Ghost and tachyon free propagation up to spin 3 in Lorentz invariant field theories

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We complete the set of spin-projector operators for fields up to rank 3 by providing all operators connecting sectors with the same spin and parity. In this way, we can broaden the search for unitary and nontachyonic particle propagation in quadratic Lagrangians with interfield mixing. We use the properties of projector algebra to reanalyze known theories and shed light on new, healthy ones. We do so with full control over the gauge constraints by determining the form of the saturated propagator in an appropriate frame of reference.

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I. INTRODUCTION

The use of fields and local Lagrangians to fit the quantum behavior of relativistic particles has met unparalleled experimental support. In the effort to craft coherent quantum field models two main approaches can be followed which carry different views for the symmetries of the theory. In a up-bottom approach, symmetries are the starting point that shape the field interactions for a given particle content. In the lower spin sector, which can be described by low-rank fields, the narrower space of invariants facilitates the building of such theories. Already for spin ≥ 2, imposed symmetries lose their major constraining power, a paradigmatic case being the many different alternatives which accompany the simplest theory of gravity. Aside from this mainstream attitude, a different scrutiny of quantum field models—more focused on a particle's free propagation and the consistency of their interactions—has been carried most notably in Refs. [1–10]. This different focus has demonstrated how some symmetries can be brought into existence by the stronger claims of unitarity and causality of the quadratic Lagrangian. Moreover, in the absence of obstructions, techniques have been developed to consistently extend the same symmetries, allowing for a nonlinear completion and hence interactions. This bottom-up approach embraces the field formalism as a pathological carrier of multispin components, and selects constraints and symmetries over the generic Lorentz-invariant Lagrangian to cure it. The path for a successful application of this method starts with an otherwise unconstrained Lorentz-invariant quadratic action, built from a selected set of fields. A spectral analysis for this theory is then performed, which illuminates the presence of ghost-like pathological propagation. The nature of ghost being twofold, both of Ostrogradsky type [11,12],[1] sourced by higher-order derivative terms, or linked to the undefined nature of Lorentz metric. On top of this, tachyonic states also populate the propagator's poles and require care. After such polishing has been realized, which is in general a difficult task, a constrained quadratic action emerges. The art of building consistent interactions on top of a linear system is aided by cohomological methods and the less systematic (but more intuitive) Noether method [4–7,17]. This will not be the subject of this paper, which focuses on the constraining power of spectral analyses. To explore the particle spectrum of a theory in a systematic way, it is favorable to explicit the link between fields, generally reducible representation of the Lorentz group, and particles, interpreted *à la* Wigner as representations of the little group. Different methods have been developed, in particular in the context of gauge theories and higher-spin model building [18–25], to pinpoint the physical degrees of freedom. We will rely on the techniques introduced by Rivers [9], and adopted in the seminal works [10,26–28], that exploit the algebra of projector operators to highlight the spin components of a given field. This formalism also smoothly solves the problem of the inversion of the equation of motion in the presence of local symmetries, which are controlled by the gauge-invariant *saturated* propagator. These powerful techniques have handed the keys to reveal the inevitability of the Fierz-Pauli action [10] and reveal the particle spectrum of diffeomorphism-invariant theories of gravity in first-order formalism (for an incomplete list of works on the subject, see

[1] Note that ways to welcome such ghosts have been explored [13–16].

Refs. [10,26–32]). The latter case is paradigmatic of the use of many different fields with quadratic mixing and simultaneous transformation under gauge symmetry. It has received increasing attention in the last years and has required the computation of projector operators up to spin 3, an operation completed only recently [29,33]. When moving to higher spin the simultaneous presence of many fields, possibly in an auxiliary role, is mandatory to provide the propagation of the wanted states. There is therefore fair expectation that future phenomenology could be tackled by models where particle propagation is encoded in multiple fields with collective gauge symmetries whose constraining power and renormalizability have not yet been given the proper attention. This is particularly interesting given that consistent couplings of higher-spin particles seem to require higher-derivative interactions and therefore a naive power-counting renormalization is no longer applicable. More promising, similarly to the case of the nonlinear sigma model, symmetries might help to constrain the form of the UV terms and provide predictive models. Surprisingly, while geometric theories of gravity have attracted most of the use of the method in question, almost no attention has been given to Lorentz-invariant higher-spin propagation involving multiple fields, with one notable exception being the study of the Singh model in Ref. [33]. A possible reason to justify such lacking can be tracked to the missing operators needed to express the mixing terms involving all the fields. Up to fields of rank 3, and therefore equal spin number, we provide the missing terms so to open the study of the spectrum for new unexplored models with possible collective symmetries. We illustrate a possible procedure (which has common roots with Refs. [30,32]) with a greater focus on the explicit form of the saturated propagator and that unambiguously establishes the nature of the propagating particles. In this work we use the mostly minus metric signature.

II. FIELDS, PARTICLES, AND THE QUADRATIC ACTION

A. Fields and particles

While attempts to emancipate particle phenomenology from local fields are an intriguing line of past and present research [34–37], their use is still mainstream when it comes to providing scattering amplitudes at relativistic energies. The main issue of this approach is represented by the clash between the field components and the particle that they describe. More precisely, while tensor fields carry representations of the Lorentz group $\mathcal{D}(s_1, s_2)$,[2] the particle's labels are linked to the little group. These are the smaller $SU(2)$ for massive particles, and $U(1)$ for massless particles, in terms of which $\mathcal{D}(s_1, s_2)$ is reducible. In the massive case the physically relevant quantum number is therefore the spin s, with representations $2s+1$ dimensional and values $-s, -s+1, \ldots s-2, s-1, +s$. Similarly, for the little group of massless particles, the representations are identified with the integer number s, connected to the eigenvalues of the conserved helicity operator, with only two different states $\pm s$. When parity invariance is meaningful, the twofold discrete values of the parity operator ($P = \pm 1$) aid in giving a full description of Poincaré-invariant systems through fields, with the Abelian subgroup of translations being trivially realized by their continuous momentum dependence. In our analysis, as is customary in the literature, we will use the decomposition of fields in the $SO(2)$ little group which not only is enough to derive conclusions about the causal and unitary propagation of massive particles, but also provides the intermediate step in describing the unitarity of helicity states within the formalism of the saturated propagator. It is standard notation to label the little group components within the reducible Lorentz tensor with the symbol S^p, with S being the $SU(2)$ spin number and p the parity eigenvalue. To this symbol we add a further subscript to distinguish multiple components of the same S^p element for the totality of cases analyzed in this paper, from rank-0 to rank-3 fields. Therefore, following the enumeration of Ref. [29] and extending it to include the vector and scalar cases, we have

$$\begin{aligned} \phi_{\mu\nu\rho} &\supset 3_1^- \oplus 2_1^+ \oplus 2_2^+ \oplus 2_3^+ \oplus 2_1^- \oplus 2_2^- \oplus 1_1^+ \oplus 1_2^+ \\ &\quad \oplus 1_3^+ \oplus 1_1^- \oplus 1_2^- \oplus 1_3^- \oplus 1_4^- \oplus 1_5^- \oplus 1_6^- \oplus 0_1^+ \\ &\quad \oplus 0_2^+ \oplus 0_3^+ \oplus 0_4^+ \oplus 0_1^-, \\ \phi_{\mu\nu} &\supset 2_4^+ \oplus 1_7^- \oplus 0_5^+ \oplus 0_6^+, \\ \phi_\mu &\supset 1_8^- \oplus 0_7^+, \\ \phi &\supset 0_8^+, \end{aligned} \tag{2.1}$$

where we only impose symmetry for the rank-2 tensor. Equation (2.1) makes explicit how a single indexed field carries multiple particles, with substantial growth in their number with the field rank. Therefore, when using fields as building blocks for the dynamics of higher-spin particles, the necessity of constraining the fields arises. Fierz and Pauli [38] presented the rules to build linear equations of motion that only propagate a spin-s particle by adopting a rank-s symmetric and traceless tensor field $\Phi_{\mu_1\mu_2\cdots\mu_s}$, which therefore carries the Lorentz representation $\mathcal{D}(s/2, s/2)$. Being such single representation already reducible in $SU(2)$ components, on top of the Klein-Gordon equation

$$(\Box + m^2)\phi_{\mu_1\mu_2\cdots\mu_s} = 0, \tag{2.2}$$

the null-divergence condition

$$\partial^{\mu_i}\phi_{\mu_1\mu_2\cdots\mu_s} = 0 \qquad (i = 1, ..s) \tag{2.3}$$

[2]We limit ourselves to the bosonic case, avoiding double-covering representations of the little group, which is suitable for fermions.

