator Eqs. (9) and (10)

$$G_0^{(k)}(\omega_k, q_k) = \frac{E_k(q_k)}{(\omega_l^2 + \omega_{0l}^2)^{3/2 - k/2}},$$
  
$$\omega_{0l} = aq_k, \quad k = 0, 1, 2.$$
(28)

and in 3R-vertices

$$\Gamma_{0}^{(k_{0},k_{1},k_{2})}(\omega,\omega_{1},\omega_{2},\vec{q},\vec{q}_{1},\vec{q}_{2}) = \gamma_{k_{0}}(\kappa,q)\gamma_{k_{1}}(\kappa_{1},q_{1}) \\
\times \gamma_{k_{2}}(\kappa_{2},q_{2}^{2}), \\
\kappa_{i} = (\omega_{i}^{2} + \omega_{0i}^{2})^{1/2} = (\omega_{i}^{2} + a^{2}q_{i}^{2})^{1/2}, \qquad \omega_{i} = \omega,\omega_{1}.\omega_{2}.$$
(29)

It should be noted that  $k_1$ ,  $k_2$  values must be equal to those in the corresponding terms of the left and the right Reggeon propagators in the given 3F-vertex. Functions  $\gamma(\omega, k)$  in Eq. (29) we choose in the form

$$\gamma_i(\omega_i, \kappa_i) = \gamma_i(0, q_i) \kappa_i^{\mu(k_i)}.$$
(30)

Equations (19) and (20) in the case when  $k_i \neq 0$  are transformed to

$$G_{1}^{(k)}(\omega,q) \stackrel{\omega,q \to 0}{\approx} G_{0}^{(k)}(\omega,q) [1 + \Delta G^{(l)}],$$
$$\Delta G^{(k)} = \sum_{k,k_{1},k_{2}=0}^{2} G_{0}^{(k)}(\omega,q_{l}) \Sigma_{0}^{(k,k_{1},k_{2})}(\omega,q_{l}).$$
(31)

Now we can take into account that  $G_0^{(k=0)} \gg G_0^{(k=1)} \gg G_0^{(k=2)}$  in the considered region of  $\omega$ , q. [For  $k \neq 0$  we have in  $\Delta G^{(l)}$  additional small factors  $(\omega^2)^n$  where n > 0 dependant on the values of  $k, k_1, k_2$ .] Thus, we come back to the results (22), (23), and (26).

In Sec. III B additional inequality is found

$$1 < \mu_0 < 3/2.$$
 (32)

For corrections to the vertex function one can obtain the following estimations

$$\Gamma_{1}^{(k,k_{1},k_{2})} \propto \Gamma_{0}^{(k,k_{1},k_{2})} [1 + \Delta \Gamma^{(k)}],$$
  

$$\Delta \Gamma^{(k)} \propto \sum_{k_{3},k_{4},k_{5}=0}^{2} (\omega^{2})^{-3+\mu(k_{3})+\mu(k_{4})+\mu(k_{5})+k_{3}+k_{4}+k_{5}}$$
  

$$= \sum_{k_{3},k_{4},k_{5}=0}^{2} (\omega^{2})^{3(\mu_{0}-1)+\mu(k,k_{1},k_{2})},$$
(33)

where

$$\mu(k, k_1, k_2) = \sum_{i=3}^{5} (\mu(k_i) - \mu_0 + k_i).$$
(34)

Function  $\mu(k, k_1, k_2)$  is positive for any values of  $0 \le k_i \le 2$  and equal to zero if  $k_i = 0$ . It is valid for both choices of  $\mu_1(k)$  in Eq. (15). Thus, in the general case a smaller corrections comparing with the main ones (at all k = 0) are added.

In the B the inequalities for parameters  $\mu_0$ ,  $\lambda$  in three choices for the function  $\mu_1(k)$  defined in the Eq. (15) are obtained. Combining all these constraints we get

$$\begin{aligned} 1 &< \mu_0 < 3/2, \\ \lambda &= 0 & \text{for the case (a),} \\ -1/2 &< \lambda < -/1/4 & \text{for the case (b),} \\ \lambda &> 0 & \text{for the case (c).} \end{aligned}$$
(35)

