

## Null surface thermodynamics

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We establish that boundary degrees of freedom associated with a generic codimension one null surface in  $D$ -dimensional pure Einstein gravity naturally admit a thermodynamical description. We expect the *null surface thermodynamics* to universally follow as a result of the diffeomorphism invariance of the theory, not relying on other special features of the null surface or the gravity theory. Using standard surface charge analysis and covariant phase-space method, we formulate laws of null surface thermodynamics which are local equations over an arbitrary null surface paralleling local versions of the zeroth- and first laws and the Gibbs-Duhem equation. This thermodynamical system is generally an open system and can be closed only when there is no flux of gravitons through the null surface. Our analysis extends the usual black hole thermodynamics to a universal feature of any area element on a generic null surface. We discuss the relevance of our study for the membrane paradigm and black hole microstates.

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### I. INTRODUCTION

Despite apparent differences, there are various hints that gravity, as formulated by Einstein's general relativity (GR), and thermodynamics are closely related to each other, both at conceptual and formulation levels. Perhaps, the first hint was already in the statement of Einstein's equivalence principle and the universality of GR and thermodynamics. In the context of black hole physics the resemblance between laws of black hole mechanics and laws of thermodynamics [1] was gradually completed into the equivalence of the two [2–10].

The connection is not limited to black holes. In a seminal paper [11], Unruh showed that there is a nonzero temperature associated with a generic accelerated observer, as required by the equivalence principle. The next remarkable step was provided by Wald, who showed that black hole entropy is a conserved charge associated with bifurcate Killing horizons [12] and derived the first law of

thermodynamics for generic probes around such black holes as a direct consequence of diffeomorphism invariance [13]. And finally, Jacobson derived Einstein's field equations from the first law of thermodynamics adapted around a null surface [14]; see also [15,16] and [17–19]. The connection between gravity and thermodynamics was also reinforced through the holographic principle [20] and the AdS/CFT duality [21] and presented bluntly in [22].

Thermodynamics aspects of black holes are generically attributed to black hole microstates and to the horizon. Horizon is typically a null surface which is the boundary of the spacetime as viewed by non-free-fall observers outside the hole. Presence of boundaries leads to existence of “boundary degrees of freedom” which reside only at the (timelike or null) boundaries and interact with bulk (graviton) modes. Black hole microstates may be sought among these boundary degrees of freedom. With this motivation, we study gravity theory on spacetimes with a null boundary. This boundary can be an arbitrary one in spacetime and is not necessarily horizon of a black hole. This has been the research program pursued in some recent works [23–26] and in particular in [27–42]. It has been established that (see [24,27] and references therein) the most general solution phase space of  $D$ -dimensional pure Einstein gravity theory around a given null surface besides the bulk graviton modes involves boundary modes parametrized by  $D$  arbitrary functions over the  $D - 1$  dimensional null boundary.

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In this work we show that the thermodynamics description is not limited to (black hole) horizons. We recast the equations used in boundary symmetry and charge analysis for a pure  $D$ -dimensional Einstein gravity in presence of a null boundary as the local first law of thermodynamics and the Gibbs-Duhem equation. We construct the solution phase space governing the boundary degrees of freedom and show it can be naturally viewed as an open thermodynamic phase space. This open thermodynamics system can be closed if we turn off the graviton flux passing through the null surface. The latter, together with an extra relation among the chemical potentials and associated surface charges (3.9), yields the statement of the local zeroth law. These relations are universal and independent of details of gravitational theory or the null surface. Our derivation only relies on diffeomorphism invariance of the theory and we expect our thermodynamical description to be true for any generally invariant theory of gravity.

## II. NULL SURFACE SOLUTION PHASE SPACE: A REVIEW

We start with a  $D$ -dimensional ( $D \geq 3$ ) generic metric adopting  $v, r, x^A$  coordinates,

$$ds^2 = -Vdv^2 + 2\eta dvdr + g_{AB}(dx^A + U^A dv)(dx^B + U^B dv), \quad (2.1)$$

such that  $r = 0$  is the null surface  $\mathcal{N}$  and metric coefficients admit the near- $\mathcal{N}$  expansion,

$$\begin{aligned} V &= -\eta \left( \Gamma - \frac{2}{D-2} \frac{\mathcal{D}_v \Omega}{\Omega} + \frac{\mathcal{D}_v \eta}{\eta} \right) r + \mathcal{O}(r^2) \\ U^A &= \mathcal{U}^A - r \frac{\eta}{\Omega} \mathcal{J}^A + \mathcal{O}(r^2) \\ g_{AB} &= \Omega_{AB} + \mathcal{O}(r), \end{aligned} \quad (2.2)$$

where all the fields are functions of  $v, x^A$  and

$$\Omega_{AB} = \Omega^{2/(D-2)} \gamma_{AB}, \quad \Omega := \sqrt{\det \Omega_{AB}}, \quad \det \gamma_{AB} = 1. \quad (2.3)$$

We use the definition

$$\mathcal{D}_v := \partial_v - \mathcal{L}_{\mathcal{U}}, \quad (2.4)$$

where  $\mathcal{L}_{\mathcal{U}}$  is the Lie derivative along the  $\mathcal{U}^A$  direction. Let  $\Theta$  be the expansion of vector field generating the null surface  $\mathcal{N}$  and  $N_{AB}$  be the *news tensor* associated with flux of gravitons through  $\mathcal{N}$ :

$$\Theta := \mathcal{D}_v \ln \Omega, \quad N_{AB} := \frac{1}{2} \Omega^{2/(D-2)} \mathcal{D}_v \gamma_{AB}. \quad (2.5)$$

We use  $\Omega^{AB}$  and  $\Omega_{AB}$ , respectively, for raising and lowering capital Latin indices. Note that  $N_{AB}$  as defined above is a symmetric-traceless tensor.