must also be imposed to prevent the propagation of particles with spin lower than s. It was immediately realized by Fierz and Pauli that the simplicity of their universal description for all integer-spin particles could not survive the introduction of interactions without leading to inconsistencies. Such inconsistencies could be healed by relaying to a Lagrangian origin of the equations, enriched with auxiliary, lower-rank fields. Since the seminal Fierz-Pauli paper, most subsequent attempts to find healthy Lagrangians for the propagation of higher-spin particles have relied on quadratic mixing among different fields. For instance, taking the simple but already involved case of a spin-3 particle, the Singh-Hagen [39] proposal uses a Lagrangian with a collective rank-3 and rank-0 dynamics, while the Klishevich-Zinoviev model [40,41] needs all fields with rank less than or equal to 3. As exemplary cases, we will study such models with our complete set of projectors in the following pages.[3]

B. Projector operators

Having recognized the relevance of Lagrangian theories with multiple fields entangled in quadratic mixing, we recap the efficient approach based on projector operators [10,26–33] to assess the nature of their particle spectrum. In what follows, when stating general properties of the action, we will hide the index structure and consider an indexless superfield formalism as in $\Phi = \{\phi_{\mu\nu\rho}, \phi_{\mu\nu}, \phi_\mu, \phi\}$, considering field contractions in the intuitive form

$$\Phi\Psi = \phi_{\mu_1\mu_2\mu_3}\psi^{\mu_1\mu_2\mu_3} + \phi_{\mu_1\mu_2}\psi^{\mu_1\mu_2} + \phi_{\mu_1}\psi^{\mu_1} + \phi\psi, \tag{2.4}$$

and similarly for supermatrices K,

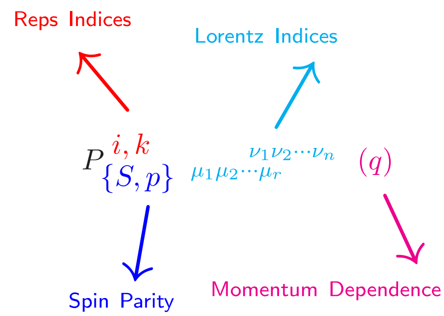
$$\begin{aligned}\Phi K\Psi = {} & \phi_{\mu_1\mu_2\mu_3}\kappa^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\psi_{\nu_1\nu_2\nu_3} + \phi_{\mu_1\mu_2\mu_3}\kappa^{\mu_1\mu_2\mu_3\nu_1\nu_2}\psi_{\nu_1\nu_2} \\ & + \phi_{\mu_1\mu_2}\kappa^{\mu_1\mu_2\nu_1\nu_2\nu_3}\psi_{\nu_1\nu_2\nu_3} + \cdots \phi_{\mu_1}\kappa^{\mu_1\nu_1}\psi_{\nu_1} + \phi_{\mu_1}\kappa^{\mu_1}\psi \\ & + \phi\kappa^{\nu_1}\psi_{\nu_1} + \phi\kappa\psi,\end{aligned} \tag{2.5}$$

where, of course, in practical examples some components might be absent.

The starting point of the algorithm is the source-dependent quadratic action, which in momentum space can be put in the economical form

$$\begin{aligned}S = \frac{1}{2}\int d^4q(\Phi(-q)K(q)\Phi(q) + \mathcal{J}(-q)\Phi(q) \\ + \mathcal{J}(q)\Phi(-q)).\end{aligned} \tag{2.6}$$

The explicit particle content of Eq. (2.6) is accessed by expanding the fields in terms of the irreducible S^p components, a process taken care of by a set of operators with a rather involved superscript and subscript index structure:



(2.7)

While it is difficult to top the clear illustrations given in Refs. [10,26,29–32], we briefly review the main properties of the operators in Eq. (2.7) to keep these pages as self-contained as possible. We start with the simple case of a single field $\phi_{\mu_1\mu_2\cdots\mu_n}(q)$; then, the subset of operators with $i = k$ in Eq. (2.7) are actually projectors onto the space of spin S and parity p, fulfilling the completeness relation

$$\phi_{\mu_1\mu_2\cdots\mu_n}(q) = \sum_{S,p,i} P^{i,i}_{\{S,p\}\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_n}(q)\phi_{\nu_1\nu_2\cdots\nu_n}(q). \tag{2.8}$$

The meaning of the "Reps Indices" i, i is therefore to keep track of the multiple representations within the spin/parity sector $\{S, p\}$, as enumerated in Eq. (2.1). The need for a double entry is then intended when considering *transitions* between different representations with the same S and p values. Being possible to realize such transition also among fields of different rank, we have considered in Eq. (2.7) the generic case with different numbers of upper and lower Lorentz indices. For instance, by looking at Eq. (2.1), we might recover a vector with spin/parity 1^- out of a rank-3 tensor via

$$P^{8,1}_{\{1,-\}\mu}{}^{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3} = P^{8,1}_{\{1,-\}\mu}{}^{\rho_1\rho_2\rho_3}P^{1,1}_{\{1,-\}\rho_1\rho_2\rho_3}{}^{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3} \tag{2.9}$$

where we neglected the momentum dependence to lighten the notation. Equations (2.8)–(2.9) are particular applications of

$$\begin{aligned}\sum_{S,p,i} P^{i,i}_{\{S,p\}\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_n} &= \hat{1}_{\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_n}, \\ P^{i,k}_{\{S,p\}\mu_1\mu_2\ldots\mu_n}{}^{\rho_1\rho_2\ldots\rho_n} P^{j,w}_{\{R,m\}\rho_1\rho_2\ldots\rho_n}{}^{\nu_1\nu_2\ldots\nu_n} &= \delta_{k,j}\delta_{S,R}\delta_{p,m}P^{i,w}_{\{S,p\}\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_n}, \\ P^{i,j}_{\{S,p\}\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_n} &= (P^{j,i}_{\{S,p\}}{}^{\nu_1\nu_2\ldots\nu_n}_{\mu_1\mu_2\ldots\mu_n})^*,\end{aligned} \tag{2.10}$$

which describe the completeness of the projectors, and the orthogonality and Hermiticity of the operators.[4] The task of computing the explicit forms of these operators for a set of rank-2 and rank-3 fields was only recently completed [29,33].

[3] The unitarity of the Singh-Hagen model was already investigated in Ref. [33] where, in order to only use rank-3 projectors, a clever integration of the auxiliary rank-0 field was employed, trading the linear Lagrangian for a nonlinear one.

[4] We introduced the unit operator $\hat{1}$ in the field space of given rank and index symmetry so that, for instance, for a symmetric rank-2 field we have $\hat{1}_{\mu_1\mu_2}{}^{\nu_1\nu_2} = \frac{1}{2}(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} + \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2})$.

This paper's original contribution is to provide the missing operators to study the mixing between all of the fields up to rank 3, including the scalar-tensor and vector-tensor transition operators. We refer to the cited literature for details about the computation of the actual projectors, in particular the role of the transverse and longitudinal operators which allows possible future generalizations to higher spins. For our computation a brute force method has been used, with an appropriate code to generate tensor-covariant combinations with given symmetries so as to impose Eq. (2.10) and solve the corresponding linear system. We provide a shortened list of such operators, which can be completed to the full set with the use of Eq. (2.10), in the Appendix. The full code-ready set can be found in the Supplemental Material [42]. For this purpose, the tools found in Refs. [43–45] have been of great help. Once the set of operators fulfilling Eq. (2.10) is known, the investigation about the nature of the particle spectrum follows almost mechanically. In particular, within this formalism, the subtle presence of gauge symmetries can be revealed and dealt with in a simple and physically intuitive way.

