$SO(5)$ Landau model and 4D quantum Hall effect in the $SO(4)$ monopole background

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We investigate the $SO(5)$ Landau problem in the $SO(4)$ monopole gauge field background by applying the techniques of the non-linear realization of quantum field theory. The $SO(4)$ monopole carries two topological invariants, the second Chern number and a generalized Euler number, specified by the $SU(2)$ monopole and antimonopole indices, I_+ and I_-. The energy levels of the $SO(5)$ Landau problem are grouped into $\mathrm{Min}(I_+, I_-) + 1$ sectors, each of which holds Landau levels. In the n-sectors, Nth Landau level eigenstates constitute the $SO(5)$ irreducible representation with $(p, q)_5 = (N + I_+ + I_- - n, N + n)_5$ whose function form is obtained from the $SO(5)$ nonlinear realization matrix. In the $n = 0$ sector, the emergent quantum geometry of the lowest Landau level is identified as the fuzzy four-sphere with radius being proportional to the difference between I_+ and I_-. The Laughlin-like wavefunction is constructed by imposing the $SO(5)$ lowest Landau level projection to the many-body wavefunction made of the Slater determinant. We also analyze the relativistic version of the $SO(5)$ Landau model to demonstrate the Atiyah-Singer index theorem in the $SO(4)$ gauge field configuration.

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I. INTRODUCTION

More than forty years ago, Yang introduced the $SU(2)$ monopole that epitomizes beautiful topological features of a non-Abelian gauge field [1,2]. The $SU(2)$ monopole on S^4 realizes a natural non-Abelian generalization of the $U(1)$ principal fiber of the Dirac monopole on S^2 [3], and the $SU(2)$ monopole charge exemplifies a physical manifestation of the second Chern number. Not only for its elegant mathematical structure, the $SU(2)$ monopole found its physical applications in the $SO(5)$ Landau model and four-dimensional (4D) quantum Hall effect [4], which, from a modern point of view, is the first theoretical model of a topological insulator in higher dimension. The underlying geometry of the system is the nested quantum Nambu geometry that does not have any counterpart in classical geometry [5], which renders the system to be quite unique also in view of the noncommutative geometry [6,7]. Tensor-type Chern-Simons theories are proposed as effective field theories [6,7] that naturally induce generalized fractional statistics of extended objects [8–10]. The theoretical formulation of the quantum Hall effect has now been generalized to even higher dimensions [11–21] and supersymmetric versions [22,23].

In recent years, studies of the higher dimensional topological phases took a new turn. The idea of the synthetic dimension and artificial gauge field allowed researchers to access higher dimensional topological phases with tabletop experiments. The artificial $SU(2)$ monopole gauge field has been implemented in systems such as cold atoms [24] and metamaterials [25]. For topological features specific to the 4D quantum Hall effect, a number of experiments have been proposed in cold atoms [26,27], photonics [28], circuit [29], and acoustics [30], and several theoretical predictions have already been confirmed [31,32]. Along with the developments, a five-dimensional (5D) Weyl semimetal with an $SU(2)$ monopole and $SU(2)$ antimonopole structure in the momentum space has been proposed [33,34] and reported to host higher order topological insulators [35,36]. Partially inspired by the recent progress of higher dimensional topological physics, we present a formulation of the 4D quantum Hall effect with an $SO(4)$ gauge structure. The $SO(4) \simeq SU(2) \otimes SU(2)$ group is only the semisimple group among all of the $SO(n)$ groups, and the $SO(4)$ monopole can be regarded as a "composite" of the $SU(2)$ monopole and the $SU(2)$ antimonopole. This notable structure is significant in the perspective of the topological insulator, because with the $SU(2)$ monopole and the $SU(2)$ antimonopole in the same magnitude, the system may realize a nonchiral topological phase in a higher dimension. This feature is quite analogous to that of the quantum spin Hall effect [37–39].[1] Also in perspectives of the string theory, the nonchiral topological insulator is interesting. The formerly constructed even dimensional quantum Hall systems are all chiral that correspond to the chiral superstring theory known as type II, while nonchiral quantum Hall systems realize rare setups that correspond to the nonchiral superstring theory known as type I [42,43].

The $SO(4)$ gauge structure naturally appears in the context of the 4D quantum Hall effect.[2] In the setup of the Landau models, the gauge group is adopted to be equal to the holonomy group of the base manifold (see [44,45] as reviews). For the $SO(5)$ Landau model, the base manifold is S^4 whose holonomy group is $SO(4)$, and in former researches, one $SU(2)$ of the $SO(4) \simeq SU(2) \otimes SU(2)$ was adopted as the gauge group. Notably, Yang applied the method of separation of variables in solving the differential equation of the $SO(5)$ Landau problem in the $SU(2)$ monopole background and successfully derived the eigenvalues and the eigenfunctions [2].[3] Though the analysis of the $SO(4)$ case is obviously significant, it is still left unexplored. It may be because the Landau problem in the $SO(4)$ monopole background is far more complicated compared to the $SU(2)$ case. To overcome such technical difficulties, we adopt the techniques of nonlinear realization. While the nonlinear realization technique has been developed in quantum field theory [46–48], the nonlinear realization is closely related to quantum mechanical systems with gauge symmetries [49] and has been successfully applied to recent analyses of the Landau models [11,14,50,51]. We use this method and completely solve the $SO(5)$ Landau model in the $SO(4)$ monopole background. With newly obtained monopole harmonics, we unveil particular properties of the $SO(5)$ Landau model and 4D quantum Hall effect.

The paper is organized as follows. In Sec. II, we present a brief review about the nonlinear realization of the $SO(3)$ Landau model. Section III explains the Yang $SU(2)$ monopole in a modern notation and derives a general form of the $SO(5)$ matrix generators. In Sec. IV, we exploit the nonlinear realization for the $SO(5)$ group. The $SO(5)$ Landau problem in the $SO(4)$ monopole background is investigated in Sec. V. In Sec. VI, we identify the noncommutative geometry and construct a Laughlin-like many-body wave function. The relativistic Landau model is discussed in Sec. VII to demonstrate the Atiyah-Singer index theorem for the $SO(4)$ gauge field. Section VIII is devoted to summary and discussions.

II. $SO(3)$ MONOPOLE HARMONICS AND NONLINEAR REALIZATION

The monopole harmonics are known as the eigenstates of the $SU(2)$ Casimir of the angular momentum in the Dirac monopole background. In the Dirac gauge, the monopole gauge field is given by

$$A_i = -g\frac{1}{r(r+z)}\epsilon_{ij3}x_j, \tag{1}$$

and the corresponding magnetic field is derived as

$$B_i = \epsilon_{ijk}\partial_j A_k = g\frac{1}{r^3}x_i. \tag{2}$$

Here, g takes an integer or a half-integer due to the Dirac quantization condition.[4] The covariant angular momentum operators are constructed as

$$\Lambda_i = -i\epsilon_{ijk}x_j(\partial_k + iA_k), \tag{4}$$

and the total angular momentum operators are

$$L_i^{(g)} = \Lambda_i + r^2 B_i. \tag{5}$$

In detail, Eq. (5) is given by

$$L_3^{(g)} = L_3^{(0)} + g, \quad L_m^{(g)} = L_m^{(0)} + g\frac{1}{r+x_3}x_m \quad (m=1,2), \tag{6}$$

with

$$L_i^{(0)} = -i\epsilon_{ijk}x_j\partial_k. \tag{7}$$

[1] For a time-reversal symmetric 3D topological insulator with Landau levels, one may consult Refs. [40,41].

[2] Strictly speaking, the universal cover of $SO(4)$, i.e., $Spin(4)$, is adopted as the gauge group.

[3] Such monopole harmonics are known as the $SU(2)$ monopole harmonics, but in the present paper, we refer to the eigenstates as the $SO(5)$ monopole harmonics with emphasis on their $SO(5)$ covariance.

[4] The $U(1)$ monopole charge is given by

$$c_1 = \frac{1}{2\pi}\int_{S^2} B = 2g, \tag{3}$$

which represents the first Chern number of integer value. The result (3) is consistent with the fact that g is either an integer or a half-integer.

We introduce a nonlinear realization of the $SU(2)$ group for the coset $SU(2)/U(1)$ as[5]

$$\Phi_l(\theta,\phi) = e^{i\theta\sum_{m,n=1}^{2}\epsilon_{mn}y_m(\phi)S_n^{(l)}}, \tag{11}$$

where

$$y_1 \equiv \cos\phi, \qquad y_2 \equiv \sin\phi, \tag{12}$$

and $S_i^{(l)}$ denote the $SU(2)$ matrices of spin magnitude l with their third component being

$$S_z^{(l)} = \mathrm{diag}(l, l-1, l-2, \ldots, -l). \tag{13}$$

We see that the nonlinear realization (11) is a $(2l+1)\times(2l+1)$ matrix that satisfies

$$L_i^{(g=S_z^{(l)})}\Phi_l(\theta,\phi) = \Phi_l(\theta,\phi)S_i^{(l)}. \tag{14}$$

By denoting the components of $\Phi_l(\theta,\phi)$ as

$$\varphi_{l,m}^{(g)}(\theta,\phi) \equiv (\Phi_l(\theta,\phi))_{g,m} \qquad (g, m = l, l-1, l-2, \ldots, -l+1, -l), \tag{15}$$