The coefficients appearing in the metric are subject to Einstein field equations. In particular, there are Raychaudhuri and Damour equations which play a crucial role in our analysis; see [27] for a more detailed treatment. These two equations, respectively, are

$$\mathcal{D}_v \Theta + \frac{1}{2} \left( \Gamma - \frac{\mathcal{D}_v \eta}{\eta} \right) \Theta + N_{AB} N^{AB} = 0, \quad (2.6a)$$

$$\mathcal{D}_v \mathcal{J}_A + \Omega \Theta \frac{\partial_A \eta}{\eta} + \Omega \partial_A (\Gamma - 2\Theta) + 2\Omega \bar{\nabla}^B N_{AB} = 0. \quad (2.6b)$$

Here  $\bar{\nabla}_A$  denotes the covariant derivative with respect to the metric  $\Omega_{AB}$ . The new variable  $\mathcal{P}$ , defined as

$$\mathcal{P} := \ln \frac{\eta}{\Theta^2}, \quad (2.7)$$

may substitute  $\eta$  and in terms of which (2.6) simplifies to

$$\mathcal{D}_v \Omega = \Omega \Theta, \quad (2.8a)$$

$$\mathcal{D}_v \mathcal{P} = \Gamma + \frac{2}{\Theta} N_{AB} N^{AB}, \quad (2.8b)$$

$$\mathcal{D}_v \mathcal{J}_A + \Omega \Theta \partial_A \mathcal{P} + \Omega \partial_A \Gamma + 2\Omega \bar{\nabla}^B N_{AB} = 0. \quad (2.8c)$$

### A. Off-shell presymplectic form

Starting from the Einstein-Hilbert action,

$$S_{\text{EH}} = \frac{1}{16\pi G} \int dr dv d^{D-2} x \eta \sqrt{\det g_{AB}} L_{\text{EH}}, \quad L_{\text{EH}} = R - 2\Lambda, \quad (2.9)$$

one can compute the usual Lee-Wald presymplectic form [43] over the set of geometries (2.1), yielding

$$\begin{aligned} \Omega_{\text{LW}} &= \frac{1}{16\pi G} \int_{\mathcal{N}} dv d^{D-2} x [\delta \mathcal{U}^A \wedge \delta \mathcal{J}_A - \delta \Gamma \wedge \delta \Omega \\ &\quad + \delta(\Omega \Theta) \wedge \delta \mathcal{P} + \delta \Omega_{AB} \wedge \delta(\Omega N^{AB})]. \end{aligned} \quad (2.10)$$

While the above expression clearly shows which variables are canonical conjugate of each other, the functions appearing there are subject to Eqs. (2.8) and not all of them are independent. In other words, the solution phase space is obtained after imposing the constraints (2.8) upon the parameter space and the symplectic form, e.g., using the Dirac bracket method or going to the reduced phase space.

### B. Null boundary symmetry generators

The vector field

$$\xi = T\partial_v + r(\mathcal{D}_v T - W)\partial_r + (Y^A - r\eta\partial^A T)\partial_A + \mathcal{O}(r^2), \quad (2.11)$$

preserves the form of metric (2.1), keeps  $r=0$  a null surface, and generates the following variations over the solution phase-space functions:

$$\delta_\xi \eta = T\partial_v \eta + 2\eta\mathcal{D}_v T - W\eta + Y^A \partial_A \eta, \quad (2.12a)$$

$$\delta_\xi \Gamma = -\mathcal{D}_v(W - \Gamma T) + (Y^A + \mathcal{U}^A T)\partial_A \Gamma, \quad (2.12b)$$

$$\delta_\xi(\Omega\Theta) = \mathcal{D}_v(T\Omega\Theta) + \mathcal{L}_{(Y+T\mathcal{U})}(\Omega\Theta), \quad (2.12c)$$

$$\delta_\xi \mathcal{U}^A = \mathcal{D}_v(Y^A + T\mathcal{U}^A), \quad (2.12d)$$

$$\delta_\xi \Omega_{AB} = 2TN_{AB} + \mathcal{L}_{(Y+T\mathcal{U})}\Omega_{AB} + \frac{2}{D-2}T\Theta\Omega_{AB}, \quad (2.12e)$$

where  $\mathcal{L}_Y$  denote the Lie derivative along  $Y^A$ , and for associated conjugate charges (see below)

$$\delta_\xi \Omega = T\Omega\Theta + \mathcal{L}_{(Y+T\mathcal{U})}\Omega, \quad (2.13a)$$

$$\delta_\xi \mathcal{P} = T\mathcal{D}_v \mathcal{P} + \mathcal{L}_{(Y+T\mathcal{U})}\mathcal{P} - W, \quad (2.13b)$$

$$\begin{aligned} \delta_\xi \mathcal{J}_A &= T\mathcal{D}_v \mathcal{J}_A + \mathcal{L}_{(Y+T\mathcal{U})}\mathcal{J}_A \\ &+ \Omega[\partial_A W - \Gamma\partial_A T - 2N_{AB}\partial^B T], \end{aligned} \quad (2.13c)$$

$$\delta_\xi N_{AB} = \mathcal{D}_v(TN_{AB}) + \mathcal{L}_{(Y+T\mathcal{U})}N_{AB}. \quad (2.13d)$$