C. Quadratic action and saturated propagator

The source-dependent quadratic action (2.6) can now be manipulated to reveal the propagating spin/parity sectors via the expansion

$$\int d^4q\Phi(-q)K(q)\Phi(q) = \int d^4q\Phi(-q)\sum_{S,p,i,j}(a^{\{S,p\}}_{i,j}P^{i,j}_{\{S,p\}})\Phi(q). \quad (2.11)$$

To get the fundamental matrices $a^{\{S,p\}}_{i,j}$ we need to get the components of the supermatrix $K(q)$ corresponding to the fields that include the ith and jth representations of spin/parity S^p [Eq. (2.1)]. The Lorentz indices of this object are then traced against the operator $P^{i,j}_{\{S,p\}}$, opportunely weighted with the dimension of the S^p representation $d_S = 2S + 1$. So, for instance, to get $a^{\{1,-\}}_{1,8}$ we use

$$a^{\{1,-\}}_{1,8} = \frac{1}{3}P^{1,8}_{\{1,-\}\mu_1\mu_2\mu_3}{}^{\nu_1}\quad \kappa^{\mu_1\mu_2\mu_3}{}_{\nu_1} \quad (2.12)$$

or similarly, for diagonal elements,

$$a^{\{2,+\}}_{4,4} = \frac{1}{5}P^{4,4}_{\{2,+\}\mu_1\mu_2}{}^{\nu_1\nu_2}\kappa^{\mu_1\mu_2}{}_{\nu_1\nu_2}, \quad (2.13)$$

and so on. The matrices $a^{\{S,p\}}_{i,j}$ are central in these spectral investigations, encoding all of the information on the particle quanta in an easy-to-manipulate form. Indeed, with the form (2.11) the computation of the propagator $\mathcal{D}(q)$ in the presence of the sources $J(q)$ translates into the inversion of the equation

$$K(q)\mathcal{D}(q) = \sum_{S,p,i,j}(a^{\{S,p\}}_{i,j}P^{i,j}_{\{S,p\}})\mathcal{D}(q) = \hat{1}, \quad (2.14)$$

which has the solution

$$\mathcal{D}(q) = \sum_{S,p,i,j} b^{\{S,p\}}_{i,j}P^{i,j}_{\{S,p\}}, \quad (2.15)$$

with $b^{\{S,p\}}_{i,j} = (a^{\{S,p\}}_{i,j})^{-1}$. The advantage in using such a decomposition is the direct connection with gauge symmetries, which arise when the determinant of $a^{\{S,p\}}_{i,j}$ is zero, making the naive inversion impossible. Then, in the presence of a set of n null vectors $X^{s=1,2\cdots n}_i$ for the matrix $a^{\{S,p\}}_{i,j}$, we have n corresponding symmetries under the gauge transformations

$$\delta\Phi = X^s_i P^{i,j}_{\{S,p\}}\Psi, \qquad (s = 1,2\cdots n), \quad (2.16)$$

with an arbitrary superfield Ψ. To fix such gauge freedom we take the nondegenerate submatrix $\tilde{a}^{\{S,p\}}_{i,j}$, which can be promptly inverted. Then, *we can recover gauge invariance* by asking the sources to solve for the gauge constraints

$$X^{*s}_j P^{i,j}_{\{S,p\}}J(q) = 0, \qquad (s = 1,2\cdots n), \quad (2.17)$$

and defining

$$\mathcal{D}_S(q) = \tilde{J}^*(q)\left(\sum_{S,p,i,j}\tilde{b}^{\{S,p\}}_{i,j}P^{i,j}_{\{S,p\}}\right)\tilde{J}(q), \quad (2.18)$$

where we used $\tilde{J}(q)$ to refer to solutions of Eq. (2.17).

D. Poles, ghosts, and tachyons

Equation (2.18) is the gauge-invariant *saturated propagator* which has the generic structure of an undefined quadratic form in the sources with possible poles in the squared momentum q^2. In the proximity of such poles we recover the structure

$$\lim_{q^2\to M_a^2} D_S(q) \sim \frac{\sum_m r_m|j^m|^2}{(q^2 - M_a^2)^n}, \quad (2.19)$$

where j^m are linear combinations of source components with definite properties under angular momentum ($M_a \neq 0$) or helicity ($M_a = 0$) transformations (for details, see Sec. II of Ref. [46]). Many factors can concur to challenge the healthy propagation of particles, providing in turn strong constraints over the parameter space of the linear model. As is known, the residue of the propagator poles has a special role in defining the unitarity of the model and the undefined Lorentz metric cannot ensure the positivity of the residue in Eq. (2.19), thus generally propagating *ghosts*. Connected to this is also the elimination of possible poles of order >1 in q^2, linked to Ostrogradsky instabilities [$n > 1$ in Eq. (2.19)]. The tribute to causality is instead paid by avoiding superluminal propagation which is triggered by complex masses $M_a^2 < 0$ providing yet another constraint to account for. A certain number of shortcuts have been developed to avoid dealing with Eq. (2.19) directly and instead reduce the problem to the simpler matrices $\tilde{b}_{i,j}^{\{S,p\}}$. Indeed, by restricting ourselves to nondegenerate massive particles, the positivity of the quadratic form over the pole $q^2 \to M^2$ can be read off the simple formula

$$\operatorname*{Res}_{q^2\to M^2} \sum_i (-1)^p \tilde{b}_{i,i}^{\{S,p\}} > 0.$$

This approach misses potentially interesting scenarios, such as massless propagation, critical cases where different spin sectors share a common mass, and (notably for our work) collective gauge-invariant descriptions of massive particles (see Sec. IV). For this reason, in the spirit of Refs. [10,32] we will directly rely on the source-dependent saturated propagator to derive any conclusion about unitarity and causality. The impairment to get such information from Eq. (2.19) is connected to the imposition of constraints on the sources, which will be difficult to exhibit when keeping an index-free formalism. Therefore, some progress can be made by exploiting Lorentz invariance and solving the gauge constraints in particular frames. In the massive case we feed our algorithm with the form of Eq. (2.17) in the frame $q = (\omega, \vec{0})$ and solve for the components; then, such a subset of independent sources is reinserted into Eq. (2.17) where the residue is easily computed. The massless case is slightly more subtle because spurious poles in q^2 are also present in the polarization operators. Therefore, we intermediately use the frame $q = (\omega, 0, 0, \kappa)$ and only later impose the light-like limit $\kappa \to \omega$.

III. APPLICATIONS: REVIEW

Before adopting the projector technology to explore uncharted territories, we review more familiar ones, stressing (with the benefits of insight) the role of quadratic mixing and gauge symmetries.

A. Proca-Stueckelberg-Goldstone

The interacting theory of the vector and scalar field is a simple yet rich playground which shows the interplay between our formalism and the physical properties of particles they accommodate. In this section, we extensively review Proca and Stueckelberg theories. This subject has been already explored in Ref. [47], but we find it necessary to present it here as well, before moving to more complex scenarios. We start by analyzing the massive vector system described by the Proca action

$$S_P = \int d^4x \left[-\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \frac{m_V^2}{2} V_\mu V^\mu \right]. \tag{3.1}$$

The decomposition (2.1) defines two spin/parity sectors 1^- and 0^+ with the (one-dimensional) matrices

$$\begin{aligned} a_{8,8}^{\{1,-\}} &= q^2 - m_V^2, \\ a_{7,7}^{\{0,+\}} &= m_V^2. \end{aligned} \tag{3.2}$$

The mass parameter removes the possible degeneracy of the scalar sector, thus breaking the corresponding gauge symmetry. The saturated propagator $\mathcal{D}_S(q) = \mathcal{D}_S^{1^-}(q) + \mathcal{D}_S^{0^+}(q)$ can therefore be trivially computed as

$$\begin{aligned} \mathcal{D}_S^{1^-}(q) &= J_\mu^*(q) \tilde{b}_{8,8}^{\{1,-\}} P_{\{1^-\}}^{8,8\ \ \mu\nu} J_\nu(q) \\ &= \frac{J^{*\mu}(q) J^\nu(q)}{q^2 - m_V^2} \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right), \\ \mathcal{D}_S^{0^+}(q) &= J_\mu^*(q) \tilde{b}_{7,7}^{\{0,+\}} P_{\{0^+\}}^{7,7\ \ \mu\nu} J_\nu(q) \\ &= \frac{J^{*\mu}(q) J^\nu(q) q_\mu q_\nu}{m_V^2 q^2}. \end{aligned} \tag{3.3}$$

We see that, while the 0^+ sector has only a spurious massless pole, the sector 1^- might exhibit healthy, non-tachyonic propagation. Going to the frame $q^\mu = (\omega, \vec{0})$, we simply find

$$\lim_{\omega^2 \to m_V^2} D_S(q) = \frac{|J^1|^2 + |J^2|^2 + |J^3|^2}{\omega^2 - m_V^2}, \tag{3.4}$$

with the three degrees of freedom as expected for a massive spin-1 particle. It is known that, to circumvent the bad $q^2 \to \infty$ behavior of the 0^+ sector, gauge symmetries must be introduced to sweep it away. In *all known cases* this will require an extension with quadratic mixing between the vector and scalar sectors. In the minimal case this can be achieved via the Stueckelberg mechanism, with only one field introduced in the simple form

$$S_S = S_P + \int d^4x \left[\frac{1}{2} \partial_\mu \phi_S \partial^\mu \phi_S + m_V \phi_S \partial_\mu V^\mu \right]. \tag{3.5}$$

