Worldsheet free fields, higher spin symmetry, and free $\mathcal{N}=4$ super-Yang-Mills theory

Changhyun Ahn *

Department of Physics, Kyungpook National University, Taegu 41566, Korea

(Received 22 September 2021; accepted 2 March 2022; published 18 March 2022)

By using the free field worldsheet realization described by Gaberdiel and Gopakumar recently, we construct the nontrivial lowest generators of the higher spin superalgebra $hs(2,2|4)$. They consist of cubic terms between the bilinears of ambitwistorlike fields. We also obtain the worldsheet description for the findings of Sezgin and Sundell twenty years ago given by the familiar oscillator construction. The first order poles of the operator product expansions (OPEs), between the conformal weight-1 generators of Lie superalgebra $PSU(2,2|4)$ and the above conformal weight-3 generators of $hs(2,2|4)$, are determined explicitly and the additional generators appear in the worldsheet theory.

DOI: 10.1103/PhysRevD.105.066006

I. INTRODUCTION

Gaberdiel and Gopakumar have described the worldsheet description for the $\mathrm{AdS}_5 \times S^5$ string theory dual to free four dimensional $\mathcal{N}=4$ super Yang-Mills theory in [1]. Their free field description is related to the ambitwistor string theory and the finite set of generalized zero modes (or wedge modes) in each spectrally flowed sector are physical. Furthermore, they impose some residual gauge constraints on the Fock space generated by these wedge oscillators, and demonstrate the matching of the physical spectrum of the string theory with that of free $\mathcal{N}=4$ super Yang-Mills theory at the planar level [2]. See also the relevant works in [3–6] where the tensionless string theory on $\mathrm{AdS}_3 \times S^3$, in the worldsheet theory with free fields, is studied.

At vanishing gauge coupling constant, the Lie superalgebra $PSU(2,2|4)$ of $\mathcal{N}=4$ super Yang-Mills theory gets enhanced to the higher spin superalgebra $hs(2,2|4)$. The fundamental unitary irreducible representation of $hs(2,2|4)$ is the singleton with vanishing central charge [7–10]. The symmetric tensor product of two singletons yields the massless AdS_5 higher spin gauge fields. The physical fields after gauging are organized by the "levels" $l = 0,1,2,\ldots,\infty$ of $PSU(2,2|4)$ multiplets [11,12]. See also the original paper [13] used in [11]. In particular, the level $l=0$ multiplet is the five dimensional $\mathcal{N}=8$ gauged supergravity multiplet [14] and the $hs(2,2|4)$ generators depending on the $U(1)$ charge are classified by the levels explicitly. See also some relevant papers on the construction of the composite operators built out of the singleton [15–17]. Moreover, the spectrum of single trace operators in the free $\mathcal{N}=4$ super Yang-Mills theory can be decomposed into the irreducible representations of the $hs(2,2|4)$ [18]. See also [19].

As pointed out by [1,2], the worldsheet realization provides the familiar oscillator construction [7] by considering each pair of modes of the free fields. In this paper, we would like to determine the worldsheet realization for the higher spin generators found in [11]. The first nontrivial case appears when the level becomes $l=1$ and the higher spin generators consist of the cubic terms between the bilinears of ambitwistorlike fields in the worldsheet approach by counting the number of oscillators [11,18]. Then the generators of $PSU(2,2|4)$ have the conformal weight-1 while the higher spin generators of $hs(2,2|4)$ have the conformal weight-3. We will obtain the complete expressions for the higher spin generators of $hs(2,2|4)$ for the level $l=1$ by using the standard operator product expansions (OPEs) in two dimensional conformal field theory.[1]

In Sec. II, we review the free field construction of the worldsheet theory in [1,2], express the $PSU(2,2|4)$ explicitly and the stress energy tensor is described.

In Sec. III, we obtain the lowest higher spin generators of $hs(2,2|4)$ by using the free field construction with the help of two dimensional conformal field theory.

*ahn@knu.ac.kr

[1] See Maldacena's comment on Gopakumar's talk in strings 2021.

In Sec. IV, we write down the complete first order poles from the OPEs between the generators of $PSU(2,2|4)$ and those of $hs(2,2|4)$.

In Sec. V, we summarize the main results of this paper and the future directions of related works are given.

In the Appendix, some details of the previous sections are presented explicitly.

II. REVIEW

A. Free fields

We consider the weight-$\frac{1}{2}$ conjugate pairs of symplectic boson [20] fields $(\lambda^{\alpha}, \mu^{\dagger}_{\alpha})$ and $(\mu^{\dot{\alpha}}, \lambda^{\dagger}_{\dot{\alpha}})$ where $\alpha, \dot{\alpha} = 1,\, 2$ and four weight-$\frac{1}{2}$ complex fermions $(\psi^{a}, \psi^{\dagger}_{a})$ where $a = 1,$ 2, 3, 4 [1,2]. The α and $\dot{\alpha}$ are spinor indices with respect to two different $SU(2)$'s and ψ^a transforms in the fundamental representation of $SU(4)$. Note that the conformal dimension-$\frac{1}{2}$ fields, $(\lambda^{\alpha}, \mu^{\dagger}_{\alpha})$ and $(\mu^{\dot{\alpha}}, \lambda^{\dagger}_{\dot{\alpha}})$, are bosonic and they satisfy "quasi" statistics. We will follow most of the notations presented in [1,2].

Their nontrivial operator product expansions (OPEs) in the left-moving sector of the worldsheet theory we are describing are given by

$$\begin{aligned}
\lambda^{\alpha}(z)\mu^{\dagger}_{\beta}(w) &= \frac{1}{(z-w)}\delta^{\alpha}_{\beta} + \cdots, \\
\mu^{\dot{\alpha}}(z)\lambda^{\dagger}_{\dot{\beta}}(w) &= \frac{1}{(z-w)}\delta^{\dot{\alpha}}_{\dot{\beta}} + \cdots, \\
\psi^{a}(z)\psi^{\dagger}_{b}(w) &= \frac{1}{(z-w)}\delta^{a}_{b} + \cdots.
\end{aligned} \tag{2.1}$$

The abbreviated parts in (2.1) are the regular terms as usual in two dimensional conformal field theory. By introducing the components of ambitwistor fields [21]

$$Z^I \equiv (\lambda^{\alpha}, \mu^{\dot{\alpha}}, \psi^{a}), \qquad Y_J \equiv (\mu^{\dagger}_{\alpha}, \lambda^{\dagger}_{\dot{\alpha}}, \psi^{\dagger}_{a}), \tag{2.2}$$

we can rewrite the above three OPEs (2.1) as a single one [22] alternatively

$$Z^I(z)Y_J(w) = \frac{1}{(z-w)}\delta^I_J + \cdots. \tag{2.3}$$

The upper and lower indices I, J stand for $\alpha, \dot{\alpha}$, and a. For the calculations of any OPEs containing the multiple of ambitwistor fields (2.2), it is useful to use (2.3) rather than (2.1) and after that we can specify the indices $I, J, K \cdots$ of these from (2.2) later.[2]

[2]If we interchange the order of the OPE in (2.3), then we have $Y_J(z)Z^I(w) = \frac{1}{(z-w)}(-1)^{d_I d_J + 1}\delta^I_J + \cdots$ where the grading $d_I = 2$ for the bosonic fields and $d_I = 1$ for the fermionic fields [23–26]. In other words, the additional factor $(-1)^{d_I d_J}$ arises. Note that the components $Z^a = \psi^a$ and $Y_a = \psi^{\dagger}_a$ are fermionic.

By constructing the quadratic terms [1,2,21]

$$J^I{}_J \equiv Y_J Z^I, \tag{2.4}$$

the current algebra version of the oscillator construction [7] of Lie superalgebra $U(2,2|4)$ can be described by (i) the generators of Lorentz symmetry, $\mathcal{L}^{\alpha}{}_{\beta}$ and $\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}$, (ii) the generator of R symmetry, $\mathcal{R}^a{}_b$, (iii) the generators of super translations, $\mathcal{Q}^{\alpha}{}_{a}$, $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}$, and $\mathcal{P}^{\dot{\alpha}}{}_{\beta}$. Moreover, the $\mathcal{N} = 4$ super Poincaré algebra obtained by these generators can be enlarged by the generators of super conformal boosts, $\mathcal{S}^{\alpha}{}_{a}, \dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}$, and $\mathcal{K}^{\alpha}{}_{\dot{\beta}}$. There exist also the $U(1)$ hyper charge $\mathcal{B}$, the central charge $\mathcal{C}$ and the dilatation generator $\mathcal{D}$. Then the generators [27] of Lie superalgebra $U(2,2|4)$ can be extended by the following generators in terms of ambitwistor fields [1,2]

$$\begin{aligned}
&\mathcal{L}^{\alpha}{}_{\beta} = Y_{\beta}Z^{\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta}Y_{\gamma}Z^{\gamma}, \qquad \dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}} = Y_{\dot{\beta}}Z^{\dot{\alpha}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}Y_{\dot{\gamma}}Z^{\dot{\gamma}}, \\
&\mathcal{R}^{a}{}_{b} = Y_{b}Z^{a} - \frac{1}{4}\delta^{a}_{b}Y_{c}Z^{c}, \\
&\mathcal{Q}^{\alpha}{}_{a} = Y_{a}Z^{\alpha}, \qquad \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a} = Y_{a}Z^{\dot{\alpha}}, \qquad \mathcal{P}^{\dot{\alpha}}{}_{\beta} = Y_{\beta}Z^{\dot{\alpha}}, \\
&\mathcal{S}^{\alpha}{}_{a} = Y_{a}Z^{\alpha}, \qquad \dot{\mathcal{S}}^{a}{}_{\dot{\alpha}} = Y_{\dot{\alpha}}Z^{a}, \qquad \mathcal{K}^{\alpha}{}_{\dot{\beta}} = Y_{\dot{\beta}}Z^{\alpha}, \\
&\mathcal{B} = \frac{1}{2}(Y_{\alpha}Z^{\alpha} + Y_{\dot{\alpha}}Z^{\dot{\alpha}}), \qquad \mathcal{C} = \frac{1}{2}(Y_{\alpha}Z^{\alpha} + Y_{\dot{\alpha}}Z^{\dot{\alpha}} + Y_{a}Z^{a}), \\
&\mathcal{D} = \frac{1}{2}(Y_{\alpha}Z^{\alpha} - Y_{\dot{\alpha}}Z^{\dot{\alpha}}).
\end{aligned} \tag{2.5}$$

As usual, the repeated indices are summed over the corresponding indices. As noted in [1,2], each pair of modes of the free fields provides two copies of the usual oscillator construction. Therefore, once we restrict to the zero modes of (2.5) in their (anti)commutator relations, the known Lie superalgebra $U(2,2|4)$ [27] can be obtained. We present their complete OPEs in Appendix A in the worldsheet theory.[3]

It is useful to introduce the following $U(1)$ generators which appear in the above $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ generators

$$\mathcal{U} \equiv Y_{\gamma}Z^{\gamma}, \qquad \dot{\mathcal{U}} \equiv Y_{\dot{\gamma}}Z^{\dot{\gamma}}, \qquad \mathcal{V} \equiv Y_{c}Z^{c}. \tag{2.6}$$

Note that the $\mathcal{V}$ appears in the second term of $\mathcal{R}^a{}_b$ in (2.5) which is traceless: $\mathcal{R}^a{}_a = 0$.

[3]We use the Thielemans package [28] with a *Mathematica* [29]. Note that the group indices $\alpha, \dot{\alpha}$ and a are fixed. All the coefficients appearing in the right hand sides of the OPEs are numerical values. Once we identify the group index structures both sides of the OPEs, then it is straightforward to calculate all these coefficients inside a Package explicitly due to the free fields.

In particular, the nonzero $\mathcal{V}$-charge for $\mathcal{Q}^{\alpha}{}_{a}$ is equal to -1 and the nonzero $\mathcal{V}$-charge for $\dot{\mathcal{Q}}^{\alpha}{}_{a}$ is equal to 1 from the observation of Eq. (A2). This corresponds to Y-charge in [11] up to sign. By simply counting the number of supersymmetry generators in the multiple product of the generators of (2.5), we can determine the $\mathcal{V}$-charge. The remaining ten generators have vanishing $\mathcal{V}$-charges.

Note that the ordering of two operators in (2.4) or (2.5) is important because sometimes we will have additional minus sign when we interchange the ambitwistor fields each other.

B. The Lie superalgebra $PSU(2,2|4)$

We can calculate the OPEs between the conformal weight-1 currents in (2.4) by using the defining relation in (2.3) with the help of the footnote and it turns out that

$$J^I{}_J(z)J^K{}_L(w) = -\frac{1}{(z-w)^2}(-1)^{d_J d_K}\delta^I_L\delta^K_J + \frac{1}{(z-w)}[\delta^I_L J^K{}_J + (-1)^{(d_L+d_K)(d_I+d_J)+1}\delta^K{}_J J^I{}_L](w) + \cdots. \quad (2.7)$$

The grading d_I is defined in the footnote 2. We can also check, from (2.7), that the second order pole of the OPE between $J^+ \equiv \mathcal{L}^1{}_2$, $J^- \equiv \mathcal{L}^2{}_1$ and $J^3 \equiv \frac{1}{2}(\mathcal{L}^2{}_2 - \mathcal{L}^1{}_1)$ implies that the level is equal to -1. Similarly, the OPE between $\dot{J}^+ \equiv \dot{\mathcal{L}}^{\dot{1}}{}_{\dot{2}}$, $\dot{J}^- \equiv \dot{\mathcal{L}}^{\dot{2}}{}_{\dot{1}}$ and $\dot{J}^3 \equiv \frac{1}{2}(\dot{\mathcal{L}}^{\dot{2}}{}_{\dot{2}} - \dot{\mathcal{L}}^{\dot{1}}{}_{\dot{1}})$ leads to the fact that the level is also equal to -1. We obtain Appendix A from this defining relation (2.7) by specifying the indices explicitly. The OPEs between the $U(1)$ generator $\mathcal{C}$ appearing in (2.5) and other generators of $U(2,2|4)$ do not have any singular terms in Eq. (A1) except the OPE $\mathcal{B}(z)\mathcal{C}(w)$. We are left with $PSU(2,2|4)$ after the $U(1)$ generator $\mathcal{C}$ is "quotiented" [1,2].

We can calculate the OPEs between the single $J^I{}_J(z)$ and the quadratic term $J^K{}_L J^M{}_N(w)$ and the OPEs between the single $J^I{}_J(z)$ and the cubic term $J^K{}_L J^M{}_N J^P{}_Q(w)$ but we do not present them in this paper because they have long expressions due to the presence of various gradings. Later we will present the first order pole of the latter explicitly in next section.

C. The stress energy tensor

By requiring that the ambitwistor fields (2.2) are weight-$\frac{1}{2}$ primary and the generators (2.5) are weight-1 primary (See also the footnote 4), we can determine the stress energy tensor from the possible quadratic terms from (2.5) completely and it is given by

$$T = \frac{1}{2}(\lambda^\alpha \partial\mu^\dagger_\alpha + \mu^{\dot\alpha}\partial\lambda^\dagger_{\dot\alpha} - \psi^a\partial\psi^\dagger_a - \partial\lambda^\alpha\mu^\dagger_\alpha - \partial\mu^{\dot\alpha}\lambda^\dagger_{\dot\alpha} + \partial\psi^a\psi^\dagger_a)$$
$$= \frac{1}{2}(-1)^{d_I}(Z^I\partial Y_I - \partial Z^I Y_I). \quad (2.8)$$

As before, the repeated indices are summed. Note that there is an additional factor for the grading when we change the order between the ambitwistor fields in the second expression of (2.8). This stress energy tensor satisfies the usual standard OPE $T(z)T(w)$ and the central charge is equal to zero. We will use the explicit expression (2.8) in order to calculate the possible (quasi)primary operators in next section.[4]

In this section, we summarize the extension of the Lie superalgebra $PSU(2,2|4)$ generated by (2.5) in the worldsheet theory. Implicitly it is given by (2.7) or explicitly it is also given by Eq. (A1). If we focus on the zero modes for these generators, then this will lead to the standard (anti) commutator relations [27].

III. CONSTRUCTION OF THE LOWEST GENERATORS OF THE HIGHER SPIN SUPERALGEBRA $hs(2,2|4)$

We would like to construct the worldsheet description for the higher spin generators of $hs(2,2|4)$ found in [11,12]. We have seen the conformal weight-1 generators which are primary under the stress energy tensor (2.8). According to the results of [11], the nontrivial lowest generators consist of cubic terms in the above weight-1 generators corresponding to the level $l = 1$ case (For $l = 0$ case, they are linear in the weight-1 generators while for $l = 2$ case they are quintic in the weight-1 generators).

We observe that the $(2l+1)$ can be identified with the conformal dimension (or weight or spin) under (2.8) in the worldsheet theory.

From the conformal field theory analysis [30–32], it is known that in the OPE between the weight-1 operator (which is a primary) and the weight-3 (quasi)primary operator, in principle, there appear a (new) weight-1 operator in the third order pole and a (new) weight-2 operator in the second order pole. By simple counting the relative coefficients for the descendant operators of these operators which will appear in the second and first order poles, they do not appear in the first order pole.