### C. Surface charge variation

One may compute the charge variation associated with the boundary symmetry generators using covariant phase-space method [13,43]. Detailed analysis yields [27]

$$\begin{aligned} \delta Q_\xi &= \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x [(W - \Gamma T)\delta\Omega + (Y^A + \mathcal{U}^A T)\delta\mathcal{J}_A \\ &+ T\Omega\Theta\delta\mathcal{P} - T\Omega\Omega^{AB}\delta N_{AB}], \end{aligned} \quad (2.14)$$

where  $\mathcal{N}_v$  is a constant  $v$  section on  $\mathcal{N}$ . This charge variation is an integral over  $\sum_{i=1}^4 \mathcal{G}_i \delta \mathcal{Q}_i$ , where  $\mathcal{Q}_i$  parametrizes the solution phase space. Among the four families,  $N_{AB}$  corresponds to the bulk degrees of freedom while three others  $\Omega, \mathcal{J}_A, \mathcal{P}$  parametrize boundary information.  $\Gamma, \mathcal{U}^A$  functions which appear in  $\mathcal{G}_i$  are subject to field

equations (2.8) and  $\delta \mathcal{Q}_i$  subject to linearized equations of motion.

The  $\mathcal{G}_i$  are ‘‘field-dependent’’ linear combinations of symmetry generators  $T, W, Y^A$ , notably  $\mathcal{G}_i$  depend on  $\Gamma, \mathcal{U}^A$  as well as  $\Omega\Theta$  and  $\Omega_{AB}$  and  $\delta_\xi \mathcal{G}_i \neq 0$ . The charge variation  $\delta Q_\xi$ , as stressed in the notation  $\delta$ , is hence not integrable.  $\Gamma, \mathcal{U}^A$  may be, respectively, solved for in terms of the charges using (2.8b) and (2.8c) and therefore all these coefficients may be represented through the charges. Note also that there are three symmetry generators and four towers of charges and these are functions over the null surface  $\mathcal{N}$ . We crucially note that  $\delta\Omega, \delta\mathcal{J}_A, \delta\mathcal{P}, \delta N_{AB}$  denote generic variations around solutions of equations of motion (EoM) and are subject to linearized field equations. These linearized equations may be viewed as equations for variations  $\delta\Gamma, \delta\mathcal{U}^A$ . The solution phase space is hence parametrized by the four tower of charges and their variations.

We close this part by giving expressions for three ‘‘zero-mode’’ charges,  $\xi = -r\partial_r$ ,  $\xi = \partial_A$ , and  $\xi = \partial_v$ . One may readily observe that the first two are integrable and the latter is not:

$$\begin{aligned} Q_{-r\partial_r} &:= \frac{\mathbf{S}}{4\pi} = \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x \Omega, \\ Q_{\partial_A} &:= \mathbf{J}_A = \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x \mathcal{J}_A, \\ \delta Q_{\partial_v} &:= \delta \mathbf{H} = \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x (-\Gamma\delta\Omega + \mathcal{U}^A \delta\mathcal{J}_A \\ &+ \Omega\Theta\delta\mathcal{P} - \Omega\Omega^{AB}\delta N_{AB}). \end{aligned} \quad (2.15)$$

### D. Surface charges and flux in thermodynamics slicing

The charge variation may be split into Noether (integrable) part  $Q^N$  and the ‘‘flux’’ part  $F$ :  $\delta Q_\xi = \delta Q_\xi^N + F_\xi(\delta g; g)$ .  $Q^N$  may be computed for the Einstein-Hilbert action using the standard Noether procedure, yielding

$$Q_\xi^N = \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x [W\Omega + Y^A \mathcal{J}_A + T(-\Gamma\Omega + \mathcal{U}^A \mathcal{J}_A)], \quad (2.16)$$

and nonintegrable flux part:

$$\begin{aligned} F_\xi(\delta g; g) &= \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x T(\Omega\delta\Gamma - \mathcal{J}_A \delta\mathcal{U}^A \\ &+ \Omega\Theta\delta\mathcal{P} - \Omega\Omega^{AB}\delta N_{AB}). \end{aligned} \quad (2.17)$$

Here we are assuming symmetry generators  $T, W, Y^A$  to be field independent, i.e.,  $\delta T = \delta W = 0 = \delta Y^A$ .