## III. DIFFRACTION PROCESSES WITH LARGE RAPIDITY GAPS

## A. Single diffraction dissociation

The SDD process is illustrated in Fig. 9. In general case the input propagator *G* and 3R-vertex with the parameters  $(\mu_0, \lambda)$  defined by Eqs. (28), (29), and (30),

$$\sigma_{SDD}(s) = \sum_{k_0, k_1, k_2=0}^{2} \sigma_{SDD}^{(k_0, k_1, k_2)}(s),$$
(36)

where

$$\sigma_{SDD}^{(k_0,k_1,k_2)}(s) = \int_{\xi_0}^{\xi-\xi_0} d\xi_X \int_0^\infty dq \, q \, \frac{d\sigma_{SDD}^{(k_0,k_1,k_2)}}{dt d\xi_X}, \quad (37)$$

and

$$\frac{d\sigma_{SDD}^{(k_0,k_1,k_2)}}{dtd\xi_X} = C_{SDD} \int \frac{d\omega}{2\pi i} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} e^{\xi_X \omega} e^{\xi_1(\omega_1 + \omega_2)} \\
\times \eta_{\omega_1} \eta_{\omega_2}^* v_{k_1}(t) v_{k_2}(t) E_{k_0}(0) E_{k_1}(t) E_{k_2}(t) \\
\times \frac{\kappa_0^{\mu_1(k_0)} \kappa_1^{\mu_1(k_1)} \kappa_2^{\mu_1(k_2)}}{\kappa_0^{3-k_0} \kappa_1^{3-k_1} \kappa_2^{3-k_2}} \\
= \tilde{C}_{SDD} \int \frac{d\omega}{2\pi i} \frac{e^{\xi_X \omega}}{\omega^{3-\mu(k_0)}} \int \frac{d\omega_1}{2\pi i} \frac{e^{\xi_1 \omega_1}}{\kappa_1^{3-\mu(k_1)}} \int \frac{d\omega_2}{2\pi i} \\
\times \frac{e^{\xi_1 \omega_2}}{\kappa_2^{3-\mu(k_2)}} \eta_{\omega_1} \eta_{\omega_2}^* v^2(t) E_{k_0}(0) E_{k_1}(t) E_{k_2}(t),$$
(38)

where

$$C_{SDD} = \frac{1}{32\pi^2} v_{k_0}(0) \gamma_{k_0}(0,0) \gamma_{k_1}(0,0) \gamma_{k_1}(0,0),$$
  

$$E_{k_i}(t) = \exp(B_{k_i}t) \cdot v_{k_i}(t) = v_{k_i}(0) \exp(B_{v,k_i}vt),$$
  

$$\mu(k) = \mu_1(k) + k,$$
(39)

and  $\mu_1(k)$  is defined in Eq. (15).

Then, after integration over  $\omega$  the differential SDD cross section can be written in the following form

$$\begin{aligned} \frac{d\sigma_{SDD}^{(k_0,k_1,k_2)}}{dtd\xi_X} \\ &= \tilde{C}_{SDD} \left( \frac{J_{1-\mu(k_1)/2}(\tilde{\xi}_1 a q)}{\Gamma((3-\mu(k_1)/2)} \left( \frac{\tilde{\xi}_1}{2aq} \right)^{1-\mu(k_1)/2} \right) E_{k_1}(q^2) v_{k_1}(q^2) \\ &\times \left( \frac{J_{1-\mu(k_2)/2}(\tilde{\xi}_1^* a q)}{\Gamma((3-\mu(k_2)/2)} \left( \frac{\tilde{\xi}_1^*}{2aq} \right)^{1-\mu(k_2)/2} \right) E_{k_2}(q^2) v_{k_2}(q^2), \end{aligned}$$

 $\tilde{C}_{SDD}$ 

$$=C_{SDD}\frac{2\pi 2^{-2+\mu(k_1)/2+\mu(k_2)/2}}{\Gamma(3-\mu(k_0))\Gamma((3-\mu(k_1))/2)\Gamma((3-\mu(k_2))/2)},$$
(40)

where  $\tilde{\xi}_1 = \xi_1 - i\pi/2$ ,  $\xi_1 = \xi - \xi_X$ . The given presentations of integrals over  $\omega_i$  are valid if  $3 - \mu_1(k_i) > 0$ .

Restrictions on the parameters  $\mu_0$ , and  $\lambda$  in function  $\mu_1(k)$  have been derived in Appendix B.

$$\begin{aligned} &\sigma_{SDD}^{(k_0,k_1,k_2)}(s) \\ &\approx \tilde{C}_{SDD} \sum_{k_0,k_1,k_2=0}^2 \int_{\xi_0}^{\xi-\xi_0} d\xi_X \int_0^{\infty} dq \, q \, \frac{d\sigma_{SDD}^{(k_0,k_1,k_2)}}{dt d\xi_X} \\ &\approx \tilde{C}_{SDD} \sum_{k,k_1,k_2=0}^2 v_{k_1} v_{k_2} \int_{\xi_0}^{\xi-\xi_0} d\xi_X \xi_X^{2-\mu(k_0)} (\xi-\xi_X)^{2-\mu(k_1)-\mu(k_2)} \\ &\qquad \times \int_0^{\infty} dx \, x \left(\frac{J_{1-\mu(k_1)/2}(\tilde{x})}{\tilde{x}^{1-\mu(k_1)/2}}\right) \left(\frac{J_{1-\mu(k_2)/2}(\tilde{x})}{\tilde{x}^{1-\mu(k_2)/2}}\right)^* \\ &\propto \sum_{k_0,k_1,k_2=0}^2 \int_{\xi_0}^{\xi-\xi_0} d\xi_X \xi_X^{2-\mu(k_0)} (\xi-\xi_X)^{2-\mu(k_1)-\mu(k_2)}, \quad (41) \end{aligned}$$

