Five-branes wrapped on topological disks from 7D $N=2$ gauged supergravity

Parinya Karndumri[*] and Patharadanai Nuchino[†]

String Theory and Supergravity Group, Department of Physics, Faculty of Science, Chulalongkorn University, 254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand

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We study supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions, for Σ being a topological disk with nontrivial $U(1)$ holonomy at the boundary or "half-spindle," in seven-dimensional $N = 2$ gauged supergravity coupled to three vector multiplets. We consider compact and noncompact gauge groups, $SO(4-p, p)$, for $p = 0, 1, 2$, and find a number of $\mathrm{AdS}_5 \times \Sigma$ solutions with $SO(2) \times SO(2)$ and $SO(2)_{\mathrm{diag}}$ residual symmetry from $SO(4)$ and $SO(2,2)$ gauge groups. We also find an $SO(2)_R \subset SO(3)_R \sim SU(2)_R$ symmetric solution, which can be regarded as a solution of pure $N = 2$ gauged supergravity with the $SO(3)_R$ gauge group and all the fields from vector multiplets vanishing. The solutions preserve $\frac{1}{2}$ of the original $N = 2$ supersymmetry and could be interpreted as supergravity duals of $N = 1$ superconformal field theories in four dimensions. In particular, some of these solutions can be embedded in ten or eleven dimensions, in which a description in terms of five-branes wrapped on a topological disk can be given.

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I. INTRODUCTION

The AdS/CFT correspondence [1–3] leads to holographic descriptions of strongly coupled superconformal field theories (SCFTs). Various aspects of these SCFTs, including their nonconformal phases, can be studied by the corresponding dual-gravity solutions in string/M theory or, at low energy, supergravity theories in ten or eleven dimensions. In this framework, the conformal field theories can be considered as world-volume theories on the branes in the near-horizon limit. A class of solutions that describes branes wrapping on particular manifolds is of particular interest, since these can lead to supersymmetric field theories in lower dimensions arising from world-volume theories of wrapped branes. These configurations describe RG flows across dimensions from higher-dimensional SCFTs to lower-dimensional ones and provide a useful holographic description of the less-known higher-dimensional SCFTs such as $N = (2,0), (1,0)$ in six dimensions via four-dimensional SCFTs, of which many aspects are better understood.

At low energy, these wrapped branes can be described by supersymmetric $\mathrm{AdS}_m \times M^n$ solutions of gauged supergravity in $m+n$ dimensions [4]. M^n is an n-dimensional compact manifold with constant curvature on which the wrapped $(m+n-2)$-branes lead to an $(m-1)$-dimensional SCFT from a compactification of the dual $(m+n-1)$-dimensional SCFT on M^n. The corresponding supergravity solutions preserve some amount of the original supersymmetry by means of a topological twist [5], implemented by turning on some gauge fields to cancel the spin connections on the compact manifold M^n. A large number of these solutions have previously been found in various dimensions; see Refs. [6–29] for an incomplete list. Recently, new classes of $\mathrm{AdS} \times \Sigma$ solutions, in which unbroken supersymmetry is not realized by a topological twist, have been found for Σ being a two-dimensional space with nonconstant curvature. These solutions describe supersymmetric branes wrapped on a spindle, which is topologically a two-sphere with orbifold singularities at the poles [30–36] (see also Ref. [37] for a more recent result), or on a topological disk with nontrivial $U(1)$ holonomy on the boundary or "half-spindle" [38–43]. These lead to new supersymmetric AdS geometries from gauged supergravities which are dual to lower-dimensional SCFTs, arising from compactifications of higher-dimensional SCFTs on a spindle or a half-spindle. In particular, supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions from the $U(1)^2$ truncation of the maximal $SO(5)$ gauged supergravity in seven dimensions obtained in Refs. [38,39] have been shown to be dual to four-dimensional $N = 2$ SCFTs of the Argyres-Douglas (AD) type [44]. Furthermore, it should be remarked that both the spindle and half-spindle can be obtained from different global extensions of the same local solutions, as pointed out recently

[*]parinya.ka@hotmail.com
[†]danai.nuchino@hotmail.com

in Ref. [45]. In this work, we are interested in supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions from matter-coupled $N = 2$ gauged supergravity in seven dimensions, as constructed in Ref. [46]; see Refs. [47–51] for earlier constructions. We mainly consider $N = 2$ gauged supergravity coupled to three vector multiplets with possible gauge groups given by $SO(4) \sim SO(3) \times SO(3)$, $SO(2,2) \sim SO(2,1) \times SO(2,1)$, and $SO(3,1)$. It is well known that only $SO(4)$ and $SO(3,1)$ gauge groups admit supersymmetric AdS_7 vacua dual to $N = (1,0)$ SCFTs in six dimensions with $SO(3)_R$ R symmetry [52–54]. Furthermore, a number of interesting holographic solutions have been found in Refs. [21,28,52,53,55]. We will add more solutions to this list by finding supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions within this $N = 2$ gauged supergravity for Σ being a half-spindle. It has been pointed out in Ref. [35] that supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions with Σ being a spindle do not exist in minimal or pure $N = 2$ gauged supergravity with the $SO(3)$ gauge group; see also Ref. [37]. It turns out that solutions with Σ being a half-spindle do exist in pure $N = 2$ gauged supergravity. These solutions preserve $\frac{1}{2}$ of the supersymmetry and $SO(2)_R \subset SO(3)_R$, and they can be obtained from a truncation of $\mathrm{AdS}_5 \times \Sigma$ solutions with $SO(2) \times SO(2)$ symmetry in the $SO(4)$ gauged supergravity considered in this work. Furthermore, we also find $SO(2)_{\mathrm{diag}}$ symmetric solutions that can be mapped to the solution found in Ref. [39] from $U(1)^2$ truncation of the maximal $SO(5)$ gauged supergravity.

However, unlike the solutions in Ref. [39] dual to $N = 2$ SCFTs of the AD type, these solutions preserve only eight supercharges and should be dual to $N = 1$ SCFTs in four dimensions. In addition, most of the solutions found in this paper currently have no known higher-dimensional origin. In particular, it has been shown in Ref. [56] that uplifting seven-dimensional $N = 2$ gauged supergravity with AdS_7 vacua to ten dimensions can be achieved only if there is no vector multiplet or just one vector multiplet. On the other hand, the uplift to eleven dimensions can be performed via an S^4 truncation if the $N = 2$ theories are truncations of the maximal $N = 4$ gauged supergravity. The embedding in this case can be obtained from the results of Refs. [57,58]. Moreover, pure $N = 2$ gauged supergravity with the $SO(3)$ gauge group can also be uplifted to type-IIA supergravity [59,60]. Finally, we will find $\mathrm{AdS}_5 \times \Sigma$ solutions in $N = 2$ gauged supergravity with a noncompact $SO(2,2) \sim SO(2,1) \times SO(2,1)$ gauge group. Since this gauged supergravity does not admit any supersymmetric AdS_7 vacua, the maximally supersymmetric vacua are given by half-supersymmetric domain walls dual to $N = (1,0)$ nonconformal field theories in six dimensions according to the DW/QFT correspondence [61–64]. In this case, the resulting $\mathrm{AdS}_5 \times \Sigma$ solutions are expected to describe four-dimensional $N = 1$ SCFTs arising from six-dimensional $N = (1,0)$ field theories compactified on a topological disk or half-spindle. To the best of our knowledge, these are the first examples of $\mathrm{AdS}_5 \times \Sigma$ solutions involving half-spindles with domain wall asymptotics. The paper is organized as follows: In Sec. II, we give a brief review of seven-dimensional $N = 2$ gauged supergravity coupled to an arbitrary number of vector multiplets. Supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions in the $SO(4)$, $SO(2,2)$, and $SO(3,1)$ gauge groups will be considered in Secs. III–V, respectively. Some conclusions and comments will be given in Sec. VI.

II. MATTER-COUPLED $N = 2$ GAUGED SUPERGRAVITY IN SEVEN DIMENSIONS

In this section, we give relevant formulas involving bosonic Lagrangian and supersymmetry transformations of fermions to find supersymmetric solutions of matter-coupled $N = 2$ gauged supergravity in seven dimensions. We follow most of the conventions and notations in Ref. [49], in which the detailed construction can be found; see also Ref. [65] for gaugings using the embedding tensor formalism in the case of three vector multiplets.

In seven dimensions, the half-maximal $N = 2$ supergravity multiplet contains the following component fields:

$$(e_\mu^{\hat{\mu}}, \psi_\mu^a, A_\mu^i, \chi^a, B_{\mu\nu}, \sigma),$$

given by the graviton $e_\mu^{\hat{\mu}}$, two gravitini ψ_μ^a, three vectors A_μ^i, two spin-$\frac{1}{2}$ fields χ^a, a two-form field $B_{\mu\nu}$, and the scalar field or dilaton σ. We denote curved and flat space-time indices by $\mu, \nu, \ldots$ and $\hat{\mu}, \hat{\nu}, \ldots$, respectively. The indices i, $j = 1, 2, 3$ and a, $b = 1, 2$ label triplets and doublets of $SO(3)_R \sim SU(2)_R$ R symmetry, respectively.

The supergravity multiplet can couple to an arbitrary number n of vector multiplets with the field content

$$(A_\mu, \lambda^a, \phi^i)^r. \tag{1}$$

Each vector multiplet, labeled by an index $r = 1, \ldots, n$, consists of a vector field A_μ, two gaugini λ^a, and three scalars ϕ^i. Together with the supergravity multiplet, there are $3 + n$ vector fields transforming in a fundamental representation of the global symmetry $SO(3,n)$, collectively denoted by $A_\mu^I = (A_\mu^i, A_\mu^r)$. The $SO(3,n)$ fundamental indices $I, J = 1, \ldots, 3 + n$ are lowered and raised, respectively, by the $SO(3,n)$ invariant tensor and its inverse

$$\eta_{IJ} = \eta^{IJ} = \mathrm{diag}(-1,-1,-1,\underbrace{1,\ldots,1}_{n}). \tag{2}$$

Similarly to the dilaton σ described by a coset manifold $SO(1,1) \sim \mathbb{R}^+$, the $3n$ scalars ϕ^{ir} from the n vector multiplets are parametrized by an $SO(3,n)/SO(3) \times SO(n)$ coset manifold. With $A = (i,r)$ being an $SO(3) \times SO(n)$ index, the associated coset representative can be written as

$$L_I{}^A = (L_I{}^i, L_I{}^r). \tag{3}$$

$L_I{}^A$ transforms under the global $SO(3,n)$ and the local $SO(3)\times SO(n)$ by left and right multiplication, respectively. The inverse of $L_I{}^A$ will be denoted by $L_A{}^I = (L_i{}^I, L_r{}^I)$ and satisfies the following relations:

$$L_j{}^I L_I{}^i = \delta^i_j, \quad L_s{}^I L_I{}^r = \delta^r_s, \quad \eta_{IJ} = -L_I{}^i L_J{}^i + L_I{}^r L_J{}^r. \tag{4}$$

Note also that the indices i, j and r, s are raised and lowered by δ_{ij} and δ_{rs}, respectively.

Gaugings of the matter-coupled $N=2$ supergravity can be obtained by promoting a subgroup $G\subset SO(3,n)$ to be a local symmetry. The embedding of G in $SO(3,n)$ is described by the $SO(3,n)$ tensor $f_{IJ}{}^K$ identified with structure constants of the gauge group G via the gauge algebra

$$[T_I, T_J] = f_{IJ}{}^K T_K, \tag{5}$$

where T_I's are the gauge generators. In the embedding tensor formalism, $f_{IJ}{}^K$ is one of the components of the embedding tensor; see Ref. [65] for more detail. In order for the gauging to be consistent with supersymmetry, $f_{IJ}{}^K$ must satisfy the conditions

$$f_{IJK} = \eta_{KL} f_{IJ}{}^L = f_{[IJK]} \quad \text{and} \quad f_{[IJ}{}^L f_{K]L}{}^M = 0. \tag{6}$$

Since η_{IJ} has only three negative eigenvalues, any gauge group can have at most three compact or three noncompact generators. Therefore, the allowed semisimple gauge groups are of the form $G\sim G_0\times H\subset SO(3,n)$, with H being a compact group of dimension $\dim H \le (n+3-\dim G_0)$ [49]. On the other hand, G_0 can only be one of six possibilities: $SO(3)$, $SO(3,1)$, $SL(3,\mathbb{R})$, $SO(2,1)$, $SO(2,2)$, and $SO(2,2)\times SO(2,1)$.

Apart from the usual gaugings, there is also a massive deformation given by adding a topological mass term to the three-form field $C_{\mu\nu\rho}$ dual to the two-form field $B_{\mu\nu}$. This additional deformation is crucial for the gauged supergravity to admit AdS_7 vacua. With both of these deformations, the bosonic Lagrangian of the matter-coupled $N=2$ gauged supergravity is given in differential form language by

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2}R*\mathbf{1} - \frac{1}{2}e^{\sigma}a_{IJ}*F^I_{(2)}\wedge F^J_{(2)} \\ & - \frac{1}{2}e^{-2\sigma}*H_{(4)}\wedge H_{(4)} - \frac{5}{8}*d\sigma\wedge d\sigma \\ & - \frac{1}{2}*P^{ir}_{(1)}\wedge P^{ir}_{(1)} + \frac{1}{\sqrt{2}}H_{(4)}\wedge\omega_{(3)} - 4hH_{(4)} \\ & \wedge C_{(3)} - \mathbf{V}*\mathbf{1}.\end{aligned} \tag{7}$$

The constant h describes the topological mass term for the three-form $C_{(3)}$ with the field strength $H_{(4)} = dC_{(3)}$. The gauge field strength is defined by

$$F^I_{(2)} = dA^I_{(1)} + \frac{1}{2}f_{JK}{}^I A^J_{(1)}\wedge A^K_{(1)}. \tag{8}$$

The scalar matrix a_{IJ} appearing in the kinetic term of vector fields is given in terms of the coset representative as follows:

$$a_{IJ} = L_I{}^i L_J{}^i + L_I{}^r L_J{}^r. \tag{9}$$

The Chern-Simons three-form satisfying $d\omega_{(3)} = F^I_{(2)}\wedge F^I_{(2)}$ is defined by

$$\omega_{(3)} = F^I_{(2)}\wedge A^I_{(1)} - \frac{1}{6}f_{IJ}{}^K A^I_{(1)}\wedge A^J_{(1)}\wedge A_{(1)K}. \tag{10}$$

The scalar potential is given by

$$\mathbf{V} = \frac{1}{4}e^{-\sigma}\left(C^{ir}C_{ir} - \frac{1}{9}C^2\right) + 16h^2 e^{4\sigma} - \frac{4\sqrt{2}}{3}he^{\frac{3\sigma}{2}}C, \tag{11}$$

where C-functions, or fermion-shift matrices, are defined as

$$C = -\frac{1}{\sqrt{2}}f_{IJ}{}^K L_i{}^I L_j{}^J L_{Kk}\varepsilon^{ijk}, \tag{12}$$

$$C^{ir} = \frac{1}{\sqrt{2}}f_{IJ}{}^K L_j{}^I L_k{}^J L_K{}^r\varepsilon^{ijk}, \tag{13}$$

$$C_{rsi} = f_{IJ}{}^K L_r{}^I L_s{}^J L_{Ki}. \tag{14}$$

The scalar kinetic term is defined in terms of the vielbein on the $SO(3,n)/SO(3)\times SO(n)$ coset as

$$P^{ir}_{(1)} = L^{rI}(\delta^K_I d + f_{IJ}{}^K A^J_{(1)})L_K{}^i. \tag{15}$$

Supersymmetry transformations of fermionic fields read

$$\begin{aligned}\delta\psi^a_\mu = {}& 2D_\mu\epsilon^a - \frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C\Gamma_\mu\epsilon^a - \frac{4}{5}he^{2\sigma}\Gamma_\mu\epsilon^a \\ & - \frac{i}{20}e^{\frac{\sigma}{2}}F^i_{\rho\sigma}(\sigma^i)^a{}_b(3\Gamma_\mu\Gamma^{\rho\sigma} - 5\Gamma^{\rho\sigma}\Gamma_\mu)\epsilon^b \\ & - \frac{1}{240\sqrt{2}}e^{-\sigma}H_{\rho\sigma\lambda\tau}(\Gamma_\mu\Gamma^{\rho\sigma\lambda\tau} + 5\Gamma^{\rho\sigma\lambda\tau}\Gamma_\mu)\epsilon^a,\end{aligned} \tag{16}$$

$$\begin{aligned}\delta\chi^a = {}& -\frac{1}{2}\Gamma^\mu\partial_\mu\sigma\epsilon^a + \frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C\epsilon^a - \frac{16}{5}e^{2\sigma}h\epsilon^a \\ & - \frac{i}{10}e^{\frac{\sigma}{2}}F^i_{\mu\nu}(\sigma^i)^a{}_b\Gamma^{\mu\nu}\epsilon^b - \frac{1}{60\sqrt{2}}e^{-\sigma}H_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma}\epsilon^a,\end{aligned} \tag{17}$$

$$\delta\lambda^{ar} = i\Gamma^\mu P^{ir}_\mu (\sigma^i)^a{}_b \epsilon^b - \frac{1}{2} e^{\frac{\sigma}{2}} F^r_{\mu\nu}\Gamma^{\mu\nu}\epsilon^a - \frac{i}{\sqrt{2}} e^{-\frac{\sigma}{2}} C^{ir}(\sigma^i)^a{}_b\epsilon^b. \quad (18)$$

In these equations, $(\sigma^i)^a{}_b$ are the usual Pauli matrices, and $\Gamma_\mu = e^{\hat{\mu}}_\mu \Gamma_{\hat{\mu}}$, in which $\Gamma_{\hat{\mu}}$'s are seven-dimensional spacetime gamma matrices satisfying the $SO(1,6)$ Clifford algebra

$$\{\Gamma_{\hat{\mu}}, \Gamma_{\hat{\nu}}\} = 2\eta_{\hat{\mu}\hat{\nu}}, \quad \eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-++++++). \quad (19)$$

The dressed field strengths $F^i_{(2)}$ and $F^r_{(2)}$ are defined by

$$F^i_{(2)} = L_I{}^i F^I_{(2)} \quad \text{and} \quad F^r_{(2)} = L_I{}^r F^I_{(2)}. \quad (20)$$

The covariant derivative of the supersymmetry parameter ϵ^a is given by

$$D_\mu \epsilon^a = \partial_\mu \epsilon^a + \frac{1}{4}\omega_\mu{}^{\hat{\nu}\hat{\rho}}\Gamma_{\hat{\nu}\hat{\rho}}\epsilon^a + \frac{1}{2\sqrt{2}} Q^i_\mu (\sigma^i)^a{}_b \epsilon^b, \quad (21)$$

where Q^i_μ is defined in terms of the $SO(3)_R$ composite connection Q^{ij}_μ as

$$Q^i_\mu = \frac{i}{\sqrt{2}} \epsilon^{ijk} Q^{jk}_\mu, \quad (22)$$

with

$$Q^{ij}_\mu = L^{jI}(\delta^K_I \partial_\mu + f_{IJ}{}^K A^J_\mu) L_K{}^i. \quad (23)$$

For convenience, we also give all the bosonic field equations derived from the Lagrangian [Eq. (7)]:

$$0 = d(e^{-2\sigma} * H_{(4)}) + 8hH_{(4)} - \frac{1}{\sqrt{2}} F^I_{(2)} \wedge F^I_{(2)}, \quad (24)$$

$$0 = D(e^\sigma a_{IJ} * F^I_{(2)}) - \sqrt{2} H_{(4)} \wedge F^J_{(2)} + *P^{ir}_{(1)} f_{IJ}{}^K L_r{}^I L_{Ki}, \quad (25)$$

$$0 = D(*P^{ir}_{(1)}) - 2e^\sigma L_I{}^i L_J{}^r * F^I_{(2)} \wedge F^J_{(2)} - \left(\frac{1}{\sqrt{2}} e^{-\sigma} C^{js} C_{rsk}\epsilon^{ijk} + 4\sqrt{2} h e^{\frac{3\sigma}{2}} C^{ir}\right)\varepsilon_{(7)}, \quad (26)$$

$$0 = \frac{5}{4} d(*d\sigma) - \frac{1}{2} e^\sigma a_{IJ} * F^I_{(2)} \wedge F^J_{(2)} + e^{-2\sigma} * H_{(4)} \wedge H_{(4)} + \left[\frac{1}{4} e^{-\sigma}\left(C^{ir}C_{ir} - \frac{1}{9} C^2\right) + 2\sqrt{2} h e^{\frac{3\sigma}{2}} C - 64 h^2 e^{4\sigma}\right]\varepsilon_{(7)}, \quad (27)$$

$$0 = R_{\mu\nu} - \frac{5}{4}\partial_\mu\sigma\partial_\nu\sigma - a_{IJ} e^\sigma \left(F^I_{\mu\rho} F^{J\rho}_\nu - \frac{1}{10} g_{\mu\nu} F^I_{\rho\sigma} F^{J\rho\sigma}\right) - P^{ir}_\mu P^{ir}_\nu - \frac{2}{5} g_{\mu\nu}\mathbf{V} - \frac{1}{6} e^{-2\sigma} \times \left(H_{\mu\rho\sigma\lambda} H_\nu{}^{\rho\sigma\lambda} - \frac{3}{20} g_{\mu\nu} H_{\rho\sigma\lambda\tau} H^{\rho\sigma\lambda\tau}\right). \quad (28)$$

We finally give a general parametrization of the $SO(3,n)/SO(3)\times SO(n)$ coset which is useful for finding explicit solutions. We first introduce $(n+3)^2$ basis elements of a general $(n+3)\times(n+3)$ matrix as follows:

$$(e_{IJ})_{KL} = \delta_{IK}\delta_{JL}. \quad (29)$$

The composite $SO(3)\times SO(n)$ generators are given by

$$SO(3)\colon\ J^{(1)}_{ij} = e_{ji} - e_{ij}, \quad i,j = 1,2,3,$$
$$SO(n)\colon\ J^{(2)}_{rs} = e_{s+3,r+3} - e_{r+3,s+3}, \quad r,s = 1,\ldots,n. \quad (30)$$

The noncompact generators corresponding to the $3n$ scalars are given by

$$Y_{ir} = e_{i,r+3} + e_{r+3,i}. \quad (31)$$

III. $SO(4)$ GAUGE GROUP

We first consider $N=2$ gauged supergravity with the $SO(4)\sim SO(3)\times SO(3)$ gauge group obtained by coupling the gravity multiplet to $n=3$ vector multiplets. The first $SO(3)$ factor is identified with the $SO(3)_R \sim SU(2)_R$ R symmetry. The corresponding structure constants are given by

$$f_{IJK} = (\tilde{g}_1\varepsilon_{ijk}, -\tilde{g}_2\varepsilon_{rst}), \qquad r,s,\ldots = 1,2,3, \quad (32)$$

in which $\tilde{g}_1$ and $\tilde{g}_2$ are coupling constants of $SO(3)_R$ and $SO(3)$ generated by $J^{(1)}_{ij}$ and $J^{(2)}_{rs}$, respectively.