Eq. (14) is recast into the following form:

$$L_i^{(g)}\varphi_{l,m}^{(g)} = \sum_{m'=-l}^{l}\varphi_{l,m'}^{(g)}(S_i^{(l)})_{m'm}, \tag{16}$$

and then

$$L_i^{(g)2}\varphi_{l,m}^{(g)} = \sum_{m'=-l}^{l}\varphi_{l,m'}^{(g)}(S_i^{(l)2})_{m'm} = l(l+1)\varphi_{l,m}^{(g)}, \tag{17}$$

which indicates that $\varphi_{l,m}^{(g)}(\theta,\phi)$ realize the monopole harmonics introduced in [52]. With normalization factors, the normalized monopole harmonics are expressed as

$$\sqrt{\frac{2l+1}{4\pi}}\varphi_{l,m}^{(g)}(\theta,\phi). \tag{18}$$

Notice that the nonlinear realization (11) is factorized as

$$\Phi_l(\theta,\phi) = e^{-i\phi S_z^{(l)}}e^{i\theta S_y^{(l)}}e^{i\phi S_z^{(l)}} = D_l(\phi,-\theta,-\phi). \tag{19}$$

Here, D is Wigner's D functions (see [53] for instance):

$$D_l(\chi,\theta,\phi) = e^{-i\chi S_z^{(l)}}e^{-i\theta S_y^{(l)}}e^{-i\phi S_z^{(l)}}. \tag{20}$$

Equation (15) is equal to $D_l(\phi,-\theta,-\phi)_{g,m} = d_{l,g,m}(-\theta)e^{i(m-g)\phi}$ with $d_{j,m,m'}$ being Wigner's small D matrix[6]:

$$d_{j,m,m'}(\theta) = (e^{-i\theta S_y^{(j)}})_{m,m'}. \tag{23}$$

With the monopole harmonics that satisfy (17), it is now feasible to solve the $SO(3)$ Landau problem on a sphere [52,54]:

$$H = \frac{1}{2M}\sum_{i=1}^{3}\Lambda_i{}^2 = \frac{1}{2M}\left(\sum_{i=1}^{3}L_i^{(g)2} - r^4\sum_i B_i{}^2\right) = \frac{1}{2M}\left(\sum_{i=1}^{3}L_i^{(g)2} - g^2\right). \tag{24}$$

While l was assumed to be a given quantity, the input parameter in the Landau Hamiltonian is the monopole charge g, and then l should be determined by g. In the following we assume $g \geq 0$ for simplicity. The $SU(2)$ spin index l is greater than or equal to g, and so l starts from g (not from 0). Therefore, the Landau level index N may be identified as

$$N \equiv l - g = 0, 1, 2, \ldots. \tag{25}$$

[5]When $l = 1/2$, Eq. (11) is represented as

$$\Phi_{1/2}(\theta,\phi) = e^{i\frac{\theta}{2}\sum_{m,n=1}^{2}\epsilon_{mn}y_m(\phi)\sigma_n} = \begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi}\\ -\sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2}\end{pmatrix} = \frac{1}{\sqrt{2(1+x_3)}}\begin{pmatrix}1+x_3 & x_1-ix_2\\ -x_1-ix_2 & 1+x_3\end{pmatrix}, \tag{8}$$

where

$$x_1 = \sin\theta\cos\phi, \qquad x_2 = \sin\theta\sin\phi, \qquad x_3 = \cos\theta. \tag{9}$$

One may readily check that $\Phi_{1/2}(\theta,\phi)$ (8) satisfies (14):

$$L_i^{(g=\frac{1}{2}\sigma_3)}\Phi_{1/2}(\theta,\phi) = \Phi_{1/2}(\theta,\phi)\frac{1}{2}\sigma_i. \tag{10}$$

[6]The explicit form of (23) is given by

$$d_{j,m,m'}(\theta) = (-1)^{m-m'}\sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\left(\cos\frac{\theta}{2}\right)^{m+m'} \times\left(\sin\frac{\theta}{2}\right)^{m-m'}P_{j-m}^{(m-m',m+m')}(\cos\theta), \tag{21}$$

where $P_n^{(\alpha,\beta)}(x)$ stand for the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta}. \tag{22}$$

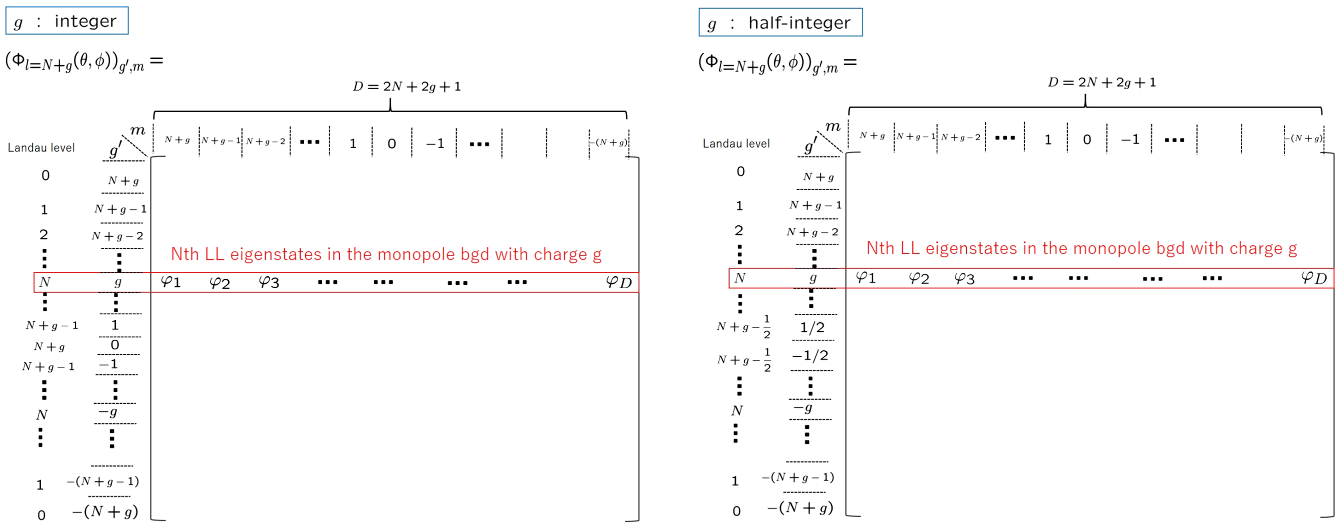

FIG. 1. The Nth Landau level eigenstates are realized as the components of the red enclosed $(N+1)$th column of the nonlinear realization.

We then identify the $SU(2)$ spin index l of the nonlinear realization (11) as

$$l = N + g. \tag{26}$$

From (15), we can now derive the $(N+1)$th column of $\Phi_{l=N+g}(\theta,\phi)$ as the set of the Nth Landau level eigenstates:

$$\Phi_{l=N+g,g,m} = \varphi^{(g)}_{l=N+g,m} \qquad (m = l, l-1, l-2, \ldots, -l). \tag{27}$$

See Fig. 1. Equation (17) implies that the eigenenergy of (24) is given by

$$\begin{aligned} E_N &= \frac{1}{2M}(S_i^{(l=N+g)2} - g^2) \\ &= \frac{1}{2M}(l(l+1)|_{l=N+g} - g^2) \\ &= \frac{1}{2M}(N(N+1) + g(2N+1)), \end{aligned} \tag{28}$$

and (27) denotes the Nth Landau level eigenstates. Notice that we first identified the Landau level eigenstates as the nonlinear realization, and later we derived the Landau energy levels from the $SU(2)$ covariance of the nonlinear realization.

Let us summarize the essence of the nonlinear realization technique. Once the nonlinear realization was constructed, we can read off the lowest and higher Landau level eigenstates from its matrix elements. In the construction of the nonlinear realization (11), what we needed was just the higher spin matrices. The explicit form of the higher spin matrices has been known, but even if we did not know them, we can derive them by sandwiching the angular momentum operators with some appropriate irreducible representation, say, the lowest Landau level (LLL) eigenstates.[7] In the following sections, we apply these observations for solving the $SO(5)$ Landau problem in the $SO(4)$ monopole background.

III. $SO(5)$ MATRIX GENERATORS FROM YANG'S MONOPOLE HARMONICS

We first need to derive the matrix generators of arbitrary $SO(5)$ irreducible representations. Fortunately, Yang already derived a complete basis set of the $SO(5)$ irreducible representations as the $SO(5)$ monopole harmonics [2]. Sandwiching the $SO(5)$ angular momentum operators with the $SO(5)$ monopole harmonics, we can in principle derive the $SO(5)$ matrix generators of arbitrary representations. In this section, we review Yang's work with a modern notation [5] and derive a general matrix form of the $SO(5)$ generators.