[4]Therefore, we have $T(z)Z^I(w) = \frac{1}{(z-w)^2}\frac{1}{2}Z^I(w) + \frac{1}{(z-w)}\partial Z^I(w) + \cdots$, $T(z)Y_I(w) = \frac{1}{(z-w)^2}\frac{1}{2}Y_I(w) + \frac{1}{(z-w)}\partial Y_I(w) + \cdots$, $T(z)J^I{}_J(w) = \frac{1}{(z-w)^2}J^I{}_J(w) + \frac{1}{(z-w)}\partial J^I{}_J(w) + \cdots$ and from these we can calculate the following OPE $T(z)J^I{}_J J^K{}_L(w) = \frac{1}{(z-w)^4}(-1)^{d_J d_K+1}\delta^I_L\delta^K_J + \frac{1}{(z-w)^3}[\delta^I_L J^K{}_J + (-1)^{(d_L+d_K)(d_I+d_J)+1}\times \delta^K_J J^I{}_L](w) + \frac{1}{(z-w)^2}2J^I{}_J J^K{}_L(w) + \frac{1}{(z-w)}\partial(J^I{}_J J^K{}_L)(w) + \cdots$ which implies that this does not produce the (quasi)primary operator in general. We can check whether this is really (quasi) primary or not after specifying the indices explicitly.

Therefore, we will focus on the first order pole in the OPE between the weight-1 operator and the weight-3 operator. This first order pole provides a new (quasi) primary operators. In doing this, we should check that the weight-3 operator should be (quasi) primary. That is, at least the third order pole of the OPE between the stress energy tensor and this weight-3 operator should vanish.

We have the following first order pole in the OPE between $J^I{}_J(z)$ and $J^K{}_L J^M{}_N J^P{}_Q(w)$ by using (2.7) successively as follows:

$$\begin{aligned} J^I{}_j(z) J^K{}_L J^M{}_N J^P{}_Q(w)|_{\frac{1}{(z-w)}} &= \delta^I_L J^K{}_J J^M{}_N J^P{}_Q(w) + (-1)^{(d_I+d_K)(d_I+d_J)+1} \delta^K_J J^I{}_L J^M{}_N J^P{}_Q(w) \\ &+ (-1)^{(d_I+d_J)(d_K+d_L)} J^K{}_L [\delta^I_N J^M{}_J J^P{}_Q + (-1)^{(d_N+d_M)(d_I+d_J)+1} \delta^M_J J^I{}_N J^P{}_Q \\ &+ (-1)^{(d_I+d_J)(d_N+d_M)} \delta^I_Q J^M{}_N J^P{}_J + (-1)^{(d_I+d_J)(d_N+d_M+d_P+d_Q)+1} \delta^P_J J^M{}_N J^I{}_Q](w). \quad (3.1) \end{aligned}$$

Let us emphasize that the right-hand side of (3.1) is a (quasi) primary operator as before as long as the third order pole of Eq. (B1) vanishes. We obtain all the information on the higher spin generators in this section from this (implicit) OPE (3.1) by imposing the explicit indices on (3.1). In other words, the first order pole can be written in terms of the known operators by collecting them appropriately or if not, then there appears in the new (quasi)primary operator. We do not have to subtract the contributions from the descendant operators as we mentioned before. Of course, there are also fourth, third and second order poles in the above OPE.

We focus on the Tables 4 and 5 of [11] with $l = 1$ case and $s = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ and 4. Their l is related to the numbers of bosonic and fermionic oscillators and is given by the Eq. (3.6) in [11] and their s is related to the numbers of bosonic oscillators and is given around Eq. (3.18) in [11]. Furthermore, their Eq. (3.19) contains all the information on the above two tables although it is not easy to read off the relevant quantities properly.[5]

A. The $s = 1$ case: $\mathbf{1}_0$ and $\mathbf{15}_0$

Because their X appearing in Eq. (2.4) in [11] corresponds to our $\mathcal{V}$ up to sign and normalization, we can observe that the $SU(4)$ singlet is a cubic in $\mathcal{V}$ which has vanishing $\mathcal{V}$-charge from Eq. (A2). Moreover, the $SU(4)$ nonsinglet contains the quadratic in X and we can identify this as a quadratic in $\mathcal{V}$ together with $\mathcal{R}^a{}_b$ which is a **15** representation of $SU(4)$. Note that by construction of (2.5), we observe the fact that $\mathcal{R}^a{}_a$ vanishes. In the tensor product of $\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15}$ [34,35], after subtracting the $\mathcal{V}$ part, we are left with the representation **15**. Once again, the $\mathcal{V}$-charge in the cubic of $\mathcal{V}\mathcal{V}\mathcal{R}^a{}_b$ vanishes.

Therefore we identify the following higher spin generators corresponding to the representations $\mathbf{1}_0$ and $\mathbf{15}_0$ respectively as follows[6]:

$$\begin{aligned} \mathcal{W} &\equiv \mathcal{V}\mathcal{V}\mathcal{V}, \\ \mathcal{W}^a{}_b &\equiv \mathcal{V}\mathcal{V}\mathcal{R}^a{}_b + \mathcal{V}\mathcal{R}^a{}_b\mathcal{V} + \mathcal{R}^a{}_b\mathcal{V}\mathcal{V}. \end{aligned} \quad (3.2)$$

We can check these higher spin generators in (3.2) are quasiprimary operators under the stress energy tensor (2.8). In other words, the OPEs between the stress energy tensor and these generators contain nonzero fourth order poles although the third order poles become zero according to Eq. (B1) by specifying the indices correctly.

Although the OPE between $\mathcal{V}$ and $\mathcal{R}^a{}_b$ is regular and they are commuting operators (the second and the third terms in the right-hand side of $\mathcal{W}^a{}_b$ are the same as the first one), we will keep its form in symmetrical way as in (3.2). When we act the supersymmetry generators on the $\mathcal{W}^a{}_b$, then we will observe that each three terms contributes differently due to the normal ordering.

In next subsections, we will determine the remaining higher spin generators by acting the supersymmetry generators $\mathcal{Q}^a{}_\alpha$ and $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a$ on (3.2) successively.

B. The $s = \frac{3}{2}$ case: $\mathbf{4}_{-1}$, $\bar{\mathbf{4}}_1$, $\mathbf{20}_{-1}$, and $\overline{\mathbf{20}}_1$

Now we move on the next column of the Table 4 with $l = 1$ of [11]. Eventually we will present all the first order poles in the OPEs between some weight-1 operators and the weight-3 operators in next section with Appendix C. However, in this section, we will focus on some of them which determine the higher spin generators completely. One way to determine these particular higher spin generators is to consider that we can calculate the first order pole in the OPE between the supersymmetry generator $\mathcal{Q}^a{}_\alpha$ which is fermionic and $\mathcal{W}^b{}_c$ which is introduced in previous subsection (3.2). Either we can use Eq. (A1) or

[5]For $l = 0$ in Table 4 of [11], there are generators $\mathcal{R}^a{}_b$, $\mathcal{Q}^\alpha{}_a$, $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a$ and $\mathcal{P}^{\dot{\alpha}}{}_\beta$ corresponding to $\mathbf{15}_0$, $\mathbf{4}_{-1}$, $\bar{\mathbf{4}}_1$ and $\mathbf{1}_0$ respectively. It is easy to see that they are closed by themselves in Eq. (A1). In the oscillator construction, the remaining generators of $PSU(2,2|4)$ acting on the physical vacuum state vanish [27,33]. We will calculate the OPEs between these weight-1 operators including the $U(1)$ operator $\mathcal{V}$ relevant to $\mathcal{R}^a{}_b$ and the weight-3 operators in next section. The algebra from these five weight-1 operators is closed.

[6]We denote the higher spin generators as the letter $\mathcal{W}$ with appropriate group indices. For the additional new higher spin generators we put a hat on $\mathcal{W}$ with some indices.

the previous OPE (3.1) can be used by selecting the corresponding indices for this particular OPE.

It turns out that by antisymmetrizing the upper indices[7]

$$\mathcal{Q}^{[a}{}_{\alpha}(z)\mathcal{W}^{b]}{}_{c}(w)|_{\frac{1}{(z-w)}} = \mathcal{W}^{[ab]}{}_{c\alpha}(w) + \delta_c^{[a}\mathcal{W}^{b]}{}_{\alpha}(w) - \frac{1}{4}\delta_c^{[b}\mathcal{W}^{a]}{}_{\alpha}(w), \quad (3.3)$$

where the right-hand side of (3.3) consists of two kinds of higher spin generators as follows:

$$\mathcal{W}^{a}{}_{\alpha} \equiv \mathcal{V}\mathcal{V}\mathcal{Q}^{a}{}_{\alpha} + \mathcal{V}\mathcal{Q}^{a}{}_{\alpha}\mathcal{V} + \mathcal{Q}^{a}{}_{\alpha}\mathcal{V}\mathcal{V},$$
$$\mathcal{W}^{[ab]}{}_{c\alpha} \equiv \mathcal{V}\mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{c} + \mathcal{Q}^{[a}{}_{\alpha}\mathcal{V}\mathcal{R}^{b]}{}_{c} + \mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{c}\mathcal{V} + \mathcal{V}\mathcal{R}^{[b}{}_{c}\mathcal{Q}^{a]}{}_{\alpha} + \mathcal{R}^{[b}{}_{c}\mathcal{V}\mathcal{Q}^{a]}{}_{\alpha} + \mathcal{R}^{[b}{}_{c}\mathcal{Q}^{a]}{}_{\alpha}\mathcal{V}. \quad (3.4)$$

Note that the first one in (3.4) is a quasiprimary operator while the second one in (3.4) is a primary operator according to Eq. (B1). Note that the second one is antisymmetric in the upper indices. As mentioned before, the weight-1 operator $\mathcal{Q}^{a}{}_{\alpha}$ has nontrivial OPE with $\mathcal{V}$ [See also Eq. (A2)] and the ordering between them is not trivial and if we interchange them, there appears a derivative term of weight-1 operator. The quasiprimary condition of the first operator requires all of three terms (this is the reason why we have three terms in (3.2)) and we can easily observe that the first operator corresponds to the representation $\mathbf{4}_{-1}$ because it contains a single weight-1 operator which has $\mathcal{V}$-charge -1 (Of course, the $\mathcal{V}$-charge of $\mathcal{V}$ is equal to zero) and it has upper index a which transforms as a fundamental representation of $SU(4)$.

In the tensor product of $\bar{\mathbf{6}}\otimes\bar{\mathbf{4}} = \mathbf{20}\oplus\mathbf{4}$ [34,35], we obtain the representation $\mathbf{20}$ by subtracting the fundamental representation $\mathbf{4}$. The second higher spin generator in (3.4) consists of the upper antisymmetric combination and the lower antifundamental one. Therefore, in total, it provides the tensor product $\bar{\mathbf{6}}\otimes\bar{\mathbf{4}}$. Now we consider the contracted one which is given by $\mathcal{W}^{ab}{}_{a\alpha}$ which transforms as a fundamental representation $\mathbf{4}$ of $SU(4)$. Then after subtracting this representation from $\bar{\mathbf{6}}\otimes\bar{\mathbf{4}}$, we will eventually obtain the representation $\mathbf{20}_{-1}$. Furthermore, it has $\mathcal{V}$-charge -1 also because there exists a single $\mathcal{Q}^{a}{}_{\alpha}$ and the operator $\mathcal{R}^{a}_{b}$ has a vanishing $\mathcal{V}$-charge. Note that the expression without the antisymmetric bracket in the second higher spin generator in (3.4) is itself a primary operator and it is obvious to see that the higher spin generator $\mathcal{W}^{ab}{}_{a\alpha}$ also transforms as a primary operator after taking antisymmetric combination.

Therefore, we should consider the particular antisymmetric combination in the OPE of (3.3). Without it, we would not obtain the corresponding right higher spin generator which transforms properly. In other words, the antisymmetric combination in the indices a and b is crucial for the presence of the representation $\mathbf{20}_{-1}$ in the oscillator construction in [11].[8]

C. The $s=2$ case: $\mathbf{1}_0,\mathbf{15}_0,\mathbf{20}'_0,\mathbf{6}_{-2},\mathbf{6}_2,\mathbf{10}_{-2}$, and $\overline{\mathbf{10}}_2$

Let us consider the next column of the Tables 4 and 5 with $l=1$ of [11]. Again, we can use either (3.1) or Eq. (A1). We can calculate the OPEs between the supersymmetry generators and the higher spin generators found in previous subsection.

It turns out, from (3.4), that we have

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\mathcal{W}^{b}{}_{\beta}(w)|_{\frac{1}{(z-w)}} = \delta_a^b\mathcal{W}^{\dot{\alpha}}{}_{\beta}(w) + \hat{\mathcal{W}}^{b\dot{\alpha}}{}_{a\beta}, \quad (3.5)$$

where the right hand side of (3.5) contains the following higher spin generators

$$\mathcal{W}^{\dot{\alpha}}{}_{\beta} \equiv \mathcal{V}\mathcal{V}\mathcal{P}^{\dot{\alpha}}{}_{\beta} + \mathcal{V}\mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{V} + \mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{V}\mathcal{V},$$
$$\hat{\mathcal{W}}^{b\dot{\alpha}}{}_{a\alpha} \equiv \mathcal{Q}^{b}{}_{\alpha}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{V} + \mathcal{Q}^{b}{}_{\alpha}\mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a} + \mathcal{V}\mathcal{Q}^{b}{}_{\alpha}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a} - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{Q}^{b}{}_{\alpha}\mathcal{V} - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{V}\mathcal{Q}^{b}{}_{\alpha} - \mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{Q}^{b}{}_{\alpha}. \quad (3.6)$$

Compared with the previous OPE, there is no (anti) symmetric combination in the $SU(4)$ indices. The first higher spin generator of (3.6) is a quasiprimary operator by using Eq. (B1). Because there is no $SU(4)$ index, the $\mathcal{V}$-charge vanishes and moreover the quadratic expression in $\mathcal{V}$ arises from the oscillator construction, we can identify this as $\mathbf{1}_0$ in [11].[9]

Let us look at the second higher spin generator in (3.6) which is a primary operator under the stress energy tensor (2.8). We can view this as the tensor product of the representation $\mathbf{4}$ corresponding to the upper index and the representation $\bar{\mathbf{4}}$ corresponding to the lower index and moreover its $\mathcal{V}$-charge vanishes because there appear two

[7]In this paper, the (anti)symmetric notations are for $SU(4)$ indices. The bracket [] stands for antisymmetric one and the bracket () stands for symmetric one without any overall numerical factors.

[8]Similarly, we obtain $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}(z)\mathcal{W}^{b}{}_{c]}(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{b\dot{\alpha}}{}_{[ac]}(w) - \delta^b_{[a}\dot{\mathcal{W}}^{\dot{\alpha}}{}_{c]}(w) + \frac{1}{4}\delta^b_{[c}\dot{\mathcal{W}}^{\dot{\alpha}}{}_{a]}(w)$, where the right-hand side has the following higher spin generators $\dot{\mathcal{W}}^{\dot{\alpha}}{}_{a} \equiv \mathcal{V}\mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a} + \mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{V} + \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}\mathcal{V}\mathcal{V}$ corresponding to the representation $\bar{\mathbf{4}}_1$, and $\dot{\mathcal{W}}^{b\dot{\alpha}}{}_{[ac]} \equiv \mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{R}^{b}{}_{c]} + \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{V}\mathcal{R}^{b}{}_{c]} + \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{R}^{b}{}_{c]}\mathcal{V} + \mathcal{V}\mathcal{R}^{b}{}_{[c}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]} + \mathcal{R}^{b}{}_{[c}\mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]} + \mathcal{R}^{b}{}_{[c}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}\mathcal{V}$ corresponding to the representation $\overline{\mathbf{20}}_1$ from the analysis of the tensor product $\mathbf{6}\otimes\mathbf{4} = \overline{\mathbf{20}}\oplus\bar{\mathbf{4}}$.