where we put  $E_{k_i}(q) \approx 1$ ,  $v_{k_i}(q) \approx v_{k_i}(0) \equiv v_{k_i}$ 

Cross section energy dependence is governed by the integral

$$\begin{split} & \int_{\xi_0}^{\xi-\xi_0} d\xi_X \xi_X^{2-\mu(k_0)-k_0} (\xi-\xi_X)^{2-\mu(k_1)-\mu(k_2)-k_1-k_2} \\ & = \xi^{5-\mu(k_0)-\mu(k_1)-\mu(k_2)-(k_0+k_1+k_2)} \\ & \times \int_{\xi_0}^{1-\xi_0/\xi} dx x^{2-\mu(k_0)-k_0} (1-x)^{2-\mu(k_1)-\mu(k_21)-k_1-k_2} \\ & = \xi^{-1+\alpha_1+\alpha_2} I_{SDD}(\xi), \end{split}$$

where  $\alpha_1 = 3 - \mu(k_0) - k_0 > 0$ ,  $\alpha_2 = 3 - \mu(k_1) - \mu(k_2) - k_1 - k_2$  (sign of  $\alpha_2$  depends on the values of  $k_i$ ).

The estimation of the  $I_{SDD}(\xi)$  is given in Appendix C. So, the final result is the following

$$\sigma_{SDD}^{(k_0,k_1,k_2)}(s) \propto \int_{0}^{\xi} d\xi_X \xi_X^{2-\mu_1(k_0)} (\xi - \xi_X)^{2-\mu_1(k_1)-\mu_1(k_2)}$$
  
$$\propto \xi^{2-3(\mu_0-1)} \times \begin{cases} b) \ \xi^{\lambda\mu_0(k_0+k_1+k_2)}, & -1/2 < \lambda < 0, \\ c) \ \xi^{-\lambda\mu_0} \sum_{i=0}^{2} \frac{k_i}{1+\lambda k_i}, & \lambda > 0. \end{cases}$$
(42)

Thus, if  $\mu_0 > 1$  then the integrated cross section  $\sigma_{SDD}^{(k_0,k_1,k_2)}(s)$  rises with energy slower than  $\xi^2$  at any  $0 \le k_0, k_1, k_2 \le 2$ .

At  $s \to \infty$  the main contribution (it has all  $k_i = 0$ ) to the SDD cross section is

$$\sigma_{SDD}(s) \approx \sigma_{SDD}^{(0,0,0)}(s) \propto \xi^{2-3(\mu_0-1)}.$$
 (43)

#### **B.** Central diffraction production

Here, and in the following subsections, we consider only the main contributions to diffraction cross sections, because similarly to the SSD case, other terms at  $s \rightarrow \infty$  rise more slowly.

Let us write the expression for the differential CDP cross section (Fig. 10) in terms of the above proposed propagators and vertices at all  $k_i = 0$ ,

$$\frac{d\sigma^{CDP}(\xi)}{dt_1 d\xi_1 dt_2 d\xi_2} = C_{CDP} \int \frac{d\omega}{2\pi i} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} \int \frac{d\omega_3}{2\pi i} \int \frac{d\omega_4}{2\pi i} \\
\times \eta_{\omega_1} \eta_{\omega_2}^* \eta_{\omega_3} \eta_{\omega_4}^* e^{\xi_{\chi}\omega} e^{\xi_1(\omega_1+\omega_2)} e^{\xi_2(\omega_3+\omega_4)} \frac{(\kappa^2 \kappa_{1+}\kappa_{2+}\kappa_{3+}\kappa_{4+})^{\mu_0}}{\kappa^3 \kappa_{1+}^3 \kappa_{3+}^3 \kappa_{4+}^3} \\
= C_{CDP} \int \frac{d\omega}{2\pi i} \frac{e^{\xi_{\chi}\omega}}{\omega^{3-2\mu_0}} \int \frac{d\omega_1}{2\pi i} \frac{\eta_{\omega_1} e^{\xi_1\omega_1}}{(\omega_1^2 + \omega_{0,1}^2)^{(3-\mu_0)/2}} \int \frac{d\omega_2}{2\pi i} \frac{\eta_{\omega_2}}{(\omega_2^2 + \omega_{0,1}^2)^{(3-\mu_0)/2}} \\
\times \int \frac{d\omega_3}{2\pi i} \frac{\eta_{\omega_3} e^{\xi_2\omega_3}}{(\omega_3^2 + \omega_{0,2}^2)^{(3-\mu_0)/2}} \int \frac{d\omega_4}{2\pi i} \frac{\eta_{\omega_4}}{(\omega_4^2 + \omega_{0,2}^2)^{(3-\mu_0)/2}},$$
(44)