We are interested in supersymmetric solutions in the form of a product space between an AdS_5 and a topological disk Σ with a nontrivial $U(1)$ holonomy at the boundary. Following Ref. [39], we take the ansatz for the seven-dimensional metric to be

$$ds^2_7 = f(r) ds^2_{AdS_5} + g_1(r) dr^2 + g_2(r) dz^2, \quad (33)$$

where the metric on AdS_5 with unit radius is given by

$$ds^2_{AdS_5} = \frac{1}{\rho^2}(dx^2_{1,3} + d\rho^2), \quad (34)$$

with $dx^2_{1,3} = \eta_{mn}dx^m dx^n$, $m,n = 0,\ldots,3$ being the flat metric on the four-dimensional Minkowski space Mkw_4. The values r and z are the radial and angular coordinates on the topological disk Σ, respectively, whose ranges will be determined later on. The seven-dimensional curved and flat space-time indices will be split into $\mu = (m, \rho, r, z)$ and $\hat{\mu} = (\hat{m}, \hat{\rho}, \hat{r}, \hat{z})$, respectively.

With the vielbein one-forms

$$e^{\hat{m}}_{(1)} = \frac{\sqrt{f(r)}}{\rho} dx^m, \qquad e^{\hat{\rho}}_{(1)} = \frac{\sqrt{f(r)}}{\rho} d\rho,$$
$$e^{\hat{r}}_{(1)} = \sqrt{g_1(r)} dr, \qquad e^{\hat{z}}_{(1)} = \sqrt{g_2(r)} dz, \tag{35}$$

we can straightforwardly compute all nonvanishing components of the spin connections:

$$\omega^{\hat{m}\hat{\rho}}_{(1)} = -\frac{e^{\hat{m}}}{\sqrt{f}}, \qquad \omega^{\hat{m}\hat{r}}_{(1)} = \frac{f' e^{\hat{m}}}{2f\sqrt{g_1}},$$
$$\omega^{\hat{\rho}\hat{r}}_{(1)} = \frac{f' e^{\hat{\rho}}}{2f\sqrt{g_1}}, \qquad \omega^{\hat{z}\hat{r}}_{(1)} = \frac{g_2' e^{\hat{z}}}{2g_2\sqrt{g_1}}. \tag{36}$$

From now on, we will use primes to denote r derivatives and mostly suppress arguments of the r-dependent functions for convenience.

A. $SO(2)_R$ and $SO(2)\times SO(2)$ symmetric solutions

We now move to the ansatz for gauge fields in the cases of $SO(2)_R \subset SO(3)_R$ and $SO(2)\times SO(2)$ symmetry. Since the former can be obtained as a truncation of the latter, we will first consider the case of $SO(2)\times SO(2)$ symmetry and perform a suitable truncation to obtain $SO(2)_R$ symmetric solutions. For $SO(2)\times SO(2)$ symmetric solutions, we will choose the $SO(2)\times SO(2)$ subgroup generated by $J^{(1)}_{12}$ and $J^{(2)}_{12}$ and turn on the following gauge fields:

$$A^I_{(1)} = [A_1(r)\delta^I_3 + A_2(r)\delta^I_6] dz. \tag{37}$$

The corresponding two-form field strengths are given by

$$F^I_{(2)} = (A_1'\delta^I_3 + A_2'\delta^I_6) dr\wedge dz. \tag{38}$$

This ansatz leads to $F^I_{(2)}\wedge F^I_{(2)} = 0$. According to the field equation of the three-form field given in Eq. (24), we can consistently set $C_{(3)} = 0$.

Among the nine scalars from the $SO(3,3)/SO(3)\times SO(3)$ coset, there is only one $SO(2)\times SO(2)$ singlet scalar. Following Ref. [52], this scalar field corresponds to the noncompact generator Y_{33}, and the coset representative can be written as

$$L = e^{\phi Y_{33}}. \tag{39}$$

This singlet scalar ϕ and the dilaton σ depend only on the radial coordinate. It is now straightforward to compute the C-functions appearing in the supersymmetry transformations:

$$C = 3\sqrt{2}\tilde{g}_1\cosh\phi, \qquad C^{ir} = -\sqrt{2}\tilde{g}_1\sinh\phi\,\delta^i_3\delta^r_3. \tag{40}$$

The scalar vielbein and $SO(2)_R$ composite connection have the following nonvanishing components:

$$P^{ir}_{(1)} = \phi'\delta^i_3\delta^r_3 dr \quad \text{and} \quad Q^{ij}_{(1)} = \tilde{g}_1 A_1 \epsilon^{ij3} dz. \tag{41}$$

It should be noted here that only A_1 appears in the composite connection, because $A^3_{(1)}$ is the vector field that gauges $SO(2)_R\subset SO(3)_R$, under which the gravitini and supersymmetry parameters are charged. With all these and a similar analysis as in Ref. [39], we can determine all the BPS equations from the supersymmetry transformations of fermionic fields. The detailed analysis and relevant results can be found in the Appendix. In the following, we will separately consider solutions with $SO(2)_R$ and $SO(2)\times SO(2)$ symmetries.

1. $SO(2)_R$ symmetric solution in pure $N=2$ gauged supergravity

We first consider a simple case of $SO(2)_R$ symmetric solutions which can be obtained by setting $A^6_{(1)} = 0$. Equation (A60) then gives

$$\mathbf{F}_1 = be^{-\sigma}\cosh\phi f^{-\frac{5}{2}} \quad \text{and} \quad \mathbf{F}_2 = -be^{-\sigma}\sinh\phi f^{-\frac{5}{2}}, \tag{42}$$

in which we have written $a_1 = a_2 = b$.

With this explicit form of $\mathbf{F}_1$ and $\mathbf{F}_2$, the BPS condition in Eq. (A51) implies that

$$bh\sinh\phi = 0. \tag{43}$$

For $h = 0$, the gauged supergravity does not admit any supersymmetric AdS_7 vacua, so we will keep $h\neq 0$. With $b = 0$, all the gauge fields vanish. This clearly does not lead to any solutions of the form $AdS_5\times\Sigma$. Therefore, to find possible $AdS_5\times\Sigma$ solutions with AdS_7 asymptotics, we need to set $\phi = 0$. Effectively, all the fields from vector multiplets are truncated out. The resulting solutions can be then considered as solutions of the minimal or pure $N = 2$ gauged supergravity with the $SO(3)_R$ gauge group.

With $\phi = 0$, Eq. (A60) implies that $A_2' = 0$. All the BPS conditions from Eqs. (A26) and (A45)–(A57), as well as the field equation [Eq. (26)] for scalars from the vector multiplets, are automatically satisfied. With all these, we are left with the algebraic conditions of Eqs. (A30)–(A44). First, we solve for f from Eq. (A40) with the solution given by

$$f = \frac{2\sqrt{bh}e^{\sigma}}{\sqrt{s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)}}. \tag{44}$$

With this result, Eq. (A32) can be solved for g_1, giving rise to

$$g_1 = \frac{200\sqrt{bh^5}e^{\sigma}(\sigma')^2}{(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)^2\left[32\sqrt{bh^5} - e^{-5\sigma}\sqrt{s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)}\right]}. \tag{45}$$

The condition in Eq. (A41) together with the solution for $A_1' = \hat{A}_1'$ in Eq. (A60) gives an ordinary differential equation of the form

$$\hat{A}_1' = \frac{5\tilde{g}_1\sigma'}{2(12he^{\frac{5\sigma}{2}} - \tilde{g}_1)}\hat{A}_1, \tag{46}$$

which can be readily solved by

$$\hat{A}_1 = c(12h - \tilde{g}_1 e^{-\frac{5\sigma}{2}}) \tag{47}$$

for an integration constant c. Substituting all these results into Eqs. (A36) or (A41) leads to the following solution for g_2:

$$g_2 = \frac{c^2\tilde{g}_1^2 e^{\sigma}\left[32\sqrt{bh^5} - e^{-5\sigma}\sqrt{s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)}\right]}{\sqrt{s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)}}. \tag{48}$$

Finally, it can be verified that all the BPS conditions of Eqs. (A30)–(A44) as well as the dilaton and Einstein field equations, Eqs. (27) and (28), are satisfied by these solutions, provided that

$$\mathrm{sign}(c\tilde{g}_1\sigma') = +1. \tag{49}$$

Furthermore, the BPS equations [Eqs. (A23)–(A25)] are satisfied by the following form of the two-component spinor

$$\eta = e^{iqz}Ye^{\frac{5\sigma}{4}}\begin{pmatrix}\sqrt{8(h^5b)^{\frac{1}{4}} + \sqrt{2}se^{-\frac{5\sigma}{2}}[s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)]^{\frac{1}{4}}} \\ -\sqrt{8(h^5b)^{\frac{1}{4}} - \sqrt{2}se^{-\frac{5\sigma}{2}}[s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)]^{\frac{1}{4}}}\end{pmatrix}, \tag{50}$$

with the function Y being the solution of an ordinary differential equation given in Eq. (A16). The explicit form of the solution for Y can be written as

$$Y = \frac{Y_0 e^{-\sigma}}{[s(16he^{\frac{5\sigma}{2}} - \tilde{g}_1)]^{\frac{1}{8}}}, \tag{51}$$

in which Y_0 is an integration constant. It should be noted that the solution is characterized by a set of functions that are determined in terms of the dilaton σ together with its derivative. However, the r-dependent function $\sigma(r)$ is not determined by the BPS equations. This is very similar to the solutions obtained in Refs. [39–42].

To further analyze the solution, we first define the parameters

$$B = 8h^2\sqrt{b}, \qquad m = \frac{\tilde{g}_1}{16h}, \qquad \mathcal{C} = 2\tilde{g}_1 hc, \tag{52}$$

together with the function

$$W = B - e^{-5\sigma}\sqrt{s(e^{\frac{5\sigma}{2}} - m)}. \tag{53}$$

In terms of these quantities, the seven-dimensional metric reads

$$ds_7^2 = \frac{Be^{\sigma}}{16h^2\sqrt{s(e^{\frac{5\sigma}{2}} - m)}}ds^2_{\mathrm{AdS}_5} + \frac{25Be^{\sigma}(\sigma')^2}{4^5h^2W(e^{\frac{5\sigma}{2}} - m)^2}dr^2 + \frac{\mathcal{C}^2We^{\sigma}}{4h^2\sqrt{s(e^{\frac{5\sigma}{2}} - m)}}dz^2, \tag{54}$$

which is singular when $W \to 0$. It turns out that the analysis near $W = 0$ is simpler if we fix the solution of σ to

$$\sigma = -\frac{2}{5}\ln r. \tag{55}$$

This choice implies $r > 0$, and the sign condition of Eq. (49) requires $\mathrm{sign}(c\tilde{g}_1) = -1$. Accordingly, the constant $\mathcal{C}$ must be negative for $h > 0$.

In terms of the radial coordinate, the seven-dimensional metric is given by

$$ds_7^2 = \frac{Br^{1/10}}{16h^2\sqrt{s(1 - mr)}}[ds^2_{\mathrm{AdS}_5} + ds^2_{\Sigma}],$$

$$\text{with} \quad ds^2_{\Sigma} = \frac{r^{-1/2}}{16W[s(1 - mr)]^{3/2}}dr^2 + \frac{4\mathcal{C}^2W}{B}dz^2 \tag{56}$$

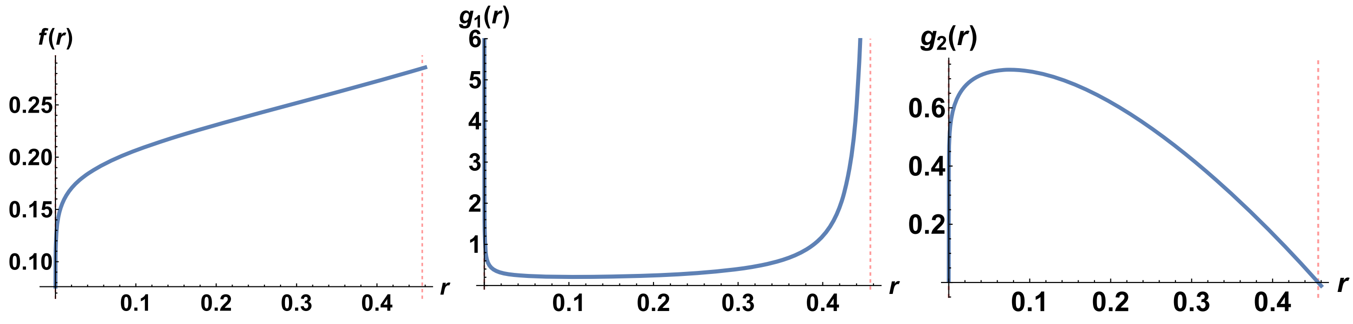
and

$$W = B - r^{3/2}\sqrt{s(1 - mr)}. \tag{57}$$

The equation $W = 0$ admits four roots, given by

$$r_{(\pm_1,\pm_2)} = \frac{1}{4m}\left[1 \pm_1 2m\sqrt{X} \pm_2 2m\sqrt{\frac{3}{4m^2} - X \pm_1 \frac{1}{4m^3\sqrt{X}}}\right] \tag{58}$$

with



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 1. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N = 2$ gauged supergravity with $s = 1$, $m = \frac{3}{4}$, $B = h = \frac{1}{4}$, and $\mathcal{C} = -1$. All the warp factors are positive in the range $0 < r < r_{(+,-)} = 0.456$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

$$X = \frac{1}{4m^2} + \frac{4(\frac{2}{3})^{1/3} s B^{4/3}}{[9s + \sqrt{81 - 768 s B^2 m^3}]^{\frac{1}{3}}} + \frac{B^{2/3}[9s + \sqrt{81 - 768 s B^2 m^3}]^{\frac{1}{3}}}{18^{1/3} m}. \tag{59}$$

These roots are all distinct due to different sign choices $\pm_1$ and $\pm_2$ appearing in Eq. (58).

Using Eq. (A22), we find the explicit form of the $SO(2)_R$ vector field:

$$A_1 = \frac{1}{8mh}[|\mathcal{C}|(4mr - 3) - q]. \tag{60}$$

In terms of the radial coordinate, the two-component spinor η is given by

$$\eta = Y_0 e^{iqz} \frac{2^{1/4} r^{1/40}}{[s(1 - mr)]^{\frac{1}{8}}} \begin{pmatrix} \sqrt{\sqrt{B} + s r^{3/4}[s(1 - mr)]^{\frac{1}{4}}} \\ -\sqrt{\sqrt{B} - s r^{3/4}[s(1 - mr)]^{\frac{1}{4}}} \end{pmatrix}. \tag{61}$$

The allowed ranges of the radial coordinate r for a regular solution are constrained by requiring that the dilaton scalar be real ($r > 0$) and that all the seven-dimensional metric functions be positive. There are seven possibilities depending on the values of the parameters s, m, and B. It should also be noted that these ranges, together with the corresponding behaviors of the solution, are very similar to those considered in Ref. [39]. We now discuss these possibilities in detail.

Case I: $s = 1$, $m > 0$, $0 < B < \frac{3\sqrt{3}}{16m\sqrt{m}}$, $0 < r < r_{(+,-)}$. For clarity, we plot a representative solution of the warp factors with $s = 1$, $m = \frac{3}{4}$, $B = h = \frac{1}{4}$, and $\mathcal{C} = -1$ in Fig. 1. As seen from the figure, as $r \to 0$, both f and g_2 approach zero, while g_1 diverges to $+\infty$. This is a curvature singularity of the seven-dimensional metric, as pointed out in Ref. [39]. Setting $r = R^{4/3}$, we find that the seven-dimensional metric becomes conformal to a product of AdS_5 and a cylinder near $R = 0$:

$$ds_7^2 \approx \frac{BR^{2/15}}{16h^2}\left[ds^2_{AdS_5} + \frac{1}{9B} dR^2 + 4\mathcal{C}^2 dz^2\right]. \tag{62}$$

As $r \to r_{(+,-)}$, the AdS_5 warp factor is smooth and the z circle shrinks due to $W(r_{(+,-)}) = B - r_{(+,-)}^{3/2}\sqrt{1 - mr_{(+,-)}} = 0$. By introducing a new radial coordinate $R = \sqrt{r_{(+,-)} - r}$, we find that g_2 is finite as $r \to r_{(+,-)}$. The seven-dimensional metric near this end point takes the form of

$$ds_7^2 \approx \frac{B r_{(+,-)}^{1/10}}{16h^2\sqrt{1 - mr_{(+,-)}}}\left[ds^2_{AdS_5} + \frac{dR^2 + 4\mathcal{C}^2[3 - 4mr_{(+,-)}]^2 R^2 dz^2}{-4W'(r_{(+,-)})\sqrt{r_{(+,-)}}(1 - mr_{(+,-)})^{3/2}}\right]. \tag{63}$$

The $R - z$ surface becomes locally an $\mathbb{R}^2/\mathbb{Z}_l$ orbifold near the point $r = r_{(+,-)}$ if we set

$$|\mathcal{C}| = \frac{1}{2l[3 - 4mr_{(+,-)}]}, \qquad l = 1, 2, 3, \ldots. \tag{64}$$

In this case, the function $3 - 4mr_{(+,-)}$ depends on two constants, m and B. However, its explicit form obtained from Eq. (58) is highly complicated, so we will only show that $3 - 4mr_{(+,-)}$ is strictly positive in the range of the r coordinate under consideration by giving a numerical plot of $3 - 4mr_{(+,-)}$ in Fig. 2.