[9]Note that the OPEs between $\mathcal{P}^{\dot{\alpha}}{}_{\beta}$ and the weight-1 operators are regular except $\mathcal{L}^{\alpha}{}_{\beta}$, $\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}$, $\mathcal{D}$, $\mathcal{S}^{\alpha}{}_{a}$, $\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}$, $\mathcal{K}^{\alpha}{}_{\dot{\beta}}$, $\mathcal{U}$, and $\dot{\mathcal{U}}$ from Eq. (A1). In other words, the OPEs between $\mathcal{P}^{\dot{\alpha}}{}_{\beta}$ and the five weight-1 operators appearing in the footnote 5 do not have the singular terms.

kinds of supersymmetry generators. We do not find this higher spin generator from the Tables 4 and 5 of [11]. As mentioned before, we put a hat on this generator because this is a new primary operator.[10]

Let us move on the following first order pole in the OPE between the supersymmetry generator and the second higher spin generator in (3.4) after antisymmetrizing for the lower two indices

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}(z)\mathcal{W}^{[bc]}{}_{d]\beta}(w)|_{\frac{1}{(z-w)}}=\delta^{[b}_{[a}\mathcal{W}^{c]\dot{\alpha}}{}_{d]\beta}(w)+\mathcal{W}^{[bc]\dot{\alpha}}{}_{[ad]\beta}(w)+\delta^{[c}_{[a}\hat{\mathcal{W}}^{b]\dot{\alpha}}{}_{d]\beta}(w)-\frac{1}{4}\delta^{[c}_{[d}\hat{\mathcal{W}}^{b]\dot{\alpha}}{}_{a]\beta}(w), \tag{3.7}$$

where the right-hand side of (3.7) contains the following higher spin generators together with the previous operator in (3.6)

$$\begin{aligned}\mathcal{W}^{a\dot{\alpha}}{}_{b\beta} &\equiv \mathcal{V}\mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{R}^{a}{}_{b}+\mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{V}\mathcal{R}^{a}{}_{b}+\mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{R}^{a}{}_{b}\mathcal{V}\\ &\quad+\mathcal{V}\mathcal{R}^{a}{}_{b}\mathcal{P}^{\dot{\alpha}}{}_{\beta}+\mathcal{R}^{a}{}_{b}\mathcal{V}\mathcal{P}^{\dot{\alpha}}{}_{\beta}+\mathcal{R}^{a}{}_{b}\mathcal{P}^{\dot{\alpha}}{}_{\beta}\mathcal{V},\\ \mathcal{W}^{[ab]\dot{\alpha}}{}_{[cd]\alpha} &\equiv \mathcal{Q}^{[a}{}_{\alpha}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\mathcal{R}^{b]}{}_{d]}+\mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{[d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]}\\ &\quad+\mathcal{R}^{[b}{}_{[d}\mathcal{Q}^{a]}{}_{\alpha}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]}-\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{d]}\\ &\quad-\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\mathcal{R}^{b]}{}_{d]}\mathcal{Q}^{a]}{}_{\alpha}-\mathcal{R}^{[b}{}_{[d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]}\mathcal{Q}^{a]}{}_{\alpha}.\end{aligned} \tag{3.8}$$

We can easily identify the first operator of (3.8) which is a primary as the representation $\mathbf{15}_0$. We have already observed that the weight-1 operator $\mathcal{R}^a{}_b$ transforms as this representation under the $SU(4)$. Moreover, there is a single $\mathcal{V}$ in this expression (again from the result of [11]) and it is obvious that the $\mathcal{V}$-charge is equal to zero.

There are two antisymmetric combinations between the upper indices and lower indices from the second operator of (3.8). It is known that in $SU(4)$, we have $\mathbf{6}=\bar{\mathbf{6}}$. In the tensor product of $\mathbf{6}\otimes\mathbf{6}=\mathbf{1}\oplus\mathbf{15}\oplus\mathbf{20}'$ [34,35], after subtracting the first two representations, we obtain the representation $\mathbf{20}'$. That is, we observe that when we contract one index from $\mathcal{W}^{[ab]\dot{\alpha}}{}_{[cd]\alpha}$, then the representation $\mathbf{15}$ corresponds to $\mathcal{W}^{[ab]\dot{\alpha}}{}_{[ad]\alpha}$. Further contraction will give us $\mathcal{W}^{[ab]\dot{\alpha}}{}_{[ab]\alpha}$ which has a representation $\mathbf{1}$. Therefore, we obtain the representation $\mathbf{20}'$ by restricting to these two conditions. It is easy to see that the $\mathcal{V}$-charge vanishes. We can check this operator is a primary under the stress energy tensor (2.8).[11]

We continue to analyze the next higher spin generators which have nonzero $\mathcal{V}$-charges. We can calculate the following OPE and obtain the first order pole, from (3.4), as follows:

$$\mathcal{Q}^{a}{}_{\alpha}(z)\mathcal{W}^{[bc]}{}_{a\beta}(w)|_{\frac{1}{(z-w)}}=\mathcal{W}^{[bc]a}{}_{a\beta\alpha}(w)-\frac{15}{4}\hat{\mathcal{W}}^{[bc]}{}_{\beta\alpha}(w), \tag{3.9}$$

where the right-hand side of (3.9) consists of the following higher spin generators

$$\begin{aligned}\mathcal{W}^{[ab]c}{}_{c\alpha\beta} &\equiv -\mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{c}\mathcal{Q}^{c}{}_{\beta}-\mathcal{R}^{[b}{}_{c}\mathcal{Q}^{a]}{}_{\alpha}\mathcal{Q}^{c}{}_{\beta}-\mathcal{Q}^{[a}{}_{\alpha}\mathcal{Q}^{c}{}_{\beta}\mathcal{R}^{b]}{}_{c}\\ &\quad+\mathcal{Q}^{c}{}_{\beta}\mathcal{R}^{[b}{}_{c}\mathcal{Q}^{a]}{}_{\alpha}+\mathcal{R}^{[b}{}_{c}\mathcal{Q}^{c}{}_{\beta}\mathcal{Q}^{a]}{}_{\alpha}+\mathcal{Q}^{c}{}_{\beta}\mathcal{Q}^{[a}{}_{\alpha}\mathcal{R}^{b]}{}_{c},\\ \hat{\mathcal{W}}^{[ab]}{}_{\alpha\beta} &\equiv \mathcal{V}\mathcal{Q}^{[a}{}_{\alpha}\mathcal{Q}^{b]}{}_{\beta}+\mathcal{Q}^{[a}{}_{\alpha}\mathcal{V}\mathcal{Q}^{b]}{}_{\beta}+\mathcal{Q}^{[a}{}_{\alpha}\mathcal{Q}^{b]}{}_{\beta}\mathcal{V}\\ &\quad-\mathcal{V}\mathcal{Q}^{[b}{}_{\beta}\mathcal{Q}^{a]}{}_{\alpha}-\mathcal{Q}^{[b}{}_{\beta}\mathcal{V}\mathcal{Q}^{a]}{}_{\alpha}-\mathcal{Q}^{[b}{}_{\beta}\mathcal{Q}^{a]}{}_{\alpha}\mathcal{V}.\end{aligned} \tag{3.10}$$

Note that the upper and lower index a is summed in the left-hand side of the OPE of (3.9). We can identify the first operator of (3.10) as $\mathbf{6}_{-2}$ because the two upper $SU(4)$ indices are antisymmetric together with the contraction for other two and due to the two supersymmetric generators, the $\mathcal{V}$-charge becomes -2 as before. On the other hands, the second operator of (3.10), which has also $\mathcal{V}$-charge -2 and consists of the tensor product of $\mathbf{4}$ and $\mathbf{4}$ (again $\mathbf{6}_{-2}$) of $SU(4)$, can be regarded as a new primary operator which is not present in [11]. We can check this is a primary operator from Eq. (B1). As done before, we can obtain the conjugated version of (3.9) with the footnote and there exists a relevant generator.[12]

Finally, by considering the following OPE from (3.4) we determine the higher spin generator having nonzero $\mathcal{V}$-charge, after symmetrizing the upper indices,

$$\mathcal{Q}^{(a}{}_{\alpha}(z)\mathcal{W}^{b)}{}_{\beta}(w)|_{\frac{1}{(z-w)}}=\mathcal{W}^{(ab)}{}_{\alpha\beta}(w), \tag{3.11}$$

where the right-hand side of (3.11) can be written as

$$\begin{aligned}\mathcal{W}^{(ab)}{}_{\alpha\beta} &\equiv \mathcal{V}\mathcal{Q}^{(a}{}_{\alpha}\mathcal{Q}^{b)}{}_{\beta}+\mathcal{Q}^{(a}{}_{\alpha}\mathcal{V}\mathcal{Q}^{b)}{}_{\beta}+\mathcal{Q}^{(a}{}_{\alpha}\mathcal{Q}^{b)}{}_{\beta}\mathcal{V}\\ &\quad-\mathcal{V}\mathcal{Q}^{(b}{}_{\beta}\mathcal{Q}^{a)}{}_{\alpha}-\mathcal{Q}^{(b}{}_{\beta}\mathcal{V}\mathcal{Q}^{a)}{}_{\alpha}-\mathcal{Q}^{(b}{}_{\beta}\mathcal{Q}^{a)}{}_{\alpha}\mathcal{V}.\end{aligned} \tag{3.12}$$

It is obvious to see that this (3.12), which is a primary, has the representation $\mathbf{10}_{-2}$ from the symmetric combination of the upper two indices. Simple counting of $\mathcal{V}$-charge implies

[10] We have similar relation $\mathcal{Q}^{a}{}_{\alpha}(z)\dot{\mathcal{W}}^{\dot{\beta}}{}_{b}(w)|_{\frac{1}{(z-w)}}=\delta^{a}_{b}\mathcal{W}^{\dot{\beta}}{}_{\alpha}(w)+\hat{\mathcal{W}}^{a\dot{\beta}}{}_{b\alpha}(w)$ where the generators of right-hand side are given by (3.6). The $SU(4)$ indices appear separately.

[11] The conjugated version of (3.7) appears as follows: $\mathcal{Q}^{[a}{}_{\alpha}(z)\dot{\mathcal{W}}^{c]\dot{\beta}}{}_{[bd]}(w)|_{\frac{1}{(z-w)}}=\delta^{[a}_{[b}\mathcal{W}^{c]\dot{\beta}}{}_{d]\alpha}(w)+\mathcal{W}^{[ac]\dot{\beta}}{}_{[bd]\alpha}(w)-\delta^{[a}_{[d}\hat{\mathcal{W}}^{c]\dot{\beta}}{}_{b]\alpha}(w)-\frac{1}{4}\delta^{[c}_{[d}\hat{\mathcal{W}}^{a]\dot{\beta}}{}_{b]\alpha}(w)$ together with the footnote 8, and the relations (3.6) and (3.8).

[12] That is, we have the following first order pole, from the footnote 8, $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\dot{\mathcal{W}}^{a\dot{\beta}}{}_{[bd]}(w)|_{\frac{1}{(z-w)}}=-\dot{\mathcal{W}}^{a\dot{\beta}\dot{\alpha}}{}_{[bd]a}(w)+\frac{15}{4}\dot{\hat{\mathcal{W}}}^{\dot{\beta}\dot{\alpha}}{}_{[bd]}(w)$ where $\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{[ab]c}\equiv-\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{R}^{c}{}_{b]}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}-\mathcal{R}^{c}{}_{[b}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}-\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}\mathcal{R}^{c}{}_{b]}+\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}\mathcal{R}^{c}{}_{[b}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}+\mathcal{R}^{c}{}_{[b}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}+\dot{\mathcal{Q}}^{\dot{\beta}}{}_{c}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{R}^{c}{}_{b]}$ corresponding to the representation $\bar{\mathbf{6}}_2$ and the new higher spin generator $\dot{\hat{\mathcal{W}}}^{\dot{\alpha}\dot{\beta}}{}_{[ab]}\equiv\mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{b]}+\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\mathcal{V}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{b]}+\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{b]}\mathcal{V}-\mathcal{V}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{[b}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}-\dot{\mathcal{Q}}^{\dot{\beta}}{}_{[b}\mathcal{V}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}-\dot{\mathcal{Q}}^{\dot{\beta}}{}_{[b}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a]}\mathcal{V}$ which transforms as $\bar{\mathbf{6}}_2$.

that this higher spin generator has -2. Furthermore, it has linear dependence of $\mathcal{V}$ as in [11].[13]

D. The $s=\frac{5}{2}$ case: $\mathbf{4}_{-1}$,$\bar{\mathbf{4}}_1$,$\bar{\mathbf{4}}_{-3}$,$\mathbf{4}_3$,$\mathbf{20}_{-1}$, and $\overline{\mathbf{20}}_1$

From now on, all the higher spin generators can be related to the corresponding multiplets in the Table 3 of [11]. In the previous three cases, there are some mismatches between the Table 3 and the Tables 4 and 5 of [11]. As done in previous subsection, we compute the following OPE from (3.8) and focus on the first order pole, after antisymmetrizing the upper indices,

$$\mathcal{Q}^{[a}{}_{\beta}(z)\mathcal{W}^{b]\dot\alpha}{}_{c\gamma}(w)|_{\frac{1}{(z-w)}} = \mathcal{W}^{[ab]\dot\alpha}{}_{c\beta\gamma}(w) + \delta_c^{[a}\mathcal{W}^{b]\dot\alpha}{}_{\beta\gamma} - \frac{1}{4}\delta_c^{[b}\mathcal{W}^{a]\dot\alpha}{}_{\beta\gamma}(w), \tag{3.13}$$

where the right-hand side of (3.13) provides the following higher spin generators

$$\begin{aligned}\mathcal{W}^{a\dot\alpha}{}_{\beta\gamma} \equiv{}& \mathcal{V}\mathcal{Q}^a{}_\beta\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{Q}^a{}_\beta\mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{Q}^a{}_\beta\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V} \\ &+ \mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{Q}^a{}_\beta + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V}\mathcal{Q}^a{}_\beta + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{Q}^a{}_\beta\mathcal{V}, \\ \mathcal{W}^{[ab]\dot\alpha}{}_{c\beta\gamma} \equiv{}& \mathcal{Q}^{[a}{}_\beta\mathcal{R}^{b]}{}_c\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{R}^{[b}{}_c\mathcal{Q}^{a]}{}_\beta\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{Q}^{[a}{}_\beta\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{R}^{b]}{}_c \\ &+ \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{R}^{[b}{}_c\mathcal{Q}^{a]}{}_\beta + \mathcal{R}^{[b}{}_c\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{Q}^{a]}{}_\beta + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{Q}^{[a}{}_\beta\mathcal{R}^{b]}{}_c.\end{aligned} \tag{3.14}$$

We can see that the first generator of (3.14) has the representation $\mathbf{4}_{-1}$ with $\mathcal{V}$-charge -1. For the second generator of (3.14), there are two upper antisymmetric indices with a single lower index. We have seen the similar structure around (3.4). As long as the $SU(4)$ representation with $\mathcal{V}$-charge is concerned, there is no difference whether there is a factor $\mathcal{V}$ in (3.4) or $\mathcal{P}^{\dot\alpha}{}_\beta$ in (3.14). This implies that the above generator transforms as the representation $\mathbf{20}_{-1}$ by subtracting the trace part (with a contraction in the indices) with $\mathcal{V}$-charge -1. They are primary under the stress energy tensor.[14]

The next case can be obtained from the following OPE result by using the higher spin generator (3.10) properly (complete antisymmetrization of the upper indices)

$$\mathcal{Q}^{[a}{}_\alpha(z)\mathcal{W}^{bcd]}{}_{d\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\delta_d^{[a}\mathcal{W}^{bcd]}{}_{\beta\gamma\alpha}(w) + \frac{1}{4}\delta_d^{[d}\mathcal{W}^{bca]}{}_{\beta\gamma\alpha}(w), \tag{3.15}$$

where the right-hand side of (3.15) contains the following higher spin generator

$$\begin{aligned}\mathcal{W}^{[abc]}{}_{\alpha\beta\gamma} \equiv{}& \mathcal{Q}^{[a}{}_\alpha\mathcal{Q}^b{}_\beta\mathcal{Q}^{c]}{}_\gamma + \mathcal{Q}^{[c}{}_\gamma\mathcal{Q}^a{}_\alpha\mathcal{Q}^{b]}{}_\beta + \mathcal{Q}^{[b}{}_\beta\mathcal{Q}^c{}_\gamma\mathcal{Q}^{a]}{}_\alpha \\ &- \mathcal{Q}^{[a}{}_\alpha\mathcal{Q}^c{}_\gamma\mathcal{Q}^{b]}{}_\beta - \mathcal{Q}^{[b}{}_\beta\mathcal{Q}^a{}_\alpha\mathcal{Q}^{c]}{}_\gamma - \mathcal{Q}^{[c}{}_\gamma\mathcal{Q}^b{}_\beta\mathcal{Q}^{a]}{}_\alpha.\end{aligned} \tag{3.16}$$

First of all, the $\mathcal{V}$-charge of (3.16) is given by -3. From the tensor product of $\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}$ [34,35] due to the three upper indices, we obtain the following decomposition $\bar{\mathbf{4}}\oplus\overline{\mathbf{20}}\oplus\overline{\mathbf{20}}\oplus\overline{\mathbf{20}''}$. Then by taking the totally antisymmetric combination of the indices, the representation $\bar{\mathbf{4}}_{-3}$ with $\mathcal{V}$-charge can be obtained and we can check this (3.16) is a primary operator.[15]

E. The $s=3$ case: $\mathbf{1}_0$,$\mathbf{15}_0$,$\mathbf{6}_{-2}$, and $\mathbf{6}_2$

Now we analyze the following OPE, from the previous result in (3.14),

$$\dot{\mathcal{Q}}^{\dot\alpha}{}_a(z)\mathcal{W}^{b\dot\beta}{}_{\gamma\delta}(w)|_{\frac{1}{(z-w)}} = \delta_a^b\mathcal{W}^{\dot\alpha\dot\beta}{}_{\gamma\delta}(w) + \hat{\mathcal{W}}^{b\dot\beta\dot\alpha}{}_{a\gamma\delta}(w), \tag{3.17}$$

where the right-hand side of (3.17) has the following higher spin generators

$$\begin{aligned}\mathcal{W}^{\dot\alpha\dot\beta}{}_{\gamma\delta} \equiv{}& \mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{P}^{\dot\beta}{}_\delta + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V}\mathcal{P}^{\dot\beta}{}_\delta + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{P}^{\dot\beta}{}_\delta\mathcal{V} \\ &+ \mathcal{V}\mathcal{P}^{\dot\beta}{}_\delta\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{P}^{\dot\beta}{}_\delta\mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{P}^{\dot\beta}{}_\delta\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V}, \\ \hat{\mathcal{W}}^{a\dot\alpha\dot\beta}{}_{b\gamma\delta} \equiv{}& \mathcal{Q}^a{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_b\mathcal{P}^{\dot\alpha}{}_\delta + \mathcal{Q}^a{}_\gamma\mathcal{P}^{\dot\alpha}{}_\delta\dot{\mathcal{Q}}^{\dot\beta}{}_b + \mathcal{P}^{\dot\alpha}{}_\delta\mathcal{Q}^a{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_b \\ &- \dot{\mathcal{Q}}^{\dot\beta}{}_b\mathcal{Q}^a{}_\gamma\mathcal{P}^{\dot\alpha}{}_\delta - \dot{\mathcal{Q}}^{\dot\beta}{}_b\mathcal{P}^{\dot\alpha}{}_\delta\mathcal{Q}^a{}_\gamma - \mathcal{Q}^a{}_\gamma\mathcal{P}^{\dot\alpha}{}_\delta\dot{\mathcal{Q}}^{\dot\beta}_b.\end{aligned} \tag{3.18}$$