where

$$C_{CDP} = \frac{1}{32\pi^2} v_0(0,0) v_0^2(t,0) \gamma_0^2(0,0) \gamma_0^2(0,q_1^2) \gamma_0^2(0,q_2^2) E^2(t_1) E^2(t_2),$$
  

$$E(t_1) = \exp(t_1 B_1) . E(t_2) = \exp(t_2 B_2).$$
(45)

The integral over  $\omega$  in (44) converges if  $\mu_0 < 3/2$ .

$$\begin{split} \frac{d\sigma^{CDP}}{dt_1 d\xi_1 dt_2 d\xi_2} &\propto \xi_X^{2-2\mu_0} \left( \frac{J_{1-\mu_0/2}(\tilde{\xi}_1 a q_1)}{\Gamma((3-\mu_0)/2)} \left( \frac{\tilde{\xi}_1}{2aq} \right)^{1-\mu_0/2} \right) \left( \frac{J_{1-\mu_0/2}(\tilde{\xi}_1^* a q_1)}{\Gamma((3-\mu_0)/2)} \left( \frac{\tilde{\xi}_1^*}{2aq} \right)^{1-\mu_0/2} \right) \\ &\times \left( \frac{J_{1-\mu_0/2}(\tilde{\xi}_2 a q_2)}{\Gamma((3-\mu_0)/2)} \left( \frac{\tilde{\xi}_2}{2aq} \right)^{1-\mu_0/2} \right) \left( \frac{J_{1-\mu_0/2}(\tilde{\xi}_2^* a q_1)}{\Gamma((3-\mu_0)/2)} \left( \frac{\tilde{\xi}_2^*}{2aq} \right)^{1-\mu_0/2} \right). \end{split}$$

Similarly to the SDD case, we can estimate the CDP cross section as

$$\sigma_{CDP}(s) \propto \int_{0}^{\xi} d\xi_1 \xi_1^{2-2\mu_0} \int_{0}^{\xi-\xi_1} d\xi_2 \xi_2^{2-2\mu_0} (\xi-\xi_1-\xi_2)^{2-2\mu_0} \approx \xi^{8-6\mu_0} \int_{0}^{1} dx_1 x_1^{2-\mu_0} \int_{0}^{1-x_1} dx_2 x_2^{2-\mu_0} (1-x_1-x_2)^{2-2\mu_0}, \quad (46)$$

where  $\xi_X = \xi - \xi_1 - \xi_2$ . The intgrated CDP cross section rises slower than  $\xi^2$  if  $8 - 6\mu_0 < 2$ . Thus,  $\sigma_{CDP}$  does not rise as fast as  $\xi^2$  if

$$1 < \mu_0 < 3/2.$$
 (47)

## C. Double diffraction dissociation

Let us consider the production of two bunches  $X_1$ ,  $X_2$  of rapidity widths  $\xi_{X_1}$   $\xi_{X_2}$ , separated by a large rapidity gap  $\xi_1 = \xi - \xi_{X_1} - \xi_{X_2}$  (Fig. 11)

$$\frac{d\sigma_{DDD}}{dtd\xi_{1X}d\xi_{2X}} = C_{DDD} \int \frac{d\omega_{01}}{2\pi i} \int \frac{d\omega_{02}}{2\pi i} \int \frac{d\omega_{1}}{2\pi i} \int \frac{d\omega_{2}}{2\pi i} \eta_{\omega_{1}} \eta_{\omega_{2}}^{*} e^{\xi_{1X}\omega_{01}} e^{\xi_{2X}\omega_{02}} e^{\xi_{1}(\omega_{1}+\omega_{2})} \frac{(k_{01}k_{02}k_{1+}^{2}k_{2+}^{2}))^{\mu_{0}}}{\kappa_{01}^{3}\kappa_{02}^{3}\kappa_{1+}^{3}\kappa_{2+}^{3}} \\
= C_{DDD} \frac{(\xi_{X1}\xi_{X2})^{2-\mu_{0}}}{\Gamma^{2}(3-\mu_{0})} \int \frac{d\omega_{1}}{2\pi} \int \frac{d\omega_{2}}{2\pi i} \frac{e^{\xi_{1}'\omega_{1}+\xi_{1}'*\omega_{2}}}{(\omega_{1}^{2}+a^{2}q^{2})^{3/2-\mu_{0}}(\omega_{2}^{2}+a^{2}q^{2})^{3/2-\mu_{0}}} \\
\propto (\xi_{X1}\xi_{X2})^{2-\mu_{0}} \left| \left(\frac{\xi_{1}'}{2aq}\right)^{1-\mu_{0}} J_{1-\mu_{0}}(\xi_{1}'aq) \right|^{2},$$
(48)