A. Basics of the $SO(5)$ representation

The $SO(5)$ algebra holds two non-negative integer Casimir indices, p and q ($p \geq q$); the $SO(5)$ Casimir eigenvalue for the $SO(5)$ irreducible representation, $(p,q)_5$, is given by

$$\lambda(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2 + 2p + q; \tag{30}$$

[7] Using the LLL eigenstates $\varphi^{(g)}_{g,m}$, we can construct the higher spin matrices with spin magnitude g by the formula:

$$\frac{2g+1}{4\pi}\int_{S^2} d\Omega_2 \varphi^{(g)*}_{g,m} L_i^{(g)} \varphi^{(g)}_{g,m'} = (S_i^{(g)})_{mm'}. \tag{29}$$

and the corresponding dimension is

$$D(p,q)=\frac{1}{6}(p+2)(q+1)(p+q+3)(p-q+1). \quad (31)$$

The $SO(4) \simeq SU(2) \otimes SU(2)$ subgroup decomposition is given by (Fig. 2)

$$(p,q)_5 = \bigoplus_{0\le n\le q} \bigoplus_{-\frac{p-q}{2}\le s\le \frac{p-q}{2}} (j,k)_4, \quad (32)$$

where

$$(j,k)_4 \equiv \left(\frac{n}{2}+\frac{p-q}{4}+\frac{s}{2}, \frac{n}{2}+\frac{p-q}{4}-\frac{s}{2}\right)_4. \quad (33)$$

The symbols, j and k, denote the bi-spin indices of the $SO(4) \simeq SU(2) \otimes SU(2)$ group, while $n = j+k-\frac{p-q}{2} (= 0, 1, 2, \ldots, q)$ and $s = j-k (= -\frac{p-q}{2}, -\frac{p-q}{+}1, -\frac{p-q}{2}+2, \ldots, \frac{p-q}{2})$ indicate the Landau level index and the chirality parameter in the $SO(4)$ Landau model [5]. The notations, $(j,k)_4$ and $[n,s]$, are both useful according to context, and we hereafter utilize them interchangeably:

$$(j,k)_4 \leftrightarrow [n,s]. \quad (34)$$

Let us call the oblique lines in Fig. 2 specified by $j+k = n+\frac{p-q}{2}$ the $SO(4)$ lines. Each filled circle represents an $SO(4)$ irreducible representation $(j,k)_4$ with dimension $(2j+1)(2k+1)$. On the nth $SO(4)$ line, there are $(p-q+1)$ $SO(4)$ irreducible representations, and the total dimension of those $SO(4)$ irreducible representations is counted as

$$d(n, p-q) \equiv \sum_{-\frac{p-q}{2}\le s\le\frac{p-q}{2}} (2j+1)(2k+1)$$
$$= \frac{1}{6}(p-q+1)((p-q)^2+(6n+5)(p-q) + 6(n+1)^2). \quad (35)$$

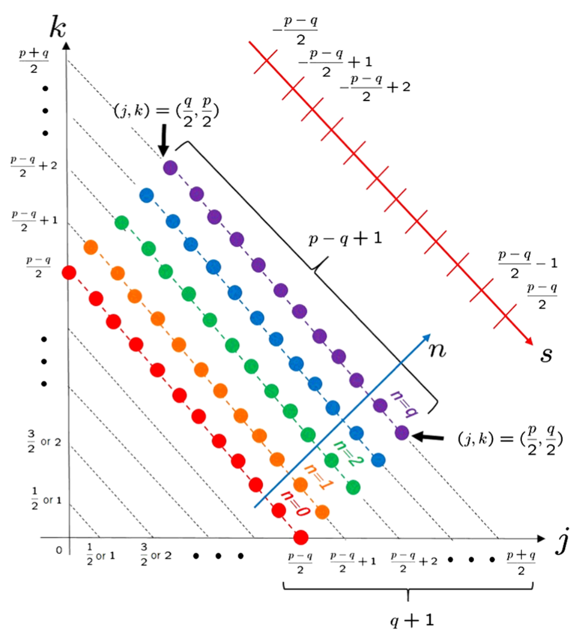

FIG. 2. Each of the filled circles represents an $SO(4)$ irreducible representation. The $SO(4)$ irreducible representations represented by the filled circles amount to the $SO(5)$ irreducible representation $(p,q)_5$. (Taken from [5].)

As depicted in Fig. 2, the $SO(4)$ irreducible representations on the $(q+1)$ $SO(4)$ lines $(n = 0, 1, 2, \ldots, q)$ constitute the $SO(5)$ irreducible representation $(p,q)_5$:

$$\sum_{n=0}^{q} d(n, p-q) = D(p,q), \quad (36)$$

where $D(p,q)$ is given by (31).

B. $SO(5)$ monopole harmonics in the $SU(2)$ background

In the Dirac gauge, the $SU(2)$ antimonopole gauge field [4] is represented as

$$A_m = -\frac{1}{r(r+x_5)}\bar{\eta}^i_{mn} x_n S_i \quad (m,n = 1,2,3,4), \quad A_5 = 0, \quad (37)$$

where S_i $(i = 1, 2, 3)$ denote the $SU(2)$ matrix of the spin $I/2$ representation,

$$S_i S_i = \frac{I}{2}\left(\frac{I}{2}+1\right)\mathbf{1}_{I+1}, \quad (38)$$

and $\bar{\eta}^i_{mn}$ signifies the 't Hooft symbol:

$$\eta^i_{mn} \equiv \epsilon_{mni4} + \delta_{mi}\delta_{n4} - \delta_{m4}\delta_{ni},$$
$$\bar{\eta}^i_{mn} \equiv \epsilon_{mni4} - \delta_{mi}\delta_{n4} + \delta_{m4}\delta_{ni}. \quad (39)$$

We construct the covariant angular momentum operators as

$$\Lambda_{ab} = -ix_a D_b + ix_b D_a \qquad (D_a = \partial_a + iA_a) \quad (40)$$

and the total $SO(5)$ angular momentum operators as

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab}. \quad (41)$$

The field strength, $F_{ab}=\partial_a A_b-\partial_b A_a+i[A_a,A_b]$ $(a,b=1,2,3,4,5)$, is derived as[8]

$$F_{mn}=-\frac{1}{r^2}x_mA_n+\frac{1}{r^2}x_nA_m+\frac{1}{r^2}\bar{\eta}^i_{mn}S_i,$$
$$F_{m5}=-F_{5m}=\frac{1}{r^2}(r+x_5)A_m, \tag{44}$$

and (41) is given by

$$L_{mn}=L^{(0)}_{mn}+\bar{\eta}^i_{mn}S_i,\qquad L_{m5}=L^{(0)}_{m5}-\frac{1}{r+x_5}\bar{\eta}^i_{mn}x_nS_i, \tag{45}$$

where $L^{(0)}_{ab}$ denote the $SO(5)$ free angular momentum operators:

$$L^{(0)}_{ab}=-ix_a\partial_b+ix_b\partial_a. \tag{46}$$

Now the eigenvalue problem of the $SO(5)$ Casimir operator reads

$$\sum_{a<b=1}^{5}L_{ab}{}^2\boldsymbol{\psi}=\lambda\boldsymbol{\psi}. \tag{47}$$

Yang showed that with a given $SU(2)$ monopole index I, p and q are related as

$$p-q=I \tag{48}$$

or

$$(p,q)_5=(N+I,N)_5. \tag{49}$$

Here N denotes a non-negative integer value that corresponds to the Landau level of the $SO(5)$ Landau model [4]. Substituting (49) into (30) and (31), respectively, we readily obtain the $SO(5)$ Casimir eigenvalues of (47) and the degeneracies as

$$\lambda(N+I,N)=N^2+N(I+3)+\frac{1}{2}I(I+4), \tag{50a}$$

$$D(N+I,N)=\frac{1}{6}(N+1)(I+1)(I+N+2)(I+2N+3). \tag{50b}$$

Thus, once the identification (49) was established, the derivation of the eigenvalues is an easy task, but the derivation of the eigenstates needs a different task. Yang used the method of the separation of variables for solving the differential equation (47) [2]. We will not here repeat that derivation but just write down the results in a modern notation [5]. With the polar coordinates on a four-sphere (with unit radius)

$$x_1=\sin\xi\sin\chi\sin\theta\cos\phi,\qquad x_2=\sin\xi\sin\chi\sin\theta\sin\phi,$$
$$x_3=\sin\xi\sin\chi\cos\theta,$$
$$x_4=\sin\xi\cos\chi,\qquad x_5=\cos\xi,$$
$$(0\le\xi\le\pi,\quad 0\le\chi\le\pi,\quad 0\le\theta\le\pi,\quad 0\le\phi<2\pi), \tag{51}$$

the normalized $SO(5)$ monopole harmonics are represented as[9]

$$\boldsymbol{\psi}_{N;j,m_j;k,m_k}(\Omega_4)=G_{N,j,k}(\xi)\cdot\boldsymbol{Y}_{j,m_j;k,m_k}(\Omega_3)\quad[\Omega_3=(\chi,\theta,\phi)], \tag{54}$$

where

$$G_{N,j,k}(\xi)=(-1)^{2j+1}\sqrt{N+\frac{I}{2}+\frac{3}{2}}\frac{1}{\sin\xi}d_{N+\frac{I}{2}+1,-j+k,j+k+1}(\xi), \tag{55a}$$

$$\boldsymbol{Y}_{j,m_j;k,m_k}(\Omega_3)=\sum_{m_R=-j}^{j}\begin{pmatrix}C^{k,m_k}_{j,m_R;\frac{I}{2},\frac{I}{2}}\Phi_{j,m_j;j,m_R}(\Omega_3)\\ C^{k,m_k}_{j,m_R;\frac{I}{2},\frac{I}{2}-1}\Phi_{j,m_j;j,m_R}(\Omega_3)\\ \vdots\\ C^{k,m_k}_{j,m_R;\frac{I}{2},-\frac{I}{2}}\Phi_{j,m_j;j,m_R}(\Omega_3)\end{pmatrix}. \tag{55b}$$

[8] The nontrivial topology of the $SU(2)$ monopole field configuration is accounted for by

$$\pi_3(SU(2))\simeq\mathbb{Z}, \tag{42}$$

and the corresponding second Chern number is evaluated as

$$c_2=\frac{1}{8\pi^2}\int_{S^4}\mathrm{tr}F^2=-\frac{1}{6}I(I+1)(I+2), \tag{43}$$

where $F=\frac{1}{2}F_{ab}dx_a\wedge dx_b$ with (44).