For the first generator of (3.18), there is no $SU(4)$ index and the $\mathcal{V}$-charge is equal to zero. Then we can identify this with the representation $\mathbf{1}_0$. For the second generator, the $\mathcal{V}$-charge vanishes also and it is given by the tensor product

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[13]We obtain $\dot{\mathcal{Q}}^{\dot\alpha}{}_{(a}(z)\dot{\mathcal{W}}^{\dot\beta}{}_{b)}(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{\dot\alpha\dot\beta}{}_{(ab)}(w)$ together with $\dot{\mathcal{W}}^{\dot\alpha\dot\beta}{}_{(ab)} \equiv \mathcal{V}\dot{\mathcal{Q}}^{\dot\alpha}{}_{(a}\dot{\mathcal{Q}}^{\dot\beta}{}_{b)} + \dot{\mathcal{Q}}^{\dot\alpha}{}_{(a}\mathcal{V}\dot{\mathcal{Q}}^{\dot\beta}{}_{b)} + \dot{\mathcal{Q}}^{\dot\alpha}{}_{(a}\dot{\mathcal{Q}}^{\dot\beta}{}_{b)}\mathcal{V} - \mathcal{V}\dot{\mathcal{Q}}^{\dot\beta}{}_{(b}\dot{\mathcal{Q}}^{\dot\alpha}{}_{a)} - \dot{\mathcal{Q}}^{\dot\beta}{}_{(b}\mathcal{V}\dot{\mathcal{Q}}^{\dot\alpha}{}_{a)} - \dot{\mathcal{Q}}^{\dot\beta}{}_{(b}\dot{\mathcal{Q}}^{\dot\alpha}{}_{a)}\mathcal{V}$ corresponding to $\overline{\mathbf{10}}_2$.

[14]We can determine the similar OPE, by antisymmetrizing the lower indices, $\dot{\mathcal{Q}}^{\dot\beta}{}_{[a}(z)\mathcal{W}^{b\dot\alpha}{}_{c]\gamma}(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{b\dot\beta\dot\alpha}{}_{[ac]\gamma}(w) - \delta^b_{[a}\dot{\mathcal{W}}^{\dot\beta\dot\alpha}{}_{c]\gamma} + \frac{1}{4}\delta^b_{[c}\dot{\mathcal{W}}^{\dot\beta\dot\alpha}{}_{c]\gamma}(w)$ with two higher spin generators $\dot{\mathcal{W}}^{\dot\beta\dot\alpha}{}_{a\gamma} \equiv \mathcal{V}\dot{\mathcal{Q}}^{\dot\beta}{}_a\mathcal{P}^{\dot\alpha}{}_\gamma + \dot{\mathcal{Q}}^{\dot\beta}{}_a\mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma + \dot{\mathcal{Q}}^{\dot\beta}{}_a\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V} + \mathcal{V}\mathcal{P}^{\dot\alpha}{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_a + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{V}\dot{\mathcal{Q}}^{\dot\beta}{}_a + \mathcal{P}^{\dot\alpha}{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_a\mathcal{V}$ transforming as $\bar{\mathbf{4}}_1$ and $\dot{\mathcal{W}}^{b\dot\beta\dot\alpha}{}_{[ac]\gamma} \equiv \dot{\mathcal{Q}}^{\dot\beta}{}_{[a}\mathcal{R}^b{}_{c]}\mathcal{P}^{\dot\alpha}{}_\gamma + \mathcal{R}^b{}_{[c}\dot{\mathcal{Q}}^{\dot\beta}{}_{a]}\mathcal{P}^{\dot\alpha}{}_\gamma + \dot{\mathcal{Q}}^{\dot\beta}{}_{[a}\mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{R}^b{}_{c]} + \mathcal{P}^{\dot\alpha}{}_\gamma\mathcal{R}^b{}_{[c}\dot{\mathcal{Q}}^{\dot\beta}{}_{a]} + \mathcal{R}^b{}_{[c}\mathcal{P}^{\dot\alpha}{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_{a]} + \mathcal{P}^{\dot\alpha}{}_\gamma\dot{\mathcal{Q}}^{\dot\beta}{}_{[a}\mathcal{R}^b{}_{c]}$ which transforms as $\overline{\mathbf{20}}_1$.

[15]In this case, we have $\dot{\mathcal{Q}}^{\dot\alpha}{}_{[a}(z)\dot{\mathcal{W}}^{d\dot\beta\dot\gamma}{}_{bcd]}(w)|_{\frac{1}{(z-w)}} = \delta^d_{[a}\dot{\mathcal{W}}^{\dot\beta\dot\gamma\dot\alpha}{}_{bcd]}(w) - \frac{1}{4}\delta^d_{[d}\dot{\mathcal{W}}^{\dot\beta\dot\gamma\dot\alpha}{}_{bca]}(w)$ together with the higher spin generator $\dot{\mathcal{W}}^{\dot\alpha\dot\beta\dot\gamma}{}_{[abc]} \equiv \dot{\mathcal{Q}}^{\dot\alpha}{}_{[a}\dot{\mathcal{Q}}^{\dot\beta}{}_b\dot{\mathcal{Q}}^{\dot\gamma}{}_{c]} + \dot{\mathcal{Q}}^{\dot\gamma}{}_{[c}\dot{\mathcal{Q}}^{\dot\alpha}{}_a\dot{\mathcal{Q}}^{\dot\beta}{}_{b]} + \dot{\mathcal{Q}}^{\dot\beta}{}_{[b}\dot{\mathcal{Q}}^{\dot\gamma}{}_c\dot{\mathcal{Q}}^{\dot\alpha}{}_{a]} - \dot{\mathcal{Q}}^{\dot\alpha}{}_{[a}\dot{\mathcal{Q}}^{\dot\gamma}{}_c\dot{\mathcal{Q}}^{\dot\beta}{}_{b]} - \dot{\mathcal{Q}}^{\dot\beta}{}_{[b}\dot{\mathcal{Q}}^{\dot\alpha}{}_a\dot{\mathcal{Q}}^{\dot\gamma}{}_{c]} - \dot{\mathcal{Q}}^{\dot\gamma}{}_{[c}\dot{\mathcal{Q}}^{\dot\beta}{}_b\dot{\mathcal{Q}}^{\dot\alpha}{}_{a]}$ transforming as $\mathbf{4}_3$.

between the representation $\mathbf{4}$ and $\bar{\mathbf{4}}$. In the construction of [11], we cannot find this higher spin generator. We can check that they (3.18) are primary operators.[16]

Now we describe the following OPE together with (3.14)

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{[bc]\dot{\beta}}{}_{d\gamma\delta}(w)|_{\frac{1}{(z-w)}} = \delta_a^{[b}\mathcal{W}^{c]\dot{\alpha}\dot{\beta}}{}_{d\gamma\delta}(w) + \delta_a^{[c}\hat{\mathcal{W}}^{b]\dot{\beta}\dot{\alpha}}{}_{d\gamma\delta}(w) - \frac{1}{4}\delta_d^{[c}\hat{\mathcal{W}}^{b]\dot{\beta}\dot{\alpha}}{}_{a\gamma\delta}(w), \quad (3.19)$$

where the right-hand side of (3.19) contains the following higher spin generator

$$\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{b\gamma\delta} \equiv \mathcal{P}^{\dot{\alpha}}{}_{\gamma}\mathcal{P}^{\dot{\beta}}{}_{\delta}\mathcal{R}^a{}_b + \mathcal{P}^{\dot{\alpha}}{}_{\gamma}\mathcal{R}^a{}_b\mathcal{P}^{\dot{\beta}}{}_{\delta} + \mathcal{R}^a{}_b\mathcal{P}^{\dot{\alpha}}{}_{\gamma}\mathcal{P}^{\dot{\beta}}{}_{\delta} + \mathcal{P}^{\dot{\beta}}{}_{\delta}\mathcal{P}^{\dot{\alpha}}{}_{\gamma}\mathcal{R}^a{}_b + \mathcal{P}^{\dot{\beta}}{}_{\delta}\mathcal{R}^a{}_b\mathcal{P}^{\dot{\alpha}}{}_{\gamma} + \mathcal{R}^a{}_b\mathcal{P}^{\dot{\beta}}{}_{\delta}\mathcal{P}^{\dot{\alpha}}{}_{\gamma}, \quad (3.20)$$

which transforms as $\mathbf{15}_0$ with a vanishing $\mathcal{V}$-charge and is a primary operator.[17]

Finally, in this subsection, we consider the following OPE with complete antisymmetric upper indices

$$\mathcal{Q}^{[a}{}_{\alpha}(z)\mathcal{W}^{bc]\dot{\alpha}}{}_{d\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\delta_d^{[a}\mathcal{W}^{bc]\dot{\alpha}}{}_{\beta\alpha\gamma}(w) + \frac{1}{4}\delta_d^{[c}\mathcal{W}^{ba]\dot{\alpha}}{}_{\beta\alpha\gamma}(w), \quad (3.21)$$

where the right-hand side of (3.21) contains the following higher spin generator

$$\mathcal{W}^{[ab]\dot{\alpha}}{}_{\beta\gamma\delta} \equiv \mathcal{Q}^{[a}{}_{\beta}\mathcal{Q}^{b]}{}_{\gamma}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{Q}^{[a}{}_{\beta}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^{b]}{}_{\gamma} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^{[a}{}_{\beta}\mathcal{Q}^{b]}{}_{\gamma} - \mathcal{Q}^{[b}{}_{\gamma}\mathcal{Q}^{[a}{}_{\beta}\mathcal{P}^{\dot{\alpha}}{}_{\delta} - \mathcal{Q}^{[b}{}_{\gamma}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^{a]}{}_{\beta} - \mathcal{Q}^{[a}{}_{\beta}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^{b]}{}_{\gamma}, \quad (3.22)$$

which transforms as $\mathbf{6}_{-2}$ (from the antisymmetric combination of upper two indices) with $\mathcal{V}$-charge -2 and is a primary operator. In this case, we have the conjugated version of this higher spin generator with corresponding OPE as follows.[18]

F. The $s = \frac{7}{2}$ case: $\mathbf{4}_{-1}$ and $\bar{\mathbf{4}}_1$

By using (3.22), we calculate the following OPE and read off the first order pole

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{[bc]\dot{\beta}}{}_{\gamma\delta\epsilon}(w)|_{\frac{1}{(z-w)}} = \delta_a^{[b}\mathcal{W}^{c]\dot{\alpha}\dot{\beta}}{}_{\delta\gamma\epsilon}(w) - \delta_a^{[c}\mathcal{W}^{b]\dot{\alpha}\dot{\beta}}{}_{\delta\gamma\epsilon}(w), \quad (3.23)$$

where the right-hand side of (3.23) contains the higher spin generator

$$\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{\gamma\delta\epsilon} \equiv \mathcal{Q}^a{}_{\gamma}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^a{}_{\gamma}\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{Q}^a{}_{\gamma} + \mathcal{Q}^a{}_{\gamma}\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{Q}^a{}_{\gamma}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{Q}^a{}_{\gamma}, \quad (3.24)$$

which transforms as $\mathbf{4}_{-1}$ from the upper index a with $\mathcal{V}$-charge -1 and is a primary operator. Furthermore, there exists a relevant OPE with the conjugated higher spin generator.[19]

G. The $s = 4$ case: $\mathbf{1}_0$

We obtain the following OPE, by using (3.24),

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{b\dot{\gamma}\dot{\delta}}{}_{\beta\epsilon\rho}(w)|_{\frac{1}{(z-w)}} = \delta_a^b\mathcal{W}^{\dot{\alpha}\dot{\gamma}\dot{\delta}}{}_{\beta\epsilon\rho}(w), \quad (3.25)$$

where the right-hand side of (3.25) has the higher spin generator

$$\mathcal{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{\delta\epsilon\rho} \equiv \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\gamma}}{}_{\rho} + \mathcal{P}^{\dot{\gamma}}{}_{\rho}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\gamma}}{}_{\rho}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\gamma}}{}_{\rho}\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\gamma}}{}_{\rho} + \mathcal{P}^{\dot{\gamma}}{}_{\rho}\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta}. \quad (3.26)$$

Again this transforms as $\mathbf{1}_0$ with $\mathcal{V}$-charge zero because there is no $SU(4)$ index. As described in the footnote 9, the OPEs between the supersymmetry generators and the $\mathcal{P}^{\dot{\alpha}}{}_{\beta}$ do not have any singular terms, we do not find any new higher spin generators from (3.26).[20]

In this section, the higher spin generators are obtained in (3.2), (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.16), (3.18), (3.20), (3.22), (3.24), (3.26), the footnotes 8, 12, 13, 14, 15, 18, and 19 explicitly. They are written in terms of the cubic terms between the weight-1 operators and are

[16] By using the higher spin generator appearing in the footnote 14, we obtain $\mathcal{Q}^a{}_{\alpha}(z)\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}}{}_{b\delta}(w)|_{\frac{1}{(z-w)}} = \delta_b^a\mathcal{W}^{\dot{\gamma}\dot{\beta}}{}_{\alpha\delta}(w) + \hat{\mathcal{W}}^{a\dot{\gamma}\dot{\beta}}{}_{b\alpha\delta}$ where the relations in (3.18) are used.

[17] Again by using the higher spin generator in the footnote 14 we determine $\mathcal{Q}^a{}_{\alpha}(z)\dot{\mathcal{W}}^{c\dot{\beta}\dot{\gamma}}{}_{[bd]\delta}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[b}\mathcal{W}^{c\dot{\beta}\dot{\gamma}}{}_{d]\alpha\delta}(w) - \delta^a_{[d}\hat{\mathcal{W}}^{c\dot{\gamma}\dot{\beta}}{}_{b]\alpha\delta}(w) + \frac{1}{4}\delta^c_{[d}\hat{\mathcal{W}}^{a\dot{\gamma}\dot{\beta}}{}_{b]\alpha\delta}(w)$ where the relations (3.18) and (3.20) are used.

[18] That is, $\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}(z)\dot{\mathcal{W}}^{c\dot{\beta}\dot{\gamma}}{}_{bd]\gamma}(w)|_{\frac{1}{(z-w)}} = \delta^c_{[a}\dot{\mathcal{W}}^{\dot{\beta}\dot{\alpha}\dot{\gamma}}{}_{bd]\gamma}(w) - \frac{1}{4}\delta^c_{[d}\dot{\mathcal{W}}^{\dot{\beta}\dot{\alpha}\dot{\gamma}}{}_{ba]\gamma}(w)$ with the higher spin generator $\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}\dot{\alpha}}{}_{[ab]\delta} \equiv \dot{\mathcal{Q}}^{\dot{\beta}}{}_{[a}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{b]}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \dot{\mathcal{Q}}^{\dot{\beta}}{}_{[a}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{b]} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{[a}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{b]} - \dot{\mathcal{Q}}^{\dot{\gamma}}{}_{[b}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{a]}\mathcal{P}^{\dot{\alpha}}{}_{\delta} - \dot{\mathcal{Q}}^{\dot{\gamma}}{}_{[b}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{a]} - \dot{\mathcal{Q}}^{\dot{\beta}}{}_{[a}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{b]}$ corresponding to the representation $\bar{\mathbf{6}}_2$.

[19] In other words, from the higher spin generator in the footnote 18, we have $\mathcal{Q}^a{}_{\alpha}(z)\dot{\mathcal{W}}^{\dot{\beta}\dot{\delta}\dot{\gamma}}{}_{[bc]\epsilon}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[b}\dot{\mathcal{W}}^{\dot{\beta}\dot{\delta}\dot{\gamma}}{}_{c]\alpha\epsilon}(w) - \delta^a_{[c}\dot{\mathcal{W}}^{\dot{\beta}\dot{\delta}\dot{\gamma}}{}_{b]\alpha\epsilon}(w)$ and $\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{a\delta\epsilon} \equiv \dot{\mathcal{Q}}^{\dot{\gamma}}{}_a\mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_a\mathcal{P}^{\dot{\beta}}{}_{\epsilon} + \mathcal{P}^{\dot{\alpha}}{}_{\delta}\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_a + \dot{\mathcal{Q}}^{\dot{\gamma}}{}_a\mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_a\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \mathcal{P}^{\dot{\beta}}{}_{\epsilon}\mathcal{P}^{\dot{\alpha}}{}_{\delta}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_a$ transforming as $\bar{\mathbf{4}}_1$.