For later use, we also present the expressions for the zero-mode Noether charges:

$$\begin{aligned} Q_{-r\partial_r}^N &= \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x \Omega, \\ Q_{\partial_A}^N &= \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x \mathcal{J}_A, \\ Q_{\partial_v}^N &:= \mathbf{E} = \frac{1}{16\pi G} \int_{\mathcal{N}_v} d^{D-2}x (-\Gamma\Omega + \mathcal{U}^A \mathcal{J}_A), \end{aligned} \quad (2.18)$$

### E. Balance or “generalized charge conservation” equation

In our general setup charges and the flux are given by integrals over codimension two surface  $\mathcal{N}_v$ . They are hence functions of “lightcone time” coordinate  $v$  and the charges are not conserved. From the expressions above, one can deduce

$$\frac{d}{dv} Q_{\xi}^N \approx -F_{\partial_v}(\delta_{\xi}g; g), \quad (2.19)$$

where  $\approx$  denotes on-shell equality. Equation (2.19) may be viewed as (i) manifestation of the boundary EoM written in terms of charges; (ii) a generalized charge conservation equation as it relates to time dependence, or nonconservation, of the charge (as viewed by the null boundary observer) to the flux passing through the boundary; (iii) how the passage of flux through the null boundary is “balanced” by the rearrangements in the charges. In this respect, it is very similar to the usual balance equation used at asymptotic null surfaces, which is now written for an arbitrary null surface in the bulk. Note also that the third viewpoint yields *null surface memory effects* discussed in [27].

## III. NULL SURFACE THERMODYNAMICS

Consider a usual thermodynamical system with chemical potentials  $\mu_i (i = 1, 2, \dots, N)$  and temperature  $T$ . This system is specified with charges  $Q_i$ , the entropy  $S$ , and the energy  $E$ ; that is, there are  $N + 2$  charges and  $N + 1$  chemical potentials. The distinction between charges and associated chemical potentials is by convention and is specified with/specifies the ensemble. In microcanonical ensemble (which we have already assumed), the first law takes the form

$$dE = TdS + \sum_{i=1}^N \mu_i dQ_i. \quad (3.1)$$

This equation implies that the lhs is an exact one-form over the thermodynamic space. Moreover, chemical potentials and the charges are related by the Gibbs-Duhem relation:

$$SdT + \sum_{i=1}^N Q_i d\mu_i = 0. \quad (3.2)$$

Together with the first law (3.1) this yields  $E = TS + \sum_i \mu_i Q_i$ . This equation relates  $E$  to the other charges and chemical potentials, e.g.,  $E = E(S, Q_i)$  (in microcanonical description) or  $E = E(T, \mu_i)$  (in grandcanonical description). Depending on the ensemble chosen,  $N + 1$  number of chemical potentials and/or charges may be taken to be “independent” variables parametrizing the thermodynamical configuration space and the rest of  $N + 1$  of them as functions of the former  $N + 1$  variables. In other words, the thermodynamic configuration space is  $(N + 1)$ -dimensional and the change of ensemble is basically a canonical transformation, the generator of which is the difference between various “energy” functions associated with each ensemble.

### A. Null boundary thermodynamical phase space

Starting at the expression of the charge variation (2.14), one can recognize that functions parametrizing the solution space come in two categories: the bulk modes  $N_{AB}$  (and its conjugate “chemical potential” determinant-free part of  $\Omega^{AB}, \gamma^{AB}$ ) and the boundary modes. The latter may also be separated into those whose variation appears  $\Omega, \mathcal{P}$ , and  $\mathcal{J}_A$ , and those which appear only in the coefficients, in chemical potentials  $\Gamma, \mathcal{U}^A$ . There are hence  $D = 1 + 1 + (D - 2)$  charges and  $D - 1 = 1 + (D - 2)$  chemical potentials.

We crucially note that if we treat  $\Gamma, \mathcal{U}^A$  and associated charges  $\Omega, \mathcal{J}_A$  as independent variables,  $\mathcal{P}$  is special as it does not appear in the integrable part of the charges (2.16) and only appears in the expression for the flux (2.17) through  $\Omega\Theta\delta\mathcal{P}$  term. Moreover, as already remarked [cf. discussions below (2.14)], the chemical potentials may be expressed in terms of the charges using field equations. Again we note at  $\Theta = 0$ ,  $\mathcal{P}$  dependence completely drops out of the analysis.

Given all the above, we are led to the following picture for the generic case.

- (I) Null boundary solution space relevant to the null boundary thermodynamics consists of three parts:
  - (I.1)  $(D - 1)$  dimensional “thermodynamic sector” parametrized by  $(\Gamma, \mathcal{U}^A)$  and conjugate charges  $(\Omega, \mathcal{J}_A)$ ;
  - (I.2)  $\mathcal{P}$ , which only appears in the flux (2.17) and not in the Noether charge (2.16);
  - (I.3) the bulk mode parametrized by determinant-free part of  $\Omega^{AB}$  and its “conjugate charge”  $N_{AB}$  which appear in the flux (2.17).
- (II)  $N_{AB}$  parametrizes effects of the bulk and how they take the boundary system out-of-thermal equilibrium (OTE), whereas  $\mathcal{P}$  parametrizes OTE within the boundary dynamics. Put differently, OTE may come from inner boundary dynamics and/or from the

gravity waves passing through the null boundary, parametrized by  $N_{AB}$ .

- (III) Expansion parameter  $\Theta$  is a measure of OTE, from both bulk and boundary viewpoints. When  $\Theta = 0$  the system is completely specified by the  $D - 1$  dimensional thermodynamic phase space.
- (IV) The rest of the in-falling graviton modes parametrized through  $\mathcal{O}(r)$  terms in  $g_{AB}$  do not enter in the boundary-/thermodynamics, as of course expected from usual causality and that the boundary is a null surface.