[9] The orthonormal relation for the $SO(5)$ monopole harmonics is given by

$$\int d\Omega_4\boldsymbol{\psi}_{N;j,m_j;k,m_k}(\Omega_4)^\dagger\boldsymbol{\psi}_{N';j',m'_j;k',m'_k}(\Omega_4)=\delta_{NN'}\delta_{jj'}\delta_{kk'}\delta_{m_jm'_j}\delta_{m_km'_k}, \tag{52}$$

where

$$d\Omega_4=\sin^3\xi\,\sin^2\chi\;\sin\theta\,d\xi d\chi d\theta d\phi. \tag{53}$$

Here, $d_{N+\frac{I}{2}+1,-j+k,j+k+1}$ in (55a) stand for the Wigner's small D matrix (23), $C^{k,m_k}_{j,m_R;I/2,s_z}$ in (55b) represent the Clebsch-Gordan coefficients, and $\Phi_{j,m_j;j,m_R}(\Omega_3)$ denote the $SO(4)$ spherical harmonics [51]. From (49), the $SO(4)$ bi-spins (33) now become

$$(j,k)_4 \equiv \left(\frac{n}{2}+\frac{I}{4}+\frac{s}{2}, \frac{n}{2}+\frac{I}{4}-\frac{s}{2}\right)_4, \qquad (56)$$

where

$$n = 0,1,2,3,\dots,N, \qquad s = \frac{I}{2}, \frac{I}{2}-1, \dots, -\frac{I}{2}. \qquad (57)$$

Equation (56) implies that the Hilbert space of the Nth $SO(5)$ Landau level consists of the smaller Hilbert spaces of the inner $SO(4)$ Landau levels:

$$\mathcal{H}^{(p=N+I,q=N)}_{SO(5)} = \bigoplus_{0\le n\le N} \bigoplus_{-\frac{I}{2}\le s\le \frac{I}{2}} \mathcal{H}^{[n,s]}_{SO(4)}. \qquad (58)$$

For instance, the LLL ($N=0$) of $I=1$ holds fourfold degeneracy made of two $SO(4)$ irreducible representations, $[n,s]=[0,1/2]$ and $[0,-1/2]$,[10]

$$\begin{aligned}
\boldsymbol{\psi}_1 &\equiv \boldsymbol{\psi}_{0;1/2,1/2;0,0} = -\frac{\sqrt{3}}{2\pi}\sin\frac{\xi}{2}\begin{pmatrix}\cos\chi - i\sin\chi\cos\theta \\ -i\sin\chi\sin\theta e^{i\phi}\end{pmatrix},\\
\boldsymbol{\psi}_2 &\equiv \boldsymbol{\psi}_{0;1/2,-1/2;0,0} = \frac{\sqrt{3}}{2\pi}\sin\frac{\xi}{2}\begin{pmatrix} i\sin\chi\sin\theta e^{-i\phi} \\ -\cos\chi - i\sin\chi\cos\theta\end{pmatrix},\\
\boldsymbol{\psi}_3 &\equiv \boldsymbol{\psi}_{0;0,0;1/2,1/2} = -\frac{\sqrt{3}}{2\pi}\begin{pmatrix}\cos\frac{\xi}{2} \\ 0\end{pmatrix},\\
\boldsymbol{\psi}_4 &\equiv \boldsymbol{\psi}_{0;0,0;1/2,-1/2} = -\frac{\sqrt{3}}{2\pi}\begin{pmatrix}0 \\ \cos\frac{\xi}{2}\end{pmatrix}.
\end{aligned} \qquad (60)$$

C. $SO(5)$ matrix generators for arbitrary irreducible representation

We next investigate the matrix form of the $SO(5)$ generators of arbitrary irreducible representations. For notational brevity, with the understanding of (49) we simply represent $\boldsymbol{\psi}_{N,;j,m_j;k,m_k}$ (54) as

$$\boldsymbol{\psi}^{(p,q)_5}_\alpha, \qquad (61)$$

[10]The states of (60) are essentially equal to those of (B3):

$$\boldsymbol{\psi}_1 = \frac{\sqrt{3}}{2\pi}\boldsymbol{\psi}^{[0,-\frac{1}{2}]}_1, \qquad \boldsymbol{\psi}_1 = \frac{\sqrt{3}}{2\pi}\boldsymbol{\psi}^{[0,-\frac{1}{2}]}_2,$$
$$\boldsymbol{\psi}_3 = -\frac{\sqrt{3}}{2\pi}\boldsymbol{\psi}^{[0,-\frac{1}{2}]}_3, \qquad \boldsymbol{\psi}_4 = -\frac{\sqrt{3}}{2\pi}\boldsymbol{\psi}^{[0,-\frac{1}{2}]}_4. \qquad (59)$$

where

$$\alpha = (j,m_j;k,m_k) = 1,2,\dots,D(p,q). \qquad (62)$$

As the $SO(5)$ monopole harmonics realize a $(p,q)_5$ irreducible representation under the transformations generated by L_{ab},

$$L_{ab}\boldsymbol{\psi}^{(p,q)_5}_\alpha = \boldsymbol{\psi}^{(p,q)_5}_\beta (\Sigma^{(p,q)_5}_{ab})_{\beta\alpha}, \qquad (63)$$

we can derive the $SO(5)$ matrix generators of $(p,q)_5$ by

$$(\Sigma^{(p,q)_5}_{ab})_{\alpha\beta} = \int_{S^4} d\Omega_4 \boldsymbol{\psi}^{(p,q)_5\dagger}_\alpha L_{ab}\boldsymbol{\psi}^{(p,q)_5}_\beta. \qquad (64)$$

For instance, from (60), $\Sigma^{(1,0)_5}_{ab}$ are derived as[11]

$$\begin{aligned}
\Sigma^{(1,0)_5}_{mn} &= \frac{1}{2}\begin{pmatrix}\eta^i_{mn}\sigma_i & 0 \\ 0 & \bar{\eta}^i_{mn}\sigma_i\end{pmatrix},\\
\Sigma^{(1,0)_5}_{m5} &= i\frac{1}{2}\begin{pmatrix}0 & -\bar{q}_m \\ q_m & 0\end{pmatrix},
\end{aligned} \qquad (67)$$

where η^i_{mn} and $\bar{\eta}^i_{mn}$ are the 't Hooft symbols (39), and q_m and $\bar{q}_m$ denote the quaternions and their quaternion conjugates:

$$q_m = \{-i\sigma_i, 1\}, \qquad \bar{q}_m = \{i\sigma_i, 1\}. \qquad (68)$$

The $SO(4)$ decomposition (58) implies

$$\Sigma^{(p,q)_5}_{mn} = \bigoplus_{0\le n\le q} \bigoplus_{-\frac{p-q}{2}\le s\le \frac{p-q}{2}} \sigma^{(j=\frac{n}{2}+\frac{p-q}{4}+\frac{s}{2}, k=\frac{n}{2}+\frac{p-q}{4}-\frac{s}{2})_4}_{mn}, \qquad (69)$$

where $\sigma^{(j,k)_4}_{mn}$ are the $SO(4)\simeq SU(2)_L\otimes SU(2)_R$ matrix generators with index $(j,k)_4$,

$$\sigma^{(j,k)_4}_{mn} \equiv \eta^i_{mn}S^{(j)}_i \otimes \mathbf{1}_{2k+1} + \mathbf{1}_{2j+1}\otimes \bar{\eta}^i_{mn}S^{(k)}_i. \qquad (70)$$

More specifically,

[11]With the $SO(5)$ gamma matrices

$$\gamma_m = \begin{pmatrix}0 & \bar{q}_m \\ q_m & 0\end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}, \qquad (65)$$

Eqs. (67) are simply given by

$$\Sigma^{(1,0)_5}_{ab} = -i\frac{1}{4}[\gamma_a, \gamma_b]. \qquad (66)$$

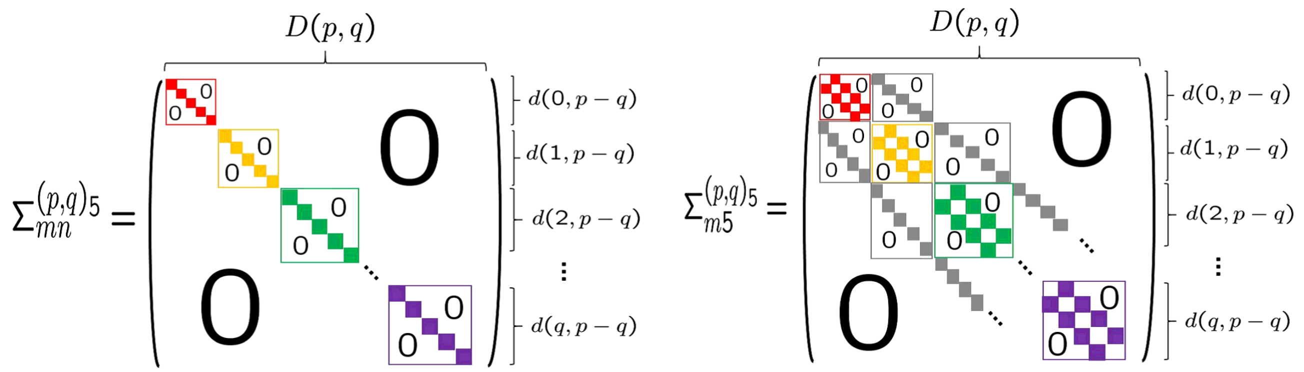

FIG. 3. General matrix form of the $SO(5)$ generators. The $SO(4)$ block matrices with nonzero elements are denoted as the filled squares and rectangles.