By using the Gauss-Bonnet theorem and the same computation as in Ref. [39], we can calculate the Euler characteristic of Σ,

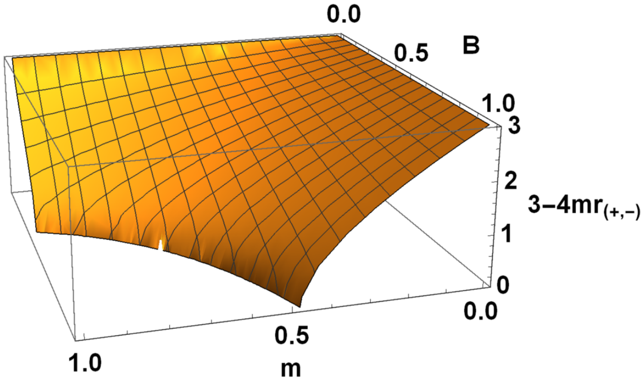


FIG. 2. A numerical plot of the function $3-4mr_{(+,-)}$ appearing in Eq. (64). Note that $3-4mr_{(+,-)}\to 3$ and $3-4mr_{(+,-)}\to 0$ as $B\to 0$ or $m\to 0$ and $B\to \frac{3\sqrt{3}}{16m\sqrt{m}}$, respectively.

$$\begin{aligned}\chi(\Sigma) &= \frac{1}{4\pi}\int_{\Sigma} R_{\Sigma}\mathrm{vol}_{\Sigma}\\ &= \frac{2|\mathcal{C}|[3-4mr_{(+,-)}]r_{(+,-)}^{3/4}[1-mr_{(+,-)}]^{1/4}}{\sqrt{B}}\\ &= 2|\mathcal{C}|[3-4mr_{(+,-)}] = \frac{1}{l}, \end{aligned} \tag{65}$$

which is a natural result for a disk with an $\mathbb{R}^2/\mathbb{Z}_l$ orbifold singularity at $r=r_{(+,-)}$. Note that the integration has been performed on the interval $0<r<r_{(+,-)}$ and $0<z<2\pi$, and we have used $B=r_{(+,-)}^{3/2}\sqrt{1-mr_{(+,-)}}$ obtained from $W(r_{(+,-)})=0$ with $s=+1$.

In order for the $SO(2)_R$ gauge field to vanish at $r=r_{(+,-)}$, at which the z circle shrinks, we fix the constant q to be

$$q=-|\mathcal{C}|[3-4mr_{(+,-)}]=-\frac{1}{2l}, \tag{66}$$

giving rise to the $SO(2)_R$ gauge field of the form

$$A_1=\frac{|\mathcal{C}|}{2h}[r-r_{(+,-)}]. \tag{67}$$

The explicit form of the Killing spinor at $r=r_{(+,-)}$ is given by

$$\eta = 2^{3/4}Y_0 e^{-\frac{iz}{2l}} r_{(+,-)}^{2/5}\begin{pmatrix}1\\0\end{pmatrix}. \tag{68}$$

Since only the upper component is nonvanishing, only $\frac{1}{4}$ of the original supersymmetry, or four supercharges, are preserved at $r=r_{(+,-)}$. Moreover, η is also well behaved near $r=r_{(+,-)}$, and hence globally defined on the disk Σ as in Ref. [39].

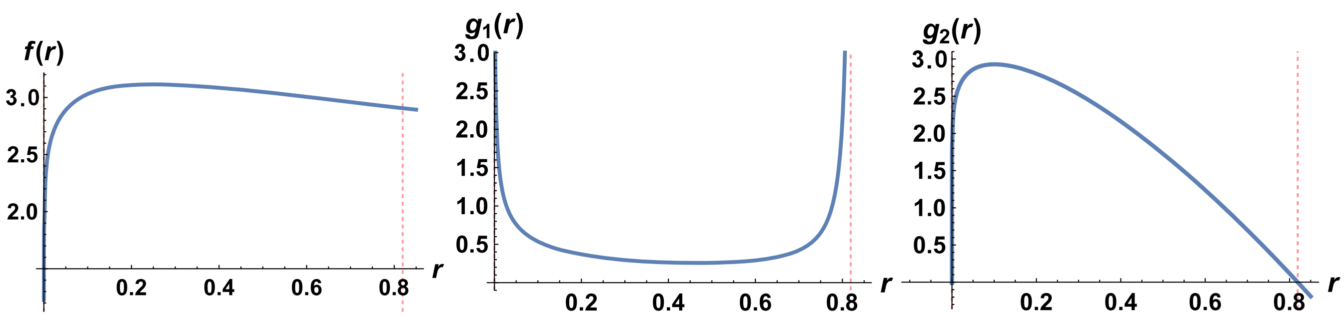
Case II: $s=1$, $m<0$, $B>0$, $0<r<r_{(-,+)}$. An example of numerical solutions with $s=1$, $m=-1$, $B=1$, $h=\frac{1}{8}$, and $\mathcal{C}=-\frac{1}{2}$ is given in Fig. 3. This case is very similar to Case I. The previous analysis at both the $r=0$ and $r=r_{(-,+)}$ end points can be repeated by formally replacing $m\to -|m|$ and $r_{(+,-)}\to r_{(-,+)}$. However, in order for the z circle to shrink smoothly at $r=r_{(-,+)}$, instead of Eq. (64), we have to impose

$$|\mathcal{C}|=\frac{1}{2l[3+4|m|r_{(-,+)}]},\qquad l=1,2,3,\ldots. \tag{69}$$

A numerical plot of $3+4|m|r_{(-,+)}$ is shown in Fig. 4. The function is very different from $3-4mr_{(+,-)}$ in Case I, since the condition on the constant B in this case is less stringent.

Case III: $s=1$, $m>0$, $0<B<\frac{3\sqrt{3}}{16m\sqrt{m}}$, $r_{(+,+)}<r<\frac{1}{m}$. As $r\to\frac{1}{m}$, the AdS_5 warp factor goes to $+\infty$, as can be seen from Fig. 5. Setting $r=\frac{1}{m}-16\mathcal{C}^4m^2B^2R^4$, we find the seven-dimensional metric of the form

$$ds_7^2\approx\frac{1}{16h^2m^{8/5}R^2}\left[\frac{1}{4\mathcal{C}^2}ds^2_{\mathrm{AdS}_5}+dR^2+dz^2\right],\qquad R\to 0. \tag{70}$$



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 3. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N=2$ gauged supergravity with $s=1$, $m=-1$, $B=1$, $h=\frac{1}{8}$, and $\mathcal{C}=-\frac{1}{2}$. The warp factors are positive in the range $0<r<r_{(-,+)}=0.819$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

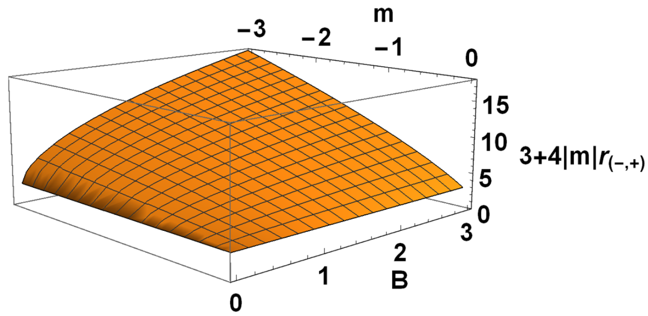


FIG. 4. A numerical plot of the function $3 + 4|m|r_{(-,+)}$ appearing in the condition of Eq. (69). We also note that $3 + 4|m|r_{(-,+)} \to 3$ as $B \to 0$ or $m \to 0$.

This metric is again conformally related to a product of AdS_5 and a cylinder.

By changing the radial coordinate to $\rho = \sqrt{r_{(+,+)} - r}$, we find the seven-dimensional metric, as $r \to r_{(+,+)}$, of the form

$$ds_7^2 \approx \frac{B r_{(+,+)}^{1/10}}{16h^2\sqrt{1 - m r_{(+,+)}}}\left[ds^2_{\mathrm{AdS}_5} + \frac{dR^2 + 4\mathcal{C}^2[3 - 4mr_{(+,+)}]^2 R^2 dz^2}{-4W'(r_{(+,+)})\sqrt{r_{(+,+)}}(1 - mr_{(+,+)})^{3/2}}\right], \quad (71)$$

which is similar to Eq. (63) with $r_{(+,-)}$ replaced by $r_{(+,+)}$. The z circle also shrinks smoothly near the end point at $r = r_{(+,+)}$, and the $R - z$ surface is locally an $\mathbb{R}^2/\mathbb{Z}_l$ if we choose

$$\mathcal{C} = \frac{1}{2l[3 - 4mr_{(+,+)}]}, \qquad l = 1, 2, 3, \ldots. \quad (72)$$

Unlike all previous cases, $3 - 4mr_{(+,+)}$ is negative in the regularity ranges of B and m, as can be seen from the numerical plot given in Fig. 6. In Fig. 7, we also plot $3 - 4mr_{(+,+)}$ (blue surface) and $3 - 4mr_{(+,-)}$ (orange surface) in Case I. The two join smoothly at $B = \frac{3\sqrt{3}}{16m\sqrt{m}}$, where $3 - 4mr_{(+,-)} = 0 = 3–4mr_{(+,+)}$.

To obtain the $SO(2)_R$ gauge field that vanishes at $r = r_{(+,+)}$, we choose

$$q = \mathcal{C}[3 - 4mr_{(+,+)}] = \frac{1}{2l}, \quad (73)$$

resulting in A_1 of the form

$$A_1 = \frac{|\mathcal{C}|}{2h}[r - r_{(+,+)}]. \quad (74)$$

Since $q = \frac{1}{2l}$ in this case, the Killing spinor η near $r = r_{(+,+)}$ is the same as Eq. (68), with $r_{(+,-)}$ and z replaced by $r_{(+,+)}$ and $-z$.

Case IV: $s = 1, m > 0, B = \frac{3\sqrt{3}}{16m\sqrt{m}}, 0 < r < r_{(+,-)}$. With $B = \frac{3\sqrt{3}}{16m\sqrt{m}}$, the quantity X in the four roots of $W = 0$ reduces to

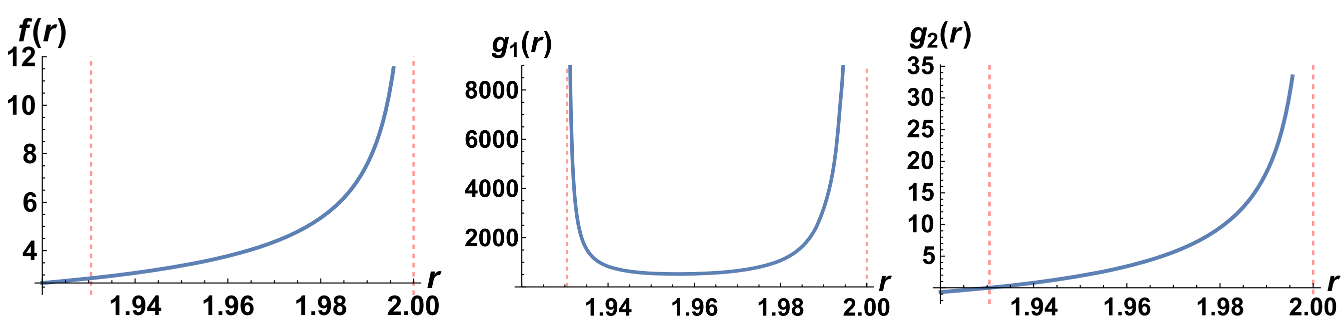
$$X = \frac{1}{m^2}, \quad (75)$$

giving rise to a much simpler form of the $r_{(\pm_1,\pm_2)}$ solution:

$$r_{(\pm_1,\pm_2)} = \frac{1}{4m}[1 \pm_1 2 \pm_2 \sqrt{-1 \pm_1 1}]. \quad (76)$$

In this case, there is only one real root of $W = 0$, given by $r_* = r_{(+,\pm_2)} = \frac{3}{4m}$.

However, as pointed out in Ref. [39], the z circle does not shrink smoothly at $r = r_*$ for any value of $\mathcal{C}$, due to the function $W(r = r_*)$ having a double zero. This can also be seen explicitly in the present work by considering Case I with $B \to \frac{3\sqrt{3}}{16m\sqrt{m}}$. In this limit, we cannot impose the condition of Eq. (64), since $|\mathcal{C}|$ diverges as $3 - 4mr_{(+,-)} \to 0$. If we set $r = \frac{3}{4m} - \frac{1}{R}$, the seven-dimensional metric is approximately given by



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 5. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N = 2$ gauged supergravity with $s = 1$, $m = \frac{1}{2}$, $B = \frac{1}{2}$, $h = \frac{1}{4}$, and $\mathcal{C} = -1$. The warp factors are all positive in the range $r_{(+,+)} = 1.93 < r < \frac{1}{m} = 2$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

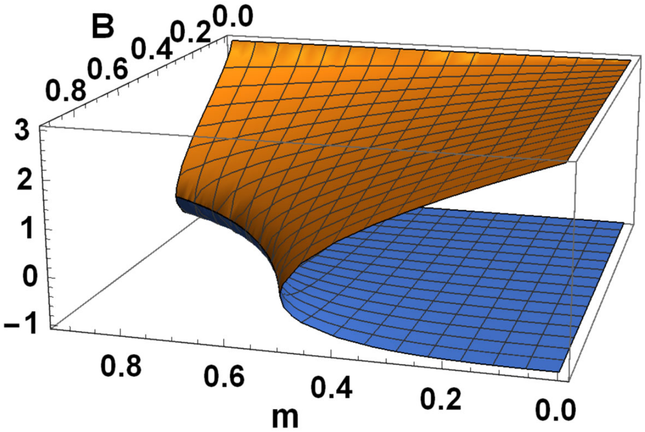


FIG. 7. A numerical plot of $3-4mr_{(+,-)}$ (orange surface) and $3-4mr_{(+,+)}$ (blue surface), which are connected to each other at $B=\frac{3\sqrt{3}}{16m\sqrt{m}}$.

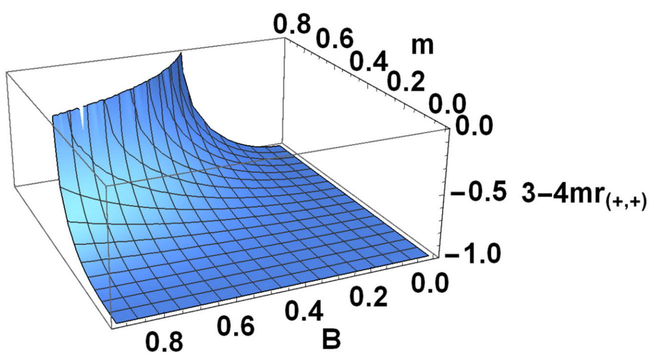


FIG. 6. A numerical plot of the function $3-4mr_{(+,+)}$ appearing in the condition of Eq. (72). The function approaches -1 as $B\to 0$ or $m\to 0$. As $B\to\frac{3\sqrt{3}}{16m\sqrt{m}}$, we find that $3-4mr_{(+,-)}\to 0$.

$$ds_7^2 \approx \frac{3^{3/5}}{128(2^{1/5})h^2m^{8/5}}\left[3ds^2_{\mathrm{AdS}_5}+\frac{dR^2+64\mathcal{C}^2m^2dz^2}{R^2}\right] \tag{77}$$

as $R\to+\infty$. On the other hand, for $r\to 0$, we find the seven-dimensional metric given in Eq. (62) as in Case I.

Case V: $s=1$, $m>0$, $B=\frac{3\sqrt{3}}{16m\sqrt{m}}$, $r_{(+,+)}<r<\frac{1}{m}$. Since $r_{(+,+)}=r_{(+,-)}=r_*=\frac{3}{4m}$ for $B=\frac{3\sqrt{3}}{16m\sqrt{m}}$, Case IV and Case V are connected at $r=r_*$. Therefore, we will give a representative numerical solution for these two cases collectively in Fig. 8. For $r\to r_*$ both from the left and from the right, the behavior of the metric is given by Eq. (77). On the other side, as $r\to\frac{1}{m}$, the metric becomes Eq. (70), as in Case III.

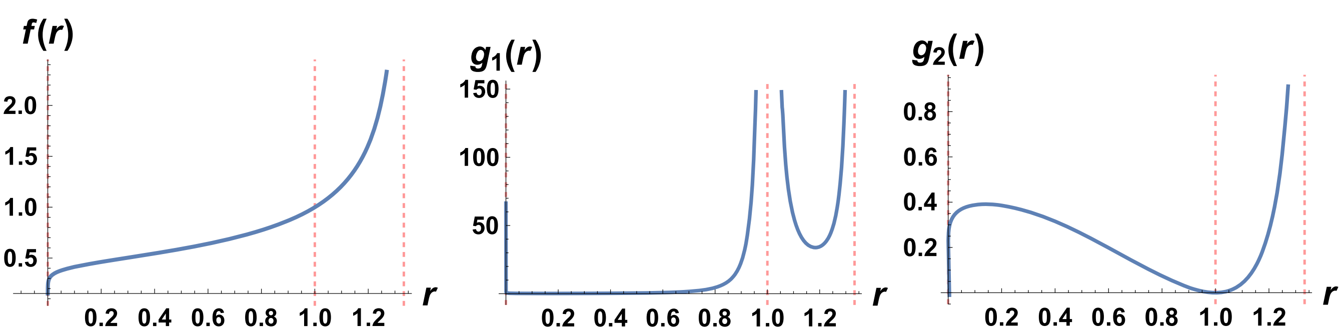
For the $SO(2)_R$ vector, we find that taking $r=r_*=\frac{3}{4m}$ in Eq. (60) results in the $SO(2)_R$ gauge field

$$A_1(r_*)=-\frac{q}{8mh}. \tag{78}$$

Therefore, we will set $q=0$ in order to make the $SO(2)_R$ gauge field vanish at $r=r_*$.

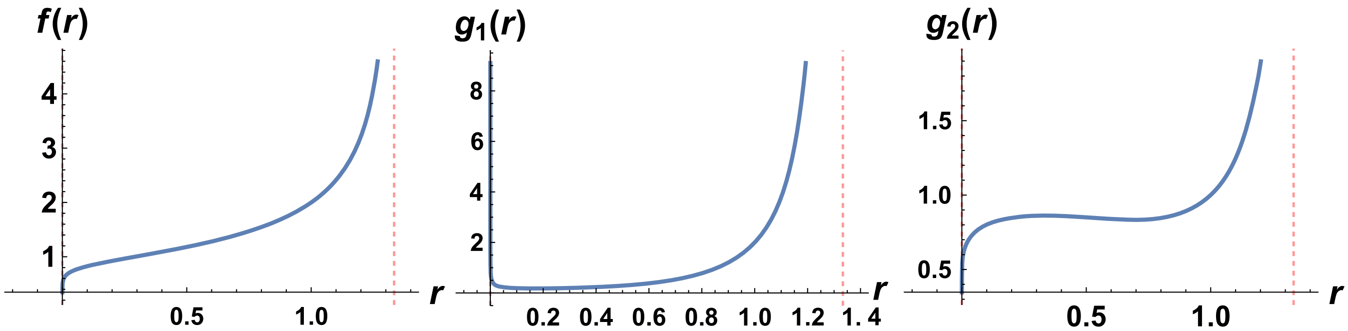
Case VI: $s=1, m>0, B>\frac{3\sqrt{3}}{16m\sqrt{m}}, 0<r<\frac{1}{m}$. This case combines the behavior near $r=0$ in Case I and the feature near $r=\frac{1}{m}$ in Case III. An example of the numerical solution with $m=\frac{3}{4}$, $B=1$, $h=\frac{1}{4}$, and $\mathcal{C}=-\frac{1}{2}$ is given in Fig. 9.

Case VII: $s=-1, m>0, B>0, \frac{1}{m}<r<r_{(+,+)}$. This is the only case with $s=-1$ with a numerical solution, given in Fig. 10. In comparison with other cases, we find that this case is very similar to Case III with the two boundaries interchanged. As $r\to\frac{1}{m}$, by setting $r=\frac{1}{m}+16\mathcal{C}^4m^2B^2R^4$,



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 8. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N=2$ gauged supergravity with $s=1$, $m=\frac{3}{4}$, $B=\frac{1}{2}$, $h=\frac{1}{4}$, and $\mathcal{C}=-\frac{1}{2}$. The warp factors are positive in the ranges $0<r<r_*=1$ (Case IV) and $r_*=1<r<\frac{1}{m}=\frac{4}{3}$ (Case V). The three vertical red dashed lines represent the three boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 9. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N=2$ gauged supergravity with $s=1$, $m=\frac{3}{4}$, $B=1$, $h=\frac{1}{4}$, and $\mathcal{C}=-\frac{1}{2}$. The warp factors are positive in the range $0<r<\frac{1}{m}=\frac{4}{3}$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

we find the seven-dimensional metric given in Eq. (70). Thus, the seven-dimensional space-time is conformal to a product of AdS_5 and a cylinder, as in Case III.