We start with the local first law, then local Gibbs-Duhem equation and come to local zeroth law, specifying the

subsectors which can be brought to a (local) equilibrium. Before moving on, we introduce a piece of useful notation. By  $\mathcal{X}$  we will denote the density of the quantity  $\mathbf{X}$ ,

$$\mathbf{X} := \int_{\mathcal{N}_v} d^{D-2}x \mathcal{X}. \quad (3.3)$$

### B. Local first law at null boundary

Defining  $\mathcal{P} := \mathcal{P}/(16\pi G)$  and  $\mathcal{N}_{AB} := (16\pi G)^{-1}N_{AB}$ , (2.15) implies

$$\delta\mathcal{H} = T_{\mathcal{N}}\delta\mathcal{S} + \mathcal{U}^A\delta\mathcal{J}_A + \Omega\Theta\delta\mathcal{P} - \Omega\Omega^{AB}\delta\mathcal{N}_{AB}, \quad T_{\mathcal{N}} := -\frac{\Gamma}{4\pi}. \quad (3.4)$$

The above is true at each  $v, x^A$  over the null surface and represents the local null boundary first law. The lhs, unlike the usual first law (3.1), is not a complete variation, as the system is describing an open thermodynamic system due to the existence of the expansion and the flux. The above reduces to a usual first law for closed systems when  $N_{AB} = 0$  or in the nonexpanding  $\Theta = 0$  case.

Note that  $\Gamma = -2\kappa + \mathcal{D}_v \ln(\eta\Omega^{2D-2})$ , where  $\kappa$  is the non-affinity parameter (surface gravity) associated the vector field generating the null surface  $\mathcal{N}$  [27].  $-\Gamma/2$  is the local acceleration of an observer for whom  $r = 0$  is locally the Rindler horizon. So,  $T_{\mathcal{N}} = \frac{\kappa}{2\pi} - \frac{1}{4\pi}\mathcal{D}_v \ln(\eta\Omega^{2D-2})$ . For non-expanding  $\Theta = 0$  case where one may put  $\eta = 1$  or when we have a Killing horizon,  $T_{\mathcal{N}}$  equals the usual Unruh/Hawking temperature; cf. Sec. III D for more discussions.

### C. Local extended Gibbs-Duhem equation at null boundary

Gibbs-Duhem equation (3.2) is a relation among the thermodynamic charges. Given the expressions for the zero mode charges (2.18) and for the densities in the same notation as in (3.3), we have

$$\mathcal{E} = T_{\mathcal{N}}\mathcal{S} + \mathcal{U}^A\mathcal{J}_A. \quad (3.5)$$

The above is an analog of the Gibbs-Duhem equation if  $\mathcal{E}$  is viewed as energy,  $\mathcal{S}$  as entropy, and  $\mathcal{J}_A$  as other conserved charges and  $\Gamma, \mathcal{U}^A$  as the respective chemical potentials. This of course manifests the picture we outlined in Sec. III A. However, one should note that (3.5) is a local equation at the null boundary, unlike its usual thermodynamic counterpart. This equation also holds for nonstationary/nonadiabatic cases when the system is OTE. So, we call (3.5) ‘‘local extended Gibbs-Duhem’’ (LEGD) equation at the null boundary.

LEGD equation, like the local first law (3.4), is a manifestation of diffeomorphism invariance of the theory. While the explicit expressions for the charges do depend on the theory, we expect (3.5) to be universally true for any diff-invariant theory of gravity in any dimension. This equation is on par with the first law of thermodynamics but extends it in two important ways: it is a local equation in  $v, x^A$  and holds also for OTE.

Since the integrable parts of the charge are (by definition) independent of the bulk flux  $N_{AB}$  and also of  $\mathcal{P}$ , the LEGD equation (3.5), also do not involve  $\mathcal{P}$  and  $N_{AB}$ . Nonetheless, the chemical potentials in (3.5),  $\Gamma$  and  $\mathcal{U}^A$ , implicitly depend on  $N_{AB}$  and  $\mathcal{P}$  through Raychaudhuri and Damour equations.

### D. Local zeroth law

Zeroth law in the usual thermodynamics is a statement of thermal equilibrium: as a consequence of the zeroth law, two (sub)systems with the same temperature and chemical potentials are in thermal equilibrium. In the usual thermodynamics flow of charges is proportional to the gradient of associated chemical potentials and hence the absence of such fluxes can be taken as a statement of the zeroth law. In our case, we are dealing with a system parametrized by chemical potentials  $\Gamma, \mathcal{U}^A$  and  $\gamma^{AB}$  which are functions of charges  $Q_\alpha \in \{\Omega, \mathcal{P}, \mathcal{J}_A, N_{AB}\}$ . This system is not in general in equilibrium but there could be special subsectors which are. The zeroth law is to specify such subsectors.