[20] Similarly, from the higher spin generator in the footnote 19, there is a relation $\mathcal{Q}^a{}_{\alpha}(z)\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}\dot{\delta}}{}_{b\epsilon\rho}(w)|_{\frac{1}{(z-w)}} = \delta_b^a\mathcal{W}^{\dot{\beta}\dot{\gamma}\dot{\delta}}{}_{\alpha\epsilon\rho}(w)$.

TABLE I. The higher spin generators with $SU(4)$ representation and $\mathcal{V}$-charge in the worldsheet theory, corresponding to the Tables 4 and 5 with the level $l = 1$ of [11]. We can observe that the two $SU(2)$ spins of the higher spin generators are given by the number of each indices $\alpha, \beta, \gamma, \cdots$ and $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$ divided by 2. For example, the higher spin generator with $s = 4$ has the corresponding spins $(j_L, j_R) = (\frac{3}{2}, \frac{3}{2})$. Note that the spin s is given by $s = 1 + j_L + j_R$. The $\mathcal{V}$-charge is given by the number of lower indices of $SU(4)$ minus the number of upper indices of $SU(4)$.

	Higher spin generators
$s = 1$	$\mathcal{W}(\mathbf{1}_0), \mathcal{W}^a{}_b(\mathbf{15}_0)$
$s = \frac{3}{2}$	$\mathcal{W}^a{}_\alpha(\mathbf{4}_{-1}), \dot{\mathcal{W}}^{\dot{\alpha}}{}_a(\bar{\mathbf{4}}_1), \mathcal{W}^{[ab]}{}_{c\alpha}(\mathbf{20}_{-1}), \dot{\mathcal{W}}^{a\dot{\alpha}}{}_{[bc]}(\overline{\mathbf{20}}_1)$
$s = 2$	$\mathcal{W}^{\dot{\alpha}}{}_\beta(\mathbf{1}_0), \mathcal{W}^{a\dot{\alpha}}{}_{b\beta}(\mathbf{15}_0), \mathcal{W}^{[ab]\dot{\alpha}}{}_{[cd]\beta}(\mathbf{20}'_0), \mathcal{W}^{[ab]c}{}_{c\alpha\beta}(\mathbf{6}_{-2}), \dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{[ab]c}(\bar{\mathbf{6}}_2), \mathcal{W}^{(ab)}{}_{\alpha\beta}(\mathbf{10}_{-2}), \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{(ab)}(\overline{\mathbf{10}}_2)$
$s = \frac{5}{2}$	$\mathcal{W}^{a\dot{\alpha}}{}_{\beta\gamma}(\mathbf{4}_{-1}), \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{a\gamma}(\bar{\mathbf{4}}_1), \mathcal{W}^{[abc]}{}_{\alpha\beta\gamma}(\bar{\mathbf{4}}_{-3}), \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[abc]}(\mathbf{4}_3), \mathcal{W}^{[ab]\dot{\alpha}}{}_{c\beta\gamma}(\mathbf{20}_{-1}), \dot{\mathcal{W}}^{a\dot{\alpha}\dot{\beta}}{}_{[bc]\gamma}(\overline{\mathbf{20}}_1)$
$s = 3$	$\mathcal{W}^{\dot{\alpha}\dot{\beta}}{}_{\gamma\delta}(\mathbf{1}_0), \mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{b\gamma\delta}(\mathbf{15}_0), \mathcal{W}^{[ab]\dot{\alpha}}{}_{\beta\gamma\delta}(\mathbf{6}_{-2}), \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[ab]\gamma}(\bar{\mathbf{6}}_2)$
$s = \frac{7}{2}$	$\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{\gamma\delta\epsilon}(\mathbf{4}_{-1}), \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{a\delta\epsilon}(\mathbf{4}_1)$
$s = 4$	$\mathcal{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{\delta\epsilon\rho}(\mathbf{1}_0)$

summarized by the Table I with $SU(4)$ representations and $\mathcal{V}$-charges.

IV. SOME OPEs BETWEEN THE GENERATORS OF $PSU(2,2|4)$ AND THE LOWEST GENERATORS OF $hs(2,2|4)$

A. Primary or quasiprimary fields

By using the explicit OPE result in Eq. (B1), we can determine the (quasi)primary fields of higher spin generators. As described before, only after checking this (quasi) primary condition, then the first order poles in the OPEs between the weight-1 operators and the weight-3 operators provide the right (quasi)primary operators of weight-3 we would like to construct.

The quasiprimary operators in the Table I are given by the higher spin generators containing the quadratic $\mathcal{V}$ terms including the cubic $\mathcal{V}$ term. The remaining higher spin generators are primary operators.

B. The OPEs between the weight-1 generators and the weight-3 generators

In Sec. III, we have computed some of the OPEs between the conformal dimension-1 generators and the conformal dimension-3 generators in order to determine the higher spin generators. In Appendix C, we will present the remaining OPEs between them. We observe that the first order poles in the right-hand sides of these OPEs (together with the symmetric or antisymmetric combinations of the left-hand sides of the OPEs) contain the higher spin generators as well as the new higher spin generators.[21] In general, in these OPEs, there are also fourth, third and second order poles we do not analyze them in this paper explicitly. In the view point of two dimensional worldsheet theory, it is important to calculate them in order to see their algebraic structures.

Of course, we can calculate the OPEs between the conformal dimension-3 generators and analyze the first order pole in order to determine the next higher spin generators which consist of the quintic terms of weight-5 operators. We will not consider all these computations in this paper although it is straightforward to do so.

C. The additional generators

We have obtained the new higher spin generators (3.6), (3.10), the footnote 12, (3.18) and Eq. (C1)

$$\hat{\mathcal{W}}^{a\dot{\alpha}}{}_{b\beta}, \qquad \hat{\mathcal{W}}^{[ab]}{}_{\alpha\beta}, \qquad \hat{\dot{\mathcal{W}}}^{\dot{\alpha}\dot{\beta}}{}_{[ab]},$$
$$\hat{\mathcal{W}}^{a\dot{\alpha}\dot{\beta}}{}_{b\gamma\delta}, \qquad \hat{\mathcal{W}}^{ab\dot{\alpha}}{}_{c\beta\gamma}, \qquad \hat{\dot{\mathcal{W}}}^{a\dot{\alpha}\dot{\beta}}{}_{bc\gamma}. \tag{4.1}$$

These also appear in the classical version of the OPEs where there are no multiple contractions between the operators. They appear in the computation of the higher spin generators of $s = 2, \frac{5}{2}$ and $s = 3$. Of course, we can further compute the OPEs between the weight-1 operators and the above higher spin generators (4.1) of weight-3 and expect that the first order poles of the right hand sides of these OPEs contain the higher spin generators in Table I and the ones of (4.1). At the moment it is not clear to observe what are the roles of (4.1). We need to calculate further OPEs between the weight-1,2,3 operators including (4.1). We do expect that when we consider the cases $l \geq 2$, the similar additional higher spin generators occur.

D. The next generators of $hs(2,2|4)$

So far, we have considered the $l = 1$ case of [11]. When $l = 2$ case, we observe that the lowest spin $s = 2$ higher spin generator contains the following expression $\mathcal{V}\mathcal{V}\mathcal{V}\mathcal{V}\mathcal{P}^{\dot{\alpha}}_{\beta} + \cdots$ corresponding to $\mathbf{1}_0$ because there is no $SU(4)$ index.

[21]In the right-hand sides of all these OPEs, the higher spin generator $\mathcal{W}$ in (3.2) does not appear at the first order poles.

According to (2.2) and (2.6), for the multiple product of $\mathcal{V}$ whose number is greater than 4, there are still various nonzero derivative terms between the fermionic fields although there is no nonderivative term between them (Of course, if we consider the classical OPEs inside the Thielemans package [28], then the above multiplet product of $\mathcal{V}$ is identically zero). On the other hands, in the oscillator construction, the corresponding X's in [11] appears only up to the quartic term because the five product of fermionic fields vanishes. The higher spin generators at $l = 2$ consist of the quintic terms in the weight-1 operators we have considered. That is, they have weight-5 operators. One way to obtain these higher spin generators is to calculate the OPEs between the weight-3 higher spin generators and look at the first order pole. It would be interesting to examine the details. Contrary to the construction of [11,12], the multiple product of $\mathcal{V}$, where the number of $\mathcal{V}$ is greater than four, can occur due to the above analysis.

V. CONCLUSIONS AND OUTLOOK

The world sheet realization of the higher spin generators of [11] at $l = 1$ is obtained. They are summarized in the Table I in addition to (4.1).

According to the Table 3 of [11], there exist various $\mathcal{N} = 8$ AdS_5 $PSU(2,2|4)$ multiplets with the levels $l = 0, 1, 2, \ldots, \infty$. As mentioned before, the $l = 0$ case is the five dimensional $\mathcal{N} = 8$ gauged supergravity multiplet. The $USp(8)$ representation in each level can be decomposed into the $SU(4)$ with $\mathcal{V}$-charge. See also [36] for this $l = 0$ multiplet in terms of two product of singletons.

The level $l = 1$ multiplet can be interpreted as the "massless" Konishi multiplet in the context of $\mathcal{N} = 4$ conformal supermultiplet in four dimensions [16]. According to the observation of [12], this multiplet can be also obtained by the tensor product of the above $l = 0$ supergravity multiplet (characterized by $\mathbf{42}_0$, $\mathbf{48}_{\frac{1}{2}}$, $\mathbf{27}_1$, $\mathbf{8}_{\frac{3}{2}}$, $\mathbf{1}_2$ with $USp(8)$ representation together with $SO(3)$ spin) with the $SU(4)$ singlet of $SO(3)$ spin-2 ($\mathbf{1}_2$). After then we obtain $\mathbf{1}_0$, $\mathbf{8}_{\frac{1}{2}}$, $\mathbf{28}_1$, $\mathbf{56}_{\frac{3}{2}}$, $\mathbf{70}_2$, $\mathbf{56}_{\frac{5}{2}}$, $\mathbf{28}_3$, $\mathbf{8}_{\frac{7}{2}}$, and $\mathbf{1}_4$ where the subscript s is the spin index appearing in the Table I.

The physical states [11,12] arise in the sectors of the master scalar field and the master gauge field (in the five dimensional higher spin gauge theory) corresponding to the higher spin generators we have described in the above Table I. Note that there exists one-to-one correspondence between the Table 3 and Tables 4 and 5 only for $s = \frac{5}{2}, 3, \frac{7}{2}$ and 4 corresponding to $\mathbf{56}_{\frac{5}{2}}$, $\mathbf{28}_3$, $\mathbf{8}_{\frac{7}{2}}$ and $\mathbf{1}_4$. That is, the representations for $s = 0, \frac{1}{2}$ ($\mathbf{1}_0$ and $\mathbf{8}_{\frac{1}{2}}$) (and the representations $\mathbf{6}$ and $\bar{\mathbf{6}}$ for $s = 1$, the representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ for $s = \frac{3}{2}$, and the representations $\mathbf{1}$ and $\bar{\mathbf{1}}$ for $s = 2$) appear in the Table 6 of [11]. See also (5.1) for their $\mathcal{V}$-charges.

There are the following future directions we can study.

(i) The complete OPEs.

In this paper, we have focused on the construction of the higher spin generators having weight-3. We understand that there are weight-2 operators in the OPEs between the weight-1 operators and the weight-3 operators. Furthermore we did not consider the OPEs between the weight-1 operators (the generators of Lorentz symmetry and the generators of super conformal boosts) (2.5) and the weight-3 operators we have constructed in the world sheet theory. It would be interesting to determine the complete OPEs between these generators of weight-1,2,3 in the context of the higher spin superalgebra $hs(2,2|4)$. Moreover, it will be interesting how they survive when we act them on the physical vacuum state by recalling the footnote on the weight-1 operators along the line of [1,2]. Eventually, we would like to construct the complete higher spin algebra which contains the higher spin generators appearing in the Tables 4 and 5 of [11] in closed form.

(ii) In the theory of $\mathcal{N} = 4$ super Yang-Mills coupled to the $\mathcal{N} = 4$ conformal supergravity.

As before, in [12], the conserved currents corresponding to the higher spin gauge theory described in the Table 3 of [11] can be described from the singleton superfield based on [15,16,36] for $l = 0$ and $l = 1$. Furthermore, in [9], their Tables 6 and 7 are related to the four dimensional $\mathcal{N} = 4$ conformal supergravity multiplet. They claim that the $l = 1$ case of the Table 3 of [11] can be obtained also from the tensor product of above Tables 6 and 7 (See also [37] on the one loop contributions of $\mathcal{N} = 4$ conformal supergravity multiplet). It would be interesting to study precise correspondence explicitly in the context of [38–40]. See also the review paper [41] for conformal supergravity and [42] for the twistor string theory description of conformal supergravity.

(iii) The action of the higher spin generators on the vacuum state.

In the oscillator construction, it is known that the $\mathcal{N} = 4$ super Yang-Mills multiplet can be identified with the multiple product of the various oscillators acting on the physical vacuum state [27]. The similar construction in the world sheet theory is obtained from the multiple product of the various zero modes of the ambitwistor fields acting on the Ramond ground state [1,2]. As we have the complete expressions for the higher spin generators, we can determine the precise action on the physical vacuum state as mentioned before.

(iv) When the coupling of $\mathcal{N} = 4$ super Yang-Mills becomes nonzero.

As the $\mathcal{N} = 4$ super Yang-Mills interaction is turned on, then the higher spin generators in the

Tables 4 and 5 of [11] with $l = 1, 2, \ldots, \infty$ will be no longer conserved. As observed in [12], the $hs(2,2|4)$ higher spin gauge theory maybe described by a string theory having a left-moving and right-moving $PSU(2,2|4)$ Kac-Moody superalgebra with a critical level $k = 1$. We have seen that this theory admits a singleton representation [1,2]. Then the question is whether the affine Kac-Moody extension of the $hs(2,2|4)$ will give us some hints in order to describe the theory for nonzero coupling of $\mathcal{N} = 4$ super Yang-Mills in four dimensions beyond the free field construction of this paper. See also the previous relevant paper [43].

(v) Any algebraic symmetries in the DDF-like operators.

In [1,2], the DDF-like operators [44] which are given by the product of the modes of ambitwistor fields (2.2) are introduced. They satisfy the nontrivial (anti)commutator relations depending on the magnitude of the sum of the two each modes. The structure constants appearing in the right-hand side of these relations are given by the ones in the superalgebra $U(2,2|4)$. They claim that the nontrivial triple products for the specific three modes vanish identically. It would be interesting to describe the above products for any three modes and observe whether there exist any nontrivial behaviors or not.

(vi) How to interpret the mismatch between the Table 3 and the Tables 4 and 5 of [11].

There are some multiplets in Table 6 of [11][22]

$$
\begin{aligned}
s = 1: &\quad \mathbf{6}_{-2}, \quad \bar{\mathbf{6}}_{2}, \\
s = \frac{3}{2}: &\quad \bar{\mathbf{4}}_{-3}, \quad \mathbf{4}_{3}, \\
s = 2: &\quad \mathbf{1}_{-4}, \quad \bar{\mathbf{1}}_{4}.
\end{aligned} \tag{5.1}
$$

These are the elements of the Table 3 but their corresponding higher spin generators do not appear in the Tables 4 and 5. However, it seems that for $l \geq 2$, we can check the sum of the representations in Table 4 [11] is given by $2 \cdot \mathbf{1} \oplus 4 \cdot \mathbf{4} \oplus 2 \cdot \mathbf{16} \oplus 4 \cdot \mathbf{24} \oplus \mathbf{36}$ and this is equal to 182 and the sum of the representations in Table 5 is given by $2 \cdot \mathbf{1} \oplus 4 \cdot \mathbf{4} \oplus 4 \cdot \mathbf{6} \oplus 2 \cdot \mathbf{16}$ and this is 74. This leads to $182 + 74 = 256$. Then there is no mismatch between the Table 3 and the Tables 4 and 5 for $l \geq 2$. It is an open problem to understand how the higher spin generators corresponding to (5.1) are not allowed for small spin s in the oscillator construction (or in the worldsheet theory).

(vii) Can the even power of oscillators survive in the world sheet description?.

In the construction of [11], the higher spin generators with equal odd numbers of oscillators can appear only. See also [18] for relevant discussion. It is not obvious to see this restriction in the world sheet theory because in the OPEs between the weight-1,2,3 operators, in general, the weight-2,4 operators as well as the weight-5 operators can appear. See also [45] for different kinds of higher spin generators. It would be interesting to study this direction in order to describe the above restriction in the world sheet theory.

ACKNOWLEDGMENTS

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1F1A1066893).