where  $\xi'_1 = \xi - i\pi/2 - \xi_{1X} - \xi_{2X}$ .

$$\sigma_{DDD} \propto \int_{\xi_{0}}^{\xi-2\xi_{0}} d\xi_{X1} \int_{\xi_{0}}^{\xi-\xi_{0}-\xi_{1}} d\xi_{X2} \xi_{X1}^{2-\mu_{0}} \xi_{X2}^{2-\mu_{0}} \int_{0}^{\infty} dqq \left| \left(\frac{\tilde{\xi}_{1}}{2aq}\right)^{1-\mu_{0}} J_{1-\mu_{0}}(\tilde{\xi}_{1}aq) \right|^{2} \\ = \int_{\xi_{0}}^{\xi-2\delta} d\xi_{X1} \int_{\xi_{0}}^{\xi-\xi_{0}-\xi_{1}} d\xi_{X2} \xi_{X1}^{2-\mu_{0}} \xi_{X2}^{2-\mu_{0}} |(\xi-\xi_{X1}-\xi_{X2})|^{2-4\mu_{0}} \int_{0}^{\infty} dzz \left| \frac{\tilde{J}_{1-\mu_{0}}(\tilde{z})}{z^{1-\mu_{0}}} \right|^{2} \\ \propto \xi^{8-6\mu_{0}} \int_{\delta}^{1-2\delta} dx_{1} x_{1}^{2-\mu_{0}} \int_{\delta}^{1-\delta-x_{1}} dx_{2} x_{2}^{2-\mu_{0}} (1-x_{1}-x_{2})^{2-4\mu_{0}} \propto \xi^{2} \xi^{-6(\mu_{0}-1)}.$$

$$(49)$$

Because of unitarity restriction on  $\sigma_{DDD}(s)$  at  $s \to \infty$  we demand  $\mu_0 > 1$ .

#### D. DDD with additional showers

Now we can generalize DDD for a production of more then two hadron showers separated by large rapidity gaps (the left diagram of Fig. 12). The right diagram comes from the generalized optical theorem for the process amplitude.

To estimate the contribution of the given process to  $\sigma_{\text{tot}}(s)$ , we can write it in the following form keeping only the dependence of  $\xi_i$ . It can be made easily by evident extension of the Eq. (49) to arbitrary  $n \ge 1$ . However, it is evident that maximal value of n is  $n_{\text{max}} = [(\xi/\xi_0 - 1)/2]$ . Here we show the result for the main contribution (all  $k_i = 0$ ) to  $\sigma_{DDD}^{(n)}(s)$ .

$$\sigma_{DDD}^{(n)}(s) \propto \int_{\Xi} \prod_{i=1}^{2n+1} d\xi_i \delta\left(\xi - \sum_{i=1}^{2n+1} \xi_i\right) \xi_1^{2-\mu_0} \xi_{2n+1}^{2-\mu_0} \prod_{i=1}^n \xi_{2i}^{2-4\mu_0}$$
$$\prod_{i=1}^{n-1} \xi_{2i+1}^{2-2\mu_0} \propto \xi^{2-6n(\mu_0-1)}, \qquad \sigma_{DDD}^{(1)}(s) \equiv \sigma_{DDD}(s). \tag{50}$$

Again, there is no a violation of *s*-channel unitarity if  $\mu_0 > 1$ . However we remind that corrections to the froissaron propagator and 3F-vertex in the DS equations are small at small  $\omega$  and *q* if  $\mu_0 > 1$ 

Similarly one can obtain for other generalized processes

$$\begin{aligned} &\sigma_{SDD}^{(n)}(s) \propto \xi^{2-3(1+2n)(\mu_0-1)}, \qquad \sigma_{SDD}^{(0)}(s) \equiv \sigma_{SDD}(s), \\ &\sigma_{CDP}^{(n)}(s) \propto \xi^{2-6(1+n)(\mu_0-1)}, \qquad \sigma_{CDP}^{(0)}(s) \equiv \sigma_{CDP}(s). \end{aligned}$$



FIG. 12. DDD with additional n - 1 "internal" hadron showers.