We see the appearance of quadratic mixing in order to entangle the two fields in a common symmetry. This symmetry can be revealed by noticing that, now, the sector 0^+ is enhanced to the *degenerate* 2×2 matrix

$$a_{i,j}^{\{0,+\}} = \begin{pmatrix} a_{7,7} & a_{7,8} \\ a_{8,7} & a_{8,8} \end{pmatrix} = \begin{pmatrix} m_V^2 & i m_V \sqrt{q^2} \\ -i m_V \sqrt{q^2} & q^2 \end{pmatrix}. \tag{3.6}$$

A direct application of Eq. (2.16) then reveals, in momentum space, the symmetry

$$\delta\Phi = (\delta V_\mu, \delta\phi_S) = (iq_\mu\psi, m_V\psi), \tag{3.7}$$

where ψ is an arbitrary scalar field. We can fix the freedom of Eq. (3.7) to allow the computation of the propagator by arbitrarily choosing either diagonal element in Eq. (3.6) as a nondegenerate submatrix. The equivalence of both choices is granted by the source constraint (2.17), which becomes

$$i\frac{q_\mu}{m_V}J^\mu(q) = J(q), \tag{3.8}$$

where the indexless $J(q)$ is the source for the scalar field ϕ.[5] Besides the role of the gauge invariance, the saturated propagator, in the limit $q^2 = \omega^2 \to m_V^2$, is formally identical to the Proca one, propagating three states of the massive 1^- spin sector. We stress that such a spectral analysis displays features that are also included in more involved Lagrangians with a Stueckelberg-like quadratic part. In particular, the presence of quadratic mixing between fields of different rank, and the corresponding inhomogeneous transformations required by gauge invariance, are manifested by Goldstone states in theories with spontaneous symmetry breaking.

B. Einstein-Palatini

The massless Fierz-Pauli Lagrangian for a symmetric rank-2 field propagates two helicity states as shown (with the help of the projector algebra) in Ref. [10]. There, the apparent propagation of a ghost-like spin-0 particle was shown to disappear in the limit $q^2 \to 0$ owing to the constrained sources. As an interesting application, we revisit the same problem by adopting a rank-3 tensor as an auxiliary field in the so-called Palatini formulation. To the best of our knowledge, the saturated propagator for this formulation has been explicitly worked out, in coordinate space, only in Ref. [30]. On top of the many interesting applications for the first-order formulation of the gravitational problem, it also gives a nontrivial display of a model requiring a rank-3 and rank-2 field cooperation in order to propagate a single helicity-2 particle. The quadratic action we target is of the form

$$\begin{aligned} S_{\mathrm{EP}} &= a_{\mathrm{EP}}\int d^4x(\delta^{\mu\nu} + H^{\mu\nu})(\partial_\alpha A^\alpha{}_{\mu\nu} - \partial_\nu A^\alpha{}_{\mu\alpha} \\ &\quad + A^\alpha{}_{\nu\mu}A^\beta{}_{\alpha\beta} - A^\alpha{}_{\beta\mu}A^\beta{}_{\alpha\mu}) \\ &= a_{\mathrm{EP}}\int d^4x[H^{\mu\nu}(\partial_\alpha A^\alpha{}_{\mu\nu} - \partial_\nu A^\alpha{}_{\mu\alpha}) + A^\alpha{}_\mu{}^\mu A^\beta{}_{\alpha\beta} \\ &\quad - A^\alpha{}_{\beta\mu}A^\beta{}_\alpha{}^\mu + O(HA^2)], \end{aligned} \tag{3.9}$$

[5] We hope to curb the inevitable profusion of symbols by leaving the number of indices to signal to which field a given source is connected to.

and we require $H^{\mu\nu} = H^{\nu\nu}$ and $A^\alpha_{\mu\nu} = A^\alpha_{\nu\mu}$, with the latter equality being required to avoid many irrelevant spin representations. For this set of fields, Eq. (2.1) is now reduced to

$$\begin{aligned} A_{\mu\nu\rho} &\supset 3_1^- \oplus 2_1^+ \oplus 2_2^+ \oplus 2_1^- \oplus 1_1^+ \oplus 1_1^- \oplus 1_2^- \oplus 1_4^- \oplus 1_5^- \\ &\quad \oplus 0_1^+ \oplus 0_2^+ \oplus 0_4^+, \\ H_{\mu\nu} &\supset 2_4^+ \oplus 1_7^- \oplus 0_5^+ \oplus 0_6^+. \end{aligned} \tag{3.10}$$

In terms of these representations the relevant spin/parity matrices become

$$a_{i,j}^{\{2,+\}} = a_{\mathrm{EP}}\begin{pmatrix} -2 & 0 & -i\frac{\sqrt{q^2}}{3} \\ 0 & 1 & -i\sqrt{\frac{2}{3}}\sqrt{q^2} \\ i\frac{\sqrt{q^2}}{3} & i\sqrt{\frac{2}{3}}\sqrt{q^2} & 0 \end{pmatrix}, \tag{3.11}$$

$$a_{i,j}^{\{1,-\}} = a_{\mathrm{EP}}\begin{pmatrix} \frac{4}{3} & -\frac{\sqrt{5}}{3} & \frac{2\sqrt{5}}{3} & -\frac{1}{3}\sqrt{\frac{5}{2}} & i\sqrt{\frac{5q^2}{6}} \\ -\frac{\sqrt{5}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{2\sqrt{2}}{3} & i\sqrt{\frac{q^2}{6}} \\ \frac{2\sqrt{5}}{3} & -\frac{1}{3} & -\frac{4}{3} & -\frac{1}{3\sqrt{2}} & -i\sqrt{\frac{q^2}{6}} \\ -\frac{1}{3}\frac{\sqrt{5}}{2} & -\frac{2\sqrt{2}}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} & -i\sqrt{\frac{q^2}{2\sqrt{3}}} \\ -i\sqrt{\frac{5q^2}{6}} & -i\sqrt{\frac{q^2}{6}} & i\sqrt{\frac{q^2}{6}} & i\sqrt{\frac{q^2}{2\sqrt{3}}} & 0 \end{pmatrix}, \tag{3.12}$$

and

$$a_{i,j}^{\{0,+\}} = a_{\mathrm{EP}}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 2 & -i\frac{\sqrt{q^2}}{\sqrt{3}} & i\sqrt{q^2} \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} & -i\sqrt{\frac{2q^2}{3}} & -i\sqrt{\frac{q^2}{2}} \\ 2 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ i\sqrt{\frac{q^2}{3}} & i\sqrt{\frac{2q^2}{3}} & 0 & 0 & 0 \\ -i\sqrt{q^2} & i\sqrt{\frac{q^2}{2}} & 0 & 0 & 0 \end{pmatrix}. \tag{3.13}$$

It easy to notice that the 1^- and 0^+ sectors provide degenerate matrices, a result familiar with the corresponding analysis of Einstein theory in second-order formulation. Different choices of the nondegenerate 4×4 submatrices are possible and a comforting consistency check reveals that the final shape of the saturated propagator is unchanged. What we think is notable in this kind of gauge invariance check is that, in some cases, spurious double poles might arise and/or disappear based on the choice of the nondegenerate submatrix. This shows that the physically relevant poles to investigate are the gauge-invariant ones of the saturated propagator, rather than those of the inverse of the spin/parity matrices $\tilde{b}_{i,j}^{\{S,p\}}$. To reveal a possible ghost-like propagation we have to follow the protocol illustrated before and arrive at the saturated propagator. In the massless case, the check entails asking the propagator to only have single poles and a definite positive residue with the appropriate number of states [see, for instance, Eq. (3.4)]. Of course, in the presence of high-rank fields, the large structure of the saturated propagator expanded in components impedes a straightforward check. We will use this example to illustrate the manipulations we adopted in order to reach unambiguous conclusions. Once the gauge constraints (2.17) are imposed on the sources, the saturated propagator is obtained by considering the inverse of *all* of the spin/parity sectors $\mathcal{D}_S(q) = D_S^{3^-}(q) + D_S^{2^+}(q) + D_S^{2^-}(q) + D_S^{1^+}(q) + D_S^{1^-}(q) + D_S^{0^+}(q)$. As said, even the sectors not showing poles have to be included to cancel the spurious ones of the projectors. The saturated propagator for the rank-3/rank-2 system has the generic form [30]

$$\lim_{q^2\to 0}\mathcal{D}_S = \frac{1}{q^2}\begin{pmatrix}\tilde{J}^{*\mu\nu} & \tilde{q}_\rho J^{*\mu\nu\rho}\end{pmatrix}\begin{pmatrix}\mathcal{M}^{1,1}_{\mu\nu\alpha\beta} & \mathcal{M}^{1,2}_{\mu\nu\alpha\beta}\\ \mathcal{M}^{2,1}_{\mu\nu\alpha\beta} & \mathcal{M}^{2,2}_{\mu\nu\alpha\beta}\end{pmatrix}\begin{pmatrix}\tilde{J}^{\alpha\beta}\\ \tilde{q}_\iota J^{\alpha\beta\iota}\end{pmatrix}. \tag{3.14}$$