On the other side, as $r\to r_{(+,+)}$, we find the following form of the seven-dimensional metric:

$$ds_7^2\approx\frac{Br_{(+,+)}^{1/10}}{16h^2\sqrt{mr_{(+,+)}-1}}\left[ds^2_{\mathrm{AdS}_5}+\frac{dR^2+4\mathcal{C}^2[4mr_{(+,+)}-3]^2R^2dz^2}{-4W'(r_{(+,+)})\sqrt{r_{(+,+)}}(mr_{(+,+)}-1)^{3/2}}\right],\qquad(79)$$

after changing the radial coordinate to $\rho=\sqrt{r_{(+,+)}-r}$ as in Case III. The z circle shrinks smoothly if we impose

$$|\mathcal{C}|=\frac{1}{2l[4mr_{(+,+)}-3]},\qquad l=1,2,3,\ldots.\qquad(80)$$

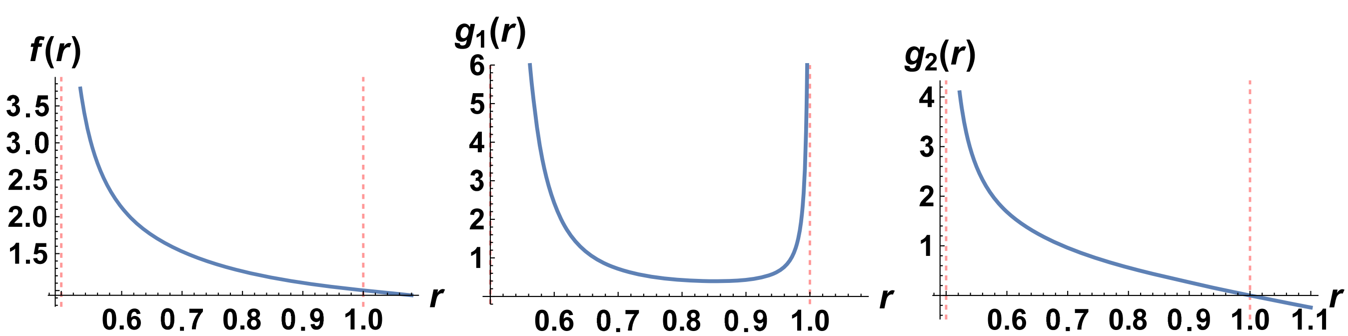
The $SO(2)_R$ gauge field is given by Eq. (74), with the constant q chosen to be

$$q=|\mathcal{C}|[4mr_{(+,+)}-3]=\frac{1}{2l}.\qquad(81)$$

Unlike all the previous cases with $s=1$, only the lower component of the Killing spinor η is nonvanishing as $r\to r_{(+,+)}$:

$$\eta=-2^{3/4}Y_0e^{\frac{iz}{2l}}r_{(+,+)}^{2/5}\begin{pmatrix}0\\1\end{pmatrix}.\qquad(82)$$

We end this section by pointing out that in the matter-coupled $N=2$ gauged supergravity with the $SO(4)$ gauge group, there are three $SO(2)_R$ singlet scalars corresponding to the $SO(3,3)$ noncompact generators Y_{31}, Y_{32}, and Y_{33}. We have also looked for $SO(2)_R$ symmetric solutions in this case. However, in order to satisfy the vector field equation [Eq. (25)], it turns out that either the $SO(2)_R$ gauge field needs to be constant, or all three-vector-multiplet scalars must vanish. This implies that the $SO(2)_R$ symmetric $AdS_5\times\Sigma$ solution can only exist in pure $N=2$ gauged supergravity.



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 10. Numerical plots of the warp factors for the $SO(2)$ symmetric solution in pure $N=2$ gauged supergravity with $s=-1$, $m=2$, $B=1$, $h=\frac{1}{4}$, and $\mathcal{C}=-\frac{1}{2}$. The warp factors are positive in the range $\frac{1}{m}=\frac{1}{2}<r<r_{(+,+)}=1$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

2. $SO(2)\times SO(2)$ symmetric solution

We now repeat the same procedure for more complicated solutions with $SO(2)\times SO(2)$ symmetry. With the explicit forms of A_1' and A_2' given in Eq. (A60), nonvanishing components of the dressed field strength tensors read

$$\mathbf{F}_1 = \frac{e^{-\sigma-\phi}}{2}(a_1 + a_2 e^{2\phi}) f^{-\frac{5}{2}},$$
$$\mathbf{F}_2 = \frac{e^{-\sigma-\phi}}{2}(a_1 - a_2 e^{2\phi}) f^{-\frac{5}{2}}. \tag{83}$$

We can now solve Eq. (A51) and find the solution for the $SO(2)\times SO(2)$ singlet scalar ϕ of the form

$$\phi = \ln\left[\frac{e^{-\frac{5\sigma}{2}}}{32a_2 h}[(a_2 - \lambda a_1)\tilde{g}_1 + K]\right], \tag{84}$$

in which

$$K = \kappa\sqrt{1024 a_1 a_2 h^2 e^{5\sigma} + (a_2 - \lambda a_1)^2 \tilde{g}_1^2} \tag{85}$$

and $\kappa = \pm 1$. We have included a sign factor $\lambda = +1, -1$ for later convenience. In the present case of an $SO(4)$ gauge group, we can set the constant $\lambda = 1$. In the next section, we will consider a noncompact $SO(2,2)$ gauge group and find that the solution takes the same form with $\lambda = -1$.

With the solution for ϕ given above, we can solve Eqs. (A40) and (A32), resulting in the following solutions for f and g_1:

$$f = \sqrt{2}\sqrt{\frac{h e^{2\sigma}[(a_2 + \lambda a_1)\tilde{g}_1 + K]}{s(256 h^2 e^{5\sigma} - \lambda \tilde{g}_1^2)}}, \tag{86}$$

$$g_1 = \frac{25{,}600\sqrt{2}h^{9/2} e^{6\sigma} K^{-2}(256 h^2 e^{5\sigma} - \lambda\tilde{g}_1^2)^{-2}[(a_2 + \lambda a_1)\tilde{g}_1 + K]^{5/2}(\sigma')^2}{[16\sqrt{2}h^{9/2}\sqrt{(a_2 + \lambda a_1)\tilde{g}_1 + K} + \lambda e^{-5\sigma}\sqrt{s(256 h^2 e^{5\sigma} - \lambda\tilde{g}_1^2)}]}. \tag{87}$$

We now determine the function $\hat{A}_1$ defined in Eq. (A22). With all the previous results, the first equation in Eq. (A60) becomes

$$\hat{A}_1' = \frac{10\tilde{g}_1 \hat{A}\sigma'[K^2(a_2 + \lambda a_1) - 2K\tilde{g}_1(a_2 - \lambda a_1)^2 + \tilde{g}_1^2(a_2 + \lambda a_1)(a_2 - \lambda a_1)^2]}{K[3K^2 - 4K\tilde{g}_1(a_2 + \lambda a_1) + \tilde{g}_1^2(a_2 - \lambda a_1)^2]}. \tag{88}$$

Solving this equation leads to the following solution:

$$\begin{aligned}\hat{A}_1 = {} & \frac{c}{8}e^{-5\sigma}\sqrt{\frac{9K^4 - 2K^2\tilde{g}_1^2(5a_1^2 + 22\lambda a_1 a_2 + 5a_2^2) + \tilde{g}_1^4(a_2 - \lambda a_1)^4}{a_1 a_2}} \\ & \times \exp\left[-\frac{(a_1^2 + 2a_1(7\lambda a_2 - y) + a_2(a_2 - 2\lambda y))\tan^{-1}(\frac{3K}{u\tilde{g}_1})}{\lambda u y}\right. \\ & \left. - \frac{(a_1^2 + 2a_1(7\lambda a_2 + y) + a_2(a_2 + 2\lambda y))\tanh^{-1}(\frac{3K}{v\tilde{g}_1})}{\lambda v y}\right],\end{aligned} \tag{89}$$

with an integration constant c and

$$u = \sqrt{4a_1 y + 4\lambda a_2 y - 5a_1^2 - 22\lambda a_1 a_2 - 5a_2^2}, \tag{90}$$

$$v = \sqrt{4a_1 y + 4\lambda a_2 y + 5a_1^2 + 22\lambda a_1 a_2 + 5a_2^2}, \tag{91}$$

$$y = \sqrt{a_1^2 + 14\lambda a_1 a_2 + a_2^2}. \tag{92}$$

Similarly, we can find the solution for g_2 as

$$g_2 = \frac{400\tilde{g}_1^2 f^5 \hat{A}_1^2 e^{2\sigma}[K^2 - (a_2 - \lambda a_1)^2\tilde{g}_1^2](\sigma')^2}{K^2 g_1[3K^2 - 4K\tilde{g}_1(a_2 + \lambda a_1) + \tilde{g}_1^2(a_2 - \lambda a_1)^2]^2}. \tag{93}$$

Finding the solution for A_2 in the other $SO(2)$ gauge field is much more difficult, since with all the previous results, the second equation in Eq. (A60) leads to a highly complicated differential equation. It should also be noted that A_2 does not appear in all of the BPS equations which depend only on A_2' and A_1'. In particular, we have verified that all the BPS conditions are satisfied by the above solution without the explicit solution for A_2. To make the subsequent analysis of the solutions more traceable, we will further simplify the ansatz for vector fields by setting

$$a_1 = -a_2 = b. \tag{94}$$

In this case, we find the nonvanishing components of the dressed field strength tensors given by

$$\mathbf{F}_1 = -be^{-\sigma}\sinh\phi f^{-\frac{5}{2}} \quad \text{and} \quad \mathbf{F}_2 = be^{-\sigma}\cosh\phi f^{-\frac{5}{2}}. \tag{95}$$

The solution found above now takes a much simpler form, with $\lambda = 1$:

$$\phi = \kappa\cosh^{-1}\left[\frac{\tilde{g}_1 e^{-\frac{5\sigma}{2}}}{16h}\right], \tag{96}$$

$$f = \frac{2\sqrt{\kappa bh}e^{\sigma}}{\sqrt{s}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}}, \tag{97}$$

$$g_1 = \frac{51,200\sqrt{\kappa bh^9}e^{6\sigma}(\sigma')^2}{(-256h^2e^{5\sigma}+\tilde{g}_1^2)^2[32\sqrt{\kappa bh^5}-\sqrt{s}e^{-5\sigma}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}]}, \tag{98}$$

$$g_2 = \frac{256c_1^2h^2\tilde{g}_1^2e^{\sigma}[32\sqrt{\kappa bh^5}-\sqrt{s}e^{-5\sigma}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}]}{\sqrt{s}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}}, \tag{99}$$

$$\hat{A}_1 = c_1(192h^2 - \tilde{g}_1^2e^{-5\sigma}). \tag{100}$$

It can also be verified that all of the BPS conditions and the field equations are satisfied if

$$\text{sign}(c_1\tilde{g}_1\sigma') = -1. \tag{101}$$

In addition, the second equation in Eq. (A60) gives the following differential equation:

$$A_2' = \frac{5c_1\tilde{g}_1(128h^2-\tilde{g}_1^2e^{-5\sigma})\sigma'}{\sqrt{-256e^{5\sigma}h^2+\tilde{g}_1^2}}, \tag{102}$$

which can be readily solved by

$$A_2 = c_1\tilde{g}_1e^{-5\sigma}\sqrt{-256h^2e^{5\sigma}+\tilde{g}_1^2} + c_2, \tag{103}$$

with an integration constant c_2. We also have an explicit form of the A_1 solution,

$$A_1 = c_1(192h^2 - \tilde{g}_1^2e^{-5\sigma}) - \frac{2q}{\tilde{g}_1}, \tag{104}$$

together with the Killing spinor

$$\eta = \frac{Y_0e^{iqz-\frac{\sigma}{4}}}{s^{1/8}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac{1}{16}}} \times \begin{pmatrix} \sqrt{4\sqrt{2}(\kappa bh^5)^{\frac{1}{4}} + s^{\frac{1}{4}}e^{-\frac{5\sigma}{2}}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac{1}{8}}} \\ -\sqrt{4\sqrt{2}(\kappa bh^5)^{\frac{1}{4}} - s^{\frac{1}{4}}e^{-\frac{5\sigma}{2}}(-256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac{1}{8}}} \end{pmatrix}, \tag{105}$$

with Y_0 being a constant.

We now turn to the regularity of the solution. From the f solution in Eq. (97), it is immediately seen that we have to impose the conditions $-256e^{5\sigma}h^2 + \tilde{g}_1^2 > 0$ and $\frac{\kappa b}{s} > 0$. The latter is clearly consistent with the condition $\text{sign}(\kappa bs) = +1$. As in the previous section, we define the following parameters:

$$B = \frac{8h^2\sqrt{\kappa b}}{\sqrt{s}}, \qquad m = \frac{\tilde{g}_1}{16h}, \qquad \mathcal{C} = 32\tilde{g}_1h^2c_1. \tag{106}$$

The complete solution then reads

$$f = \frac{Be^{\sigma}}{16h^2(-e^{5\sigma}+m^2)^{1/4}}, \tag{107}$$

$$g_1 = \frac{25Be^{6\sigma}(\sigma')^2}{1024h^2(-e^{5\sigma}+m^2)^2[B-e^{-5\sigma}(-e^{5\sigma}+m^2)^{1/4}]}, \tag{108}$$

$$g_2 = \frac{\mathcal{C}^2e^{\sigma}[B-e^{-5\sigma}(-e^{5\sigma}+m^2)^{1/4}]}{4h^2(-e^{5\sigma}+m^2)^{1/4}}, \tag{109}$$

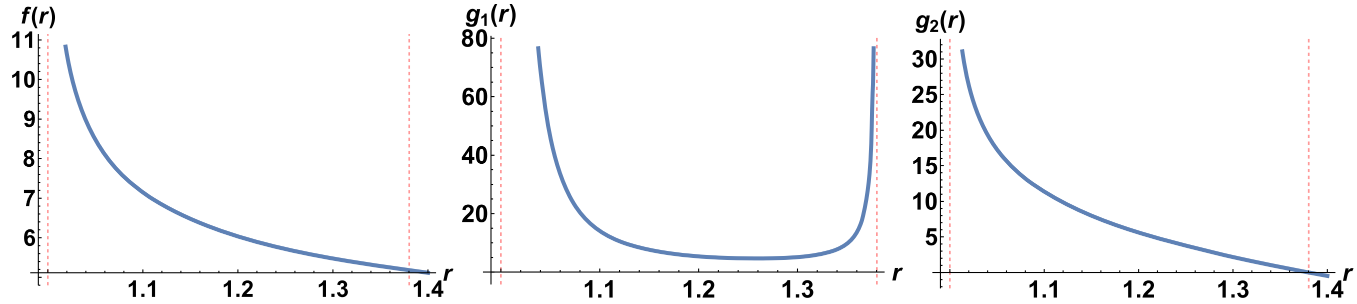
$$A_1 = \frac{1}{8mh}[\mathcal{C}(3-4m^2e^{-5\sigma}) - q], \tag{110}$$

$$A_2 = \frac{\mathcal{C}e^{-5\sigma}}{2h}\sqrt{-e^{5\sigma}+m^2} + c_2, \tag{111}$$

$$\phi = \kappa\cosh^{-1}[me^{-\frac{5\sigma}{2}}]. \tag{112}$$

We will also fix the form of the σ solution to

$$\sigma = -\frac{1}{5}\ln r \tag{113}$$



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 11. Numerical plots of the warp factors for the $SO(2)\times SO(2)$ symmetric solution in $SO(4)$ gauged supergravity with $m=1$, $B=1$, $h=\frac{1}{8}$, and $\mathcal{C}=1$. The solution is regular in the range $\frac{1}{m^2}=1<r<r_1=1.38$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

for $r>0$. This form of σ implies that the constant $\mathcal{C}$ can only be a positive integer due to Eq. (101).

In terms of the radial coordinate, the seven-dimensional metric can be written as

$$ds_7^2=\frac{Br^{1/20}}{16h^2(m^2r-1)^{1/4}}\times\left[ds^2_{\mathrm{AdS}_5}+\frac{r^{-5/4}}{64W(m^2r-1)^{7/4}}dr^2+\frac{4\mathcal{C}^2W}{B}dz^2\right],\qquad(114)$$

with

$$W=B-r^{3/4}(m^2r-1)^{1/4},\qquad(115)$$

while the solutions for ϕ, A_1, and A_2 are given by

$$\phi=\kappa\cosh^{-1}[m\sqrt{r}],\qquad(116)$$

$$A_1=\frac{1}{8mh}[\mathcal{C}(3-4m^2r)-q],\qquad(117)$$

$$A_2=\frac{\mathcal{C}}{2h}\sqrt{r(m^2r-1)}+c_2.\qquad(118)$$

Finally, the Killing spinor η takes the form

$$\eta=Y_0e^{iqz}\frac{r^{1/80}}{(m^2r-1)^{\frac{1}{16}}}\begin{pmatrix}\sqrt{\sqrt{B}+r^{3/8}(m^2r-1)^{\frac{1}{8}}}\\ -\sqrt{\sqrt{B}-r^{3/8}(m^2r-1)^{\frac{1}{8}}}\end{pmatrix}.\qquad(119)$$

In the present case, we find only one possible range of the radial coordinate r that leads to a regular solution:

$$m\neq0,\qquad B>0,\qquad \frac{1}{m^2}<r<r_1,\qquad(120)$$

with r_1 determined from $W(r_1)=0$. The explicit form of r_1 is given by

$$r_1=\frac{1}{4m^2}+\frac{1}{2}\sqrt{X_1}+\frac{1}{2}\sqrt{\frac{3}{4m^4}-X_1+\frac{1}{4m^6\sqrt{X_1}}},\qquad(121)$$

in which

$$X_1=\frac{1}{4m^4}-\frac{4(\frac{2}{3})^{1/3}B^{8/3}}{(-9+\sqrt{3(27+256B^4m^6)})^{1/3}}+\frac{B^{4/3}(-9+\sqrt{3(27+256B^4m^6)})^{1/3}}{18^{1/3}m^2}.\qquad(122)$$

Note also that the two possibilities with $m>0$ and $m<0$ are equivalent. For definiteness, we will only consider the $m>0$ case. An example of numerical plots of the three warp factors for $m=1$, $B=1$, $h=\frac{1}{8}$, and $\mathcal{C}=1$ is given in Fig. 11.

The result is very similar to Case VII in the previous section. As $r\to\frac{1}{m^2}$, the seven-dimensional metric becomes conformal to a product of AdS_5 and a cylinder. With the new radial coordinate R given by $r=\frac{1}{m^2}+256\mathcal{C}^8m^4B^4R^8$, the metric near $R=0$ is given by Eq. (70). On the other side, as $r\to r_1$, the seven-dimensional metric is approximately given by

$$ds_7^2\approx\frac{Br_1^{1/20}}{16h^2(m^2r_1-1)^{1/4}}\times\left[ds^2_{\mathrm{AdS}_5}+\frac{dR^2+4\mathcal{C}^2[3-4m^2r_1]^2R^2dz^2}{-16W'(r_1)r_1^{5/4}(m^2r_1-1)^{7/4}}\right],\qquad(123)$$

in which R is the new radial coordinate, defined by $R=\sqrt{r_1-r}$. The z circle shrinks smoothly, giving rise to an $\mathbb{R}^2/\mathbb{Z}_l$ orbifold at $r=r_1$ if we impose

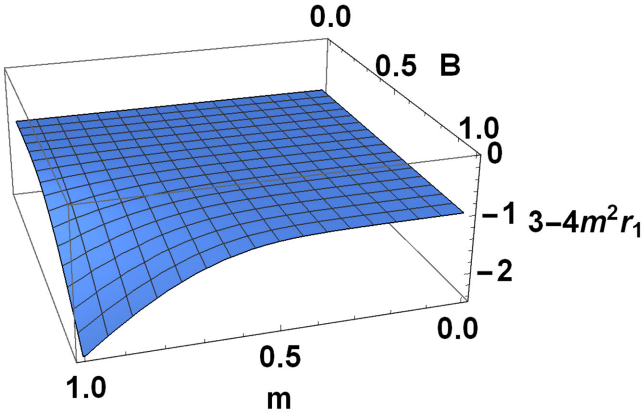


FIG. 12. A numerical plot of the function $3-4m^2r_1$ appearing in the condition of Eq. (124). Note also that $3-4m^2r_1\leq -1$ in the regularity range of Eq. (120).

$$\mathcal{C}=-\frac{1}{2l[3-4m^2r_1]},\qquad l=1,2,3,\ldots. \tag{124}$$

As in the previous section, the explicit form of $3-4m^2r_1$ is very complicated, so we will not give it here, but only show that this function is negative and less than -1 in the regularity range by a numerical plot in Fig. 12.