## **IV. CONCLUSION**

We have considered an alternative approach to solving the problem of s-channel unitarity bounds (in particular the Finkelstein-Kajantie paradox) on diffraction production amplitudes in the froissaron model, in which  $\sigma_{tot}(s) \propto$  $\ln^2(s/s_0)$ . Our main assumption concerns the vertex of the three froissarons interaction, which in our approach depends on the angular and spatial momenta of the froissarons. The basic requirements for the properties of a 3F vertex are formulated and a model is constructed in which the corrections to the Dyson-Schwinger Reggeon equations for s to  $\infty$  are small in the region of small angular and spatial momenta of froissarons at the vertex. In this paper, the Dyson-Schwinger equations are considered in the leading approximation, in which only 3F vertices are taken into account. We believe that the corrections with higher-order vertices will be even smaller in this case. They will be considered in a separate work.

The constructed model of a 3F vertex contains two parameters, one of which determines the behavior of the diffraction cross sections, which grow with energy more slowly than  $\ln^2(s/s_0)$  and the smallness of corrections to the propagator and the vertex in the Dyson-Schwinger equations. The second parameter takes into account the magnitude of the corrections, which also arise from the preasymptotic terms in the froissaron propagator.

The model is applied to asymptotic estimates of the contribution to the total cross section of the three main processes of diffraction hadron production (SDD, CDP, DDD) and their generalizations to an arbitrary number of produced hadron beams with large gap rapidities between them. All these cross sections in our approach do not functionally exceed the Froissart-Martin boundary.

Thus, in the developed approach, the Finkelstein-Kajantie contradiction does not arise at asymptotic energies.

In the diffraction interaction of protons with protons and protons with antiprotons, it is necessary to take into account the contribution of odderon along with froissaron. For strong scattering, this is done in the froissaron and maximal Odderon model [27–29]. The results of comparing the model with experimental data showed that odderon effects are visible at high energies, but remain small. The properties of differential and total pp and  $\bar{p}p$  cross sections are determined by the dominant contribution of the froissaron. We are sure that this property also holds for the processes of diffraction production in pp and  $\bar{p}p$  collisions at  $s \to \infty$ .

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## APPENDIX A: MODIFIED PROPAGATOR FOR REGGEON

Let us consider the possibility of having a smooth behavior of  $\text{Im}H^{(k)}(s, b)$  in *b* and to guarantee a divergence of integrals over momenta  $\vec{q_i}$ . For that we write a Reggeon propagators in the following form

$$G^{(k)}(\omega,q) = \frac{g_k e^{-Bq^2}}{(\omega^2 + a^2 q^2)^{3/2 - k/2}}, \quad k = 0, 1, 2, \quad (A1)$$

$$\operatorname{Im} H^{(k)}(s,b) = \frac{v^2 g_k}{4s} \operatorname{Im} \left( i \int_0^\infty dq q J_0(bq) e^{-Bq^2} \int \frac{d\omega}{2\pi i} \frac{e^{(1+\omega)\xi'}}{(\omega^2 + a^2 q^2)^{3/2-k/2}} \right)$$
$$= C \operatorname{Im} \left( i\xi'^{1-k/2} \int_0^\infty dx x^{k/2} J_0(bx/a) e^{-Bx^2/a^2} J_{1-k/2}(\xi'x) \right),$$
$$C = \frac{v^2 g_k \sqrt{\pi} 2^{-k/2}}{8s_0 a^2 \Gamma(3/2 - k/2)}, \qquad \xi' = \xi - i\pi/2.$$
(A2)

Then considering  $\xi \to \infty$  we neglect the imaginary part in  $\xi'$  and make use the integral

$$\int_{0}^{\infty} dx x^{\lambda - 1} e^{-\alpha x^{2}} J_{\mu}(\beta x) J_{\nu}(\gamma x) = \frac{\beta^{\mu} \gamma^{\nu}}{2^{\nu + \mu + 1} \Gamma(\nu + 1)} \alpha^{-(\lambda + \nu + \mu)/2} \sum_{m=0}^{\infty} \frac{\Gamma(m + (\lambda + \mu + \nu)/2)}{m! \Gamma(m + \mu + 1)} \left( -\frac{\beta^{2}}{4\alpha^{2}} \right)^{m} \times F\left(-m, -m - \mu; \nu + 1; \frac{\gamma^{2}}{\beta^{2}}\right).$$