$$\Sigma_{mn}^{(p,q)_5} \equiv \begin{pmatrix} \sigma_{mn}^{[n=0]} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{mn}^{[n=1]} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{mn}^{[n=2]} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \sigma_{mn}^{[n=q]} \end{pmatrix}, \tag{71}$$

where $\sigma_{mn}^{[n]}$ denotes the $d(n, p-q) \times d(n, p-q)$ square matrix that is further block-diagonalized:

$$\sigma_{mn}^{[n]} \equiv \begin{pmatrix} \sigma_{mn}^{(\frac{p-q}{2}+\frac{n}{2},\frac{n}{2})_4} & 0 & 0 & 0 \\ 0 & \sigma_{mn}^{(\frac{p-q}{2}+\frac{n}{2}-\frac{1}{2},\frac{n}{2}+\frac{1}{2})_4} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_{mn}^{(\frac{n}{2},\frac{n}{2}+\frac{p-q}{2})_4} \end{pmatrix}. \tag{72}$$

See the left of Fig. 3. Since L_{m5} behave as an $SO(4)$ vector of the $SO(4)$ bi-spins,

$$(j,k)_4 = \left(\frac{1}{2},\frac{1}{2}\right)_4, \tag{73}$$

the $SU(2)$ selection rule indicates that the matrix elements of L_{m5} take nonzero values only for

$$(\Delta j, \Delta k)_4 = \left(\frac{1}{2},\frac{1}{2}\right)_4, \quad \left(-\frac{1}{2},\frac{1}{2}\right)_4, \quad \left(\frac{1}{2},-\frac{1}{2}\right)_4, \quad \left(-\frac{1}{2},-\frac{1}{2}\right)_4. \tag{74}$$

In other words, $\Sigma_{m5}^{(p,q)}$ have finite matrix elements only between nearest $SO(4)$ irreducible representations in Fig. 2, and the matrix form of the $\Sigma_{m5}^{(p,q)}$ is depicted in Fig. 3. The matrices (67) actually fit the general matrix form of Fig. 3. It should be emphasized that while we utilized Yang's monopole harmonics, the obtained $SO(5)$ matrix generators do *not* depend on the functional forms specific to Yang's monopole harmonics and are universal for any $SO(5)$ irreducible representations.

IV. $SO(5)$ MONOPOLE HARMONICS AS NONLINEAR REALIZATION

Here, we discuss how the nonlinear realization is related to quantum mechanics with gauge symmetry. While we focus on the $SO(5)$ case, the obtained results can easily be generalized to arbitrary groups.

A. $SO(5)$ nonlinear realization and $SO(4)$ gauge symmetry

Let us consider the nonlinear realization of the $SO(5)$ group for the coset manifold

$$S^4 \simeq SO(5)/SO(4). \tag{75}$$

In the context of quantum field theory, the coset represents the field manifold associated with the spontaneous symmetry breaking of $SO(5) \to SO(4)$. With the broken generators

$$\Sigma_{m5}^{(p,q)_5} \quad (m = 1,2,3,4) \tag{76}$$

we can construct the associated nonlinear realization matrix

$$\Psi^{(p,q)_5}(\Omega_4) = e^{i\sum_{m=1}^{4}\alpha_m(\Omega_4)\Sigma_{m5}^{(p,q)_5}}, \tag{77}$$

where α_m are parameters to be determined. With an element of the unbroken $SO(4)$ group,

$$H = e^{\frac{1}{2}\sum_{m,n=1}^{4}\omega_{mn}\Sigma_{mn}^{(p,q)_5}}, \tag{78}$$

the $SO(5)$ group element is locally represented as

$$H^\dagger \cdot \Psi^{(p,q)_5}. \tag{79}$$

Equation (71) implies that H (78) is expressed as a completely reducible representation of the $SO(4)$,

$$H = \begin{pmatrix} h^{[0]} & 0 & 0 & 0 \\ 0 & h^{[1]} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h^{[q]} \end{pmatrix} = \bigoplus_{n=0}^{q} h^{[n]}, \tag{80}$$

and each of the block matrices is further block-diagonalized,

$$h^{[n]} = \begin{pmatrix} h^{[n,\frac{p-q}{2}]} & 0 & 0 & 0 \\ 0 & h^{[n,\frac{p-q}{2}-1]} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h^{[n,-\frac{p-q}{2}]} \end{pmatrix} = \bigoplus_{-\frac{p-q}{2}\le s\le \frac{p-q}{2}} h^{[n,s]}. \tag{81}$$

Recall that $[n,s]$ specifies the $SO(4)$ bi-spin indices (34). Assume that the unbroken $SO(4)$ transformation acts as a "gauge" transformation,[12]

$$\Psi^{(p,q)_5} \to H^\dagger \cdot \Psi^{(p,q)_5}, \tag{82}$$

while the global transformation $G \in SO(5)$ acts as a right action,

$$\Psi^{(p,q)_5} \to \Psi^{(p,q)_5} \cdot G. \tag{83}$$

The corresponding connection is introduced as

$$\mathcal{A}_a = -i\Psi^{(p,q)_5}\partial_a \Psi^{(p,q)_5\dagger} = \begin{pmatrix} A_a^{[0]} & A_a^{[0,1]} & \cdots & A_a^{[0,q]} \\ A_a^{[1,0]} & A_a^{[1]} & \cdots & A_a^{[1,q]} \\ \vdots & \vdots & \ddots & \vdots \\ A_a^{[q,0]} & A_a^{[q,1]} & \cdots & A_a^{[q]} \end{pmatrix}. \tag{84}$$

Under the transformation (82), Eq. (84) transforms as an $SO(4)$ gauge field as anticipated:

$$\mathcal{A}_a \to H^\dagger \mathcal{A}_a H - iH^\dagger \partial_a H. \tag{85}$$

However, note that $\mathcal{A}_a$ (84) is a pure gauge whose curvature identically vanishes. To realize a physical gauge field, we utilize the block-diagonal parts of (84),

$$A_a \equiv \begin{pmatrix} A_a^{[0]} & 0 & 0 & 0 \\ 0 & A_a^{[1]} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_a^{[q]} \end{pmatrix} = \bigoplus_{n=0}^{q} A_a^{[n]}, \tag{86}$$

and each of the block matrices is given by

$$A_a^{[n]} = \begin{pmatrix} A_a^{[n,\frac{p-q}{2}]} & 0 & 0 & 0 \\ 0 & A_a^{[n,\frac{p-q}{2}-1]} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_a^{[n,-\frac{p-q}{2}]} \end{pmatrix} = \bigoplus_{-\frac{p-q}{2}\le s\le \frac{p-q}{2}} A_a^{[n,s]}. \tag{87}$$

Under the transformation (82), A_a transforms similarly to (85):

$$A_a \to H^\dagger A_a H - iH^\dagger \partial_a H. \tag{88}$$

We see that A_a is no longer a pure gauge field in the sense that the corresponding curvature, $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$, does not vanish. It is also obvious that A_a are invariant under the global $SO(5)$ transformation (83). With the A_a, we can introduce the covariant derivatives and angular momentum operators for the nonlinear representation as[13]

$$D_a \Psi^{(p,q)_5} \equiv \partial_a \Psi^{(p,q)_5} + iA_a \Psi^{(p,q)_5},$$
$$J_{ab}\Psi^{(p,q)_5} \equiv (-ix_a D_b + ix_b D_a + r^2 F_{ab})\Psi^{(p,q)_5}. \tag{91}$$

Let us focus on the smaller $SO(4)$ gauge transformations denoted by $h^{[n,s]}$ of (81) that carry the $SO(4)$ bi-spin indices:

$$(j,k)_4 = \left(\frac{n}{2} + \frac{p-q}{4} + \frac{s}{2}, \frac{n}{2} + \frac{p-q}{4} - \frac{s}{2}\right)_4. \tag{92}$$

We represent Ψ (77) as

[12]In the context of field theory, Eq. (82) is called the hidden local symmetry of nonlinear realization.

[13]Under the gauge and the global transformations, the quantities defined by (91), respectively, transform as

$$D_a \Psi^{(p,q)_5} \to H^\dagger \cdot D_a \Psi^{(p,q)_5}, \quad J_{ab}\Psi^{(p,q)_5} \to H^\dagger \cdot J_{ab}\Psi^{(p,q)_5}, \tag{89}$$

and

$$D_a \Psi^{(p,q)_5} \to D_a \Psi^{(p,q)_5} \cdot G, \quad J_{ab}\Psi^{(p,q)_5} \to J_{ab}\Psi^{(p,q)_5} \cdot G. \tag{90}$$

$$\Psi^{(p,q)_5} = \begin{pmatrix} \Psi_1^{[0]} & \Psi_2^{[0]} & \cdots & \Psi_{D(p,q)}^{[0]} \\ \Psi_1^{[1]} & \Psi_2^{[1]} & \cdots & \Psi_{D(p,q)}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_1^{[q]} & \Psi_2^{[q]} & \cdots & \Psi_{D(p,q)}^{[q]} \end{pmatrix}, \tag{93}$$

and each block $\Psi^{[n]}_{(\alpha=1,2,\ldots,D(p,q))}$ $(n = 0, 1, 2, \ldots, q)$ which we call the n sector of $\Psi^{(p,q)_5}$ takes the form of

$$\Psi_\alpha^{[n]} = \begin{pmatrix} \boldsymbol{\psi}_\alpha^{[n,\frac{p-q}{2}]} \\ \boldsymbol{\psi}_\alpha^{[n,\frac{p-q}{2}-1]} \\ \vdots \\ \boldsymbol{\psi}_\alpha^{[n,s]} \\ \vdots \\ \boldsymbol{\psi}_\alpha^{[n,-\frac{p-q}{2}]} \end{pmatrix}. \tag{94}$$

The gauge (82) and the global transformations (83), respectively, act to the $\boldsymbol{\psi}_\alpha^{[n,s]}$ $(-\frac{p-q}{2} \le s \le \frac{p-q}{2})$ as

$$\boldsymbol{\psi}_\alpha^{[n,s]} \to h^{[n,s]\dagger}\boldsymbol{\psi}_\alpha^{[n,s]}, \qquad \boldsymbol{\psi}_\alpha^{[n,s]} \to \sum_{\beta=1}^{D(p,q)} \boldsymbol{\psi}_\beta^{[n,s]} G_{\beta\alpha}. \tag{95}$$