In our approach we break Lorentz covariance by expanding Eq. (3.14) in components, having $q = (\omega, 0, 0, \kappa)$, and imposing $\kappa \to \omega$ in the massless case. It is therefore more appropriate to collect all of the components of $\tilde{J}^{\mu\nu\alpha}$, $\tilde{J}^{\mu\nu}$ (which are less than the unconstrained $J^{\mu\nu\alpha}$, $J^{\mu\nu}$) in a long row vector $X = (J^{01}, J^{02}, \cdots \omega J^{001}, \omega J^{002}, \cdots)$ so that Eq. (3.14) becomes

$$\lim_{q^2\to 0}\mathcal{D}_S = \frac{1}{q^2}\tilde{X}^\dagger \mathcal{M}\tilde{X}. \tag{3.15}$$

With the matrix $\mathcal{M}$ we can promptly check the number of propagating states by computing its rank which, luckily, is equal to two. Then, the constraint over the only free parameter a_{EP} is derived by diagonalizing $\mathcal{M}$ which, in our computation, reveals

$$\lim_{q^2\to 0}\mathcal{D}_S = -\frac{1}{a_{\mathrm{EP}}}\frac{21|j_1|^2 + 10|j_2|^2}{\omega^2 - \kappa^2}, \tag{3.16}$$

where j_1 and j_2 are the two eigenvectors. As expected, the conclusion is that the dimensional a_{EP} coupling allows unitarity if negative.

C. Singh-Hagen

The same rationale we introduced in the computation of the mixed rank-3/rank-2 Einstein-Palatini model can (with some minor changes) help to reveal the possible presence of ghosts and tachyons in models with propagation of massive particles. In this and the next section we will analyze two alternative models that describe a massive spin-3 particle for which the collective cooperation of many different fields seems to be mandatory to preserve locality. The first model was introduced in Ref. [39] and was recently studied in Ref. [33] by integrating out the auxiliary scalar field. We will instead rely on the complete set of operators that we computed for this purpose. The Singh-Hagen action is given by

$$\begin{aligned}S_{\mathrm{SA}} = \int d^4x\Big[x_{\mathrm{SA}}\Big(&A_{\mu\nu\rho}\Box A^{\mu\nu\rho} - 3A_{\mu\nu\rho}\partial^\mu\partial_\alpha A^{\alpha\nu\rho}\\ &-\frac{3}{2}A^\sigma{}_{\nu\sigma}\partial^\nu\partial_\mu A^\mu{}_\rho{}^\rho + -3A^\nu{}_\sigma{}^\sigma\Box A_\nu{}^\rho{}_\rho\\ &+6A_{\rho\sigma}{}^\sigma\partial_\mu\partial_\nu A^{\mu\nu\rho} + m_A^2 A^{\mu\nu\rho}A_{\mu\nu\rho} - 3m_A^2 A^{\mu\rho}{}_\rho A_{\mu\sigma}{}^\sigma\Big)\\ &+y_{\mathrm{SA}}\Big(\frac{1}{2}\phi\Box\phi + 2m_A^2\phi^2\Big) + z_{\mathrm{SA}}m_A A^\sigma{}_{\mu\sigma}\partial^\mu\phi\Big],\end{aligned} \tag{3.17}$$

where $A^{\mu\nu\rho}$ is totally symmetric and we included four free parameters x_{SA}, y_{SA}, z_{SA}, and m_A for illustrative reasons. We notice the presence of a term with a quadratic mixing between the rank-3 tensor $A^{\mu\nu\rho}$ and the scalar ϕ. A symmetric rank-3 tensor contains the little group representations

$$A_{\mu\nu\rho} \supset 3_1^- \oplus 2_1^+ \oplus 1_1^- \oplus 1_4^- \oplus 0_1^+ \oplus 0_4^+, \tag{3.18}$$

which are completed by the trivial one 0_8^+ carried by the scalar field ϕ. We then proceed to compute the spin/parity matrices of the Singh-Hagen model, which are

$$a_{i,j}^{\{3,-\}} = -2x_{\mathrm{SA}}(q^2 - m_A^2), \tag{3.19}$$

$$a_{i,j}^{\{2,+\}} = 2x_{\mathrm{SA}}m_A^2, \tag{3.20}$$

$$a_{i,j}^{\{1,-\}} = x_{\mathrm{SA}}\begin{pmatrix}8(q^2 - m_A^2) & -2\sqrt{5}m_A^2\\ -2\sqrt{5}m_A^2 & 0\end{pmatrix}, \tag{3.21}$$

and finally,

$$a_{i,j}^{\{0,+\}} = \begin{pmatrix} x_{\mathrm{SA}}(9q^2-4m_A^2) & 3x_{\mathrm{SA}}(q^2-2m_A^2) & iz_{\mathrm{SA}}m_A\sqrt{q^2} \\ 3x_{\mathrm{SA}}(q^2-2m_A^2) & x_{\mathrm{SA}}(q^2-4m_A^2) & iz_{\mathrm{SA}}m_A\sqrt{q^2} \\ -iz_{\mathrm{SA}}m_A\sqrt{q^2} & -iz_{\mathrm{SA}}m_A\sqrt{q^2} & -y_{\mathrm{SA}}(q^2-4m_A^2) \end{pmatrix}. \tag{3.22}$$

In the Singh-Hagen model the propagation of the massive spin-3 particle does not require gauge symmetries to prevent the propagation of dangerous ghosts, and this is manifested by all of the spin/parity matrices being nondegenerate. No constraints are therefore imposed on the sources and we can proceed to the direct inversion of the matrices (3.19)–(3.22) to find the saturated propagator $\mathcal{D}_S(q) = D_S^{3^-}(q) + D_S^{2^+}(q) + D_S^{1^-}(q) + D_S^{0^+}(q)$. This object inherits the massive pole, which is displayed by inverting the $a_{i,j}^{\{3,-\}}$ matrix. In looking for the propagating massive states, we put $\mathcal{D}_S(q)$ in a form similar to the massless one (3.15). This time we arrange the row vector as $X = (J, J^{001}, J^{002}, \cdots)$ and immediately recover the form

$$\lim_{q^2\to m_A^2} \mathcal{D}_S = \frac{1}{q^2-m_A^2} X^\dagger \mathcal{M} X. \tag{3.23}$$

The matrix $\mathcal{M}$ has rank 7, which is equal to the $2s+1$ states of a massive spin-3 particle, and the (nonzero) eigenvalues are

$$\mathcal{M}_{\mathrm{res}} = x_{\mathrm{SA}}^{-1}\left\{3, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{11}{10}, \frac{11}{10}, \frac{11}{10}\right\}, \tag{3.24}$$

which can be made positive if we keep $x_{\mathrm{SA}} > 0$. As already discussed in Ref. [33], not all of the parameters we used are physical and only x_{SA} and m_A are constrained by causality and unitarity.