For  $a\xi \ll b$  we choose  $(\mu = 0, \nu = 1 - k, \lambda = k + 1, \gamma = \xi, \beta = b/a)$ 

$$\begin{split} \mathrm{Im} H^{(k)}(s,b) &= C_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} F\bigg(-m,-m;2-k;\frac{\xi^2}{(b/a)^2}\bigg) \bigg(\frac{b^2}{4(B/a)^2}\bigg)^m \\ &\approx C_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \bigg(\frac{b^2}{4(B/a)^2}\bigg)^m = C_1 \exp\bigg(-\frac{b^2}{4(B/a)^2}\bigg), \\ &C_1 &= C \frac{a^2 \xi^{1-k}}{2^{2-k} \Gamma(2-k)B}. \end{split}$$

The estimation is valid for any considered k.

For  $a\xi \gg b$  we choose  $(\nu = 0, \mu = 1 - k, \lambda = k +, \alpha = B/(a)^2, \gamma = b/a, \beta = \xi)$ . Then

$$\begin{split} \mathrm{Im} H^{(k)}(s,b) &= C_1 \sum_{m=0}^{\infty} \frac{(-)^m}{\Gamma(m+2-k)} \left(\frac{\xi^2}{4B^2/a^4}\right)^m F\left(-m,-m+k-1;1;\frac{(b/a)^2}{\xi^2}\right) \\ &\approx C_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+2-k)} \left(\frac{\xi^2}{4B^2/a^4}\right)^m. \end{split}$$

The series in the last equation can be summed separately for k = 0, 1, 2

$$\mathrm{Im}H^{(k)}(s,b) = \begin{cases} \frac{v^2 g_k B}{4 s_0 a^4} (1 - e^{-a^4 \xi^2 / 4B^2}), & k = 0, \\ \frac{v^2 g_k B}{2 s_0 a^4} e^{-a^4 \xi^2 / 4B^2} \mathrm{erfi}(\xi a^2 / 2B), & k = 1, \\ \frac{v^2 g_k}{4 s_0 B a^2} e^{-a^4 \xi^2 / 4B^2}, & k = 2. \end{cases}$$



FIG. 13. Amplitude  $\text{Im}H^{(0)}(s, b)$  calculated for propagator (A1) at C = 1, a = 0.3 Gev<sup>-1</sup>, B = 2 Gev<sup>-2</sup>.

Such a suppression factor does not allow us to calculate the integrals for H(s, b) analytically. However, integral for  $H^{(0)}$  was calculated numerically. The results are given in Fig. 13 for  $e^{-Bq^2}$ . At high energies, the tripole contribution  $G^{(0)}(\omega, q)$  is dominating; the dipole and simple pole give just small corrections.

Similar estimations and results one can obtain for the propagator

$$G^{(k)}(\omega,q) = \frac{g_k e^{-Bq}}{(\omega^2 + a^2 q^2)^{3/2-k}}, \qquad k = 0, 1, 2.$$
(A3)

# APPENDIX B: RESTRICTIONS ON THE PARAMETERS $\mu_0, \lambda$

From the properties of 3F-vertex considered in Sec. II we have (1)

(a) 
$$\mu(k) \propto \mu_0 + k$$
,  
(b)  $\mu(k) = \mu_0 (1 + \lambda k)$ ,  
(c)  $\mu(k) = \mu_0 \frac{1}{1 + \lambda k}$ , (B1)

(2)

 $\mu(k) > 0, \quad k = 0, 1, 2$ 

because the vertex has no infinity at any k (B2)

(3)

 $\mu(k) < 3 - k \quad k = 0, 1, 2$ 

because the vertex does not cancel zero in

$$(G_0^{(k)})^{-1} \propto \kappa^{(3-k)}.$$
 (B3)

We know from the smallness of corrections to main contribution (all k = 0) that  $\mu_0 > 1$ . Thus, for all variants (a), (b), (c) in Eq. (B1)

$$0 < \mu(k) < 3 - k, \qquad 1 < \mu_0 < 3.$$
 (B4)

It follows from the above inequalities that at any  $k \neq 0$ ,

$$1 + \lambda k > 0 \Rightarrow \lambda > -1/k, \Rightarrow \lambda > -1/2.$$
 (B5)

Variant (b)

$$0 < \mu_0 (1 + \lambda k) < 3 - k,$$
  

$$1 < \mu_0 < 3, \Rightarrow 1 < \mu_0 < \frac{3 - k}{1 + \lambda k}.$$
 (B6)

It follows from the last inequality that

$$\frac{3-k}{1+\lambda k} > 1, \Rightarrow \lambda < \frac{2-k}{k} = 2/k - 1.$$
 (B7)