In order for the $SO(2)\times SO(2)$ gauge fields to vanish at $r=r_1$, we fix the constants q and c_2 to be

$$q=-\mathcal{C}[3-4m^2r_1]=\frac{1}{2l}$$
$$\text{and}\quad c_2=-\frac{\mathcal{C}}{2h}\sqrt{r_1(1+m^2r_1)}=-\frac{\mathcal{C}B}{2h}, \tag{125}$$

leading to

$$A_1=\frac{m\mathcal{C}}{2h}(r_1-r)$$
$$\text{and}\quad A_2=\frac{\mathcal{C}}{2h}\left[\sqrt{r(m^2r-1)}-\sqrt{r_1(m^2r_1-1)}\right]. \tag{126}$$

We also note that A_1 is well defined for all values of r, while A_2 is complex for $r<\frac{1}{m^2}$.

The Killing spinor η at the end point $r=r_1$ is given by

$$\eta=\sqrt{2}Y_0e^{\frac{iz}{2l}}r_1^{1/5}\begin{pmatrix}1\\0\end{pmatrix}. \tag{127}$$

B. $SO(2)_{\mathrm{diag}}$ symmetric solution

We now consider $\mathrm{AdS}_5\times\Sigma$ solutions with $SO(2)_{\mathrm{diag}}\subset SO(2)\times SO(2)$ symmetry generated by $J^{(1)}_{12}+J^{(2)}_{12}$. In this case, the two $SO(2)$ gauge fields are related by

$$\tilde{g}_2A_2=\tilde{g}_1A_1, \tag{128}$$

resulting in the following ansatz for the gauge fields:

$$A^I_{(1)}=A_1\left(\delta^I_3+\frac{\tilde{g}_1}{\tilde{g}_2}\delta^I_6\right)dz. \tag{129}$$

The corresponding two-form field strength is given by

$$F^I_{(2)}=A'\left(\delta^I_3+\frac{\tilde{g}_1}{\tilde{g}_2}\delta^I_6\right)dr\wedge dz. \tag{130}$$

We can also consistently set $C_{(3)}=0$, as in the previous cases.

Apart from the $SO(2)\times SO(2)$ singlet scalar ϕ corresponding to the noncompact generator Y_{33}, there are two additional scalars from the $SO(3,3)/SO(3)\times SO(3)$ coset that are invariant under $SO(2)_{\mathrm{diag}}$. These scalars correspond to the noncompact generators

$$\hat{Y}_1=Y_{11}+Y_{22}\quad\text{and}\quad\hat{Y}_2=Y_{12}-Y_{21}. \tag{131}$$

The coset representative then takes the form

$$L=e^{\varphi_1\hat{Y}_1}e^{\phi Y_{33}}e^{\varphi_2\hat{Y}_2}. \tag{132}$$

This leads to the following nonvanishing C-functions:

$$C=\frac{3}{\sqrt{2}}[\tilde{g}_1\cosh\phi-\tilde{g}_2\sinh\phi+\cosh 2\varphi_1\cosh 2\varphi_2(\tilde{g}_1\cosh\phi+\tilde{g}_2\sinh\phi)],$$
$$C^{11}=C^{22}=-\frac{1}{\sqrt{2}}(\tilde{g}_1\cosh\phi+\tilde{g}_2\sinh\phi)\sinh 2\varphi_1,$$
$$C^{12}=-C^{21}=-\frac{1}{\sqrt{2}}(\tilde{g}_1\cosh\phi+\tilde{g}_2\sinh\phi)\cosh 2\varphi_1\sinh 2\varphi_2,$$
$$C^{33}=\frac{1}{\sqrt{2}}[\tilde{g}_2\cosh\phi-\tilde{g}_1\sinh\phi-\cosh 2\varphi_1\cosh 2\varphi_2(\tilde{g}_2\cosh\phi+\tilde{g}_1\sinh\phi)]. \tag{133}$$

The scalar vielbein is given by

$$P^{ir}_{(1)} = \begin{pmatrix} \varphi_1' \cosh 2\varphi_2 & \varphi_2' & 0 \\ -\varphi_2' & \varphi_1' \cosh 2\varphi_2 & 0 \\ 0 & 0 & \phi' \end{pmatrix} dr, \qquad (134)$$

while the composite connection takes the form of

$$Q^{ij}_{(1)} = -\epsilon^{ij3}(\varphi_1' \sinh 2\varphi_2 dr - \tilde{g}_1 A_1 dz). \qquad (135)$$

With all these, the vector field equation [Eq. (25)] gives rise to the following equations:

$$\varphi_2' = \frac{\varphi_1'}{2} \coth 2\varphi_1 \sinh 4\varphi_2, \qquad (136)$$

$$A_1'' = -\left[\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma' + \frac{2(\tilde{g}_1 \cosh 2\phi + \tilde{g}_2 \sinh 2\phi)\phi'}{\tilde{g}_2 \cosh 2\phi + \tilde{g}_1 \sinh 2\phi}\right] A_1', \qquad (137)$$

$$A_1'' = -\left[\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma' + \frac{2(\tilde{g}_2 \cosh 2\phi + \tilde{g}_1 \sinh 2\phi)\phi'}{\tilde{g}_1 \cosh 2\phi + \tilde{g}_2 \sinh 2\phi}\right] A_1'. \qquad (138)$$

Consistency between the last two equations for $\phi' \neq 0$ requires

$$\tilde{g}_2 = \pm \tilde{g}_1, \qquad (139)$$

leading to a differential equation for the $SO(2)_{\mathrm{diag}}$ gauge field:

$$A_1'' = -\left(\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma' \pm 2\phi'\right) A_1'. \qquad (140)$$

The plus/minus sign arises from that in the relation $\tilde{g}_2 = \pm \tilde{g}_1$. From now on, we will choose $\tilde{g}_2 = \tilde{g}_1$ for definiteness and find

$$A_1' = A_2' = b e^{-\sigma - 2\phi} \sqrt{g_1 g_2} f^{-\frac{5}{2}}. \qquad (141)$$

Nonvanishing components of the dressed field strength tensors are given by

$$\mathbf{F}_1 = \mathbf{F}_2 = b e^{-\sigma-\phi} f^{-\frac{5}{2}}. \qquad (142)$$

Unlike the previous cases, there are additional BPS conditions arising from the transformations $\delta\lambda^{a1} = 0$ and $\delta\lambda^{a2} = 0$, resulting in

$$0 = \left[\frac{\varphi_1' \cosh 2\varphi_2}{\sqrt{g_1}} \Gamma^{\hat{r}} - \frac{1}{\sqrt{2}} e^{-\frac{\sigma}{2}} C^{11}\right] (\sigma^1)^a{}_b \epsilon^b + \left[\frac{\varphi_2'}{\sqrt{g_1}} \Gamma^{\hat{r}} - \frac{1}{\sqrt{2}} e^{-\frac{\sigma}{2}} C^{12}\right] (\sigma^2)^a{}_b \epsilon^b. \qquad (143)$$

We note here that the projector in Eq. (A14) implies

$$(\sigma^2)^a{}_b \epsilon^b = i(\sigma^1)^a{}_b \epsilon^b. \qquad (144)$$

Therefore, with the supersymmetry parameter [Eq. (A9)], we can rewrite the above condition as

$$0 = \frac{1}{\sqrt{g_1}} (\varphi_1' \cosh 2\varphi_2 + i\varphi_2') \sigma^3 \eta - \frac{1}{\sqrt{2}} e^{-\frac{\sigma}{2}} (C^{11} + iC^{12})(i\sigma^2 \eta). \qquad (145)$$

There are 17 BPS conditions obtained from the vanishing of the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ derived from this BPS equation. Two of these are given by

$$0 = \frac{s}{\sqrt{f g_1}} (\varphi_1' \cosh 2\varphi_2 + i\varphi_2'), \qquad (146)$$

$$0 = -\frac{\tilde{g}_1}{\sqrt{2}} e^{\phi} A_1' (\sinh 2\varphi_1 + i \cosh 2\varphi_1 \sinh 2\varphi_2). \qquad (147)$$

For nonvanishing $\tilde{g}_1$ and A_1', the second condition is satisfied only by setting $\varphi_1 = \varphi_2 = 0$. All the remaining conditions are also satisfied by this result.

In addition, repeating the same procedure as in the previous sections with $\varphi_1 = \varphi_2 = 0$, we find the same set of BPS conditions [Eqs. (A30) to (A57)]. The resulting $SO(2)_{\mathrm{diag}}$ symmetric solution is then given by

$$f = \frac{2\sqrt{2}\sqrt{\tilde{g}_1 b h} e^{\sigma}}{\sqrt{s(256 h^2 e^{5\sigma} - \tilde{g}_1^2)}}, \qquad (148)$$

$$g_1 = \frac{204,800\sqrt{2}\sqrt{\tilde{g}_1 b h^9} e^{6\sigma} (\sigma')^2}{(256 h^2 e^{5\sigma} - \tilde{g}_1^2)^2 \left[32\sqrt{2}\sqrt{b h^5} - e^{-5\sigma}\sqrt{s(256 h^2 e^{5\sigma} - \tilde{g}_1^2)}\right]}, \qquad (149)$$

$$g_2 = \frac{1,024 c^2 \tilde{g}_1^2 h^2 e^{\sigma} \left[32\sqrt{2}\sqrt{\tilde{g}_1 b h^5} - e^{-5\sigma}\sqrt{s(256 h^2 e^{5\sigma} - \tilde{g}_1^2)}\right]}{\sqrt{s(256 h^2 e^{5\sigma} - \tilde{g}_1^2)}}, \qquad (150)$$

$$\phi = \frac{5\sigma}{2} + \ln\left[\frac{16h}{\tilde{g}_1}\right], \tag{151}$$

$$A_1 = c(128h^2 - \tilde{g}_1^2 e^{-5\sigma}) - \frac{2q}{\tilde{g}_1}, \tag{152}$$

with the usual sign condition $\mathrm{sign}(c\tilde{g}_1\sigma') = +1$. The Killing spinor is given by

$$\eta = \frac{Y_0 e^{iqz+\frac{\sigma}{4}}}{[s(256h^2e^{5\sigma} - \tilde{g}_1^2)]^{\frac{1}{8}}} \times \begin{pmatrix} \sqrt{8(\tilde{g}_1 bh^5)^{\frac{1}{4}} + se^{-\frac{5\sigma}{2}}[2s(256h^2e^{5\sigma} - \tilde{g}_1^2)]^{\frac{1}{4}}} \\ -\sqrt{8(\tilde{g}_1 bh^5)^{\frac{1}{4}} - se^{-\frac{5\sigma}{2}}[2s(256h^2e^{5\sigma} - \tilde{g}_1^2)]^{\frac{1}{4}}} \end{pmatrix}. \tag{153}$$

It should be remarked that the half-spindle solution found in Ref. [39] from the $U(1)^2$ truncation of the maximal seven-dimensional gauged supergravity can also be recovered from the above solution by the following identification:

$$\sigma = -\frac{2}{3}\lambda, \qquad \tilde{g}_1 = 2m, \qquad b = \frac{8B^2}{m^4}, \qquad c = \frac{\mathcal{C}^2}{2m^2}, \qquad h = \frac{m}{8} \tag{154}$$

where λ, B, m, and $\mathcal{C}$ are the scalar field and constants used in Ref. [39]. In more detail, the nonvanishing gauge field in the solution of Ref. [39] is given by

$$A^{(1)} = A^{12} = \frac{1}{2}(A^3 + A^6) = A^3, \tag{155}$$

with $A^6 = A^3$. The scalar fields are related by the following form of the unimodular matrix:

$$T_{ab} = \mathrm{diag}\left(-\frac{\sigma}{2} - \phi, -\frac{\sigma}{2} - \phi, -\frac{\sigma}{2} + \phi, -\frac{\sigma}{2} + \phi, 2\sigma\right), \tag{156}$$

with $\sigma = -2(\lambda_1 + \lambda_2)$ and $\phi = \lambda_2 - \lambda_1$. In this equation, λ_1 and λ_2 are the two scalars of the $U(1)^2$ truncation of $N = 4$ gauged supergravity used in Ref. [39]. Therefore, in a sense, the $SO(2)_{\mathrm{diag}}$ symmetric solution found in this work contains the solution of Ref. [39] for a special value of the gauge coupling constant, $\tilde{g}_1 = 16h$. However, the present solution preserves only eight supercharges, corresponding to $N = 1$ superconformal symmetry in four dimensions.

To further analyze the solution, we introduce the following constants, as in the previous cases:

$$B = 2\sqrt{2}h^{3/2}\sqrt{\tilde{g}_1 b}, \qquad m = \frac{\tilde{g}_1}{16h}, \qquad \mathcal{C} = 32\tilde{g}_1 h^2 c. \tag{157}$$

We will also take the solution for σ as given in Eq. (113). The seven-dimensional metric reads

$$ds_7^2 = \frac{Br^{3/10}}{16h^2\sqrt{s(1-m^2r)}}\left[ds^2_{\mathrm{AdS}_5} + \frac{r^{-3/2}}{16W[s(1-m^2r)]^{3/2}}dr^2 + \frac{16\mathcal{C}^2W}{B}dz^2\right], \tag{158}$$

with

$$W = B - \sqrt{sr(1-m^2r)}. \tag{159}$$

The scalar field ϕ and the $SO(2)_{\mathrm{diag}}$ gauge field are given by

$$\phi = -\frac{1}{2}\ln r - \ln m \quad \text{and} \quad A_1 = -\frac{1}{8mh}[2|\mathcal{C}|(1-2m^2r) + q], \quad \text{respectively.} \tag{160}$$

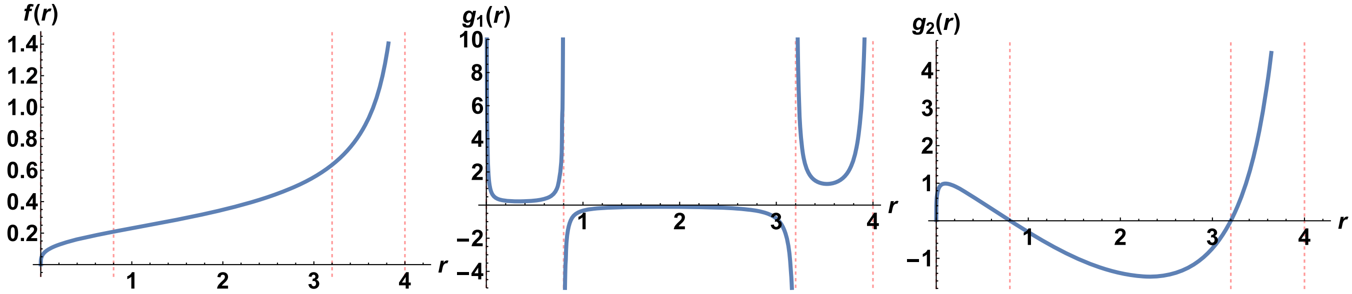
Since we have used $\sigma < 0$, it follows that the constant $\mathcal{C}$ must be negative due to the condition of Eq. (49). The Killing spinor takes the form of

$$\eta = Y_0 e^{iqz}\frac{2^{1/8}r^{3/40}}{[s(1-m^2r)]^{\frac{1}{8}}}\begin{pmatrix} \sqrt{\sqrt{B} + sr^{1/4}[s(1-m^2r)]^{\frac{1}{4}}} \\ -\sqrt{\sqrt{B} - sr^{1/4}[s(1-m^2r)]^{\frac{1}{4}}} \end{pmatrix}. \tag{161}$$

The behaviors of the solution are very similar to those given in Sec. III A 1 and in Ref. [39]. However, the first equation in Eq. (160) implies that r and m must be positive in order to find a real solution for ϕ. Therefore, there is no regular solution with $m < 0$ as in Case II of Sec. III A 1. For convenience, the remaining six possibilities with $m > 0$ will be combined into four different cases as follows:

$$\begin{aligned} &\text{(i)}\ s = +1, \quad 0 < B < \frac{1}{2m}, \quad r \in (0, r_-) \cup \left(r_+, \frac{1}{m^2}\right), \\ &\text{(ii)}\ s = +1, \quad B = \frac{1}{2m}, \quad r \in (0, r_*) \cup \left(r_*, \frac{1}{m^2}\right), \\ &\text{(iii)}\ s = +1, \quad B > \frac{1}{2m}, \quad r \in \left(0, \frac{1}{m^2}\right), \\ &\text{(iv)}\ s = -1, \quad B > 0, \quad r \in \left(\frac{1}{m^2}, r_+\right), \end{aligned} \tag{162}$$

in which



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 13. Numerical plots of the warp factors for the $SO(2)_{\mathrm{diag}}$ symmetric solution in the $SO(4)$ gauge group in Case i with $s=1$, $m=\frac{1}{2}$, $B=\frac{4}{5}$, $h=\frac{1}{2}$, and $\mathcal{C}=-1$. All the warp factors are positive in the ranges $0<r<r_-=0.8$ and $r_+=3.2<r<\frac{1}{m^2}=4$, with the four vertical red dashed lines representing the four boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

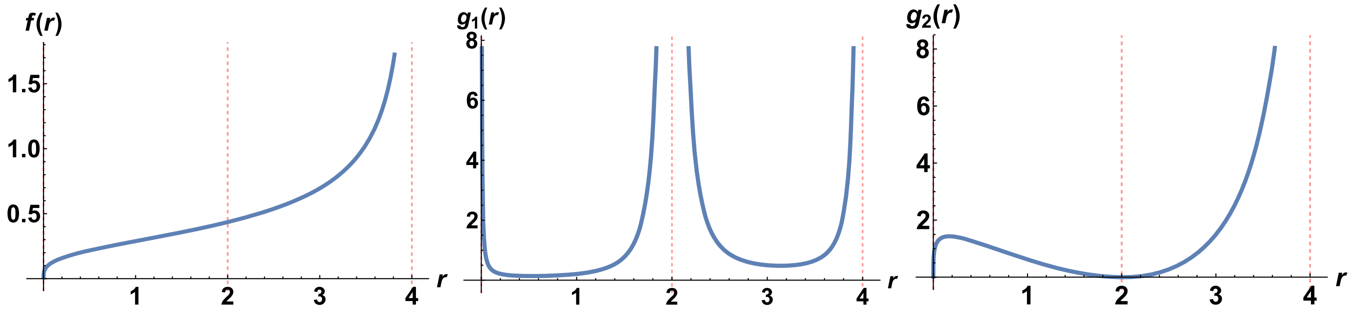
$$r_\pm=\frac{1\pm\sqrt{1-4sB^2m^2}}{2m^2} \qquad (163)$$

are the two roots of $W=B-\sqrt{sr(1-m^2r)}=0$. For $B=\frac{1}{2m}$ with $s=+1$, the two roots are equal, $r_+=r_-=r_*=\frac{1}{2m^2}$.

Examples of numerical solutions for the warp factors in these four cases are given in Figs. 13–16, respectively.

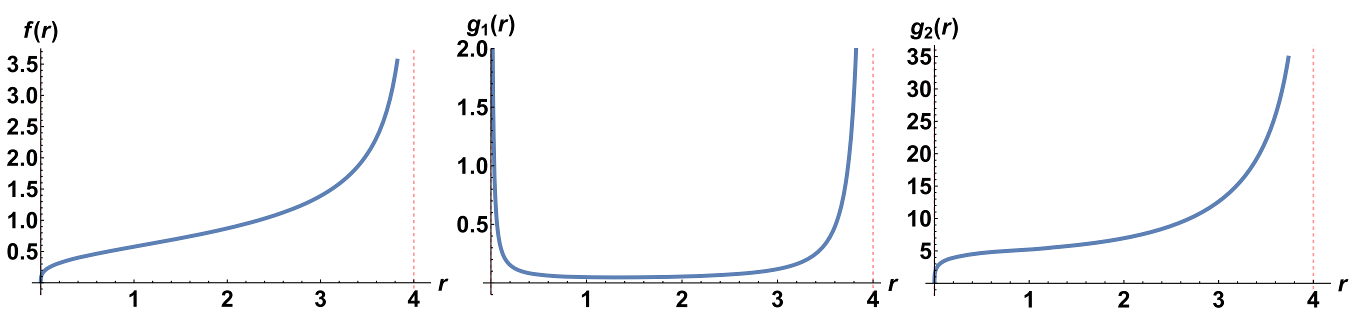
The end-point behaviors are also similar to the $SO(2)_R$ symmetric solution. As $r\to0$ or $r\to\frac{1}{m^2}$, the seven-dimensional metric is conformal to a product of AdS_5 and a cylinder. The explicit form of the metric in these limits can be found in the same way as in Case I with $r\to0$ and in Case III with $r\to\frac{1}{m^2}$.