IV. APPLICATIONS: SPECTRUM OF THE KLISHEVICH-ZINOVIEV MODEL

The Singh-Hagen model is one of the primitive attempts to define a linear Lagrangian for higher-spin particles. Further efforts to build a full dynamic around these germinal models were met with the numerous constraints which upset the definition of a nontrivial S matrix (see, for instance, Refs. [48,49] and references therein). In order to overcome the same constraints, alternative approaches have been investigated, a notable one being Vasiliev's full nonlinear theory [50–52]. The presence in these models of a nonzero cosmological constant introduces a series of peculiar phenomena, like novel stability constraints and the appearance of partial masslessness for propagating particles [53–57]. In particular, in order to investigate partially massless theories in anti–de Sitter spaces, a convenient Lagrangian setup describing massive higher-spin propagation was developed in Refs. [40,41,58]. For our purposes, it is notable that such a formulation admits a flat limit and contributes to an alternative description of higher-spin particles in Minkowski space. Moreover, differently from the Singh-Hagen case, this new formulation is inherently gauge invariant, requiring the entire set of auxiliary Stueckelberg fields with rank ≤3 to mix in the quadratic action. This model is therefore the perfect arena to test our operators and provide an alternative check of unitarity and causality violations where, moreover, a shortcut like Eq. (2.4) would be ineffective. Again, we rely on totally symmetric tensors so that the set of little group representations carried by the fields are

$$\begin{aligned} A_{\mu\nu\rho} &\supset 3_1^- \oplus 2_1^+ \oplus 1_1^- \oplus 1_4^- \oplus 0_1^+ \oplus 0_4^+, \\ H_{\mu\nu} &\supset 2_4^+ \oplus 1_7^- \oplus 0_5^+ \oplus 0_6^+, \\ V_\mu &\supset 1_8^- \oplus 0_7^+, \\ \phi &\supset 0_8^+. \end{aligned} \tag{4.1}$$

In terms of these fields, the action can be written as a sum of distinctive parts:

$$S_{KZ} = S_{AA} + S_{HH} + S_{VV} + S_{\phi\phi} + S_{\mathrm{mix}}, \tag{4.2}$$

with

$$\begin{aligned} S_{AA} = \int d^4x \Bigg[\Bigg(&-\frac{1}{2}\partial_\alpha A_{\mu\nu\rho}\partial^\alpha A^{\mu\nu\rho} \\ &+ \frac{3}{2}\partial^\mu A_{\mu\nu\rho}\partial^\alpha A_\alpha{}^{\nu\rho} + \frac{3}{2}\partial^\alpha A_\mu{}^{\mu\nu}\partial_\alpha A_\rho{}^\rho{}_\nu \\ &+ \frac{3}{4}\partial_\nu A_\mu{}^{\mu\nu}\partial_\alpha A_\rho{}^{\rho\alpha} - 3\partial_\nu A_\sigma{}^\sigma{}_\rho \partial_\mu A^{\mu\nu\rho} + \Bigg) \\ &+ m_A^2\left(\frac{1}{2}A^{\mu\nu\rho}A_{\mu\nu\rho} - \frac{3}{2}A^\mu{}_\alpha{}^\alpha A_{\mu\beta}{}^\beta\right)\Bigg], \end{aligned} \tag{4.3}$$

$$\begin{aligned} S_{HH} = \int d^4x \Bigg[3\Bigg(&\frac{1}{2}\partial^\mu H^{\alpha\beta}\partial_\mu H_{\alpha\beta} - \partial_\beta H^{\mu\beta}\partial_\alpha H_\mu{}^\alpha \\ &+ \partial_\beta H^{\mu\beta}\partial_\mu H_\alpha{}^\alpha - \frac{1}{2}\partial_\mu H^\alpha{}_\alpha \partial^\mu H^\beta{}_\beta \Bigg) \\ &+ \frac{9m_A^2}{4} H^\alpha{}_\alpha H^\beta{}_\beta \Bigg], \end{aligned} \tag{4.4}$$

$$S_{VV} = \int d^4x \left[-\frac{15}{2}(\partial^\mu V^\nu \partial_\mu V_\nu - \partial^\mu V^\nu \partial_\nu V_\mu) - \frac{45m_A^2}{4} V_\mu V^\mu \right], \tag{4.5}$$

$$S_{\phi\phi} = \int d^4x \left(-\frac{45}{2} \partial^\mu \phi \partial_\mu \phi + 225 m_A^2 \phi^2 \right), \tag{4.6}$$

and

$$S_{\rm mix} = \int d^4x \Bigg[m_A \Bigg(-\frac{3}{2} (2A^{\mu\nu\alpha}\partial_\mu H_{\nu\alpha} - 4A^\alpha{}_\alpha{}^\mu \partial_\beta H^\beta{}_\mu + A^\alpha{}_\alpha{}^\mu \partial_\mu H_\beta{}^\beta) + 15(H^{\mu\nu}\partial_\mu V_\nu - H^\alpha{}_\alpha \partial_\mu V^\mu) + -225 V^\mu \partial_\mu \phi \Bigg) + m_A^2 \left(\frac{15}{2} A^\mu{}_\alpha{}^\alpha V_\mu - 45 H^\alpha{}_\alpha \phi \right) \Bigg]. \tag{4.7}$$

Normalizations of the various terms are taken from Ref. [41]. The four spin/parity matrices are

$$a_{i,j}^{\{3,-\}} = m_A^2 - q^2, \tag{4.8}$$

$$a_{i,j}^{\{2,+\}} = \begin{pmatrix} m_A^2 & -i\sqrt{3} m_A \sqrt{q^2} \\ i\sqrt{3} m_A \sqrt{q^2} & 3q^2 \end{pmatrix}, \tag{4.9}$$

$$a_{i,j}^{\{1,-\}} = \begin{pmatrix} 4(q^2 - m_A^2) & -\sqrt{5} m_A^2 & i\sqrt{30} m_A \sqrt{q^2} & +m_A^2 \frac{5\sqrt{15}}{2} \\ -\sqrt{5} m_A^2 & 0 & 0 & m_A^2 \frac{5\sqrt{3}}{2} \\ -i\sqrt{30}\sqrt{q^2} & 0 & 0 & i m_A^2 \frac{15}{\sqrt{2}} \\ m_A^2 \frac{5\sqrt{15}}{2} & m_A^2 \frac{5\sqrt{3}}{2} & -i m_A^2 \frac{15}{\sqrt{2}} & -\frac{15}{2}(2q^2 + 3m_A^2) \end{pmatrix}, \tag{4.10}$$

$$a_{i,j}^{\{0,+\}} = \begin{pmatrix} \frac{9}{2} q^2 - 2m_A^2 & \frac{3}{2} q^2 - 3m_A^2 & -m_A i \frac{5\sqrt{3}}{2} \sqrt{q^2} & +m_A \frac{9i}{2} \sqrt{q^2} & \frac{15}{2} m_A & 0 \\ \frac{3}{2} q^2 - 2m_A^2 & \frac{1}{2} q^2 - 2m_A^2 & -m_A i \frac{3\sqrt{3}}{2} \sqrt{q^2} & +m_A \frac{3i}{2} \sqrt{q^2} & \frac{15}{2} m_A & 0 \\ m_A i \frac{5\sqrt{3}}{2} \sqrt{q^2} & m_A i \frac{3\sqrt{3}}{2} \sqrt{q^2} & \frac{27}{2} m_A^2 - 6q^2 & \frac{9\sqrt{3}}{2} m_A^2 & -i 15\sqrt{3} m_A \sqrt{q^2} & -45\sqrt{3} m_A^2 \\ m_A i \frac{9}{2} \sqrt{q^2} & -m_A i \frac{3}{2} \sqrt{q^2} & \frac{9\sqrt{3}}{2} m_A^2 & \frac{9}{2} m_A^2 & 0 & -45 m_A^2 \\ m_A^2 \frac{15}{2} & m_A^2 \frac{15}{2} & 15\sqrt{3} i m_A \sqrt{q q^2} & 0 & -\frac{45}{2} m_A^2 & -225 i m_A \sqrt{q^2} \\ 0 & 0 & -45\sqrt{3} m_A^2 & -45 m_A^2 & 225 i m_A \sqrt{q^2} & 90(q^2 + 5m_A^2) \end{pmatrix}. \tag{4.11}$$

The pervasiveness of gauge invariance is revealed by noticing that the ranks of the matrices $a_{i,j}^{\{2,+\}}$, $a_{i,j}^{\{1,-\}}$, and $a_{i,j}^{\{0,+\}}$ are 1, 2, and 5, respectively. The set of null vectors will therefore generate a long list of constraints which, again, we will solve for the source components in the frame $q = (\omega, \vec{0})$. After this step we can again discover the structure (3.23) with the eigenvalues of the large matrix M being

$$\mathcal{M}_{\rm res} = \left\{ \frac{1}{15}(533 - 2\sqrt{61591}), \frac{1}{15}(533 - 2\sqrt{61591}), \frac{1}{15}(533 + 2\sqrt{61591}), \frac{1}{15}(533 + 2\sqrt{61591}), 3, 5, \frac{683}{45} \right\},$$

which, while hideous to look at, are all positive.