Recalling (2.19), flow of charges vanishes on subsystems over which  $F_{\partial_v}(\delta_\xi g, g)$  vanishes. On a closely related account, one can show that [44] this flux has the same expression as the on-shell variation of the action. Nonetheless, while the charge variation (2.14) is invariant under the addition of a total derivative term to the Lagrangian, the Noether charge and hence the flux are not. In particular, upon addition of a boundary Lagrangian  $L_B = \partial_\mu \mathcal{B}^\mu$ , the on-shell

action variation and hence the flux  $F$  are shifted by  $\delta\mathcal{B}^r$ . For later convenience, let us call  $\mathcal{B}^r = \mathcal{G}$ . This opens up the possibility of (partially) removing the flux by an appropriate boundary term. The question is hence what are the subsectors in the solution phase space for which flux can be removed by an appropriate boundary term.

So, we start with the variation of on-shell action. A direct computation leads to

$$\begin{aligned} \delta S_{\text{EH}}|_{\text{on-shell}} &= \frac{1}{16\pi G} \int_{\mathcal{N}} dv d^{D-2}x (\Omega\Theta\delta\mathcal{P} + \Omega\delta\Gamma - \mathcal{J}_A\delta\mathcal{U}^A \\ -\Omega N^{AB}\delta\Omega_{AB}) &= \int dv F_{\partial_v}(\delta g; g), \end{aligned} \quad (3.6)$$

where  $F_{\partial_v}$  may be readily read from (2.17). Next, let us add a boundary term to the Lagrangian upon which  $\delta S_{\text{EH}}|_{\text{on-shell}} \rightarrow \delta S_{\text{EH}}|_{\text{on-shell}} + \int_{\mathcal{N}} \delta\mathcal{G}$ . As the statement of the zeroth law we require there exists a  $\mathcal{G} = \mathcal{G}(\Omega, \mathcal{P}, \mathcal{J}_A, N_{AB})$  such that

$$\delta\mathcal{G} = -\mathcal{S}(\delta T_{\mathcal{N}} - 4G\Theta\delta\mathcal{P}) - \mathcal{J}_A\delta\mathcal{U}^A + \Omega\mathcal{N}_{AB}\delta\Omega^{AB} \quad (3.7)$$

admits nonzero solutions. Integrability condition for the zeroth law (3.7) is  $\delta(\delta\mathcal{G}) = 0$ ,<sup>1</sup> which yields an equation like  $\sum_{\alpha,\beta} C_{\alpha\beta}\delta\mathcal{Q}_\alpha \wedge \delta\mathcal{Q}_\beta = 0$ , where  $\mathcal{Q}_\alpha$  are generic charges and  $C_{\alpha\beta}$  is skew symmetric. This equation is satisfied only for  $C_{\alpha\beta} = 0$ . One can immediately see  $N_{AB} = 0 = \delta N_{AB}$  is a necessary (but not sufficient) condition for (3.7) to have nontrivial solutions.

Before discussing the solutions in more detail, let us note that when (3.7) is fulfilled the charge  $\mathcal{H}$ , which appears in

the lhs of the local first law (3.4), becomes integrable and we obtain

$$\boxed{\mathcal{H} = \mathcal{G} + T_{\mathcal{N}}\mathcal{S} + \mathcal{U}^A\mathcal{J}_A}. \quad (3.8)$$

Besides  $N_{AB} = 0$ , in terms of  $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{J}_A, \mathcal{P})$  local zeroth law implies

$$\boxed{T_{\mathcal{N}} = \frac{\delta\mathcal{H}}{\delta\mathcal{S}}, \quad \mathcal{U}^A = \frac{\delta\mathcal{H}}{\delta\mathcal{J}_A}, \quad \mathcal{D}_v\mathcal{S} = \mathcal{S}\Theta = \frac{1}{4G}\frac{\delta\mathcal{H}}{\delta\mathcal{P}}}, \quad (3.9)$$

where the last equation may be seen as the *equation of state*. For the special case of  $\Theta = 0$ , one simply deduces that  $\mathcal{H}$  does not depend on  $\mathcal{P}$ . Equation (3.7) ensures that total energy and angular momentum are conserved on shell,  $\frac{d}{dv}\mathbf{H} = \frac{d}{dv}\mathbf{J}_A = 0$ , where  $\frac{d}{dv}\mathbf{X} := \int_{\mathcal{N}_v} d^{D-2}x \partial_v \mathcal{X}$ . Total entropy, on the other hand, is not conserved as  $\frac{d}{dv}\mathcal{S} = \int_{\mathcal{N}_v} d^{D-2}x \Theta \mathcal{S}$ ;  $\frac{d}{dv}\mathcal{S}$  is zero only when expansion vanishes,  $\Theta = 0$ .

### E. Generic

$\Theta \neq 0$  case The zeroth law requires  $N_{AB} = 0$  for which (2.8) reduce to

$$T_{\mathcal{N}} = -4GD_v\mathcal{P}, \quad \mathcal{D}_v[\mathcal{J}_A + 4G\bar{\nabla}_A(\mathcal{S}\mathcal{P})] = 0. \quad (3.10)$$