The more interesting behaviors as $r\to r_\pm$ are in Case i and Case iv. In terms of the new radial coordinate



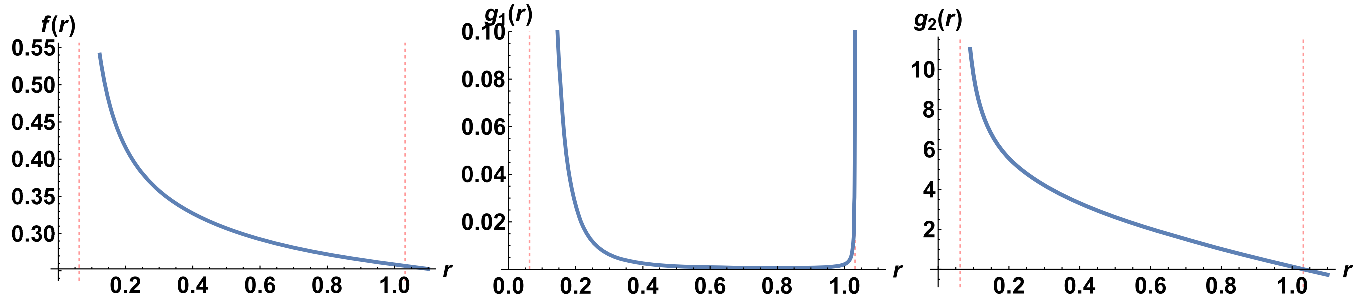
(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 14. Numerical plots of the warp factors for the $SO(2)_{\mathrm{diag}}$ symmetric solution in the $SO(4)$ gauge group in Case ii with $s=1$, $m=\frac{1}{2}$, $B=1$, $h=\frac{1}{2}$, and $\mathcal{C}=-1$. The warp factors are positive in the ranges $0<r<r_*=2$ and $r_*=2<r<\frac{1}{m^2}=4$, with the three vertical red dashed lines representing the three boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 15. Numerical plots of the warp factors for the $SO(2)_{\mathrm{diag}}$ symmetric solution in the $SO(4)$ gauge group in Case iii with $s=1$, $m=\frac{1}{2}$, $B=2$, $h=\frac{1}{2}$, and $\mathcal{C}=-1$. The warp factors are positive in the range $0<r<\frac{1}{m^2}=4$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 16. Numerical plots of the warp factors for the $SO(2)_{\rm diag}$ symmetric solution in the $SO(4)$ gauge group in Case iv with $s=-1$, $m=4$, $B=4$, $h=\frac{1}{2}$, and $\mathcal{C}=-1$. The warp factors are positive in the range $\frac{1}{m^2}=0.0625<r<r_+=1.03$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

$R=\sqrt{r_\pm - r}$, the seven-dimensional metric, as $r\to r_\pm$, is approximately given by

$$ds_7^2 \approx \frac{Br_\pm^{3/10}}{16h^2\sqrt{s(1-m^2r_\pm)}} \times\left[ds^2_{\rm AdS_5}+\frac{dR^2+16\mathcal{C}^2[1-2m^2r_\pm]^2R^2dz^2}{-4W'(r_\pm)r_\pm^{3/2}(1-m^2r_\pm)^{3/2}}\right]. \tag{164}$$

The z circle shrinks smoothly near these end points $r=r_\pm$ if we impose

$$|\mathcal{C}|=\frac{1}{4l|1-2m^2r_\pm|}=\frac{1}{4l\sqrt{1-4sB^2m^2}},\quad l=1,2,3,\ldots. \tag{165}$$

Unlike in the previous solutions, this condition can be written explicitly in terms of B and m, since $r_\pm$ takes a much simpler form than $r_{(\pm_1,\pm_2)}$ and r_1.

In order for the $SO(2)_{\rm diag}$ gauge field to vanish at $r=r_\pm$, we fix the constant q such that

$$q=\mp 2|\mathcal{C}||1-2m^2r_\pm|=\mp\frac{1}{2l}\quad\text{and}\quad A_1=\frac{m|\mathcal{C}|}{2h}(r-r_\pm), \tag{166}$$

which gives rise to the Killing spinor η at $r=r_\pm$ of the form

$$\eta=2^{-3/8}Y_0e^{\mp\frac{iz}{2l}}r_\pm^{1/5}\begin{pmatrix} s+1\\ s-1\end{pmatrix}. \tag{167}$$

Finally, it should be pointed out that if $B=\frac{1}{2m}$, we find that $|\mathcal{C}|$ diverges due to $\sqrt{1-4B^2m^2}=0$. This implies that the z circle does not shrink smoothly at $r=r_*=\frac{1}{2m^2}$ in Case ii as in Cases IV and V of the $SO(2)_R$ symmetric solution. As $r\to r_*$, the seven-dimensional metric is approximately given by

$$ds_7^2\approx\frac{1}{32(2^{4/5})h^2m^{8/5}}\left[2ds^2_{\rm AdS_5}+\frac{dR^2+64\mathcal{C}^2m^4dz^2}{R^2}\right], \tag{168}$$

with the new radial coordinate R given by $r=\frac{3}{4m}-\frac{1}{R}$ and $R\to+\infty$. In order for the $SO(2)_{\rm diag}$ gauge field to vanish at $r=r_*$, we have to set $q=0$ resulting in the z-independent Killing spinor as in Case IV and Case V of the pure supergravity solution.

IV. $SO(2,2)$ GAUGE GROUP

In this section, we consider $N=2$ gauged supergravity with the noncompact $SO(2,2)\sim SO(2,1)\times SO(2,1)$ gauge group. The embedding of this gauge group in $SO(3,n)$ requires at least $n=3$ vector multiplets. We will work with the minimal number of $n=3$ and choose the following $SO(2,2)$ structure constants:

$$f_{IJK}=(\tilde{g}_1\epsilon_{\bar{i}\bar{j}\bar{k}},-\tilde{g}_2\epsilon_{\bar{r}\bar{s}\bar{t}}). \tag{169}$$

Indices $\bar{i},\bar{j},\ldots=1,\ 2,\ 6$ and $\bar{r},\bar{s},\ldots=3,\ 4,\ 5$ are two sets of $SO(2,1)$ indices raised and lowered by $\eta_{\bar{i}\bar{j}}=\mathrm{diag}(-1,-1,1)$ and $\eta_{\bar{r}\bar{s}}=\mathrm{diag}(-1,1,1)$, respectively. Moreover, $\tilde{g}_1$ and $\tilde{g}_2$ are coupling constants for the two $SO(2,1)$ subgroups, respectively, generated by the generators

$$T_1=Y_{23},\qquad T_2=-Y_{13},\qquad T_6=J^{(1)}_{12}, \tag{170}$$

$$T_3=J^{(2)}_{12},\qquad T_4=Y_{32},\qquad T_5=-Y_{31}. \tag{171}$$

These satisfy $SO(2,1)\times SO(2,1)$ algebra:

$$[T_{\bar{i}}, T_{\bar{j}}] = -\varepsilon_{\bar{i}\bar{j}}{}^{\bar{k}} T_{\bar{k}}, \quad [T_{\bar{r}}, T_{\bar{s}}] = \varepsilon_{\bar{r}\bar{s}}{}^{\bar{t}} T_{\bar{t}}, \quad [T_{\bar{i}}, T_{\bar{r}}] = 0. \tag{172}$$

The compact $SO(2)\times SO(2)$ subgroup is generated by T_3 and T_6. In this case, T_6 generates the $SO(2)_R$ subgroup of the $SO(3)_R$ R symmetry, while T_3 generates the other $SO(2)$ in the $SO(3)$ symmetry of the vector multiplets. We now repeat the same procedure of finding $\mathrm{AdS}_5\times\Sigma$ solutions as in the previous section. Unlike the $SO(4)$ gauge group, this gauged supergravity does not admit any supersymmetric AdS_7 vacua. The maximally supersymmetric vacua are given by half-supersymmetric domain walls dual to nonconformal $N=(1,0)$ field theories in six dimensions [53]. In this case, $\mathrm{AdS}_5\times\Sigma$ solutions would describe conformal fixed points in four dimensions of six-dimensional $N=(1,0)$ field theories on a half-spindle.

A. $SO(2)\times SO(2)$ symmetric solution

As in Sec. III, supersymmetric $\mathrm{AdS}_5\times\Sigma$ solutions preserving $SO(2)\times SO(2)\subset SO(2,1)\times SO(2,1)\sim SO(2,2)$ can be found by using the same metric ansatz [Eq. (33)], $SO(2)\times SO(2)$ gauge fields [Eq. (37)], and $SO(2)\times SO(2)$ singlet scalar ϕ corresponding to the noncompact generator Y_{33}. As in the $SO(4)$ gauge group, we can consistently set $C_{(3)}=0$ due to $F^I_{(2)}\wedge F^I_{(2)}=0$.

With the coset representative [Eq. (39)], the scalar vielbein is still given by Eq. (41). However, the composite connection for $SO(2)_R$ is now given by

$$Q^{ij}_{(1)} = \tilde{g}_1 A_2 \varepsilon^{ij3} dz. \tag{173}$$

The C-functions take the form of

$$C = -3\sqrt{2}\tilde{g}_1 \sinh\phi, \qquad C^{ir} = \sqrt{2}\tilde{g}_1 \cosh\phi\,\delta^i_3\delta^r_3. \tag{174}$$

By the same analysis, we can derive a similar set of BPS equations with A_1 and A_2 interchanged. General solutions for ϕ, f, g_1, $\hat{A}_2$, and g_2 are given in Eqs. (84), (86), (87), (89), and (93), respectively, with $\lambda=-1$. As in the case of the $SO(4)$ gauge group, we are not able to determine an analytic solution for the second $SO(2)$ gauge field in this case given by A_1. Therefore, to explicitly write down a complete solution and discuss some properties of the solution, we further simplify the BPS equations by choosing $a_1=-a_2=b$ as in the $SO(4)$ case. It should be noted here that setting $a_1=a_2=b$ is also possible in this case but does not lead to any new solutions.

Repeating the same analysis as in the $SO(4)$ case, we can eventually solve all the BPS conditions and find the $SO(2)\times SO(2)$ symmetric solution:

$$\phi = \sinh^{-1}\left[\frac{\tilde{g}_1 e^{-\frac{5\sigma}{2}}}{16h}\right], \tag{175}$$

$$f = \frac{2\sqrt{bh}e^{\sigma}}{\sqrt{s}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}}, \tag{176}$$

$$g_1 = \frac{51,200\sqrt{bh^9}e^{6\sigma}(\sigma')^2}{(256h^2e^{5\sigma}+\tilde{g}_1^2)^2[32\sqrt{bh^5}-\sqrt{s}e^{-5\sigma}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}]}, \tag{177}$$

$$g_2 = \frac{256c_1^2h^2\tilde{g}_1^2e^{\sigma}[32\sqrt{bh^5}-\sqrt{s}e^{-5\sigma}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}]}{\sqrt{s}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{1/4}}, \tag{178}$$

$$A_1 = -c_1\tilde{g}_1e^{-5\sigma}\sqrt{256h^2e^{5\sigma}+\tilde{g}_1^2}+c_2, \tag{179}$$

$$A_2 = c_1(192h^2+\tilde{g}_1^2e^{-5\sigma})-\frac{2q}{\tilde{g}_1}, \tag{180}$$

together with the sign condition $\mathrm{sign}(c\tilde{g}_1\sigma')=+1$. The explicit form of the Killing spinor is given by

$$\eta = \frac{Y_0e^{iqz-\frac{\sigma}{4}}}{s^{1/8}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac{1}{16}}} \times\begin{pmatrix}\sqrt{4\sqrt{2}(h^5b)^{\frac14}+s^{1/4}e^{-\frac{5\sigma}{2}}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac18}}\\ -\sqrt{4\sqrt{2}(h^5b)^{\frac14}-s^{1/4}e^{-\frac{5\sigma}{2}}(256h^2e^{5\sigma}+\tilde{g}_1^2)^{\frac18}}\end{pmatrix}, \tag{181}$$

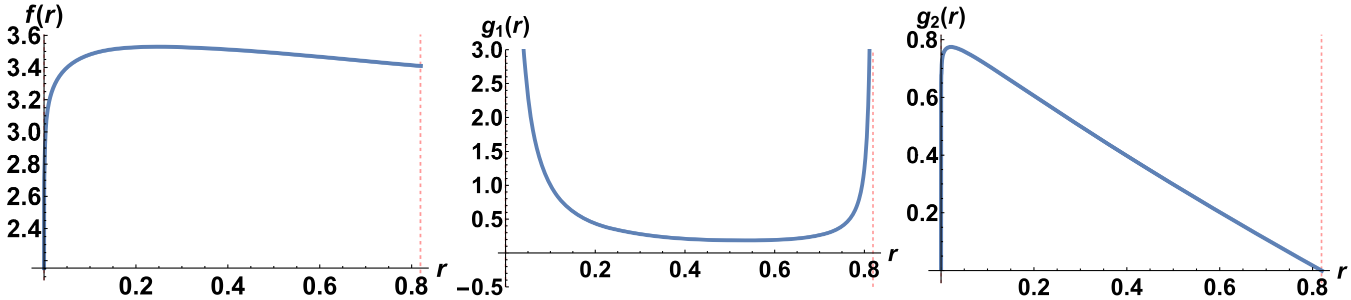
with Y_0 being a constant.

The analysis of the regularity of the solution proceeds as in the previous cases. We first take the solution for the dilaton as in Eq. (113) and introduce the following parameters:

$$B=\frac{8h^2\sqrt{b}}{\sqrt{s}}, \qquad m=\frac{\tilde{g}_1}{16h}, \qquad \mathcal{C}=32\tilde{g}_1h^2c_1. \tag{182}$$

The seven-dimensional metric reads

$$ds_7^2 = \frac{Br^{1/20}}{16h^2(1+m^2r)^{1/4}}\left[ds^2_{\mathrm{AdS}_5} + \frac{r^{-5/4}}{64W(1+m^2r)^{7/4}}dr^2+\frac{4\mathcal{C}^2W}{B}dz^2\right], \tag{183}$$



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 17. Numerical plots of the warp factors for the $SO(2)\times SO(2)$ symmetric solution in the $SO(2,2)$ gauge group with $m=1$, $B=1$, $h=\frac{1}{8}$, and $\mathcal{C}=-\frac{1}{4}$. The solution is regular in the range $0<r<r_2=0.819$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

with

$$W=B-r^{3/4}(1+m^2r)^{1/3}, \qquad (184)$$

while the solutions for ϕ, A_1, and A_2 are given by

$$\phi=\sinh^{-1}[m\sqrt{r}], \qquad (185)$$

$$A_1=\frac{|\mathcal{C}|}{2h}\sqrt{r(1+m^2r)}+c_2, \qquad (186)$$

$$A_2=-\frac{1}{8mh}[|\mathcal{C}|(4m^2r+3)-q]. \qquad (187)$$

The Killing spinor η takes the form

$$\eta=Y_0e^{iqz}\frac{r^{1/80}}{(1+m^2r)^{\frac{1}{16}}}\begin{pmatrix}\sqrt{\sqrt{B}+r^{3/8}(1+m^2r)^{\frac{1}{8}}}\\ -\sqrt{\sqrt{B}-r^{3/8}(1+m^2r)^{\frac{1}{8}}}\end{pmatrix}. \qquad (188)$$

It should be noted that since we have chosen $\mathrm{sign}(\sigma)=-1$, we need to impose the condition $\mathrm{sign}(c_1\tilde{g}_1)=-1$, resulting in $\mathrm{sign}(\mathcal{C})=-1$.

There is only one possible range for the radial coordinate r in order to obtain a regular solution:

$$m\neq 0, \qquad B>0, \qquad 0<r<r_2, \qquad (189)$$

where r_2 is determined from $W(r_2)=0$ as

$$r_2=-\frac{1}{4m^2}-\frac{1}{2}\sqrt{X_2}+\frac{1}{2}\sqrt{\frac{3}{4m^4}-X_2+\frac{1}{4m^6\sqrt{X_2}}}, \qquad (190)$$

with

$$X_2=\frac{1}{4m^4}-\frac{4(2/3)^{1/3}B^{8/3}}{(\sqrt{81+768B^4m^6}-9)^{1/3}}+\frac{B^{4/3}(\sqrt{81+768B^4m^6}-9)^{1/3}}{18^{1/3}m^2}. \qquad (191)$$

There are two possibilities with $m>0$ and $m<0$. However, these are related to each other by a sign change in the ϕ solution. In the following, we will choose $m>0$ for definiteness. An example of numerical plots for the three warp factors is given in Fig. 17.

This is very similar to Case II in Sec. III A 1. As $r\to 0$, the seven-dimensional metric is again conformal to a product of AdS_5 and a cylinder. With the new radial coordinate $R=r^{3/8}$, the metric near $R=0$ is given in Eq. (62). As $r\to r_2$, the seven-dimensional metric is approximately given by

$$ds_7^2\approx\frac{Br_2^{1/20}}{16h^2(1+m^2r_2)^{1/4}}\left[ds^2_{\mathrm{AdS}_5}+\frac{dR^2+4\mathcal{C}^2[3+4m^2r_2]^2R^2dz^2}{-16W'(r_2)r_2^{5/4}(1+m^2r_2)^{7/4}}\right], \qquad (192)$$

with the coordinate R defined by $R=\sqrt{r_2-r}$. The z circle shrinks smoothly, giving rise to an $\mathbb{R}^2/\mathbb{Z}_l$ orbifold at $r=r_2$ after imposing the condition

$$|\mathcal{C}|=\frac{1}{2l[3+4m^2r_2]}, \qquad l=1,2,3,\ldots. \qquad (193)$$

Since the explicit form of $3+4m^2r_2$ derived from Eq. (190) is rather complicated, we only numerically show that this function is always greater than 3 in the regularity range in Fig. 18.

Fixing the constants

$$q=|\mathcal{C}|[3+4m^2r_2]=\frac{1}{2l},\qquad c_2=-\frac{|\mathcal{C}|}{2h}\sqrt{r_2(1+m^2r_2)}=-\frac{|\mathcal{C}|B}{2h}, \qquad (194)$$

we obtain the $SO(2)\times SO(2)$ gauge fields

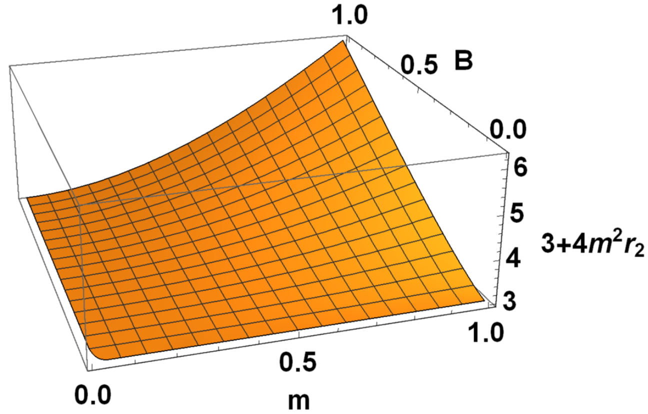


FIG. 18. A numerical plot of the function $3+4m^2r_2$ appearing in the condition of Eq. (193).

$$A_1=\frac{|\mathcal{C}|}{2h}\left[\sqrt{r(1+m^2r)}-\sqrt{r_2(1+m^2r_2)}\right],$$
$$A_2=\frac{m|\mathcal{C}|}{2h}(r_2-r), \qquad (195)$$

which vanish at $r=r_2$. The Killing spinor at the end point $r=r_2$ is explicitly given by

$$\eta=\sqrt{2}Y_0e^{\frac{iz}{2l}}r_2^{1/5}\begin{pmatrix}1\\0\end{pmatrix}. \qquad (196)$$

B. $SO(2)_{\mathrm{diag}}$ symmetric solution

To find solutions preserving $SO(2)_{\mathrm{diag}}$ symmetry generated by T_3+T_6 in $SO(2,2)$ gauge group, we use the following ansatz for vector fields:

$$A^I_{(1)}=A_2\left(\frac{\tilde{g}_1}{\tilde{g}_2}\delta^I_3+\delta^I_6\right)dz. \qquad (197)$$

There are three $SO(2)_{\mathrm{diag}}$ singlets corresponding to noncompact generators Y_{33} together with $\hat{Y}_1$ and $\hat{Y}_2$ given in Eq. (131). As in the $SO(4)$ gauge group, solving the vector field equations [Eq. (25)] requires $\tilde{g}_2=\pm\tilde{g}_1$. We again choose $\tilde{g}_2=\tilde{g}_1$, giving rise to nonvanishing components of the dressed field strength tensors in Eq. (142).