V. APPLICATIONS: COLLECTIVE PROPAGATION FOR HIGHER-SPIN PARTICLES

The previous examples displayed how the quadratic mixing between fields of different rank helps modeling manifestly local higher-spin propagation. It is therefore natural to expect that the corresponding Lagrangians might lead to a broader spectrum than that of a single particle, and indeed this is the direction taken by the numerous studies around extensions of the minimal Einstein-Palatini model of Sec. III B; see, for instance, Refs. [26–32,59–63]. The full set of operators calculated in this work opens a computational opportunity for the derivation of parameter constraints for such collective systems. It is outside of this paper's purpose to provide a survey of new models achievable with the projector formalism, an activity whose extent asks for dedicated future efforts. Nevertheless, for the sake of completeness, we conclude by showing an explicit (and seemingly new) model with collective higher-spin propagation of massive states. It has to be clarified, once again, that the shaping of a unitary and causal Lagrangian is a first step that needs to be supported by the modeling of possible consistent interactions. This latter process might relegate the linear theory to irrelevance in the presence of obstructions, confining the theory to the dull status of a free one. The building of consistent interactions on top of the free theory calls for a meticulous analysis which, again, evades the scope of our work.

A. Collective modes in massive spin-3/spin-1 propagation

With the privilege of arbitrariness, we direct our focus to the spectrum of the mixed system for the three fields,

$$\begin{aligned} A_{\mu\nu\rho} &\supset 3_1^- \oplus 2_1^+ \oplus 1_1^- \oplus 1_4^- \oplus 0_1^+ \oplus 0_4^+, \\ V_\mu &\supset 1_8^- \oplus 0_7^+, \\ \phi &\supset 0_8^+, \end{aligned} \tag{5.1}$$

and we set the goal of finding a propagating spin-3 particle and, possibly, other accompanying healthy massive particles. By starting with all of the parameters unconstrained, we are forced to repeat—for each of our attempts—the process illustrated in the previous sections, ultimately leading to the saturated propagator. In every case we would repeat such steps, and in the presence of pathologies we would look for a coupling texture in order to prevent them. To avoid redundancies and the corresponding tenfold growth of the volume of this paper, we will refrain (when not strictly necessary) from showing all of the details of the process. As in the previous examples, we want to study the action

$$S = S_{AA} + S_{VV} + S_{\phi\phi} + S_{\text{mix}}, \tag{5.2}$$

which, at the beginning, has the generic unconstrained form defined by

$$\begin{aligned} S_{AA} &= \int d^4x[a_1\partial_\alpha A_\mu{}^\sigma{}_\sigma \partial^\mu A^\rho{}_\rho{}^\alpha + a_2\partial_\alpha A_\nu{}^\sigma{}_\sigma \partial^\alpha A^\mu{}_\mu{}^\nu \\ &\quad + a_3\partial_\alpha A^{\alpha\mu\nu}\partial_\beta A_{\mu\nu}{}^\beta + a_4\partial^\nu A^\alpha{}_\alpha{}^\beta \partial_\beta A_{\beta\nu}{}^\mu \\ &\quad + a_7\partial_\nu A_{\mu\nu\rho}\partial^\nu A^{\mu\nu\rho} + m_1^2 A^{\mu\nu\rho}A_{\mu\nu\rho} + m_2^2 A^\mu{}_\mu{}^\nu A_\nu{}^\rho{}_\rho], \\ S_{VV} &= \int d^4x[v_1\partial^\mu V^\nu \partial_\mu V_\nu + v_2\partial^\mu V^\nu \partial_\nu V_\mu + m_V^2 V_\mu V^\mu], \\ S_{\phi\phi} &= \int d^4x(s_1\partial^\mu\phi\partial_\mu\phi + m_S^2\phi^2), \end{aligned} \tag{5.3}$$

and

$$S_{\text{mix}} = \int d^4x[m_{AS}\phi\partial_\beta A^\alpha{}_\alpha{}^\beta + m_{AV}V_\mu A^{\mu\alpha}{}_\alpha + m_{VS}\phi\partial_\alpha V^\alpha]. \tag{5.4}$$

The spin/parity matrices are easily found to be

$$a_{i,j}^{\{3,-\}} = 2(m_1^2 + a_7 q^2), \tag{5.5}$$

$$a_{i,j}^{\{2,+\}} = 2\left[m_1^2 + \left(a_7 + \frac{a_3}{3}\right)q^2\right], \tag{5.6}$$

$$a_{i,j}^{\{1,-\}} = \begin{pmatrix} \frac{2}{3}(3m_1^2 + 5m_2^2 + q^2(5a_2 + 3a_7)) & \frac{\sqrt{5}}{3}(2m_2^2 + q^2(2a_2 + a_4)) & \sqrt{\frac{5}{3}}m_{AV} \\ \frac{\sqrt{5}}{3}(2m_2^2 + q^2(2a_2 + a_4)) & \frac{2}{3}(3m_1^2 + m_2^2 + q^2(a_2 + 2a_3 + a_4 + 3a_7)) & \frac{m_{AV}}{\sqrt{3}} \\ \sqrt{\frac{5}{3}}m_{AV} & \frac{m_{AV}}{\sqrt{3}} & 2(m_V^2 + v_1 q^2) \end{pmatrix}, \tag{5.7}$$

and

$$a_{i,j}^{\{0,+\}} = \begin{pmatrix} 2(m_1^2+m_2^2+q^2(a_1+a_2+\frac{a_3}{3}+a_7)) & 2(m_2^2+q^2(a_1+a_2+\frac{a_4}{2})) & m_{AV} & -im_{AS}\sqrt{q^2} \\ 2(m_2^2+q^2(a_1+a_2+\frac{a_4}{2})) & 2(m_1^2+m_2^2+q^2(a_1+a_2+a_3+a_4+a_7)) & m_{AV} & -im_{AS}\sqrt{q^2} \\ m_{AV} & m_{AV} & 2(m_V^2+q^2(v_1+v_2)) & -im_{AS}\sqrt{q^2} \\ im_{AS}\sqrt{q^2} & im_{AS}\sqrt{q^2} & im_{AS}\sqrt{q^2} & 2(m_S^2+s_1q^2) \end{pmatrix}. \tag{5.8}$$

We notice immediately that a unique mass parameter appears in the combinations defining the masses of the 3^- and 2^+ sectors. Once we compute the saturated propagator above and the two massive poles $m_{3^-}^2 = -\frac{a_7}{m_1^2}$ and $m_{2^+}^2 = -\frac{a_7 + a_3/3}{m_1^2}$, we find (as is common when spin sectors stem from the same field) that it is impossible to simultaneously require causality and unitarity for both (see also Ref. [64]). We solve this by requiring

$$a_7 = -1, \qquad m_1^2 > 0, \quad \text{and} \quad a_3 = 3. \tag{5.9}$$

These conditions will allow the nonpathological propagation of a massive spin-3 particle and make the 2^+ sector innocuous. For the larger 1^- and 0^+ sectors the first obstacle is represented by higher-order poles q^{2n}, which show up already when computing the determinant of the corresponding spin/parity matrices. In looking for physical propagation, we can explore different ways to reduce the degree of such polynomials in q^2 and check, for each possibility, if they welcome unitarity and causality. After trial and error we find a workable parameter space by asking the poles of the 0^+ sector to disappear and 1^- to have a single massive pole in q^2. These requirements define a limited set of different parameter textures which we can explore in full detail. Again, this choice was dictated by an apparent simplicity of the constraints and is in no way unique. In principle, it is possible to let the determinant be higher order in q^2 without this translating into higher-order poles of the propagator (for instance, n massive scalar fields would generate a q^{2n} polynomial in the $a_{i,j}^{\{0,+\}}$ determinant). Moreover, to coherently follow the initial choice of the model's degrees of freedom $A_{\mu\nu\rho}$, V_μ, and ϕ, we have only explored cases with nonzero quadratic mixing. Once we narrow the parameter space, the constraining power of unitarity and causality can fully unfold and remove all but one surviving possibility. This is greatly simplified by setting the free parameter $a_4 = 0$ and, by simply solving the algebraic equations that nullify the unwanted coefficients of higher-order poles, getting the restrictions

$$a_1 = -\frac{3}{4}, \quad a_2 = \frac{3}{4}, \quad m_2^2 = -\frac{9}{8} m_1^2,$$
$$m_{AS} = \frac{m_S^2 - 4m_1^2 s_1}{4\sqrt{2m_S^2}}, \quad m_V^2 = 0, \quad m_{AV} = -\frac{m_1^2 m_{VS}}{\sqrt{2m_S^2}}. \tag{5.10}$$

Once these conditions are adopted, the saturated propagator over the mass

$$m_{1^-}^2 = -\frac{3m_1^2 m_{VS}^2}{2m_{VS}^2 + 15m_S^2 v_1} \tag{5.11}$$

can be promptly computed. We find three propagating states with the same residue $\mathrm{Res}_{q^2\to m_{1^-}^2}$,

$$\mathrm{Res}_{q^2\to m_{1^-}^2} = -\frac{3(56m_{VS}^6 + 5625m_S^6 v_1^2 + 20m_{VS}^4 m_S^2(5+6v_1) + 300m_{VS}^2 m_S^4 v_1(5+6v_1))}{10(2m_{VS}^2 + 15m_S^2 v_1)^3}. \tag{5.12}$$