APPENDIX A: THE $PSU(2,2|4)_1$ CURRENT ALGEBRA

1. The algebra from the generators in (2.5)

We present the various OPEs between the generators in (2.5) which did not appear in the literature before (although some of (anti)commutator relations between them are in [2]) and we take the order of generators as in $(\mathcal{L}^{\alpha}{}_{\beta}, \dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathcal{R}^{a}{}_{b}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{Q}^{a}{}_{\alpha}, \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}, \mathcal{P}^{\dot{\alpha}}{}_{\beta}, \mathcal{S}^{\alpha}{}_{a}, \dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}, \mathcal{K}^{\alpha}{}_{\dot{\beta}})$ as follows:

$$
\begin{aligned}
\mathcal{L}^{\alpha}{}_{\beta}(z)\mathcal{L}^{\gamma}{}_{\delta}(w) &= \frac{1}{(z-w)^2}\left[\frac{1}{2}\delta^{\alpha}_{\beta}\delta^{\gamma}_{\delta} - \delta^{\alpha}_{\delta}\delta^{\gamma}_{\beta}\right] + \frac{1}{(z-w)}[\delta^{\alpha}_{\delta}\mathcal{L}^{\gamma}_{\beta} - \delta^{\gamma}_{\beta}\mathcal{L}^{\alpha}_{\delta}](w) + \cdots, \\
\mathcal{L}^{\alpha}{}_{\beta}(z)\mathcal{Q}^{a}{}_{\gamma}(w) &= \frac{1}{(z-w)}\left[\delta^{\alpha}_{\gamma}\mathcal{Q}^{a}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{Q}^{a}{}_{\gamma}\right](w) + \cdots, \\
\mathcal{L}^{\alpha}{}_{\beta}(z)\mathcal{P}^{\dot{\alpha}}{}_{\gamma}(w) &= \frac{1}{(z-w)}\left[\delta^{\alpha}_{\gamma}\mathcal{P}^{\dot{\alpha}}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{P}^{\dot{\alpha}}_{\gamma}\right](w) + \cdots, \\
\mathcal{L}^{\alpha}{}_{\beta}(z)\mathcal{S}^{\gamma}{}_{a}(w) &= \frac{1}{(z-w)}\left[-\delta^{\gamma}_{\beta}\mathcal{S}^{\alpha}{}_{a} + \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{S}^{\gamma}{}_{a}\right](w) + \cdots,
\end{aligned}
$$

[22] In addition to these, there are $\mathbf{1}_0$ for $s = 0$ and $\mathbf{4}_1 \oplus \bar{\mathbf{4}}_{-1}$ for $s = \frac{1}{2}$ as before.

$$\mathcal{L}^{\alpha}{}_{\beta}(z)\mathcal{K}^{\gamma}{}_{\dot{\beta}}(w) = \frac{1}{(z-w)}\left[-\delta^{\gamma}_{\beta}\mathcal{K}^{\alpha}{}_{\dot{\beta}} + \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{K}^{\gamma}{}_{\dot{\beta}}\right](w) + \cdots,$$

$$\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}(z)\dot{\mathcal{L}}^{\dot{\gamma}}{}_{\dot{\delta}}(w) = \frac{1}{(z-w)^2}\left[\frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{\dot{\gamma}}_{\dot{\delta}} - \delta^{\dot{\alpha}}_{\dot{\delta}}\delta^{\dot{\gamma}}_{\dot{\beta}}\right] + \frac{1}{(z-w)}[\delta^{\dot{\alpha}}_{\dot{\delta}}\dot{\mathcal{L}}^{\dot{\gamma}}_{\dot{\beta}} - \delta^{\dot{\gamma}}_{\dot{\beta}}\dot{\mathcal{L}}^{\dot{\alpha}}_{\dot{\delta}}](w) + \cdots,$$

$$\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}(z)\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{a}(w) = \frac{1}{(z-w)}\left[-\delta^{\dot{\gamma}}_{\dot{\beta}}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\dot{\mathcal{Q}}^{\dot{\gamma}}{}_{a}\right](w) + \cdots,$$

$$\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}(z)\mathcal{P}^{\dot{\gamma}}{}_{\delta}(w) = \frac{1}{(z-w)}\left[-\delta^{\dot{\gamma}}_{\dot{\beta}}\mathcal{P}^{\dot{\alpha}}{}_{\delta} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{P}^{\dot{\gamma}}{}_{\delta}\right](w) + \cdots,$$

$$\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}(z)\dot{\mathcal{S}}^{a}{}_{\dot{\gamma}}(w) = \frac{1}{(z-w)}\left[\delta^{\dot{\alpha}}_{\dot{\gamma}}\mathcal{S}^{a}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\dot{\mathcal{S}}^{a}{}_{\dot{\gamma}}\right](w) + \cdots,$$

$$\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}}(z)\mathcal{K}^{\gamma}{}_{\dot{\delta}}(w) = \frac{1}{(z-w)}\left[\delta^{\dot{\alpha}}_{\dot{\delta}}\mathcal{K}^{\gamma}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{K}^{\gamma}{}_{\dot{\delta}}\right](w) + \cdots,$$

$$\mathcal{R}^{a}{}_{b}(z)\mathcal{R}^{c}{}_{d}(w) = \frac{1}{(z-w)^2}\left[-\frac{1}{4}\delta^{a}_{b}\delta^{c}_{d} - \delta^{a}_{d}\delta^{c}_{b}\right] + \frac{1}{(z-w)}[\delta^{a}_{d}\mathcal{R}^{c}{}_{b} - \delta^{c}_{b}\mathcal{R}^{a}{}_{d}](w) + \cdots,$$

$$\mathcal{R}^{a}{}_{b}(z)\mathcal{Q}^{c}{}_{\alpha}(w) = \frac{1}{(z-w)}\left[-\delta^{c}_{b}\mathcal{Q}^{a}{}_{\alpha} + \frac{1}{4}\delta^{a}_{b}\mathcal{Q}^{c}{}_{\alpha}\right](w) + \cdots,$$

$$\mathcal{R}^{a}{}_{b}(z)\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c}(w) = \frac{1}{(z-w)}\left[\delta^{a}_{c}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{b} - \frac{1}{4}\delta^{a}_{b}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c}\right](w) + \cdots,$$

$$\mathcal{R}^{a}{}_{b}(z)\mathcal{S}^{\alpha}{}_{c}(w) = \frac{1}{(z-w)}\left[\delta^{a}_{c}\mathcal{S}^{\alpha}{}_{b} - \frac{1}{4}\delta^{a}_{b}\mathcal{S}^{\alpha}{}_{c}\right](w) + \cdots,$$

$$\mathcal{R}^{a}{}_{b}(z)\dot{\mathcal{S}}^{c}{}_{\dot{\alpha}}(w) = \frac{1}{(z-w)}\left[-\delta^{c}_{b}\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}} + \frac{1}{4}\delta^{a}_{b}\dot{\mathcal{S}}^{c}{}_{\dot{\alpha}}\right](w) + \cdots,$$

$$\mathcal{B}(z)\mathcal{B}(w) = -\frac{1}{(z-w)^2} + \cdots, \qquad \mathcal{B}(z)\mathcal{C}(w) = -\frac{1}{(z-w)^2} + \cdots,$$

$$\mathcal{B}(z)\mathcal{Q}^{a}{}_{\alpha}(w) = \frac{1}{(z-w)}\frac{1}{2}\mathcal{Q}^{a}{}_{\alpha}(w) + \cdots, \qquad \mathcal{B}(z)\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(w) = -\frac{1}{(z-w)}\frac{1}{2}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(w) + \cdots,$$

$$\mathcal{B}(z)\mathcal{S}^{\alpha}{}_{a}(w) = -\frac{1}{(z-w)}\frac{1}{2}\mathcal{S}^{\alpha}{}_{a}(w) + \cdots, \qquad \mathcal{B}(z)\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}(w) = \frac{1}{(z-w)}\frac{1}{2}\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}(w) + \cdots,$$

$$\mathcal{D}(z)\mathcal{D}(w) = -\frac{1}{(z-w)^2} + \cdots, \qquad \mathcal{D}(z)\mathcal{Q}^{a}{}_{\alpha}(w) = \frac{1}{(z-w)}\frac{1}{2}\mathcal{Q}^{a}{}_{\alpha}(w) + \cdots,$$

$$\mathcal{D}(z)\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(w) = \frac{1}{(z-w)}\frac{1}{2}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(w) + \cdots, \qquad \mathcal{D}(z)\mathcal{P}^{\dot{\alpha}}{}_{\beta}(w) = \frac{1}{(z-w)}\mathcal{P}^{\dot{\alpha}}{}_{\beta}(w) + \cdots,$$

$$\mathcal{D}(z)\mathcal{S}^{\alpha}{}_{a}(w) = -\frac{1}{(z-w)}\frac{1}{2}\mathcal{S}^{\alpha}{}_{a}(w) + \cdots, \qquad \mathcal{D}(z)\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}(w) = \frac{1}{(z-w)}\frac{1}{2}\dot{\mathcal{S}}^{a}{}_{\dot{\alpha}}(w) + \cdots,$$

$$\mathcal{D}(z)\mathcal{K}^{\alpha}{}_{\dot{\beta}}(w) = -\frac{1}{(z-w)}\mathcal{K}^{\alpha}{}_{\dot{\beta}}(w) + \cdots, \qquad \mathcal{Q}^{a}{}_{\beta}(z)\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{b}(w) = \frac{1}{(z-w)}\delta^{a}_{b}\mathcal{P}^{\dot{\alpha}}{}_{\beta}(w) + \cdots,$$

$$\mathcal{Q}^{a}{}_{\alpha}(z)\mathcal{S}^{\beta}{}_{b}(w) = \frac{1}{(z-w)^2}\delta^{a}_{b}\delta^{\beta}_{\alpha} + \frac{1}{(z-w)}\left[\delta^{\beta}_{\alpha}\mathcal{R}^{a}{}_{b} + \delta^{a}_{b}\mathcal{L}^{\beta}{}_{\alpha} + \frac{1}{2}\delta^{a}_{b}\delta^{\beta}_{\alpha}(\mathcal{C}+\mathcal{D})\right](w) + \cdots,$$

$$\mathcal{Q}^{a}{}_{\alpha}(z)\mathcal{K}^{\beta}{}_{\dot{\gamma}}(w) = -\frac{1}{(z-w)}\delta^{\beta}_{\alpha}\dot{\mathcal{S}}^{a}{}_{\dot{\gamma}}(w) + \cdots,$$

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\dot{\mathcal{S}}^{b}{}_{\dot{\beta}}(w) = \frac{1}{(z-w)^2}\delta^{b}_{a}\delta^{\dot{\alpha}}_{\dot{\beta}} + \frac{1}{(z-w)}\left[\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{R}^{b}{}_{a} + \delta^{b}_{a}\dot{\mathcal{L}}^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{2}\delta^{b}_{a}\delta^{\dot{\alpha}}_{\dot{\beta}}(\mathcal{C}-\mathcal{D})\right](w) + \cdots,$$

$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\mathcal{K}^{\beta}{}_{\dot{\gamma}}(w) = \frac{1}{(z-w)}\delta^{\dot{\alpha}}_{\dot{\gamma}}\mathcal{S}^{\beta}{}_{a}(w) + \cdots, \qquad \mathcal{P}^{\dot{\alpha}}{}_{\beta}(z)\mathcal{S}^{\gamma}{}_{a}(w) = -\frac{1}{(z-w)}\delta^{\gamma}_{\beta}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(w) + \cdots,$$

$$
\begin{aligned}
\mathcal{P}^{\dot\alpha}{}_{\beta}(z)\dot{\mathcal{S}}^{a}{}_{\dot\gamma}(w) &= \frac{1}{(z-w)}\delta^{\dot\alpha}_{\dot\gamma}\mathcal{Q}^{a}{}_{\beta}(w)+\cdots,\\
\mathcal{P}^{\dot\alpha}{}_{\beta}(z)\mathcal{K}^{\gamma}{}_{\dot\delta}(w) &= -\frac{1}{(z-w)^2}\delta^{\gamma}_{\beta}\delta^{\dot\alpha}_{\dot\delta}+\frac{1}{(z-w)}[-\delta^{\gamma}_{\beta}\dot{\mathcal{L}}^{\dot\alpha}{}_{\dot\delta}+\delta^{\dot\alpha}_{\dot\delta}\mathcal{L}^{\gamma}{}_{\beta}+\delta^{\gamma}_{\beta}\delta^{\dot\alpha}_{\dot\delta}\mathcal{D}](w)+\cdots,\\
\mathcal{S}^{\alpha}{}_{a}(z)\dot{\mathcal{S}}^{b}{}_{\dot\beta}(w) &= \frac{1}{(z-w)}\delta^{b}_{a}\mathcal{K}^{\alpha}{}_{\dot\beta}(w)+\cdots. \qquad (A1)
\end{aligned}
$$

The OPEs between the generators of $(\mathcal{R}^{a}{}_{b},\mathcal{V},\mathcal{Q}^{a}{}_{\alpha},\dot{\mathcal{Q}}^{\dot\alpha}{}_{a},\mathcal{P}^{\dot\alpha}{}_{\beta})$ with $\mathcal{V}\equiv 2(\mathcal{C}-\mathcal{B})$ are closed by themselves. See also Eq. (A2).

2. Some OPEs with different $U(1)$ generators in (2.6)

By using (2.6), we can rewrite some OPEs in Eq. (A1) as follows:

$$
\begin{aligned}
&\mathcal{U}(z)\mathcal{U}(w) = -\frac{1}{(z-w)^2}2+\cdots, \qquad \mathcal{U}(z)\mathcal{B}(w) = -\frac{1}{(z-w)^2}+\cdots,\\
&\mathcal{U}(z)\mathcal{C}(w) = -\frac{1}{(z-w)^2}+\cdots, \qquad \mathcal{U}(z)\mathcal{D}(w) = -\frac{1}{(z-w)^2}+\cdots,\\
&\mathcal{U}(z)\mathcal{Q}^{a}{}_{\alpha}(w) = \frac{1}{(z-w)}\mathcal{Q}^{a}{}_{\alpha}(w)+\cdots, \qquad \mathcal{U}(z)\mathcal{P}^{\dot\alpha}{}_{\beta}(w) = \frac{1}{(z-w)}\mathcal{P}^{\dot\alpha}{}_{\beta}(w)+\cdots,\\
&\mathcal{U}(z)\mathcal{S}^{\alpha}{}_{a}(w) = -\frac{1}{(z-w)}\mathcal{S}^{\alpha}{}_{a}(w)+\cdots, \qquad \mathcal{U}(z)\mathcal{K}^{\alpha}{}_{\dot\beta}(w) = -\frac{1}{(z-w)}\mathcal{K}^{\alpha}{}_{\dot\beta}(w)+\cdots,\\
&\dot{\mathcal{U}}(z)\dot{\mathcal{U}}(w) = -\frac{1}{(z-w)^2}2+\cdots, \qquad \dot{\mathcal{U}}(z)\mathcal{B}(w) = -\frac{1}{(z-w)^2}+\cdots,\\
&\dot{\mathcal{U}}(z)\mathcal{C}(w) = -\frac{1}{(z-w)^2}+\cdots, \qquad \dot{\mathcal{U}}(z)\mathcal{D}(w) = \frac{1}{(z-w)^2}+\cdots,\\
&\dot{\mathcal{U}}(z)\dot{\mathcal{Q}}^{\dot\alpha}{}_{a}(w) = -\frac{1}{(z-w)}\dot{\mathcal{Q}}^{\dot\alpha}{}_{a}(w)+\cdots, \qquad \dot{\mathcal{U}}(z)\mathcal{P}^{\dot\alpha}{}_{\beta}(w) = -\frac{1}{(z-w)}\mathcal{P}^{\dot\alpha}{}_{\beta}(w)+\cdots,\\
&\dot{\mathcal{U}}(z)\dot{\mathcal{S}}^{a}{}_{\alpha}(w) = \frac{1}{(z-w)}\dot{\mathcal{S}}^{a}{}_{\alpha}(w)+\cdots, \qquad \dot{\mathcal{U}}(z)\mathcal{K}^{\alpha}{}_{\dot\beta}(w) = \frac{1}{(z-w)}\mathcal{K}^{\alpha}{}_{\dot\beta}(w)+\cdots,\\
&\mathcal{V}(z)\mathcal{V}(w) = \frac{1}{(z-w)^2}4+\cdots, \qquad \mathcal{V}(z)\mathcal{C}(w) = \frac{1}{(z-w)^2}2+\cdots,\\
&\mathcal{V}(z)\mathcal{Q}^{a}{}_{\alpha}(w) = -\frac{1}{(z-w)}\mathcal{Q}^{a}{}_{\alpha}(w)+\cdots, \qquad \mathcal{V}(z)\dot{\mathcal{Q}}^{\dot\alpha}{}_{a}(w) = \frac{1}{(z-w)}\dot{\mathcal{Q}}^{\dot\alpha}{}_{a}(w)+\cdots,\\
&\mathcal{V}(z)\mathcal{S}^{\alpha}{}_{a}(w) = \frac{1}{(z-w)}\mathcal{S}^{\alpha}{}_{a}(w)+\cdots, \qquad \mathcal{V}(z)\dot{\mathcal{S}}^{a}{}_{\dot\alpha}(w) = -\frac{1}{(z-w)}\dot{\mathcal{S}}^{a}{}_{\dot\alpha}(w)+\cdots. \qquad (A2)
\end{aligned}
$$

Note that the nonzero $\mathcal{V}$-charge can be obtained from the last four OPEs of (A2).