Repeating the same procedure as in Sec. III B, we can solve all the BPS conditions and find the solution

$$\phi=\frac{5\sigma}{2}+\ln\left[-\frac{16h}{\tilde{g}_1}\right],\qquad \varphi_1=\varphi_2=0, \qquad (198)$$

$$f=-\frac{2\sqrt{2}\sqrt{-\tilde{g}_1bh}e^{\sigma}}{\sqrt{s(256h^2e^{5\sigma}+\tilde{g}_1^2)}}, \qquad (199)$$

$$g_1=\frac{204,800\sqrt{2}\sqrt{-\tilde{g}_1bh^9}e^{6\sigma}(\sigma')^2}{(256h^2e^{5\sigma}+\tilde{g}_1^2)^2\left[32\sqrt{2}\sqrt{-\tilde{g}_1bh^5}+e^{-5\sigma}\sqrt{s(256h^2e^{5\sigma}+\tilde{g}_1^2)}\right]}, \qquad (200)$$

$$g_2=-\frac{1,024c^2h^2e^{\sigma}\left[32\sqrt{2}\sqrt{-\tilde{g}_1bh^5}+e^{-5\sigma}\sqrt{s(256h^2e^{5\sigma}+\tilde{g}_1^2)}\right]}{\sqrt{s(256h^2e^{5\sigma}+\tilde{g}_1^2)}}, \qquad (201)$$

$$A_2=c(128h^2+\tilde{g}_1^2e^{-5\sigma})-\frac{2q}{\tilde{g}_1}, \qquad (202)$$

together with the Killing spinor

$$\eta=\frac{Y_0e^{iqz+\frac{\sigma}{4}}}{[2s(256h^2e^{5\sigma}+\tilde{g}_1^2)]^{\frac{1}{8}}}\times\begin{pmatrix}\sqrt{8(-\tilde{g}_1bh^5)^{\frac{1}{4}}+e^{-\frac{5\sigma}{2}}[2s(256h^2e^{5\sigma}+\tilde{g}_1^2)]^{\frac{1}{4}}}\\-\sqrt{8(-\tilde{g}_1bh^5)^{\frac{1}{4}}-e^{-\frac{5\sigma}{2}}[2s(256h^2e^{5\sigma}+\tilde{g}_1^2)]^{\frac{1}{4}}}\end{pmatrix} \qquad (203)$$

and the sign condition $\mathrm{sign}(c\tilde{g}_1\sigma')=+1$. We also note that the ϕ solution in Eq. (198) requires $\frac{16h}{\tilde{g}_1}<0$. Defining the parameters

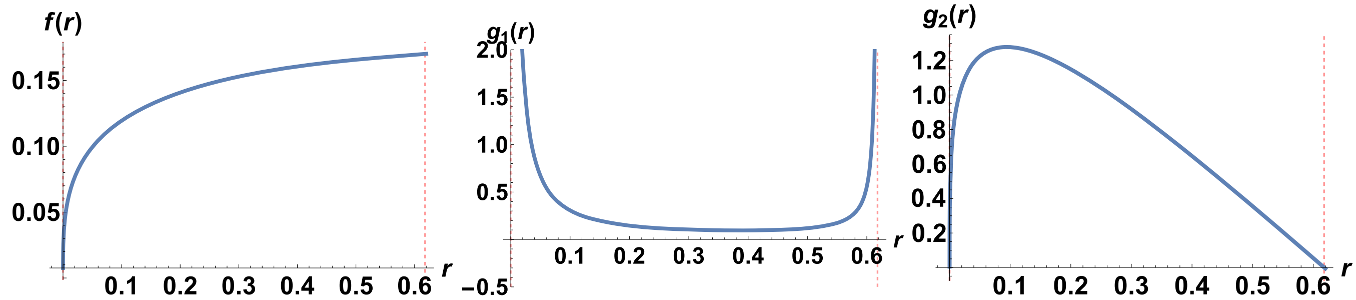
$$B=-\frac{2\sqrt{2}h\sqrt{-\tilde{g}_1bh}}{\sqrt{s}},\qquad m=\frac{\tilde{g}_1}{16h},\qquad \mathcal{C}=32\tilde{g}_1h^2c \qquad (204)$$

and using the solution for σ from Eq. (113), we find the seven-dimensional metric given by

$$ds_7^2=\frac{Br^{3/10}}{16h^2\sqrt{1+m^2r}}\left[ds^2_{\mathrm{AdS}_5}+\frac{r^{-3/2}}{16W(1+m^2r)^{3/2}}dr^2+\frac{16\mathcal{C}^2W}{B}dz^2\right], \qquad (205)$$

with

$$W=B-\sqrt{r(1+m^2r)}. \qquad (206)$$



(a) f solution (b) g_1 solution (c) g_2 solution

FIG. 19. Numerical plots of the warp factors for the $SO(2)_{\mathrm{diag}}$ symmetric solution in the $SO(2,2)$ gauge group with $m=-1$, $B=1$, $h=\frac{1}{2}$, and $\mathcal{C}=-1$. The solution is regular in the range $0<r<r_3=0.618$, with the two vertical red dashed lines representing the two boundaries. (a) f solution, (b) g_1 solution, and (c) g_2 solution.

The scalar ϕ_2 and the gauge field can be written as

$$\phi_2 = -\frac{1}{2}\ln r - \ln[-m] \quad \text{and}$$
$$A_2 = -\frac{1}{8mh}[2|\mathcal{C}|(1+2m^2 r)+q]. \tag{207}$$

Furthermore, since we have chosen $\sigma<0$, the constant $\mathcal{C}$ must be negative. Finally, the Killing spinor reads

$$\eta = Y_0 e^{iqz}\frac{2^{1/8}r^{3/40}}{(1+m^2r)^{\frac{1}{8}}}\begin{pmatrix}\sqrt{\sqrt{B}+[r(1+m^2r)]^{\frac{1}{4}}}\\ -\sqrt{\sqrt{B}-[r(1+m^2r)]^{\frac{1}{4}}}\end{pmatrix}. \tag{208}$$

As in the $SO(2)\times SO(2)$ symmetric solution, there is only one possible range of the radial coordinate r in order to obtain regular solutions. This is given by

$$m<0,\quad B>0,\quad 0<r<r_3,\quad r_3=\frac{-1+\sqrt{1+4B^2m^2}}{2m^2}, \tag{209}$$

with r_3 determined from $W(r_3)=0$. An example of numerical plots for the three warp factors is given in Fig. 19.

As $r\to 0$, the seven-dimensional metric becomes a conformal rescaling of a product between AdS_5 and a cylinder, as in Case I of the $SO(4)$ gauge group. As $r\to r_3$, the metric is approximately given by

$$ds_7^2 \approx \frac{Br_3^{3/10}}{16h^2\sqrt{1+m^2r_3}}\left[ds^2_{\mathrm{AdS}_5} + \frac{dR^2+16\mathcal{C}^2[1+2m^2r_3]^2R^2dz^2}{-4W'(r_3)r_3^{3/2}(1+m^2r_3)^{3/2}}\right], \tag{210}$$

with the new radial coordinate R defined by $R=\sqrt{r_3-r}$. The z circle shrinks smoothly, giving rise to an $\mathbb{R}^2/\mathbb{Z}_l$ orbifold at $r=r_3$ for

$$|\mathcal{C}| = \frac{1}{4l[1+2m^2r_3]} = \frac{1}{4l\sqrt{1+4B^2m^2}},\qquad l=1,2,3,\ldots. \tag{211}$$

With the constant q chosen to be

$$q=-2|\mathcal{C}|[1+2m^2r_2] = -\frac{1}{2l}, \tag{212}$$

the $SO(2)_{\mathrm{diag}}$ gauge field takes the form

$$A_2 = \frac{m|\mathcal{C}|}{2h}(r_3-r). \tag{213}$$

The Killing spinor at the end point $r=r_2$ is given by

$$\eta = 2^{5/8}Y_0 e^{\frac{-iz}{2l}} r_2^{1/5}\begin{pmatrix}1\\0\end{pmatrix}. \tag{214}$$

V. $SO(3,1)$ GAUGE GROUP

In this section, we consider the noncompact $SO(3,1)$ gauge group, which can also be embedded in $SO(3,3)$. The gauge structure constants are chosen to be

$$f_{IJK} = -\tilde{g}(\varepsilon_{ijk},\varepsilon_{rsi}),\qquad r,s,\ldots=1,2,3, \tag{215}$$

with a coupling constant $\tilde{g}$. The $SO(3,1)$ generators can be written as $T_I=(T_i,\hat{T}_r)$, with the explicit form in terms of $SO(3,3)$ generators given by

$$T_1 = J_{23}^{(1)} - J_{23}^{(2)}, \quad T_2 = J_{31}^{(1)} + J_{31}^{(2)}, \quad T_3 = J_{12}^{(1)} - J_{12}^{(2)},$$
$$\hat{T}_1 = Y_{23} + Y_{32}, \quad \hat{T}_2 = Y_{13} - Y_{31}, \quad \hat{T}_3 = -Y_{12} - Y_{21}, \tag{216}$$

satisfying $SO(3,1)$ algebra:

$$[T_i, T_j] = \varepsilon_{ij}{}^k T_k, \quad [T_i, \hat{T}_r] = \varepsilon_{ir}{}^s \hat{T}_s, \quad [\hat{T}_r, \hat{T}_s] = \varepsilon_{rs}{}^i T_i. \tag{217}$$

The T_i's are compact generators of the maximal compact subgroup $SO(3) \subset SO(3,1)$, while $\hat{T}_r$'s are noncompact generators.

We now look for supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions preserving $SO(2) \subset SO(3)$ symmetry by using the same analysis as in the previous two gauge groups. The metric is still given by Eq. (33), and the nonvanishing $SO(2)$ gauge field takes the form of

$$A_{(1)}^3 = A(r)dz. \tag{218}$$

There are three $SO(2)$ singlet scalars, corresponding to the noncompact generators

$$\bar{Y}_1 = Y_{11} - Y_{22}, \quad \bar{Y}_2 = Y_{33}, \quad \bar{Y}_3 = Y_{12} + Y_{21}, \tag{219}$$

and the coset representative is given by

$$L = e^{\phi_1 \bar{Y}_1} e^{\phi_2 \bar{Y}_2} e^{\phi_3 \bar{Y}_3}. \tag{220}$$

With all these, the vector field equation [Eq. (25)] gives rise to

$$\phi_3' = \frac{\phi_1'}{2} \tanh 2\phi_1 \sinh 4\phi_3, \tag{221}$$

$$A'' = -\left[\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma' + 2\phi_2' \tanh 2\phi_2\right] A', \tag{222}$$

$$A'' = -\left[\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma' + 2\phi_2' \coth 2\phi_2\right] A'. \tag{223}$$

Consistency between Eqs. (222) and (223) requires $\phi_2' = 0$ for $A' \neq 0$. We will write the constant ϕ_2 as c. This leads to a single differential equation for A:

$$A'' = -\left[\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma'\right] A, \tag{224}$$

with the solution

$$A' = be^{-\sigma} \sqrt{g_1 g_2} f^{-\frac{5}{2}} \tag{225}$$

for an integration constant b. The dressed field strength tensors are now given by

$$\mathbf{F}_1 = b \cosh c e^{-\sigma} f^{-\frac{5}{2}} \quad \text{and} \quad \mathbf{F}_2 = b \sinh c e^{-\sigma} f^{-\frac{5}{2}}. \tag{226}$$

In this case, nonvanishing conponenets of the C-functions read

$$C = -3\sqrt{2}\tilde{g}(\cosh c - \sinh c \sinh 2\phi_1 \cosh 2\phi_3),$$
$$C^{11} = -C^{22} = -\sqrt{2}\tilde{g} \sinh c \cosh 2\phi_1,$$
$$C^{12} = C^{21} = -\sqrt{2}\tilde{g} \sinh c \sinh 2\phi_1 \sinh 2\phi_3,$$
$$C^{33} = \sqrt{2}\tilde{g}(\sinh c - \cosh c \sinh 2\phi_1 \cosh 2\phi_3). \tag{227}$$

The scalar vielbein and the $SO(2)$ composite connection are given by

$$P_{(1)}^{ir} = \begin{pmatrix} \phi_1' \cosh 2\phi_3 & \phi_3' & 0 \\ \phi_3' & -\phi_1' \cosh 2\phi_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} dr \tag{228}$$

and

$$Q_{(1)}^{ij} = \varepsilon^{ij3}(\phi_1' \sinh 2\phi_3 dr - \tilde{g} A dz). \tag{229}$$

With the supersymmetry parameter [Eq. (A9)] subject to the projector [Eq. (A14)], the supersymmetry transformations $\delta\lambda^{a1} = 0$ and $\delta\lambda^{a2} = 0$ give the same BPS condition, while $\delta\lambda^{a3} = 0$ gives another equation. These equations take the form

$$0 = \tilde{g} \sinh c e^{-\frac{\sigma}{2}}[\cosh 2\phi_1 + i \sinh 2\phi_1 \sinh 2\phi_3](i\sigma^2\eta) + \frac{1}{\sqrt{g_1}}[\phi_1' \cosh 2\phi_3 - i\phi_3']\sigma^3\eta, \tag{230}$$

$$0 = \tilde{g} e^{-\frac{\sigma}{2}}[\sinh c - \cosh c \sinh 2\phi_1 \cosh 2\phi_3]\eta - e^{\frac{\sigma}{2}} b e^{-\sigma} \sinh c f^{-\frac{5}{2}} \sigma^3 \eta. \tag{231}$$

Solving these conditions gives $\phi_1 = 0$, $c = 0$, and $\phi_3' = 0$. Since the constant ϕ_3 does not appear in other equations, we can simply set $\phi_3 = 0$ without losing any generality. However, with $\phi_1 = \phi_2 = \phi_3 = 0$, all matter fields from the vector multiplets vanish. The resulting solution turns out to be the same as the $SO(2)_R$ symmetric solution given in pure $N = 2$ gauged supergravity discussed in Sec. III A 1.

VI. CONCLUSIONS

We have found supersymmetric $\mathrm{AdS}_5 \times \Sigma$ solutions in which Σ is a topological disk with a nontrivial $U(1)$ holonomy at the boundary or a half-spindle from matter-coupled $N = 2$ gauged supergravity in seven dimensions with $SO(4)$ and $SO(2,2)$ gauge groups. These solutions preserve eight supercharges and $SO(2)_R$, $SO(2)\times SO(2)$, and $SO(2)_{\rm diag}$ symmetries. These solutions represent a new class of supersymmetric solutions of $N=2$ gauged supergravity in seven dimensions and might be useful in holographic studies. We have also extensively discussed various possible ranges of the radial coordinate in which the resulting solutions are regular. The $SO(2)_R$ symmetric solution requires the vanishing of all fields from the vector multiplets and can be considered as a solution of pure $N=2$ gauged supergravity with the $SO(3)$ gauge group.

For the $SO(3,1)$ gauge group, we have found only solutions with $SO(2)_R\subset SO(3)_R$ symmetry. Furthermore, we have also considered other possible gauge groups—namely the $SL(3,\mathbb{R})$, $SO(2,1)$, and $SO(2,2)\times SO(2,1)\sim SO(2,1)\times SO(2,1)\times SO(2,1)$ gauge groups. However, all of these gauge groups do not lead to $\mathrm{AdS}_5\times\Sigma$ solutions with nonvanishing fields from vector multiplets. On the other hand, these gauge groups do admit a solution with $SO(2)_R$ symmetry which appears to be a universal solution to all gauge groups.

Similar to the result of Ref. [35], the solutions should be dual to $N=1$ SCFTs in four dimensions obtained from compactifications on a half-spindle of six-dimensional $N=(1,0)$ SCFTs in the case of the $SO(4)$ gauge group or $N=(1,0)$ nonconformal field theory in the case of the $SO(2,2)$ gauge group. Solutions of pure $N=2$ and $SO(4)$ gauged supergravity with equal $SO(3)$ coupling constants, $\tilde{g}_1=\tilde{g}_2$, can be embedded in eleven-dimensional supergravity via consistent truncations constructed in Refs. [57,58]. In this case, the uplifted solutions would describe M5-branes wrapped on a half-spindle as in Ref. [35], but the brane configurations preserve only $\frac{1}{4}$ of the maximal supersymmetry rather than $\frac{1}{2}$. For the $SO(2,2)$ gauge group, an embedding in ten-dimensional type-I or heterotic theories via a truncation on a hyperbolic space $H^{2,2}$ might be obtained by extending the result of Ref. [66] to $SO(2,2)$ gauged supergravity with a nonvanishing topological mass for the three-form field. In this case, the uplifted solutions would describe D5-branes or NS5-branes wrapped on half-spindles.

It would be interesting to explicitly identify the four-dimensional $N=1$ SCFTs that are dual to the supergravity solutions found in this paper. Uplifting the solutions in the $SO(2,2)$ gauge group and in the $SO(4)$ gauge group with different $SO(3)$ coupling constants to ten or eleven dimensions is of particular interest in the holographic context and could lead to new configurations of D5-/NS5-branes or M5-branes wrapped on half-spindles. This could be done along the lines of Ref. [67], in which the embedding of maximal $N=4$ gauged supergravity with various gauge groups has been given by using $SL(5)$ exceptional field theory. In the present case of half-maximal gauged supergravity, the results on $SO(3,n)$ double field theory and nongeometric fluxes in Refs. [56,68–70] would be very useful. We hope to come back to these issues in future works.

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APPENDIX: DERIVATION OF BPS EQUATIONS IN THE $SO(4)$ GAUGE GROUP

In this Appendix, we analyze first-order BPS conditions derived from fermionic supersymmetry transformations. We begin with the first condition, $\delta\psi^a_\mu=0$, along the m, ρ, r, and z directions. These are given, respectively, by

$$0=2\partial_m\epsilon^a-\frac{1}{\sqrt{f}}\Gamma_m\Gamma^{\hat{\rho}}\epsilon^a+\frac{f'}{2f\sqrt{g_1}}\Gamma_m\Gamma^{\hat{r}}\epsilon^a-\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\Gamma_m\epsilon^a+\frac{i}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma_m\Gamma^{\hat{r}\hat{z}}\epsilon^b,\tag{A1}$$

$$0=2\partial_\rho\epsilon^a+\frac{f'}{2f\sqrt{g_1}}\Gamma_\rho\Gamma^{\hat{r}}\epsilon^a-\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\Gamma_\rho\epsilon^a+\frac{i}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma_\rho\Gamma^{\hat{r}\hat{z}}\epsilon^b,\tag{A2}$$

$$0=2\partial_r\epsilon^a-\sqrt{g_1}\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\Gamma^{\hat{r}}\epsilon^a-\frac{4i}{5}\sqrt{g_1}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma^{\hat{z}}\epsilon^b,\tag{A3}$$

$$0=2\partial_z\epsilon^a-\frac{g_2'}{2\sqrt{g_1g_2}}\Gamma^{\hat{r}\hat{z}}\epsilon^a+i\tilde{g}_1A_1(\sigma^3)^a{}_b\epsilon^b-\sqrt{g_2}\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\Gamma^{\hat{z}}\epsilon^a+\frac{4i}{5}\sqrt{g_2}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma^{\hat{r}}\epsilon^b.\tag{A4}$$

The next condition, $\delta\chi^a=0$, yields

$$0=-\frac{\sigma'}{2\sqrt{g_1}}\Gamma^{\hat{r}}\epsilon^a+\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C-\frac{16}{5}he^{2\sigma}\right]\epsilon^a-\frac{i}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma^{\hat{r}\hat{z}}\epsilon^b.\tag{A5}$$

From the supersymmetry transformation of the gaugini, only $\delta\lambda^{a3}=0$ gives rise to nontrivial conditions, given by

$$0=\frac{\phi'}{\sqrt{g_1}}(\sigma^3)^a{}_b\Gamma^{\hat{r}}\epsilon^b-\frac{1}{\sqrt{2}}e^{-\frac{\sigma}{2}}C^{33}(\sigma^3)^a{}_b\epsilon^b+ie^{\frac{\sigma}{2}}\mathbf{F}_2\Gamma^{\hat{r}\hat{z}}\epsilon^a. \tag{A6}$$

In these equations, for convenience, we have introduced the notations $\mathbf{F}_1$ and $\mathbf{F}_2$ for nonvanishing components of the dressed field strength tensors via

$$F^i_{(2)}=L_I{}^iF^I_{(2)}=\mathbf{F}_1\delta^i_3e^{\hat{r}}\wedge e^{\hat{z}}\quad\text{and}\quad F^r_{(2)}=L_I{}^rF^I_{(2)}=\mathbf{F}_2\delta^r_3e^{\hat{r}}\wedge e^{\hat{z}}. \tag{A7}$$

Explicitly, in the case of $SO(2)\times SO(2)$ symmetric solutions, these are given by

$$\mathbf{F}_1=\frac{1}{\sqrt{g_1g_2}}(A_1'\cosh\phi+A_2'\sinh\phi)\quad\text{and}\quad \mathbf{F}_2=\frac{1}{\sqrt{g_1g_2}}(A_1'\sinh\phi+A_2'\cosh\phi). \tag{A8}$$

To solve the resulting BPS equations, we follow Ref. [39] and use the supersymmetry parameters of the form

$$\epsilon^a=n^a\vartheta\otimes\eta. \tag{A9}$$

Here, n^a signifies two components of a constant object in the doublet representation of $SO(3)_R$, while η is a two-component spinor depending on the r and z coordinates. On the other hand, ϑ is a four-component Killing spinor on AdS_5, satisfying

$$\nabla^{AdS_5}_{\hat{\alpha}}\vartheta=\frac{s}{2}\gamma_{\hat{\alpha}}\vartheta, \tag{A10}$$

where $\hat{\alpha}=(\hat{m},\hat{\rho})=0,1,\ldots,4$ is a flat space-time index on AdS_5, $s=\pm1$ is an arbitrary sign, and $\gamma_{\hat{\alpha}}$ are five-dimensional 4×4 gamma matrices satisfying the Clifford algebra

$$\{\gamma_{\hat{\alpha}},\gamma_{\hat{\beta}}\}=2\eta_{\hat{\alpha}\hat{\beta}},\qquad \eta_{\hat{\alpha}\hat{\beta}}=\mathrm{diag}(-++++). \tag{A11}$$

In terms of $\gamma_{\hat{\alpha}}$, we further decompose the seven-dimensional gamma matrices as

$$\Gamma_{\hat{\alpha}}=\gamma_{\hat{\alpha}}\otimes\sigma^3,\quad \Gamma_{\hat{r}}=\mathbb{1}_4\otimes\sigma^1,\quad \Gamma_{\hat{z}}=\mathbb{1}_4\otimes\sigma^2, \tag{A12}$$

in which $\mathbb{1}_n$ is an $n\times n$ identity matrix.