The above imply that zeroth law (3.9) is satisfied for any  $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{P}, \mathcal{J}_A)$ , when  $\mathcal{S}, \mathcal{P}$ , and  $\mathcal{J}_A$  have the following basic Poisson brackets [27]:

$$\begin{aligned} \{\mathcal{S}(x, v), \mathcal{P}(y, v)\} &= \frac{1}{4G}\delta^{D-2}(x-y), & \{\mathcal{S}(x, v), \mathcal{S}(y, v)\} &= \{\mathcal{P}(x, v), \mathcal{P}(y, v)\} = 0, \\ \{\mathcal{S}(x, v), \mathcal{J}_A(y, v)\} &= \mathcal{S}(y, v)\frac{\partial}{\partial x^A}\delta^{D-2}(x-y), \\ \{\mathcal{P}(x, v), \mathcal{J}_A(y, v)\} &= \left(\mathcal{P}(y, v)\frac{\partial}{\partial x^A} + \mathcal{P}(x, v)\frac{\partial}{\partial y^A}\right)\delta^{D-2}(x-y), \\ \{\mathcal{J}_A(x, v), \mathcal{J}_B(y, v)\} &= \frac{1}{16\pi G}\left(\mathcal{J}_A(y, v)\frac{\partial}{\partial x^B} - \mathcal{J}_B(x, v)\frac{\partial}{\partial y^A}\right)\delta^{D-2}(x-y), \end{aligned} \quad (3.11)$$

and  $\partial_v\mathcal{X} = \{\mathcal{H}, \mathcal{X}\}$ . That is,  $\mathcal{H}$  is the Hamiltonian over this phase space and (3.9) do not impose any restrictions on  $\mathcal{H}$  which is a scalar over  $\mathcal{N}$ .

### F. $\Theta = 0$ case

In this case trace of the extrinsic curvature of the null surface  $\mathcal{N}$  vanishes, hence it is an extremal null surface.

<sup>1</sup> $\delta(\delta\mathcal{G})$  is the presymplectic two-form (2.10); therefore, (3.7) implies vanishing symplectic form over the null surface  $\mathcal{N}$ .

Vanishing of the expansion  $\Theta$  has some important consequences. (i) Raychaudhuri equation implies  $N_{AB} = 0$ . So, again we arrive at the vanishing flux; (ii)  $\eta$  drops out from the charge variation (2.14). (iii) We lose one tower of the charge  $\mathcal{P}$ , and the associated symmetry generator becomes a pure gauge. (iv) We may fix the  $\eta = 1$  gauge which yields  $W = 2\mathcal{D}_v T$ . We hence remain with  $T, Y^A$  generators which form  $\text{Diff}(\mathcal{N})$  symmetry algebra. (v) EoM (2.6) reduces to  $\mathcal{D}_v\mathcal{J}_A = \mathcal{S}\partial_A T_{\mathcal{N}}$  and  $\mathcal{D}_v\mathcal{S} = 0$ , which may be viewed as equations for spatial derivatives of the chemical potentials.

Local zeroth law (3.9) is satisfied by any scalar Hamiltonian  $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{J}_A)$ , together with basic Poisson brackets (3.11) but with  $\mathcal{P}$  dropped [27] and again with  $\partial_\nu \mathcal{X} = \{\mathcal{H}, \mathcal{X}\}$ .

**G. Closing remarks**

(i) Local zeroth law (3.9) is just defining the Poisson bracket structure over our charge space and existence of Hamiltonian dynamics, but does not specify a Hamiltonian. (ii) Choice of Hamiltonian fixes a boundary Lagrangian and the boundary dynamical equations which in turn specifies local dynamics of charges on the null boundary  $\mathcal{N}$ . (iii) In analogy with isolated horizon [45] of black holes, if the zeroth law holds the null surface may be called an “isolated null surface”.<sup>2</sup> (iv) Our zeroth law is a weaker condition than stationarity as  $\partial_\nu$  of the chemical potentials need not vanish. (v) The usual zeroth law of black hole mechanics (for Killing horizons) that  $\mathcal{U}^A$  and  $\Gamma$  are constants over the horizon (our null boundary  $\mathcal{N}$ ) is a very special case which obeys our local zeroth law. For the stationary asymptotic flat black hole solutions to the vacuum Einstein gravity, i.e., the Myers-Perry solutions, we get  $\mathcal{E} = \left(\frac{D-3}{D-2}\right)\mathcal{H}$ , and we have the usual Smarr formula.

**IV. OUTLOOK**

Building upon the analyses of [23–26] and in particular [27], we established that the solution phase space around an arbitrary null surface in pure  $D$ -dimensional Einstein gravity naturally admits a thermodynamical description and the charges and corresponding chemical potentials form a “thermodynamical phase space.” The laws of thermodynamics are all local equations over the  $D - 1$  dimensional null surface  $\mathcal{N}$  and our analysis does not fix the boundary dynamics, boundary Hamiltonian, which may still be chosen.

As discussed, the zeroth law necessitates vanishing of the flux of bulk gravitons through  $\mathcal{N}$  and establishes basic Poisson brackets on the thermodynamic phase space. The same condition, absence of  $N_{AB}$ , has been discussed as the condition for the existence of a slicing in the solution phase space in which the charges become integrable [24,26,27]. The physics of change of phase-space slicing and null surface thermodynamics developed here is an interesting direction to for further investigations.