The gauge field $A_a^{[n,s]}$ in (87) is represented as

$$A_a^{[n,s]} = -i \sum_{\alpha=1}^{D(p,q)} \boldsymbol{\psi}_\alpha^{[n,s]} \partial_a \boldsymbol{\psi}_\alpha^{[n,s]\dagger}, \tag{96}$$

which transforms as

$$A_a^{[n,s]} \to h^{[n,s]\dagger} A_a^{[n,s]} h^{[n,s]} - i h^{[n,s]\dagger}\partial_a h^{[n,s]}. \tag{97}$$

Using (96), we can construct the covariant derivatives and the angular momentum operators as

$$\begin{aligned} D_a^{[n,s]}\boldsymbol{\psi}_\alpha^{[n,s]} &= \partial_a \boldsymbol{\psi}_\alpha^{[n,s]} + i A_a^{[n,s]}\boldsymbol{\psi}_\alpha^{[n,s]}, \\ J_{ab}^{[n,s]}\boldsymbol{\psi}_\alpha^{[n,s]} &\equiv (-i x_a D_b^{[n,s]} + i x_b D_a^{[n,s]} + r^2 F_{ab}^{[n,s]})\boldsymbol{\psi}_\alpha^{[n,s]}. \end{aligned} \tag{98}$$

The second equation of (95) implies that the set $\boldsymbol{\psi}^{[n,s]}_{\alpha=1,2,\ldots,D(p,q)}$ constitutes an $SO(5)$ irreducible representation with $(p,q)_5$, and at the same time, $\boldsymbol{\psi}_\alpha^{[n,s]}$ enjoys the $SO(4)$ gauge symmetry of the $SO(4)$ bi-spin indices (92). The physical quantities that hold such features are nothing but the $SO(5)$ monopole harmonics.

B. Determination of the $SO(5)$ nonlinear realization

Our next task is to determine the parameters α_m of the nonlinear realization (77). For this purpose, it is sufficient to consider the simplest case $\Sigma_{ab}^{(1,0)_5}$ (67), in which the nonlinear realization (77) reads

$$\Psi^{(1,0)_5}(\Omega_4) = \begin{pmatrix} \cos(\frac{\alpha}{2})1_2 & \sin(\frac{\alpha}{2})\frac{1}{\alpha}\alpha_m\bar{q}_m \\ -\sin(\frac{\alpha}{2})\frac{1}{\alpha}\alpha_m q_m & \cos(\frac{\alpha}{2})1_2 \end{pmatrix} \tag{99}$$

with $\alpha \equiv \sqrt{{\alpha_m}^2}$. According to the discussions of Sec. IV A, we rewrite (99) in the following form:

$$\Psi^{(1,0)_5}(\Omega) = \begin{pmatrix} \boldsymbol{\psi}_1^{[0,\frac{1}{2}]} & \boldsymbol{\psi}_2^{[0,\frac{1}{2}]} & \boldsymbol{\psi}_3^{[0,\frac{1}{2}]} & \boldsymbol{\psi}_4^{[0,\frac{1}{2}]} \\ \boldsymbol{\psi}_1^{[0,-\frac{1}{2}]} & \boldsymbol{\psi}_2^{[0,-\frac{1}{2}]} & \boldsymbol{\psi}_3^{[0,-\frac{1}{2}]} & \boldsymbol{\psi}_4^{[0,-\frac{1}{2}]} \end{pmatrix} \tag{100}$$

to see that the set of the upper and lower two columns, respectively, represents the monopole harmonics of $(p,q)_5 = (1,0)_5$ in the $SU(2)$ monopole background and in the $SU(2)$ antimonopole background. Recall the (anti)monopole harmonics (60) to construct

$$\begin{aligned} &\frac{2\pi}{\sqrt{3}}(\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2 \quad -\boldsymbol{\psi}_3 \quad -\boldsymbol{\psi}_4) \\ &\quad = \frac{1}{\sqrt{2(1+x_5)}}(-x_m q_m \quad (1+x_5)1_2), \end{aligned} \tag{101}$$

which should be identified as the lower two columns of (99). Now α_m can be identified as

$$\alpha_m(\Omega_4) = \xi y_m, \tag{102}$$

where y_m $(m = 1, 2, 3, 4)$ denote the coordinates on the hyperlatitude at the azimuthal angle ξ on S^4:

$$\begin{aligned} y_m &\equiv \frac{1}{\sin\xi}x_m \\ &= \{\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \cos\chi\} \in S^3. \end{aligned} \tag{103}$$

The nonlinear realization (99) is represented as

$$\Psi^{(1,0)_5}(\Omega_4) = \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix} (1+x_5)1_2 & x_m\bar{q}_m \\ -x_m q_m & (1+x_5)1_2 \end{pmatrix}. \tag{104}$$

For general representation $(p,q)_5$, the nonlinear realization is given by

$$\Psi^{(p,q)_5}(\Omega_4) = e^{i\xi\sum_{m=1}^4 y_m \Sigma_{m5}^{(p,q)_5}}, \tag{105}$$

which naturally generalizes the $SO(3)$ case (11). It is straightforward to check that (105) covariantly transforms under the $SO(5)$ rotations generated by J_{ab} (91),

$$J_{ab}\Psi^{(p,q)_5}(\Omega_4) = \Psi^{(p,q)_5}(\Omega_4)\Sigma_{ab}^{(p,q)_5}, \qquad (106)$$

which implies

$$\sum_{a<b} {J_{ab}}^2\Psi^{(p,q)_5}(\Omega_4) = \Psi^{(p,q)_5}(\Omega_4)\sum_{a<b}\Sigma_{ab}^{(p,q)_5 2} = \lambda(p,q)\Psi^{(p,q)_5}(\Omega_4). \qquad (107)$$

In the language of $\boldsymbol{\psi}_\alpha^{[n,s]}$, Eq. (107) is translated as

$$\sum_{a<b} J_{ab}^{[n,s]2}\boldsymbol{\psi}_\alpha^{[n,s]} = \lambda(p,q)\boldsymbol{\psi}_\alpha^{[n,s]}. \qquad (108)$$

Note that (108) signifies that $\boldsymbol{\psi}_\alpha^{[n,s]}$ are the $SO(5)$ monopole harmonics with the eigenvalue value $\lambda(p,q)$ in the $SO(4)$ monopole background with $(\frac{I_+}{2},\frac{I_-}{2})_4 = (\frac{n}{2}+\frac{p-q}{4}+\frac{s}{2},\frac{n}{2}+\frac{p-q}{4}-\frac{s}{2})_4$.

V. $SO(5)$ LANDAU PROBLEM IN THE $SO(4)$ MONOPOLE BACKGROUND

We now apply the techniques of the nonlinear realization to the $SO(5)$ Landau problem in the $SO(4)$ monopole background. In the context of the Landau model, p and q are quantities to be determined.

A. The $SO(4)$ monopole and $SO(5)$ Landau Hamiltonian

Before proceeding to the $SO(5)$ Landau problem, we explain topological features of the $SO(4)$ monopole gauge field. The $SO(4)$ monopole is simply introduced with replacement of the $SU(2)$ spin matrices of the Yang monopole (37) with the $SO(4)$ bi-spin matrices:

$$A_m = -\frac{1}{r(r+x_5)}\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4}x_n, \qquad A_5 = 0, \qquad (109)$$

where

$$\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4} = \eta_{mn}^i S_i^{(\frac{I_+}{2})}\otimes \mathbf{1}_{I_-+1} + \mathbf{1}_{I_++1}\otimes \bar{\eta}_{mn}^i S_i^{(\frac{I_-}{2})}. \qquad (110)$$

The $SO(4)$ monopole is conformally equivalent to the $SO(4)$ instanton on $\mathbb{R}^4$ that is a solution of the pure Yang-Mills field equations [10,55,56]. The $SO(4)$ monopole gauge field (109) can be expressed as

$$A = A_a dx_a = A^{(+)}\oplus \mathbf{1}_{I_-+1} + \mathbf{1}_{I_++1}\otimes A^{(-)}, \qquad (111)$$

where $A^{(+)}$ and $A^{(-)}$ denote the $SU(2)$ monopole field and the $SU(2)$ antimonopole field, respectively,

$$A^{(+)} = -\frac{1}{r(r+x_5)}\eta_{mn}^i S_i^{(\frac{I_+}{2})}x_n dx_m,$$
$$A^{(-)} = -\frac{1}{r(r+x_5)}\bar{\eta}_{mn}^i S_i^{(\frac{I_-}{2})}x_n dx_m. \qquad (112)$$

The corresponding field strength, $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$, is derived by

$$F_{mn} = -\frac{1}{r^2}x_m A_n + \frac{1}{r^2}x_n A_m + \frac{1}{r^2}\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4},$$
$$F_{m5} = -F_{5m} = \frac{1}{r^2}(r+x_5)A_m, \qquad (113)$$

which satisfy

$$\sum_{a<b} {F_{ab}}^2 = \frac{1}{r^4}\sum_{m<n}\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4 2} = \frac{1}{2r^4}(I_+(I_++2)+I_-(I_-+2))\mathbf{1}_{(I_++1)(I_-+1)}. \qquad (114)$$

With the vierbein e^m of S^4, Eq. (113) can be concisely expressed as

$$F = \frac{1}{2}F_{ab}dx_a\wedge dx_b = \frac{1}{2}e^m\wedge e^n\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4}. \qquad (115)$$