APPENDIX B: THE OPEs BETWEEN THE STRESS ENERGY TENSOR AND $J^{I}{}_{J}J^{K}{}_{L}J^{M}{}_{N}$

We write down the OPE between the stress energy tensor and the cubic term as follows:

$$
\begin{aligned}
T(z)J^{I}{}_{J}J^{K}{}_{L}J^{M}{}_{N}(w) = &\frac{1}{(z-w)^5}[(-1)^{d_J d_M+1}\delta^{I}_{L}\delta^{K}_{N}\delta^{M}_{J}+(-1)^{(d_L+d_K)(d_I+d_J)+d_L d_M}\delta^{K}_{J}\delta^{I}_{N}\delta^{M}_{L}]\\
&+\frac{1}{(z-w)^4}[(-1)^{d_L d_M+1}\delta^{K}_{N}\delta^{M}_{L}J^{I}{}_{J}+(-1)^{d_J d_K+1}\delta^{I}_{L}\delta^{K}_{J}J^{M}{}_{N}\\
&+(-1)^{(d_M+d_N)(d_K+d_J)+1}\delta^{I}_{L}\delta^{M}_{J}J^{K}{}_{N}
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{(d_L+d_K)(d_I+d_J)+1}\delta^K_J\delta^I_N J^M{}_L+(-1)^{(d_L+d_K)(d_I+d_J)+(d_M+d_N)(d_I+d_J)}\delta^K_J\delta^M_L J^I_N \\
&+(-1)^{(d_I+d_J)(d_L+d_K)+d_Jd_M+1}\delta^I_N\delta^M_J J^K{}_L+\delta^I_L\delta^K_N J^M{}_J](w) \\
&+\frac{1}{(z-w)^3}[\delta^I_L J^K_J J^M{}_N+(-1)^{(d_L+d_K)(d_I+d_J)+1}\delta^K_J J^I_L J^M{}_N \\
&+(-1)^{(d_I+d_J)(d_L+d_K)}\delta^I_N J^K{}_L J^M{}_J \\
&+(-1)^{(d_I+d_J)(d_L+d_K+d_M+d_N)+1}\delta^M_J J^K{}_L J^I{}_N+\delta^K_N J^I{}_J J^M{}_L \\
&+(-1)^{(d_N+d_M)(d_K+d_L)+1}\delta^M_L J^I{}_J J^K{}_N](w)+\frac{1}{(z-w)^2}3J^I{}_J J^K{}_L J^M{}_N(w) \\
&+\frac{1}{(z-w)}\partial(J^I{}_J J^K{}_L J^M{}_N)(w)+\cdots. \qquad \text{(B1)}
\end{aligned}
$$

We should obtain the weight-3 operators which transform as a quasiprimary.

APPENDIX C: THE REMAINING FIRST ORDER POLES IN THE OPEs DESCRIBED IN SEC. III

In this Appendix, we present the remaining first order poles in the OPEs between the weight-1 operators and the weight-3 operators.

Let us classify according to the spin of weight-3 operators.

(i) The $s=1$ case.

In addition to the corresponding OPEs of Sec. III B, there are the following OPEs with first order poles

$$
\begin{aligned}
&\mathcal{Q}^a{}_\alpha(z)\mathcal{W}(w)|_{\frac{1}{(z-w)}}=\mathcal{W}^a_\alpha(w), \qquad \dot{\mathcal{Q}}^{\dot\alpha}{}_a(z)\mathcal{W}(w)|_{\frac{1}{(z-w)}}=-\dot{\mathcal{W}}^{\dot\alpha}_a(w), \\
&\mathcal{R}^{[a}{}_b(z)\mathcal{W}^{c]}{}_d(w)|_{\frac{1}{(z-w)}}=\delta^{[a}_d\mathcal{W}^{c]}{}_b(w)-\delta^{[c}_b\mathcal{W}^{a]}{}_d(w).
\end{aligned}
$$

(ii) The $s=\frac{3}{2}$ case.

There are the following first order poles in the OPEs as well as the ones in Sec. III C

$$
\begin{aligned}
&\mathcal{V}(z)\mathcal{W}^a{}_\alpha(w)|_{\frac{1}{(z-w)}}=-\mathcal{W}^a{}_\alpha(w), \\
&\mathcal{R}^a{}_b(z)\mathcal{W}^c{}_\alpha(w)|_{\frac{1}{(z-w)}}=-\delta^c_b\mathcal{W}^a{}_\alpha(w)+\frac{1}{4}\delta^a_b\mathcal{W}^c{}_\alpha(w), \\
&\mathcal{V}(z)\mathcal{W}^{[ab]}{}_{c\alpha}(w)|_{\frac{1}{(z-w)}}=-\mathcal{W}^{[ab]}{}_{c\alpha}(w), \\
&\mathcal{R}^{[a}{}_b(z)\mathcal{W}^{cd]}{}_{e\alpha}(w)|_{\frac{1}{(z-w)}}=-\delta^{[c}_b\mathcal{W}^{ad]}{}_{e\alpha}(w)+\frac{1}{4}\delta^{[a}_b\mathcal{W}^{cd]}{}_{e\alpha}(w)+\delta^{[a}_e\mathcal{W}^{cd]}{}_{b\alpha}(w)-\delta^{[d}_b\mathcal{W}^{ca]}{}_{e\alpha}(w), \\
&\mathcal{V}(z)\dot{\mathcal{W}}^{b\dot\alpha}{}_{[ac]}(w)|_{\frac{1}{(z-w)}}=\dot{\mathcal{W}}^{b\dot\alpha}{}_{[ac]}(w), \\
&\mathcal{R}^a{}_{[b}(z)\dot{\mathcal{W}}^{d\dot\alpha}{}_{ce]}(w)|_{\frac{1}{(z-w)}}=\delta^a_{[c}\dot{\mathcal{W}}^{d\dot\alpha}{}_{be]}(w)-\frac{1}{4}\delta^a_{[b}\dot{\mathcal{W}}^{d\dot\alpha}{}_{ce]}(w)+\delta^a_{[e}\dot{\mathcal{W}}^{d\dot\alpha}{}_{cb]}(w)-\delta^d_{[b}\dot{\mathcal{W}}^{d\dot\alpha}{}_{cb]}(w), \\
&\mathcal{V}(z)\dot{\mathcal{W}}^{\dot\alpha}{}_a(w)|_{\frac{1}{(z-w)}}=\dot{\mathcal{W}}^{\dot\alpha}{}_a(w), \\
&\mathcal{R}^a{}_b(z)\dot{\mathcal{W}}^{\dot\alpha}{}_c(w)|_{\frac{1}{(z-w)}}=\delta^a_c\dot{\mathcal{W}}^{\dot\alpha}{}_b(w)-\frac{1}{4}\delta^a_b\dot{\mathcal{W}}^{\dot\alpha}{}_c(w).
\end{aligned}
$$

(iii) The $s=2$ case.

There are the first order poles of the following OPEs in addition to the ones in Sec. III D

$$
\begin{aligned}
\mathcal{R}^a{}_b(z)\mathcal{W}^{c\dot{\alpha}}{}_{d\alpha}(w)|_{\frac{1}{(z-w)}} &= \delta^a_d\mathcal{W}^{c\dot{\alpha}}{}_{b\alpha}(w) - \delta^c_b\mathcal{W}^{a\dot{\alpha}}{}_{d\alpha}(w),\\
\mathcal{V}(z)\mathcal{W}^{(ab)}{}_{\alpha\beta}(w)|_{\frac{1}{(z-w)}} &= -2\mathcal{W}^{(ab)}{}_{\alpha\beta}(w),\\
\mathcal{R}^{(a}{}_b(z)\mathcal{W}^{cd)}{}_{\alpha\beta}(w)|_{\frac{1}{(z-w)}} &= -\delta^{(c}_b\mathcal{W}^{ad)}{}_{\alpha\beta}(w) + \frac{1}{4}\delta^{(a}_b\mathcal{W}^{cd)}{}_{\alpha\beta}(w) + \delta^{(d}_b\mathcal{W}^{ca)}{}_{\alpha\beta}(w) - \frac{1}{4}\delta^{(a}_b\mathcal{W}^{cd)}{}_{\alpha\beta}(w),\\
\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{(bc)}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} &= -\hat{\mathcal{W}}^{(bc)\dot{\alpha}}{}_{a\beta\gamma}(w) + \delta^{(b}_a\mathcal{W}^{c)\dot{\alpha}}{}_{\beta\gamma}(w) - \delta^{(c}_a\mathcal{W}^{b)\dot{\alpha}}{}_{\beta\gamma}(w),\\
\mathcal{Q}^a{}_\alpha(z)\mathcal{W}^{\dot{\beta}}{}_\gamma(w)|_{\frac{1}{(z-w)}} &= \mathcal{W}^{a\dot{\beta}}{}_{\alpha\gamma}(w), \qquad \dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{\dot{\beta}}{}_\gamma(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{a\gamma}(w),\\
\mathcal{R}^{[a}{}_{[b}(z)\mathcal{W}^{ce]\dot{\alpha}}{}_{df]\beta}(w)|_{\frac{1}{(z-w)}} &= -\delta^{[c}_{[b}\mathcal{W}^{ce]\dot{\alpha}}{}_{df]\beta}(w) + \delta^{[a}_{[d}\mathcal{W}^{ce]\dot{\alpha}}{}_{df]\beta}(w) + \delta^{[a}_{[f}\mathcal{W}^{ce]\dot{\alpha}}{}_{db]\beta}(w) - \delta^{[e}_{[b}\mathcal{W}^{ca]\dot{\alpha}}{}_{df]\beta}(w),\\
\mathcal{Q}^{[a}{}_\alpha(z)\mathcal{W}^{bd]\dot{\beta}}{}_{[ce]\gamma}(w)|_{\frac{1}{(z-w)}} &= -\delta^{[a}_{[c}\mathcal{W}^{bd]\dot{\beta}}{}_{[c]\gamma\alpha}(w) + \delta^{[a}_{[e}\hat{\mathcal{W}}^{bd]\dot{\beta}}{}_{c]\gamma\alpha}(w) - \frac{1}{4}\delta^{d}_{[e}\hat{\mathcal{W}}^{ba]\dot{\beta}}{}_{c]\gamma\alpha}(w),\\
\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}(z)\mathcal{W}^{[bd]\dot{\beta}}{}_{ce]\gamma}(w)|_{\frac{1}{(z-w)}} &= \delta^{[b}_{[a}\dot{\mathcal{W}}^{d]\dot{\beta}\dot{\alpha}}{}_{ce]\gamma}(w) - \delta^{[d}_{[a}\dot{\hat{\mathcal{W}}}^{b]\dot{\alpha}\dot{\beta}}{}_{ec]\gamma}(w) + \frac{1}{4}\delta^{[d}_{[e}\dot{\hat{\mathcal{W}}}^{b]\dot{\alpha}\dot{\beta}}{}_{ac]\gamma}(w),\\
\mathcal{V}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{(ab)}(w)|_{\frac{1}{(z-w)}} &= 2\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{(ab)}(w),\\
\mathcal{R}^a{}_{(b}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{cd)}(w)|_{\frac{1}{(z-w)}} &= \delta^a_{(c}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{bd)}(w) - \frac{1}{2}\delta^a_{(b}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{cd)}(w) + \delta^a_{(d}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{cb)}(w),\\
\mathcal{Q}^a{}_\alpha(z)\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}}{}_{(bc)}(w)|_{\frac{1}{(z-w)}} &= \dot{\hat{\mathcal{W}}}^{a\dot{\gamma}\dot{\beta}}{}_{(bc)\alpha}(w) + \delta^a_{(b}\dot{\mathcal{W}}^{\dot{\gamma}\dot{\beta}}{}_{c)\alpha}(w) - \delta^a_{(c}\dot{\mathcal{W}}^{\dot{\gamma}\dot{\beta}}{}_{b)\alpha}(w),\\
\mathcal{V}(z)\mathcal{W}^{[ab]c}{}_{c\alpha\beta}(w)|_{\frac{1}{(z-w)}} &= -2\mathcal{W}^{[ab]c}{}_{c\alpha\beta}(w),\\
\mathcal{R}^{[a}{}_b(z)\mathcal{W}^{cd]e}{}_{e\alpha\beta}(w)|_{\frac{1}{(z-w)}} &= -\delta^{[c}_b\mathcal{W}^{ad]e}{}_{e\alpha\beta}(w) + \frac{1}{2}\delta^{[a}_b\mathcal{W}^{cd]e}{}_{e\alpha\beta}(w) - \delta^{[d}_b\mathcal{W}^{ca]e}{}_{e\alpha\beta}(w),\\
\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a(z)\mathcal{W}^{[cd]e}{}_{e\alpha\beta}(w)|_{\frac{1}{(z-w)}} &= \delta^{[c}_a\mathcal{W}^{de]\dot{\alpha}}{}_{e\beta\alpha}(w) - \delta^{[d}_a\mathcal{W}^{ce]\dot{\alpha}}{}_{e\alpha\beta}(w) - \delta^{[e}_a\hat{\mathcal{W}}^{cd]\dot{\alpha}}{}_{e\alpha\beta}(w) + \frac{1}{4}\delta^{[e}_e\hat{\mathcal{W}}^{cd]\dot{\alpha}}{}_{a\alpha\beta}(w),\\
\mathcal{V}(z)\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{[ab]c}(w)|_{\frac{1}{(z-w)}} &= 2\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{[ab]c}(w),\\
\mathcal{R}^a_{[b}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{de]c}(w)|_{\frac{1}{(z-w)}} &= \delta^a_{[d}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{be]c}(w) - \frac{1}{2}\delta^a_{[b}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{de]c}(w) + \delta^a_{[e}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{db]c}(w),
\end{aligned}
$$

where the new higher spin generators are given by

$$
\begin{aligned}
\hat{\mathcal{W}}^{(bc)\dot{\alpha}}{}_{a\beta\gamma} &\equiv \dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{(b}{}_\beta\mathcal{Q}^{c)}{}_\gamma - \mathcal{Q}^{(b}{}_\beta\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{c)}{}_\gamma + \mathcal{Q}^{(b}{}_\beta\mathcal{Q}^{c)}{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{c)}{}_\gamma\mathcal{Q}^{b)}{}_\beta + \mathcal{Q}^{c)}{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{b)}{}_\beta - \mathcal{Q}^{c)}{}_\gamma\mathcal{Q}^{b)}{}_\beta\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a,\\
\hat{\mathcal{W}}^{[bc]}{}_{a\beta\gamma} &\equiv \dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{[b}{}_\beta\mathcal{Q}^{c]}{}_\gamma - \mathcal{Q}^{[b}{}_\beta\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{c]}{}_\gamma + \mathcal{Q}^{[b}{}_\beta\mathcal{Q}^{c]}{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{[c}{}_\gamma\mathcal{Q}^{[b}{}_\beta + \mathcal{Q}^{[c}{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a\mathcal{Q}^{[b}{}_\beta - \mathcal{Q}^{[c}{}_\gamma\mathcal{Q}^{[b}{}_\beta\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a,\\
\dot{\hat{\mathcal{W}}}^{b\dot{\alpha}\dot{\beta}}{}_{[cd]\gamma} &\equiv \mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\beta}}{}_{[d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]} - \dot{\mathcal{Q}}^{\dot{\beta}}{}_{[d}\mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]} + \dot{\mathcal{Q}}^{\dot{\beta}}{}_{[d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c]}\mathcal{Q}^b{}_\gamma - \mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d]} + \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d]} - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[c}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d]}\mathcal{Q}^b{}_\gamma,\\
\dot{\hat{\mathcal{W}}}^{b\dot{\alpha}\dot{\beta}}{}_{(cd)\gamma} &\equiv \mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\beta}}{}_{(d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c)} - \dot{\mathcal{Q}}^{\dot{\beta}}{}_{(d}\mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c)} + \dot{\mathcal{Q}}^{\dot{\beta}}{}_{(d}\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{c)}\mathcal{Q}^b{}_\gamma - \mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{(c}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d)} + \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{(c}\mathcal{Q}^b{}_\gamma\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d)} - \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{(c}\dot{\mathcal{Q}}^{\dot{\beta}}{}_{d)}\mathcal{Q}^b{}_\gamma. \qquad \text{(C1)}
\end{aligned}
$$