In this work we focused on the zeroth- and first laws and the Gibbs-Duhem equation. However, the second law of thermodynamics is an important part of any thermodynamic description. In the black hole thermodynamics, there is the “generalized second law” stating that the sum of entropies of black hole and the outside do not decrease; see [10] and

references therein. A simplified version of the second law is Hawking’s area theorem that in black hole merger processes the area of horizon does not decrease which uses Rchyaudhuri equation (focusing theorem) and null energy condition for the matter fields; see, e.g., [46]. In our setting, one may look for a local version of the second law, recalling that flux of gravitons can only move through the null boundary to the “inside region” and nothing comes out. See in particular the analysis in Sec. 8 of [27]. As the first relevant step towards a local second law we have worked through “null surface focusing theorem” in the Appendix. Further analysis and discussion on this very important issue is left for future work.

Our analysis is based on covariant phase-space formulation and hence readily generalizes to any diffeomorphism invariant theory in any dimension. Given all the previous literature, especially [13], it is reasonably certain that the same “local” thermodynamical description with exactly the same equations should also hold for this generic case. In other words, diffeomorphism invariance yields the local thermodynamical picture. One may do the reverse and show that the thermodynamical description results in diffeomorphism invariance. We should emphasize that this is already weaker than Einstein’s equivalence principle and nonminimal coupling and generic modified gravity theories follow the same analysis. The connection between thermodynamics and gravity is nothing new, e.g., see [14]. Our analysis here derives the local thermodynamics relation which is assumed in [14] from first principles and does not require the null boundary to have any extra properties. Our approach here, in contrast to the usual viewpoint, puts the emphasis on the boundary phase space, rather than the bulk graviton modes. In our local thermodynamics description, the latter appear as the flux (news) through the boundary. We hope our new “boundary-based” viewpoint and framework shed new light on thermodynamics/gravity relations and can be pushed to the quantum level.

The local thermodynamical description in its basics and general ideas reminds one of the membrane paradigm [47–49]. It is interesting to relate the two more systematically. The first steps in this direction were outlined in [50]. The interesting question is whether the membrane picture restricts the boundary Hamiltonian.

Among other things, our analysis here very clearly shows how the boundary and bulk degrees of freedom interact and that the boundary phase space admits the local thermodynamical description. This is expected to be very relevant for the black hole microstate problem in that the boundary degrees of freedom and quantization thereof can account for the microstates, whereas the interactions with bulk modes would be relevant for the information loss problem. Our analysis permits a semiclassical setting in which boundary phase space is quantized while bulk modes are treated (semi) classically and hence potentially gives a better handle on both microstate and information loss problems.

<sup>2</sup>One may show by direct computation that, upon zeroth law in the  $\Theta = 0$  sector,  $\mathcal{D}_\nu \mathcal{J}_A = \mathcal{S} \partial_A T_{\mathcal{N}}$  and  $\mathcal{D}_\nu \mathcal{S} = 0$  and also  $\mathcal{D}_\nu \mathcal{E} = -\partial_\nu \mathcal{G}$ ,  $\mathcal{D}_\nu \mathcal{H} = -\partial_A (\mathcal{U}^A \mathcal{G})$ .

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## APPENDIX: NULL BOUNDARY FOCUSING THEOREM

From Raychaudhuri equation (2.6) one learns that

$$\mathcal{D}_v \Theta - \kappa \Theta + \frac{1}{D-2} \Theta^2 \leq 0, \quad \kappa := -\frac{\Gamma}{2} + \frac{1}{2} \mathcal{D}_v \ln(\eta \Omega^{\frac{2}{D-2}}). \quad (\text{A1})$$

In terms of variable

$$X(v) := \exp\left(\int_{v_0}^v \kappa\right), \quad \mathcal{D}_v X - \kappa X = 0, \quad X(v_0) = 1, \quad (\text{A2})$$

assuming  $\Theta(v) \neq 0$ , (A1) implies

$$\Theta \leq \frac{\Theta^0 X(v)}{1 + \frac{\Theta^0}{D-2} \int_{v_0}^v X(v)}, \quad (\text{A3})$$

where  $\Theta^0 = \Theta(v = v_0)$  and without loss of generality we have taken  $v_0$  such that the equality is saturated. Since  $X(v) \geq 0$ , then  $\int_{v_0}^v X(v)$  is a growing function of  $v$ . If  $\Theta^0 < 0$  (that is if we start in a contracting phase) then there will always be a “trapping time”  $v_1 > v_0$  where  $\int_{v_0}^{v_1} X(v) = -\frac{1}{\Theta^0(D-2)}$ , and  $\Theta(v_1) \rightarrow -\infty$ .

If at some  $v$ ,  $\hat{v}_0$ ,  $\Theta = 0$  then its derivative should be nonpositive at that point  $\mathcal{D}_v \Theta \leq 0$ . For the case of equality we have a nonexpanding case and the  $N_{AB}$  should also vanish and if  $\mathcal{D}_v \Theta < 0$  then at  $\hat{v}_0 + \delta v$ ,  $\Theta < 0$  and again (A3) implies existence of a trapping time. We should stress that all the above analysis is *local* on the codimension 2, constant  $v$ ,  $r$  surfaces and all quantities are functions of  $x^A$ .

In the absence of bulk modes,  $N_{AB} = 0$ , the above inequality is replaced with equality. In this case (A3) shows internal null boundary dynamics which is of course due to gravity effects. In other words, gravity dynamics is relating thermodynamical sector of the solution phase space to the other two parts, the  $\eta$  part and the bulk modes, and this dynamics is featured in the focusing equation.

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