The inequality for  $\lambda$  should be valid for any *k*, hence  $\lambda < 0$ . Thus for the variant (b) we have

$$-1/2 < \lambda < 0. \tag{B8}$$

However, it follows from the convergence of the integrals over  $\omega$  for cut Reggeons at q = 0 in evaluation of CDP cross sections, that

$$\begin{aligned} 3 - 2\mu_0 &> 0, \Rightarrow \mu_0 < 3/2, \\ 3 - 2\mu(k) - k &> 0, k = 1, \Rightarrow 2 - 2\mu(1) > 0, \Rightarrow \mu(1) < 1, \\ 3 - 2\mu(k) - k > 0, k = 2, \Rightarrow 1 - 2\mu(2) > 0, \Rightarrow \mu(2) < 1/2. \end{aligned}$$
(B9)

Hence, at  $\lambda \neq 0$ 

$$\mu(1) = \mu_0(1+1\lambda) < 1 \Rightarrow 1 < \mu_0 < \frac{1}{(1+\lambda)} \Rightarrow \lambda < 0,$$
  

$$\mu(2) = \mu_0(1+2\lambda) < 1/2 \Rightarrow 1 < \mu_0 < \frac{1}{2(1+2\lambda)}$$
  

$$\Rightarrow \lambda < -1/4.$$
(B10)

So, the final result for  $\lambda$  in case (b) is

$$-1/2 < \lambda < -1/4.$$
 (B11)

$$> 0.$$
 (B12)

## APPENDIX C: ESTIMATION OF SDD CROSS SECTION

λ

To estimate  $I_{SDD}(\xi)$ , we rewrite it as

Similarly for variant (c)

$$I_{SDD}(\xi) = \int_{\delta}^{1-\delta} dx x^{\alpha_1-1} (1-x)^{\alpha_2-1}$$

where  $\delta = \xi_0/\xi \ll 1, \neq 0$  and  $\alpha_1 = 3 - \mu(k_0) - k_0 > 0$ ,  $\alpha_2 = 3 - \mu(k_1) - \mu(k_2) - k_1 - k_2$  (sign of  $\alpha_2$  depends on the values of  $k_i$ ).

Let us split the integration domain  $(\delta, \epsilon), (\epsilon, 1 - \epsilon), (1 - \epsilon, 1 - \delta)$  where  $\epsilon$  is constant,  $\delta \ll \epsilon \ll 1$ . Then

$$\begin{split} I_{SDD}(\xi) &\approx \ = \int_{\delta}^{\epsilon} dx (x^{\alpha_1 - 1} + x^{\alpha_2 - 1}) + \mathbf{C} \\ &= \frac{c_1}{\alpha_1} (\epsilon^{\alpha_1} - \delta^{\alpha_1}) - \frac{c_2}{\alpha_2} (\epsilon^{\alpha_2} - \delta^{\alpha_2}) + \mathbf{C} \\ &\approx \begin{cases} \text{constant} & \text{if} \quad \alpha_2 > 0 \\ \frac{c_2}{\alpha_2} (\xi/\xi_0)^{-\alpha_2}, & \text{if} \quad \alpha_2 < 0 \end{cases} \end{split}$$

where C is a constant.

We have obtained in Sec. II that the smallness of the corrections in the DS equations requires  $\mu_0 > 1$ . Therefore, the following inequalities should be satisfied for the cases (b) and (c) from the Eq. (B1)

$$\mu_{1}(k) = \mu(k) + k, \quad \text{if } \lambda \neq 0 \quad \text{then } 1 + \lambda k > 0$$
  
and  $\lambda > -1/2,$   
(a)  $1 < \mu_{0} < 3,$   
(b)  $1 < \mu_{0} < \frac{3-k}{1+\lambda k},$   
(c)  $1 < \mu_{0} < (3-k)(1+\lambda k),$  (C1)

where k = 0, 1, 2. Taking into account inequalities  $\mu(k) > 0, \mu_0 > 1$ , and consequently  $1 + \lambda k > 0$ , it follows that

(b) 
$$1 < \mu_0 < \frac{1}{1+2\lambda}$$
,  $-1/2 < \lambda < 0$ ,  
(c)  $1 < \mu_0 < 1+2\lambda$ ,  $\lambda > 0$ . (C2)

Thus, with obtained restrictions on  $\lambda$  we have

$$\sigma_{SDD}(s) \propto \xi^{2-3(\mu_0-1)} \times \begin{cases} b) \ \xi^{\lambda\mu_0(k_0+k_1+k_2),} & -1/2 < \lambda < 0, \\ c) \ \xi^{-\lambda\mu_0 \sum_{i=0}^2 \frac{k_i}{1+\lambda k_i},} & \lambda > 0. \end{cases}$$
(C3)

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