(iv) The $s = \frac{5}{2}$ case.
There exist the first order poles of the following OPEs (and the ones of Sec. III E)

$$
\begin{aligned}
\mathcal{V}(z)\mathcal{W}^{[ab]\dot{\alpha}}{}_{c\beta\gamma}(w)|_{\frac{1}{(z-w)}} &= -\mathcal{W}^{[ab]\dot{\alpha}}{}_{c\beta\gamma}(w),\\
\mathcal{R}^{[a}{}_b\mathcal{W}^{cd]\dot{\alpha}}{}_{e\beta\gamma}(w)|_{\frac{1}{(z-w)}} &= -\delta^{[c}_b\mathcal{W}^{cd]\dot{\alpha}}{}_{e\beta\gamma}(w) + \frac{1}{4}\delta^{[a}_b\mathcal{W}^{cd]\dot{\alpha}}{}_{e\beta\gamma}(w) + \delta^{[a}_e\mathcal{W}^{cd]\dot{\alpha}}{}_{b\beta\gamma}(w) - \delta^{[d}_b\mathcal{W}^{ca]\dot{\alpha}}{}_{e\beta\gamma}(w),\\
\mathcal{V}(z)\mathcal{W}^{a\dot{\alpha}}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} &= -\mathcal{W}^{a\dot{\alpha}}{}_{\beta\gamma}(w),\\
\mathcal{R}^a{}_b(z)\mathcal{W}^{c\dot{\alpha}}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} &= -\delta^c_b\mathcal{W}^{a\dot{\alpha}}{}_{\beta\gamma}(w) + \frac{1}{4}\delta^a_b\mathcal{W}^{c\dot{\alpha}}{}_{\beta\gamma}(w),
\end{aligned}
$$

$$\mathcal{Q}^{[a}{}_{\alpha}(z)\mathcal{W}^{b]\dot{\alpha}}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} = \mathcal{W}^{[ab]\dot{\alpha}}{}_{\alpha\beta\gamma}(w), \qquad \mathcal{V}(z)\dot{\mathcal{W}}^{a\dot{\alpha}\dot{\beta}}{}_{[bc]\gamma}(w)|_{\frac{1}{(z-w)}} = \dot{\mathcal{W}}^{a\dot{\alpha}\dot{\beta}}{}_{[bc]\gamma}(w),$$
$$\mathcal{R}^{a}{}_{[b}(z)\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{de]\gamma}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[d}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{be]\gamma}(w) - \frac{1}{4}\delta^a_{[b}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{de]\gamma}(w) + \delta^a_{[e}\dot{\mathcal{W}}^{c\dot{\alpha}\dot{\beta}}{}_{db]\gamma}(w) - \delta^c_{[e}\dot{\mathcal{W}}^{a\dot{\alpha}\dot{\beta}}{}_{db]\gamma}(w),$$
$$\mathcal{V}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{a\gamma}(w)|_{\frac{1}{(z-w)}} = \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{a\gamma}(w),$$
$$\mathcal{R}^{a}{}_{b}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{c\gamma}(w)|_{\frac{1}{(z-w)}} = \delta^a_c\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{b\gamma}(w) - \frac{1}{4}\delta^a_b\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}}{}_{c\gamma}(w),$$
$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{[a}(z)\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}}{}_{b]\gamma}(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[ab]\gamma}(w), \qquad \mathcal{V}(z)\mathcal{W}^{[abc]}{}_{\alpha\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -3\mathcal{W}^{[abc]}{}_{\alpha\beta\gamma}(w),$$
$$\mathcal{R}^{[a}{}_{b}(z)\mathcal{W}^{cde]}{}_{\alpha\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\delta^{[c}_b\mathcal{W}^{ade]}{}_{\alpha\beta\gamma}(w) - \delta^{[d}_b\mathcal{W}^{cae]}{}_{\alpha\beta\gamma}(w) - \delta^{[e}_b\mathcal{W}^{cda]}{}_{\alpha\beta\gamma}(w) + \frac{3}{4}\delta^{[a}_b\mathcal{W}^{cde]}{}_{\alpha\beta\gamma}(w),$$
$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\mathcal{W}^{[bcd]}{}_{\alpha\beta\gamma}(w)|_{\frac{1}{(z-w)}} = \delta^{[b}_a\mathcal{W}^{cd]\dot{\alpha}}{}_{\beta\gamma\alpha}(w), \qquad \mathcal{V}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[abc]}(w)|_{\frac{1}{(z-w)}} = 3\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[abc]}(w),$$
$$\mathcal{R}^{a}{}_{[b}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cde]}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[c}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{bde]}(w) + \delta^a_{[d}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cbe]}(w) + \delta^a_{[e}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cdb]}(w) - \frac{3}{4}\delta^a_{[b}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cde]}(w),$$
$$\mathcal{Q}^{a}{}_{\alpha}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[bcd]}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[b}\dot{\mathcal{W}}^{\dot{\beta}\dot{\gamma}\dot{\alpha}}{}_{cd]\alpha}(w).$$

(v) The $s = 3$ case.
We have the following OPEs with first order poles in addition to the ones of the subsection III. 6

$$\mathcal{V}(z)\mathcal{W}^{[ab]\dot{\alpha}}{}_{\beta\gamma\delta}(w)|_{\frac{1}{(z-w)}} = -2\mathcal{W}^{[ab]\dot{\alpha}}{}_{\beta\gamma\delta}(w),$$
$$\mathcal{R}^{[a}{}_{[b}(z)\mathcal{W}^{cd]\dot{\alpha}}{}_{\beta\gamma\alpha}(w)|_{\frac{1}{(z-w)}} = -\delta^{[c}_b\mathcal{W}^{ad]\dot{\alpha}}{}_{\beta\gamma\alpha}(w) + \frac{1}{2}\delta^{[a}_b\mathcal{W}^{cd]\dot{\alpha}}{}_{\beta\gamma\alpha}(w) - \delta^{[d}_b\mathcal{W}^{ca]\dot{\alpha}}{}_{\beta\gamma\alpha}(w),$$
$$\mathcal{V}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[ab]\delta}(w)|_{\frac{1}{(z-w)}} = 2\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{[ab]\delta}(w),$$
$$\mathcal{R}^{a}{}_{[b}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cd]\delta}(w)|_{\frac{1}{(z-w)}} = \delta^a_{[c}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{bd]\delta}(w) - \frac{1}{2}\delta^a_{[b}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cd]\delta}(w) + \delta^a_{[d}\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{cb]\delta}(w),$$
$$\mathcal{R}^{a}{}_{b}(z)\mathcal{W}^{c\dot{\alpha}\dot{\beta}}{}_{d\gamma\delta}(w)|_{\frac{1}{(z-w)}} = \delta^a_d\mathcal{W}^{c\dot{\alpha}\dot{\beta}}{}_{b\gamma\delta}(w) - \delta^c_b\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{d\gamma\delta}(w),$$
$$\mathcal{Q}^{a}{}_{\alpha}(z)\mathcal{W}^{c\dot{\alpha}\dot{\beta}}{}_{d\gamma\delta}(w)|_{\frac{1}{(z-w)}} = \delta^a_d\mathcal{W}^{c\dot{\alpha}\dot{\beta}}{}_{\alpha\gamma\delta}(w) - \frac{1}{4}\delta^c_d\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{\alpha\gamma\delta}(w),$$
$$\dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\mathcal{W}^{c\dot{\beta}\dot{\gamma}}{}_{d\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\delta^c_a\mathcal{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{d\beta\gamma}(w) + \frac{1}{4}\delta^c_d\mathcal{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{a\beta\gamma}(w),$$
$$\mathcal{Q}^{a}{}_{\alpha}(z)\mathcal{W}^{\dot{\beta}\dot{\gamma}}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} = \mathcal{W}^{a\dot{\beta}\dot{\gamma}}{}_{\alpha\beta\gamma}(w), \qquad \dot{\mathcal{Q}}^{\dot{\alpha}}{}_{a}(z)\mathcal{W}^{\dot{\beta}\dot{\gamma}}{}_{\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{a\beta\gamma}(w).$$

(vi) The $s = \frac{7}{2}$ case.
As well as the OPEs in Sec. III G there are following OPEs with first order poles

$$\mathcal{V}(z)\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{\gamma\delta\epsilon}(w)|_{\frac{1}{(z-w)}} = -\mathcal{W}^{a\dot{\alpha}\dot{\beta}}{}_{\gamma\delta\epsilon}(w),$$
$$\mathcal{R}^{a}{}_{b}(z)\mathcal{W}^{c\dot{\beta}\dot{\gamma}}{}_{\alpha\beta\gamma}(w)|_{\frac{1}{(z-w)}} = -\delta^c_b\mathcal{W}^{a\dot{\beta}\dot{\gamma}}{}_{\alpha\beta\gamma}(w) + \frac{1}{4}\delta^a_b\mathcal{W}^{c\dot{\beta}\dot{\gamma}}{}_{\alpha\beta\gamma}(w).$$

(vii) The $s = 4$ case.
Finally, we have the following first order poles

$$\mathcal{V}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{b\delta\epsilon}(w)|_{\frac{1}{(z-w)}} = \dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{b\delta\epsilon}(w),$$
$$\mathcal{R}^{a}{}_{b}(z)\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{c\delta\epsilon}(w)(w)|_{\frac{1}{(z-w)}} = \delta^a_c\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{b\delta\epsilon}(w) - \frac{1}{4}\delta^a_b\dot{\mathcal{W}}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{c\delta\epsilon}(w).$$

Therefore, we have calculated the first order poles in the OPEs between five weight-1 operators and the weight-3 operators appearing in the Table I.

APPENDIX D: THE COMPLETE OPE BETWEEN $J^I{}_J(z)$ AND $J^K{}_L J^M{}_N J^P{}_Q(w)$

From the defining OPE in (2.7), we can calculate the remaining fourth, third, and second order poles of the OPE $J^I{}_J(z) J^K{}_L J^M{}_N J^P{}_Q(w)$. The first order pole is given by (3.1).

The fourth order pole can be written as

$$J^I{}_J(z) J^K{}_L J^M{}_N J^P{}_Q(w)|_{\frac{1}{(z-w)^4}} = (-1)^{d_J d_P+1} \delta^I{}_L \delta^K{}_N \delta^M{}_Q \delta^P{}_J + (-1)^{(d_N+d_M)(d_K+d_J)+d_N d_P} \delta^I{}_L \delta^M{}_J \delta^K{}_Q \delta^P{}_N + (-1)^{(d_L+d_K)(d_I+d_J)+1} \times [(-1)^{d_L d_P+1} \delta^K{}_J \delta^I{}_N \delta^M{}_Q \delta^P{}_L + (-1)^{(d_N+d_M)(d_I+d_L)+d_N d_P} \delta^K{}_J \delta^M{}_L \delta^I{}_Q \delta^P{}_N]. \tag{D1}$$

The third order pole is summarized by

$$\begin{aligned} J^I{}_J(z) J^K{}_L J^M{}_N J^P{}_Q(w)|_{\frac{1}{(z-w)^3}} = {} & [(-1)^{d_J d_M+1} \delta^I{}_L \delta^K{}_N \delta^M{}_J J^P{}_Q + \delta^I{}_L \delta^K{}_N \delta^M{}_Q J^P{}_J \\ & + (-1)^{(d_P+d_Q)(d_M+d_J)+1} \delta^I{}_L \delta^K{}_N \delta^P{}_J J^M{}_Q + (-1)^{(d_N+d_M)(d_K+d_J)+1} \delta^I{}_L \delta^M{}_J \delta^M{}_J \delta^K{}_Q J^P{}_N \\ & + (-1)^{(d_N+d_M)(d_K+d_J)+(d_P+d_Q)(d_K+d_N)} \delta^I{}_L \delta^M{}_J \delta^P{}_N J^K{}_Q \\ & + (-1)^{(d_K+d_J)(d_N+d_M)+d_J d_P+1} \delta^I{}_L \delta^K{}_Q \delta^P{}_J J^M{}_N \\ & + (-1)^{(d_L+d_K)(d_I+d_J)+1} ((-1)^{d_L d_M+1} \delta^K{}_J \delta^I{}_N \delta^M{}_L J^P{}_Q + \delta^K{}_J \delta^I{}_N \delta^M{}_Q J^P{}_L \\ & + (-1)^{(d_P+d_Q)(d_M+d_L)+1} \delta^K{}_J \delta^I{}_N \delta^P{}_L J^M{}_Q + (-1)^{(d_N+d_M)(d_I+d_L)+1} \delta^K{}_J \delta^M{}_L \delta^I{}_Q J^P{}_N \\ & + (-1)^{(d_N+d_M)(d_I+d_L)+(d_P+d_Q)(d_I+d_N)} \delta^K{}_J \delta^M{}_L \delta^P{}_N J^I{}_Q \\ & + (-1)^{(d_I+d_L)(d_N+d_M)+d_L d_P+1} \delta^K{}_J \delta^I{}_Q \delta^P{}_L J^M{}_N) \\ & + (-1)^{(d_I+d_J)(d_K+d_L)} ((-1)^{d_J d_P+1} \delta^I{}_N \delta^M{}_Q \delta^P{}_J J^K{}_L \\ & + (-1)^{(d_N+d_M)(d_I+d_J)+d_N d_P} \delta^M{}_J \delta^I{}_Q \delta^P{}_N J^K{}_L)](w). \end{aligned} \tag{D2}$$

Finally, the second order pole is described by

$$\begin{aligned} J^I{}_J(z) J^K{}_L J^M{}_N J^P{}_Q(w)|_{\frac{1}{(z-w)^2}} = {} & [(-1)^{d_J d_K+1} \delta^I{}_L \delta^K{}_J J^M{}_N J^P{}_Q + \delta^I{}_L \delta^K{}_N J^M{}_J J^P{}_Q \\ & + (-1)^{(d_N+d_M)(d_K+d_J)+1} \delta^I{}_L \delta^M{}_J J^K{}_N J^P{}_Q + (-1)^{(d_K+d_J)(d_N+d_M)} \delta^I{}_L \delta^K{}_Q J^M{}_N J^P{}_J \\ & + (-1)^{(d_K+d_J)(d_N+d_M+d_P+d_Q)+1} \delta^I{}_L \delta^P{}_J J^M{}_N J^K{}_Q \\ & + (-1)^{(d_L+d_K)(d_I+d_J)+1} \delta^K{}_J (\delta^I{}_N J^M{}_L J^P{}_Q + (-1)^{(d_N+d_M)(d_I+d_L)+1} \delta^M{}_L J^I{}_N J^P{}_Q \\ & + (-1)^{(d_I+d_L)(d_N+d_M)} \delta^I{}_Q J^M{}_N J^P{}_L + (-1)^{(d_I+d_L)(d_N+d_M+d_P+d_Q)+1} \delta^P{}_L J^M{}_N J^I{}_Q) \\ & + (-1)^{(d_I+d_J)(d_K+d_L)} J^K{}_L ((-1)^{d_J d_M+1} \delta^I{}_N \delta^M{}_J J^P{}_Q + \delta^I{}_N \delta^M{}_Q J^P{}_J \\ & + (-1)^{(d_P+d_Q)(d_M+d_J)+1} \delta^I{}_N \delta^P{}_J J^M{}_Q + (-1)^{(d_N+d_M)(d_I+d_J)+1} \delta^M{}_J \delta^I{}_Q J^P{}_N \\ & + (-1)^{(d_N+d_M)(d_I+d_J)+(d_P+d_Q)(d_I+d_N)} \delta^M{}_J \delta^P{}_N J^I{}_Q \\ & + (-1)^{(d_I+d_J)(d_N+d_M)+d_J d_P+1} \delta^I{}_Q \delta^P{}_J J^M{}_N)](w). \end{aligned} \tag{D3}$$

Therefore, the complete OPE is given by Eqs. (D1)–(D3) and (3.1).

We can also express the various (anti)commutator relations by using the above OPE. See the Ref. [31] for explicit formula. Let us consider the first OPE in (3.5) having an extra generator. It is obvious to obtain that the third order pole is given by $\delta^b_a \mathcal{P}^{\dot\alpha}{}_\beta$ from Eq. (D2) and the second order pole is given by $-\frac{1}{2}\delta^b_a \partial \mathcal{P}^{\dot\alpha}{}_\beta + \mathcal{V}^{b\dot\alpha}{}_{a\beta}$ with $\mathcal{V}^{b\dot\alpha}{}_{a\beta} \equiv -3\delta^b_a \mathcal{V}\mathcal{P}^{\dot\alpha}{}_\beta - 3\mathcal{Q}^b{}_\beta \dot{\mathcal{Q}}^{\dot\alpha}{}_a + \frac{3}{2}\delta^b_a \partial \mathcal{P}^{\dot\alpha}{}_\beta$ from Eq. (D3). Here we intentionally split the second order pole into the descendant of the weight-1 operator $\delta^b_a \mathcal{P}^{\dot\alpha}{}_\beta$ and the (quasi) primary operator. The first order pole is again given in (3.5).

Then we obtain the following anticommutator relation by using the formula in [31] or performing the two contour integrals in conformal field theory explicitly

$$\{(\dot{\mathcal{Q}}^{\dot{\alpha}}{}_a)_m, (\mathcal{W}^b{}_\beta)_n\} = \frac{1}{2}m(2m+n)\delta^b_a(\mathcal{P}^{\dot{\alpha}}{}_\beta)_{m+n} + m(\mathcal{V}^{b\dot{\alpha}}{}_{a\beta})_{m+n} + \delta^b_a(\mathcal{W}^{\dot{\alpha}}{}_\beta)_{m+n} + (\hat{\mathcal{W}}^{b\dot{\alpha}}{}_{a\beta})_{m+n}. \tag{D4}$$

Note that the coefficients, $\frac{1}{2}m(2m+n)$, m, 1 and 1, appearing in the right-hand side of Eq. (D4) hold for any (anti)commutator relations we are considering in the OPEs between the weight-1 operator and the weight-3 operator. The nonzero central terms can appear in the corresponding (anti)commutator relations. We should subtract the right descendant terms with coefficient $-\frac{1}{2}$ in the second order pole explained before in order to use the above general behavior. The weight-3 operator is not a quasiprimary operator, in general, from Eq. (B1). In order to use the formula in [31], we should check the quasiprimary condition on the weight-3 operator.

Compared to the result of [11,13], the first three terms of Eq. (D4) should appear and the last term reflects the new generator coming from the world sheet symmetry algebra. We expect that all the other (anti)commutator relations like as Eq. (D4) with possible central terms or new generators can be obtained and they (without new generators) with some normalizations should appear in $hs(2,2|4)$ in the work of [11,13]. Although we observe that there are no vanishing terms of the right-hand sides in Eq. (D4) under the restriction of wedge modes, it is an open problem to check whether the possibility of vanishings for the right-hand sides in the (anti)commutator relations under the wedge constraints (when we consider other OPEs for higher weights) arises or not.

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