Solutions that are causal and unitary exist provided that

$$m_{VS} \neq 0, \quad m_V^2 < 0, \quad m_S^2 > 0, \quad v_1 < 0,$$
$$m_1^2 > 0, \quad \text{and} \quad 2m_{VS}^2 + 15m_S^2 v_1 < 0. \tag{5.13}$$

VI. CONCLUSIONS

With this work we have completed the full set of operators needed to uncover the propagating spin/parity components expressed in quadratic Lagrangians of tensor fields up to rank 3. We did so by supporting the already known projectors with the tensor functions connecting different representations of the same spin/parity sector. We have illustrated how a dedicated sequence of manipulations can help to assess, unambiguously and in a gauge-invariant way, the physical properties of the particle quanta. By focusing on the saturated propagator, all possible cases—massless and critical ones—can be tackled, and the only drawback is represented by the inversion of large matrices when multiple components belong to the same sector. With such machinery, we have studied the spectrum of well known theories exhibiting quadratic mixing, like Stueckelberg and Einstein gravity in the Palatini formalism, as well as more exotic ones like the Singh-Hagen model and the gauge-based Klishevich-Zinoviev system. The latter for the first time subjected to a spectral analysis based on the use of projector algebra. Finally, after a long scrutiny through the strong constraints of ghost and tachyon elusion, we have defined a new model collectively propagating a massive vector together with a massive spin-3 particle.

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APPENDIX: SPIN-PROJECTOR OPERATORS INCLUDING RANK-0 AND RANK-1 FIELDS

We list here the independent set of spin-projector operators which completes the ones of Refs. [29,33] by including representations carried by scalar and vector fields. The operators missing from this list can be recovered via Eq. (2.10).

1. $P^{0,p}_{\{i=1\cdots 7,7\}}$

$$
\begin{aligned}
P^{0,p}_{\{1,7\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{(q^2 g^{\beta\gamma} q^\alpha + q^2 g^{\alpha\gamma} q^\beta + (q^2 g^{\alpha\beta} - 3q^\alpha q^\beta) q^\gamma) q_\mu}{3q^4}, \\
P^{0,p}_{\{2,7\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{(2g^{\beta\gamma} q^\alpha - g^{\alpha\gamma} q^\beta - g^{\alpha\beta} q^\gamma) q_\mu}{3\sqrt{2} q^2}, \\
P^{0,p}_{\{3,7\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{(g^{\alpha\gamma} q^\beta - g^{\alpha\beta} q^\gamma) q_\mu}{\sqrt{6} q^2}, \\
P^{0,p}_{\{4,7\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{q^\alpha q^\beta q^\gamma q_\mu}{q^4}, \\
P^{0,p}_{\{5,7\}}{}^{\alpha\beta}{}_{\mu} &= \frac{(q^2 g^{\alpha\beta} - q^\alpha q^\beta) q_\mu}{\sqrt{3} (q^2)^{3/2}}, \\
P^{0,p}_{\{6,7\}}{}^{\alpha\beta}{}_{\mu} &= \frac{q^\alpha q^\beta q_\mu}{(q^2)^{3/2}}, \\
P^{0,p}_{\{7,7\}}{}^{\alpha}{}_{\beta} &= \frac{q^\alpha q_\beta}{q^2}.
\end{aligned} \tag{A1}
$$

2. $P^{1,m}_{\{i=1\cdots 8,8\}}$

$$
\begin{aligned}
P^{1,m}_{\{1,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{1}{\sqrt{15} q^4} (q^4 g^{\alpha\beta} \delta^\gamma_\mu - q^2 \delta^\gamma_\mu q^\alpha q^\beta - q^2 \delta^\beta_\mu q^\alpha q^\gamma + q^2 \delta^\alpha_\mu (q^2 g^{\beta\gamma} - q^\beta q^\gamma) - q^2 g^{\beta\gamma} q^\alpha q_\mu - q^2 g^{\alpha\beta} q^\gamma q_\mu \\
&\quad + 3q^\alpha q^\beta q^\gamma q_\mu + q^2 g^{\alpha\gamma} (q^2 g^\beta_\mu - q^\beta q_\mu)), \\
P^{1,m}_{\{2,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{1}{2\sqrt{3} q^2} (q^2 g^{\alpha\beta} \delta^\gamma_\mu - \delta^\gamma_\mu q^\alpha q^\beta - \delta^\beta_\mu q^\alpha q^\gamma - 2\delta^\alpha_\mu (q^2 g^{\beta\gamma} - q^\beta q^\gamma) \\
&\quad + 2g^{\beta\gamma} q^\alpha q_\mu - g^{\alpha\beta} q^\gamma q_\mu - g^{\alpha\gamma} (-q^2 \delta^\beta_\mu + q^\beta q_\mu)), \\
P^{1,m}_{\{3,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{q^\alpha (-\delta^\gamma_\mu q^\beta + \delta^\beta_\mu q^\gamma) + g^{\alpha\gamma} (-q^2 g^\beta_\mu + q^\beta q^\mu) + g^{\alpha\beta} (q^2 \delta^\gamma_\mu - q^\gamma q_\mu)}{2q^2}, \\
P^{1,m}_{\{4,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{q^2 \delta^\gamma_\mu q^\alpha q^\beta + q^\gamma (q^2 \delta^\beta_\mu q^\alpha + q^\beta (q^2 \delta^\alpha_\mu - 3q^\alpha q_\mu))}{\sqrt{3} q^4}, \\
P^{1,m}_{\{5,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{\delta^\gamma_\mu q^\alpha q^\beta + \delta^\beta_\mu q^\alpha q^\gamma - 2\delta^\alpha_\mu q^\beta q^\gamma}{\sqrt{6} q^2}, \\
P^{1,m}_{\{6,8\}}{}^{\alpha\beta\gamma}{}_{\mu} &= \frac{q^\alpha (\delta^\gamma_\mu q^\beta - \delta^\beta_\mu q^\gamma)}{\sqrt{2} q^2}, \\
P^{1,m}_{\{7,8\}}{}^{\alpha\beta}{}_{\mu} &= \frac{q^2 \delta^\beta_\mu q^\alpha + q^\beta (q^2 \delta^\alpha_\mu - 2q^\alpha q_\mu)}{\sqrt{2} (q^2)^{3/2}}, \\
P^{1,m}_{\{8,8\}}{}^{\alpha}{}_{\beta} &= \delta^\alpha_\beta - \frac{q^\alpha q_\beta}{q^2}.
\end{aligned} \tag{A2}
$$

3. $P^{0,p}_{\{i=1\cdots 8,8\}}$

$$
\begin{aligned}
P^{0,p}_{\{1,8\}}{}^{\alpha\beta\gamma} &= \frac{q^2 g^{\beta\gamma} q^\alpha + q^2 g^{\alpha\gamma} q^\beta + (q^2 g^{\alpha\beta} - 3 q^\alpha q^\beta) q^\gamma}{3(q^2)^{3/2}}, \\
P^{0,p}_{\{2,8\}}{}^{\alpha\beta\gamma} &= \frac{2 g^{\beta\gamma} q^\alpha - g^{\alpha\gamma} q^\beta - g^{\alpha\beta} q^\gamma}{3\sqrt{2}(q^2)^{1/2}}, \\
P^{0,p}_{\{3,8\}}{}^{\alpha\beta\gamma} &= \frac{g^{\alpha\gamma} q^\beta - g^{\alpha\beta} q^\gamma}{\sqrt{6}(q^2)^{1/2}}, \\
P^{0,p}_{\{4,8\}}{}^{\alpha\beta\gamma} &= \frac{q^\alpha q^\beta q^\gamma}{(q^2)^{3/2}}, \\
P^{0,p}_{\{5,8\}}{}^{\alpha\beta} &= \frac{q^2 g^{\alpha\beta} - q^\alpha q^\beta}{\sqrt{3} q^2}, \\
P^{0,p}_{\{6,8\}}{}^{\alpha\beta} &= \frac{q^\alpha q^\beta}{q^2}, \\
P^{0,p}_{\{7,8\}}{}^{\alpha} &= \frac{q^\alpha}{(q^2)^{1/2}}, \\
P^{0,p}_{\{8,8\}} &= 1. \qquad \text{(A3)}
\end{aligned}
$$

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