The $SO(4)$ group hosts two invariant tensors, i.e., Kronecker delta symbol and Levi-Civita four-rank tensor, which allow us to introduce two $SO(4)$ gauge invariant topological invariants [57], the (total) second Chern number and a generalized Euler number (see Appendix A for details):

$$c_2 \equiv \frac{1}{8\pi^2}\int \mathrm{tr}(F^2) = \frac{1}{8\pi^2}\int \mathrm{tr}(\mathcal{F}^2) = \frac{1}{32\pi^2}\int F^{m_1m_2}F^{m_3m_4}\mathrm{tr}(\sigma_{m_1m_2}\sigma_{m_3m_4}), \qquad (116a)$$

$$\tilde{c}_2 \equiv \frac{1}{8\pi^2}\int \mathrm{tr}(F\mathcal{F}) = \frac{1}{8\pi^2}\int \mathrm{tr}(\mathcal{F}F) = \frac{1}{64\pi^2}\int \epsilon^{m_3m_4m_5m_6}F^{m_1m_2}F^{m_3m_4}\mathrm{tr}(\sigma_{m_1m_2}\sigma_{m_5m_6}), \qquad (116b)$$

where

$$F \equiv \frac{1}{2}F^{m_1m_2}\sigma_{m_1m_2}, \quad \mathcal{F} \equiv \frac{1}{4}\epsilon^{m_1m_2m_3m_4}F_{m_1m_2}\sigma_{m_3m_4}. \qquad (117)$$

For $F_{mn} = e_m \wedge e_n$ and $\sigma_{mn} = \sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4}$, Eq. (116) is evaluated as[14]

$$c_2^{(\frac{I_+}{2},\frac{I_-}{2})} = \frac{1}{6}(I_+ + 1)(I_- + 1)(I_+(I_+ + 2) - I_-(I_- + 2)), \tag{121a}$$

$$\tilde{c}_2^{(\frac{I_+}{2},\frac{I_-}{2})} = \frac{1}{6}(I_+ + 1)(I_- + 1)(I_+(I_+ + 2) + I_-(I_- + 2)). \tag{121b}$$

Meanwhile, from the homotopy theorem

$$\pi_3(SO(4)) \simeq \pi_3(SU(2)) \oplus \pi_3(SU(2)) \simeq \mathbb{Z} \oplus \mathbb{Z}, \tag{122}$$

we can introduce two distinct second Chern numbers corresponding to the monopole and the antimonopole,

$$c_2^+ = \frac{1}{2(2\pi)^2}\int_{S^4} \mathrm{tr}(F_+^2) = +\frac{1}{6}I_+(I_+ + 1)(I_+ + 2), \tag{123a}$$

$$c_2^- = \frac{1}{2(2\pi)^2}\int_{S^4} \mathrm{tr}(F_-^2) = -\frac{1}{6}I_-(I_- + 1)(I_- + 2), \tag{123b}$$

which are related to c_2 (121a) and $\tilde{c}_2$ (121b) as

$$\begin{aligned} c_2^{(\frac{I_+}{2},\frac{I_-}{2})} &= c_2^+(I_- + 1) + c_2^-(I_+ + 1), \\ \tilde{c}_2^{(\frac{I_+}{2},\frac{I_-}{2})} &= c_2^+(I_- + 1) - c_2^-(I_+ + 1). \end{aligned} \tag{124}$$

The second Chern number c_2 essentially represents the sum of the two monopole charges, while the generalized Euler number $\tilde{c}_2$ represents their difference. They may be reminiscent of the topological invariants of (S_z conserved) quantum spin Hall effect [37–39]; the sum of two Chern number signifies quantized charge Hall conductance, while their difference indicates quantized spin Hall conductance. In the nonchiral case $I_+ = I_- = \frac{I}{2}$ $(I = 0, 2, 4, 6, \ldots)$, though the second Chern number is trivial, the generalized Euler number is finite,

$$c_2^{(\frac{I}{4},\frac{I}{4})} = 0, \qquad \tilde{c}_2^{(\frac{I}{4},\frac{I}{4})} = \frac{1}{48}I(I+2)^2(I+4), \tag{125}$$

and $\tilde{c}_2$ is the unique topological quantity of the system.

Replacing the $SU(2)$ gauge field with the $SO(4)$ gauge field, we introduce the $SO(5)$ angular momentum operators in the $SO(4)$ monopole background in a similar manner to Sec. III B:

$$\begin{aligned} L_{mn} &= L_{mn}^{(0)} + \sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4}, \\ L_{m5} &= -L_{5m} = L_{m5}^{(0)} - \frac{1}{r + x_5}\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4} x_n. \end{aligned} \tag{126}$$

With covariant angular momentum operators $\Lambda_{ab} = -ix_a D_b + ix_b D_a$, we construct the $SO(5)$ Landau Hamiltonian in the $SO(4)$ monopole background,

$$\begin{aligned} H &= -\frac{1}{2M}(\partial_a + iA_a)^2\Big|_{r=1} \\ &= \frac{1}{2M}\sum_{a<b}{\Lambda_{ab}}^2 \\ &= \frac{1}{2M}\left(\sum_{a<b}{L_{ab}}^2 - \sum_{a<b}{F_{ab}}^2\right) \\ &= \frac{1}{2M}\left(\sum_{a<b}{L_{ab}}^2 - \frac{1}{2}(I_+(I_+ + 2) + I_-(I_- + 2))\right), \end{aligned} \tag{127}$$

and hence the energy eigenvalues of (127) are expressed as

$$E = \frac{1}{2}\left(\lambda(p,q) - \frac{1}{2}(I_+(I_+ + 2) + I_-(I_- + 2))\right). \tag{128}$$

Since the gauge field was introduced as an external gauge field that does not change its sign under the time-reversal transformation, the Landau Hamiltonian (127) does not respect the time-reversal symmetry even in the nonchiral case.

B. $SO(5)$ Landau level eigenstates

Let us first address how the $SO(5)$ Landau level eigenstates can be identified as the nonlinear realization. As discussed in Sec. IV A, $\boldsymbol{\psi}^{[n,s]}_{\alpha=1,2,\ldots,D(p,q)}$ enjoy the $SO(4)$

[14]For $(j,k) = (1/2,0), (0,1/2), (1/2,0) \oplus (0,1/2), (1/2,1/2)$, the $SO(4)$ matrix generators are, respectively, given by

$$\begin{aligned} \sigma_{mn} = \frac{1}{2}\eta^i_{mn}\sigma_i, \frac{1}{2}\bar{\eta}^i_{mn}\sigma_i, \frac{1}{2}\begin{pmatrix} \eta^i_{mn}\sigma_i & 0 \\ 0 & \bar{\eta}^i_{mn}\sigma_i \end{pmatrix}, \\ \frac{1}{2}\eta^i_{mn}\sigma_i \otimes 1_2 + 1_2 \oplus \frac{1}{2}\bar{\eta}^i_{mn}\sigma_i, \end{aligned} \tag{118}$$

and the topological invariants (116) are evaluated as

$$(c_2, \tilde{c}_2) = (1,1), (-1,1), (0,2), (0,4). \tag{119}$$

In deriving (121), we used the formula

$$\begin{aligned} \mathrm{tr}(\sigma^{(j,k)_4}_{m_1m_2}\sigma^{(j,k)_4}_{m_3m_4}) = \frac{(2j+1)(2k+1)}{3}&((j(j+1) + k(k+1)) \\ &\times(\delta_{m_1m_3}\delta_{m_2m_4} - \delta_{m_1m_4}\delta_{m_2m_3}) \\ &+ (j(j+1) - k(k+1))\epsilon_{m_1m_2m_3m_4}). \end{aligned} \tag{120}$$

$\Psi^{(p,q)}|_{(p,q)=(N+I-n,N+n)} =$

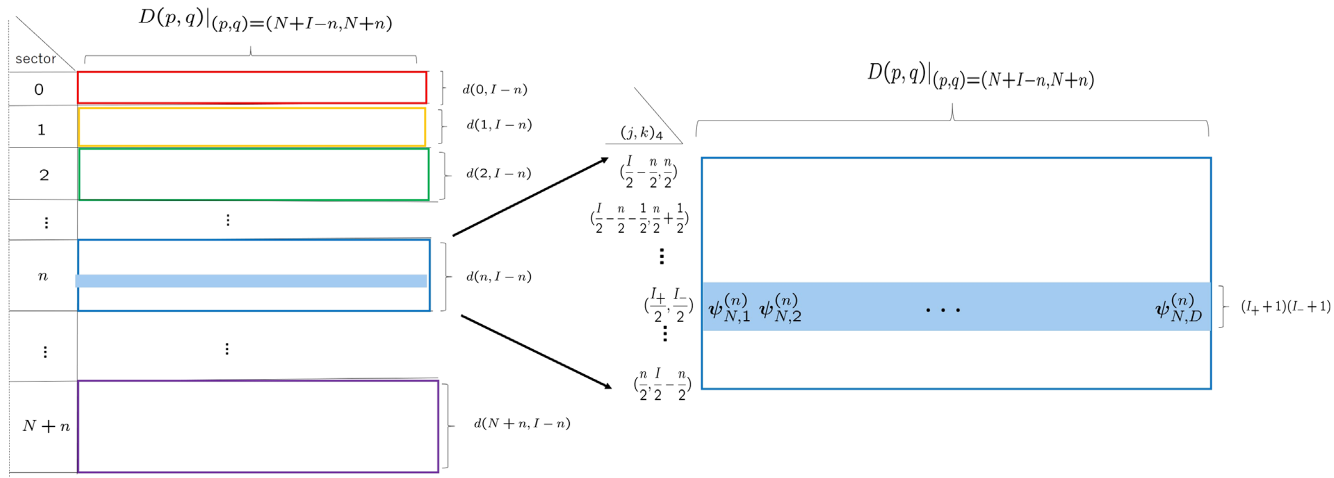

FIG. 4. The Nth Landau level eigenstates in the n sector, $\psi^{(n)}_{N,1},\psi^{(n)}_{N,2},\ldots,\psi^{(n)}_{N,D}$, can be found as the block matrix (the blue shaded region) in the n sector of $\Psi^{(p,q)}|_{p=N+I-n,q=N+n}$.

gauge symmetry with the $SO(4)$ bi-spin indices (92), which in the context of the Landau model are identified with the $SO(4)$ monopole indices,

$$\left(\frac{I_+}{2},\frac{I_-}{2}\right)_4 = \left(\frac{n}{2}+\frac{p-q}{4}+\frac{s}{2},\frac{n}{2}+\frac{p-q}{4}-\frac{s}{2}\right)_4. \tag{129}$$

Since n runs from 0 to q, q should be greater than or equal to n,[15] so we can define non-negative integers N for *each* n,

$$N \equiv q-n = 0,1,2,\ldots. \tag{130}$$

The non-negative integer N indicates the Landau level index in the n sector, and then $\psi^{[n,s]}_{\alpha=1,2,\ldots,D(p,q)}$ represent the Nth Landau level eigenstates of the n sector. We will discuss the energy levels in Sec. V C.