With the supersymmetry parameter in Eq. (A9), the first two BPS equations given in Eqs. (A1) and (A2) reduce to a single equation:

$$\begin{aligned}0&=\frac{s}{\sqrt{f}}(\gamma_{\hat{\alpha}}\otimes\mathbb{1}_2)\epsilon^a+\frac{f'\Gamma_{\hat{\alpha}}\Gamma^{\hat{r}}\epsilon^a}{2f\sqrt{g_1}}-\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\Gamma_{\hat{\alpha}}\epsilon^a\\&\quad+\frac{i}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1(\sigma^3)^a{}_b\Gamma_{\hat{\alpha}}\Gamma^{\hat{r}\hat{z}}\epsilon^b\\&=n^a(\gamma_{\hat{\alpha}}\vartheta)\otimes\left\{\frac{s}{\sqrt{f}}\eta+\frac{f'}{2f\sqrt{g_1}}(i\sigma^2\eta)\right.\\&\quad\left.-\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right](\sigma^3\eta)-\frac{1}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1\eta\right\},\end{aligned} \tag{A13}$$

in which we have expressed all the supersymmetry parameters in terms of two- and four-component spinors and imposed the projector

$$(\sigma^3)^a{}_bn^b=n^a. \tag{A14}$$

By using the same procedure in the remaining conditions, we find the following set of BPS equations on the two-component spinor η:

$$0=\frac{s}{\sqrt{f}}\eta+\frac{f'}{2f\sqrt{g_1}}(i\sigma^2\eta)-\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right](\sigma^3\eta)-\frac{1}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1\eta, \tag{A15}$$

$$0=2\partial_r\eta-\sqrt{g_1}\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right]\sigma^1\eta-\frac{4}{5}\sqrt{g_1}e^{\frac{\sigma}{2}}\mathbf{F}_1(i\sigma^2\eta), \tag{A16}$$

$$\begin{aligned}0&=-2i\partial_z\eta+\tilde{g}_1A_1\eta-\frac{g_2'}{2\sqrt{g_1g_2}}\sigma^3\eta\\&\quad+\sqrt{g_2}\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C+\frac{4}{5}he^{2\sigma}\right](i\sigma^2\eta)\\&\quad+\frac{4}{5}\sqrt{g_2}e^{\frac{\sigma}{2}}\mathbf{F}_1\sigma^1\eta,\end{aligned} \tag{A17}$$

$$0=-\frac{\sigma'}{2\sqrt{g_1}}\sigma^1\eta+\left[\frac{\sqrt{2}}{30}e^{-\frac{\sigma}{2}}C-\frac{16}{5}he^{2\sigma}\right]\eta+\frac{1}{5}e^{\frac{\sigma}{2}}\mathbf{F}_1\sigma^3\eta, \tag{A18}$$

$$0=\frac{\phi'}{\sqrt{g_1}}\sigma^1\eta-\frac{1}{\sqrt{2}}e^{-\frac{\sigma}{2}}C^{33}\eta-e^{\frac{\sigma}{2}}\mathbf{F}_2\sigma^3\eta. \tag{A19}$$

As in Ref. [39], we assume a definite charge under the $U(1)_z$ isometry for the two-component spinor leading to the ansatz for η of the form

$$\eta(r,z)=e^{iqz}\hat{\eta}(r) \tag{A20}$$

with a constant q. With this explicit form of η, it follows that the combination $(-2i\partial_z+\tilde{g}_1A_1)\eta=(2q+\tilde{g}_1A_1)\eta$ in Eq. (A17) is invariant under the transformations

$$A_1 \to A_1 - \frac{2\alpha_0}{\tilde{g}_1}, \qquad \eta \to e^{i\alpha_0 z}\eta, \tag{A21}$$

where α_0 is an arbitrary constant. It is then convenient to define

$$\tilde{g}_1 \hat{A}_1 = 2q + \tilde{g}_1 A_1 \tag{A22}$$

with $\hat{A}_1' = A_1'$. With suitable left-multiplications by Pauli matrices and additions of Eq. (A18) to Eqs. (A15) and (A17), we find the following set of algebraic equations:

$$0 = \frac{s}{\sqrt{f}}\eta + \frac{1}{2\sqrt{g_1}}\left[\frac{f'}{f} - \sigma'\right](i\sigma^2\eta) - 4he^{2\sigma}\sigma^3\eta, \tag{A23}$$

$$0 = \frac{\tilde{g}_1\hat{A}_1}{\sqrt{g_2}}\sigma^1\eta + \frac{1}{2\sqrt{g_1}}\left[\frac{g_2'}{g_2} - \sigma'\right](i\sigma^2\eta) - 4he^{2\sigma}\sigma^3\eta + e^{\frac{\sigma}{2}}\mathbf{F}_1\eta, \tag{A24}$$

$$0 = \frac{5\sigma'}{2\sqrt{g_1}}\sigma^3\eta - e^{-\frac{\sigma}{2}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\sigma}{2}}\right](i\sigma^2\eta) + e^{\frac{\sigma}{2}}\mathbf{F}_1\sigma^1\eta, \tag{A25}$$

$$0 = \frac{\phi'}{\sqrt{g_1}}\sigma^3\eta - \frac{1}{\sqrt{2}}e^{-\frac{\sigma}{2}}C^{33}(i\sigma^2\eta) + e^{\frac{\sigma}{2}}\mathbf{F}_2\sigma^1\eta. \tag{A26}$$

These equations are of the form $M^{(x)}\eta = 0$, where $x = 1, 2, 3, 4$ label the BPS equations in Eqs. (A23)–(A26), respectively. The four 2×2 matrices $M^{(x)}$ can be parametrized by

$$M^{(x)} = X_0^{(x)}\mathbb{1}_2 + X_1^{(x)}\sigma^1 + X_2^{(x)}(i\sigma^2) + X_3^{(x)}\sigma^3. \tag{A27}$$

Following [39], for each matrix $M^{(x)}$, we define the two-component vectors

$$v^{(x)} = \begin{pmatrix} X_1^{(x)} + X_2^{(x)} \\ -X_0^{(x)} - X_3^{(x)} \end{pmatrix}, \qquad w^{(x)} = \begin{pmatrix} X_0^{(x)} - X_3^{(x)} \\ -X_1^{(x)} + X_2^{(x)} \end{pmatrix} \tag{A28}$$

together with

$$\mathcal{A}^{xy} = \det(v^{(x)}|w^{(y)}), \qquad \mathcal{B}^{xy} = \det(v^{(x)}|v^{(y)}),$$
$$\mathcal{C}^{xy} = \det(w^{(x)}|w^{(y)}). \tag{A29}$$

The notation $(a|b)$ denotes a 2×2 matrix obtained from a juxtaposition of the two-column vectors a and b. As pointed out in Ref. [39], the vanishing of $\mathcal{A}^{xy}$, $\mathcal{B}^{xy}$, and $\mathcal{C}^{xy}$ gives a number of necessary conditions for the existence of a nontrivial solution for η. We will separately determine the conditions from the supergravity and vector multiplets by splitting the index $x = (\bar{x}, 4)$ with $\bar{x} = 1, 2, 3$.

Starting from the first matrix $\mathcal{A}^{xy}$, the vanishing of the diagonal components $\mathcal{A}^{\bar{x}\bar{x}}$ gives the following conditions:

$$0 = \frac{1}{f} - 16h^2e^{4\sigma} + \frac{1}{4g_1}\left[\frac{f'}{f} - \sigma'\right]^2, \tag{A30}$$

$$0 = e^{\sigma}(\mathbf{F}_1)^2 + \frac{1}{4g_1}\left[\frac{g_2'}{g_2} - \sigma'\right]^2 - 16h^2e^{4\sigma} - \frac{(\tilde{g}_1\hat{A}_1)^2}{g_2}, \tag{A31}$$

$$0 = \frac{25\sigma'^2}{4g_1} + e^{\sigma}(\mathbf{F}_1)^2 - e^{-\sigma}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\sigma}{2}}\right]^2. \tag{A32}$$

The vanishing of the off-diagonal symmetric components $\mathcal{A}^{\bar{x}\bar{y}} + \mathcal{A}^{\bar{y}\bar{x}}$ ($\bar{x} \neq \bar{y}$) yields

$$0 = \frac{1}{4g_1}\left[\frac{f'}{f} - \sigma'\right]\left[\frac{g_2'}{g_2} - \sigma'\right] + \frac{se^{\frac{\sigma}{2}}}{\sqrt{f}}\mathbf{F}_1 - 16h^2e^{4\sigma}, \tag{A33}$$

$$0 = 20he^{2\sigma}\sigma' - e^{-\frac{\sigma}{2}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\sigma}{2}}\right]\left[\frac{f'}{f} - \sigma'\right], \tag{A34}$$

$$0 = \frac{20he^{2\sigma}\sigma'}{\sqrt{g_1}} - \frac{e^{-\frac{\sigma}{2}}}{\sqrt{g_1}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\sigma}{2}}\right]\left[\frac{g_2'}{g_2} - \sigma'\right] - \frac{2\tilde{g}_1e^{\frac{\sigma}{2}}}{\sqrt{g_2}}\hat{A}_1\mathbf{F}_1, \tag{A35}$$

while the off-diagonal antisymmetric components $\mathcal{A}^{\bar{x}\bar{y}} - \mathcal{A}^{\bar{y}\bar{x}} = 0$ give

$$0 = 8he^{\frac{5\sigma}{2}}\mathbf{F}_1 - \frac{8she^{2\sigma}}{\sqrt{f}} + \frac{\tilde{g}_1}{\sqrt{g_1g_2}}\left[\frac{f'}{f} - \sigma'\right]\hat{A}_1, \tag{A36}$$

$$0 = e^{\frac{\sigma}{2}}\left[\frac{f'}{f} - \sigma'\right]\mathbf{F}_1 + \frac{5s}{\sqrt{f}}\sigma', \tag{A37}$$

$$0 = \frac{e^{\sigma}}{\sqrt{g_1}}\left[\frac{g_2'}{g_2} + 4\sigma'\right]\mathbf{F}_1 + \frac{2\tilde{g}_1}{\sqrt{g_2}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right]\hat{A}_1. \tag{A38}$$

Matrices $\mathcal{B}^{xy}$ and $\mathcal{C}^{xy}$ are both antisymmetric in xy indices. It turns out that the resulting BPS conditions arising from the combinations $\mathcal{B}^{xy} \pm \mathcal{C}^{xy} = 0$ take a simpler form. The combinations $\mathcal{B}^{\bar{x}\bar{y}} + \mathcal{C}^{\bar{x}\bar{y}} = 0$ give the following conditions:

$$0 = \frac{e^{\frac{\sigma}{2}}}{\sqrt{g_1}}\left[\frac{f'}{f} - \sigma'\right]\mathbf{F}_1 - \frac{s}{\sqrt{fg_1}}\left[\frac{g_2'}{g_2} - \sigma'\right] + \frac{8\tilde{g}_1he^{2\sigma}}{\sqrt{g_2}}\hat{A}_1, \tag{A39}$$

$$0 = 4he^{3\sigma}\mathbf{F}_1 + \frac{s}{\sqrt{f}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right], \tag{A40}$$

$$0 = \frac{5\tilde{g}_1\sigma'}{\sqrt{g_1 g_2}}\hat{A}_1 + 2\left[\frac{C}{3\sqrt{2}} - 12he^{\frac{5\phi}{2}}\right]\mathbf{F}_1, \quad \text{(A41)}$$

while the combinations $\mathcal{B}^{\bar{x}\bar{y}} - \mathcal{C}^{\bar{x}\bar{y}} = 0$ lead to

$$0 = \frac{2he^{2\sigma}}{\sqrt{g_1}}\left[\frac{f'}{f} - \frac{g_2'}{g_2}\right] + \frac{s\tilde{g}_1}{\sqrt{f g_2}}\hat{A}_1, \quad \text{(A42)}$$

$$0 = \frac{2se^{\frac{\sigma}{2}}}{\sqrt{f}}\mathbf{F}_1 - \frac{5\sigma'}{2g_1}\left[\frac{f'}{f} - \sigma'\right] + 8he^{\frac{3\sigma}{2}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right], \quad \text{(A43)}$$

$$0 = 2e^{\sigma}(\mathbf{F}_1)^2 - \frac{5\sigma'}{2g_1}\left[\frac{g_2'}{g_2} - \sigma'\right] + 8he^{\frac{3\sigma}{2}}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right]. \quad \text{(A44)}$$

There are, in total, 15 algebraic conditions obtained from the BPS equations of the supergravity multiplet in Eqs. (A23)–(A25).

Extending this procedure to the BPS equation (A26) from vector multiplets gives additional BPS conditions derived from the vanishing of $\mathcal{A}^{x4}$, $\mathcal{B}^{\bar{x}4}$, and $\mathcal{C}^{\bar{x}4}$. The first condition obtained from $\mathcal{A}^{44} = 0$ takes the form

$$0 = \frac{(\phi')^2}{g_1} + e^{\sigma}(\mathbf{F}_2)^2 - \frac{e^{-\sigma}}{2}(C^{33})^2. \quad \text{(A45)}$$

The symmetric part $\mathcal{A}^{\bar{x}4} + \mathcal{A}^{4\bar{x}} = 0$ gives

$$0 = 16he^{\frac{5\sigma}{2}}\phi' - \sqrt{2}C^{33}\left[\frac{f'}{f} - \sigma'\right], \quad \text{(A46)}$$

$$0 = \frac{16he^{\frac{5\sigma}{2}}\phi'}{\sqrt{g_1}} - \frac{\sqrt{2}C^{33}}{\sqrt{g_1}}\left[\frac{g_2'}{g_2} - \sigma'\right] - \frac{4\tilde{g}_1 e^{\sigma}}{\sqrt{g_2}}\mathbf{F}_2\hat{A}_1, \quad \text{(A47)}$$

$$0 = \frac{5\sigma'\phi'}{g_1} - \sqrt{2}e^{-\sigma}C^{33}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right] + 2e^{\sigma}\mathbf{F}_1\mathbf{F}_2, \quad \text{(A48)}$$

while the antisymmetric part $\mathcal{A}^{\bar{x}4} - \mathcal{A}^{4\bar{x}} = 0$ results in

$$0 = \frac{2s\phi'}{\sqrt{f}} + e^{\frac{\sigma}{2}}\mathbf{F}_2\left[\frac{f'}{f} - \sigma'\right], \quad \text{(A49)}$$

$$0 = \frac{2e^{\sigma}\phi'}{\sqrt{g_1}}\mathbf{F}_1 + \frac{e^{\sigma}}{\sqrt{g_1}}\left[\frac{g_2'}{g_2} - \sigma'\right]\mathbf{F}_2 + \frac{\sqrt{2}\tilde{g}_1 C^{33}}{\sqrt{g_2}}\hat{A}_1, \quad \text{(A50)}$$

$$0 = C^{33}\mathbf{F}_1 - \sqrt{2}\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right]\mathbf{F}_2. \quad \text{(A51)}$$

Moreover, the combinations $\mathcal{B}^{\bar{x}4} + \mathcal{C}^{\bar{x}4} = 0$ and $\mathcal{B}^{\bar{x}4} - \mathcal{C}^{\bar{x}4} = 0$ lead to the following sets of extra conditions, respectively:

$$0 = \frac{2\tilde{g}_1\phi'}{\sqrt{g_1 g_2}}\hat{A}_1 + \sqrt{2}C^{33}\mathbf{F}_1 + 8he^{\frac{5\sigma}{2}}\mathbf{F}_2, \quad \text{(A52)}$$

$$0 = 2\phi'\mathbf{F}_1 - 5\sigma'\mathbf{F}_2, \quad \text{(A53)}$$

$$0 = \frac{\sqrt{2}sC^{33}}{\sqrt{f}} + 8he^{3\sigma}\mathbf{F}_2 \quad \text{(A54)}$$

and

$$0 = \frac{\phi'}{g_1}\left[\frac{f'}{f} - \sigma'\right] - \frac{2se^{\frac{\sigma}{2}}}{\sqrt{f}}\mathbf{F}_2 - 4\sqrt{2}he^{\frac{3\sigma}{2}}C^{33}, \quad \text{(A55)}$$

$$0 = \frac{\phi'}{g_1}\left[\frac{g_2'}{g_2} - \sigma'\right] - 2e^{\sigma}\mathbf{F}_1\mathbf{F}_2 - 4\sqrt{2}he^{\frac{3\sigma}{2}}C^{33}, \quad \text{(A56)}$$

$$0 = 5\sqrt{2}\sigma'C^{33} - 4\phi'\left[\frac{C}{3\sqrt{2}} - 16he^{\frac{5\phi}{2}}\right]. \quad \text{(A57)}$$

Finally, the BPS equation [Eq. (A16)] will be used to determine the explicit form of $\eta(r,z)$.

We now turn to the vector fields and determine the explicit forms of A_1' and A_2' which are relevant for solving the BPS conditions obtained previously. With only the $SO(2)\times SO(2)$ singlet scalar being nonvanishing, we find that $*P^{ir}_{(1)}f_{IJ}{}^{K}L_r{}^{I}L_{Ki} = 0$. Together with $C_{(3)} = 0$, the field equation [Eq. (25)] reduces to

$$A_1'' = -2\phi'A_2' - \left(\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma'\right)A_1', \quad \text{(A58)}$$

$$A_2'' = -2\phi'A_1' - \left(\frac{5f'}{2f} - \frac{g_1'}{2g_1} - \frac{g_2'}{2g_2} + \sigma'\right)A_2'. \quad \text{(A59)}$$

The most general solution to these equations is given by

$$A_1' = \frac{e^{-\sigma-2\phi}}{2}(a_1 + a_2 e^{4\phi})\sqrt{g_1 g_2}f^{-\frac{5}{2}} \quad \text{and}$$
$$A_2' = \frac{e^{-\sigma-2\phi}}{2}(a_1 - a_2 e^{4\phi})\sqrt{g_1 g_2}f^{-\frac{5}{2}}, \quad \text{(A60)}$$

where a_1 and a_2 are constants.

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