In the Landau problem, the $SO(4)$ monopole indices, I_+ and I_-, are input parameters, and we need to specify p and q for the given I_+ and I_-. The former two conditions, Eqs. (129) and (130), uniquely specify the $SO(5)$ indices as

$$p = N+I-n, \qquad q = N+n, \tag{131}$$

where

$$I \equiv I_+ + I_-. \tag{132}$$

Since $p \geq q$, Eq. (131) implies that n has an upper limit and the range of n may be given by

$$n = 0,1,2,\ldots,\mathrm{Min}(I_+,I_-). \tag{133}$$

We give a precise prescription for deriving the Nth Landau level eigenstates in the n sector. We first need to derive the $SO(5)$ matrix generators, $\Sigma^{(p,q)_5}_{ab}$, with $(p,q)_5 = (N+I-n,N+n)_5$. That is doable by taking the matrix elements of the $SO(5)$ angular momentum operators with Yang's monopole harmonics as discussed in Sec. III C. Next from the matrix generators, we construct the nonlinear realization using the formula

$$\Psi(\Omega_4) = \exp\left(i\xi\sum_{m=1}^{4} y_m \Sigma^{(p,q)_5}_{m5}\right)\Bigg|_{p=N+I-n,q=N+n}. \tag{134}$$

Finally, as indicated in Fig. 4, we extract an appropriate block matrix from the n sector of Ψ. The components $\psi^{(n)}_{N,\alpha}$ denote the Nth Landau level eigenstates in the n sector, which are normalized as[16]

$$\sqrt{\frac{D(p,q)}{(I_++1)(I_-+1)A(S^4)}}\psi^{(n)}_{N,\alpha}(\Omega_4), \tag{136}$$

[15] Recall the similar discussions in the $SO(3)$ Landau model around (25). Equation (130) is a generalization of (25).

[16] The connection of $\psi^{(n)}_{N,\alpha}$ yields the $SO(4)$ monopole gauge field (109),

$$A = -i\sum_{\alpha=1}^{D(p,q)} \psi^{(n)}_{N,\alpha} d\psi^{(n)\dagger}_{N,\alpha} = -\frac{1}{1+x_5}\sigma^{(\frac{I_+}{2},\frac{I_-}{2})_4}_{mn} x_n dx_m. \tag{135}$$

Note that the A in (135) does not depend on either N or n.

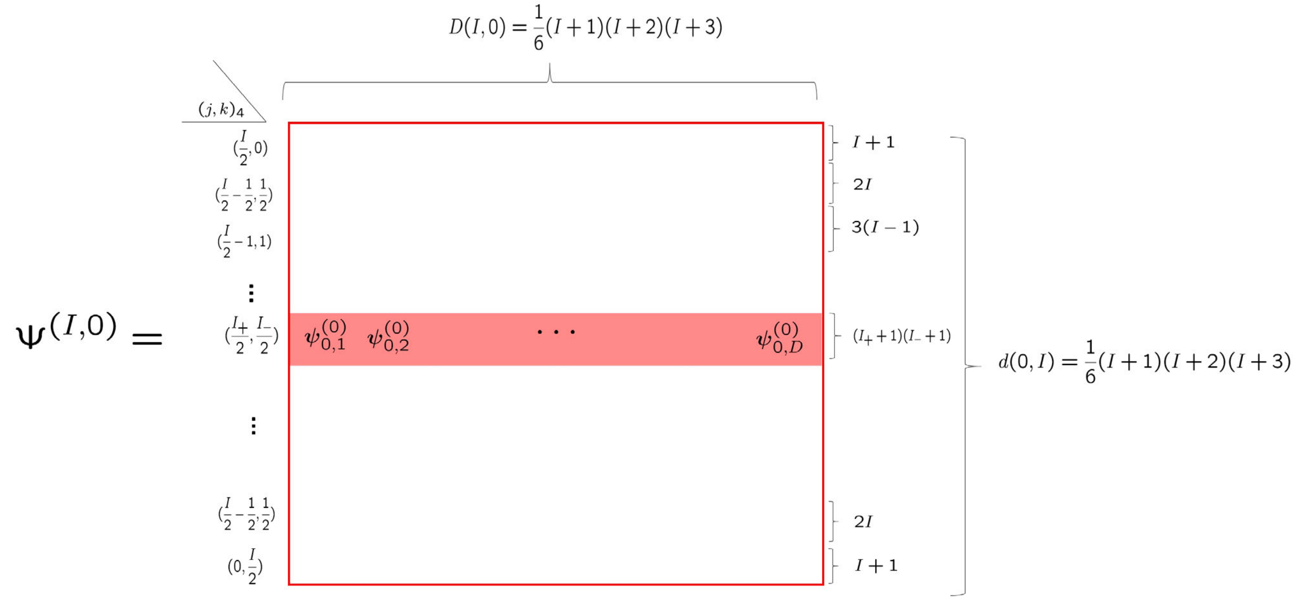

FIG. 5. For the $SO(4)$ monopole with $(\frac{I_+}{2}, \frac{I_-}{2})$, the eigenstates of the LLL in the $n = 0$ sector are realized as the red shaded region of the nonlinear realization $\Psi^{(I,0)}$.

with $A(S^4) = 8\pi^2/3$. Especially for the LLL ($N = 0$) in the $n = 0$ sector, the eigenstates are given by the red shaded region in Fig. 5.

Mathematical software is highly efficient in practically deriving the nonlinear realization. Computation time will be significantly reduced using the Euler decomposition form of (134):

$$\Psi(\Omega_4) = H(\Omega_3)^\dagger e^{i\xi\Sigma_{45}} H(\Omega_3), \tag{137}$$

where

$$H(\Omega_3) = e^{-i\chi\Sigma_{34}} e^{i\theta\Sigma_{31}} e^{i\phi\Sigma_{12}}. \tag{138}$$

Following the above prescription, we have derived the $SO(5)$ monopole harmonics in several $SO(4)$ monopole backgrounds (see Appendix B also), and their probability densities are depicted in Fig. 6.

C. $SO(5)$ Landau levels

With (131), we may derive the energy levels of (128) as

$$\begin{aligned} E_N^{(n)} &= \frac{1}{2M}\Big(\lambda(p,q)|_{(p,q)=(N+I-n,N+n)} \\ &\quad - \frac{1}{2}(I_+(I_+ + 2) + I_-(I_- + 2))\Big) \\ &= \frac{1}{2M}(N(N+3) + I(N-n) + n(n-1)) \\ &\quad + \frac{1}{2M}(I + I_+ I_-), \end{aligned} \tag{139}$$

where

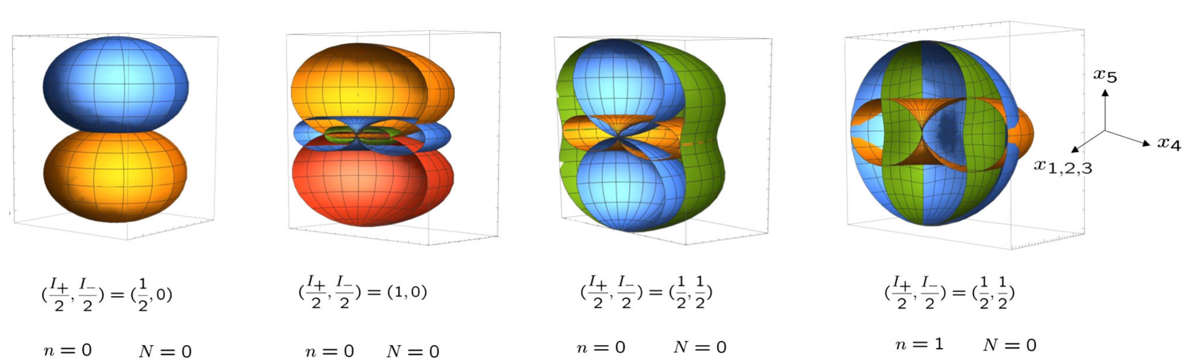

FIG. 6. Probability densities of the $SO(5)$ monopole harmonics. The different colored probability densities correspond to distinct $SO(5)$ monopole harmonics. For $I_+ \neq I_-$ (the left two) each of the probability distributions is asymmetric with respect to $x_5 = 0$ in general, while for $I_+ = I_-$ (the right two) each probability distribution is symmetric with respect to $x_5 = 0$.

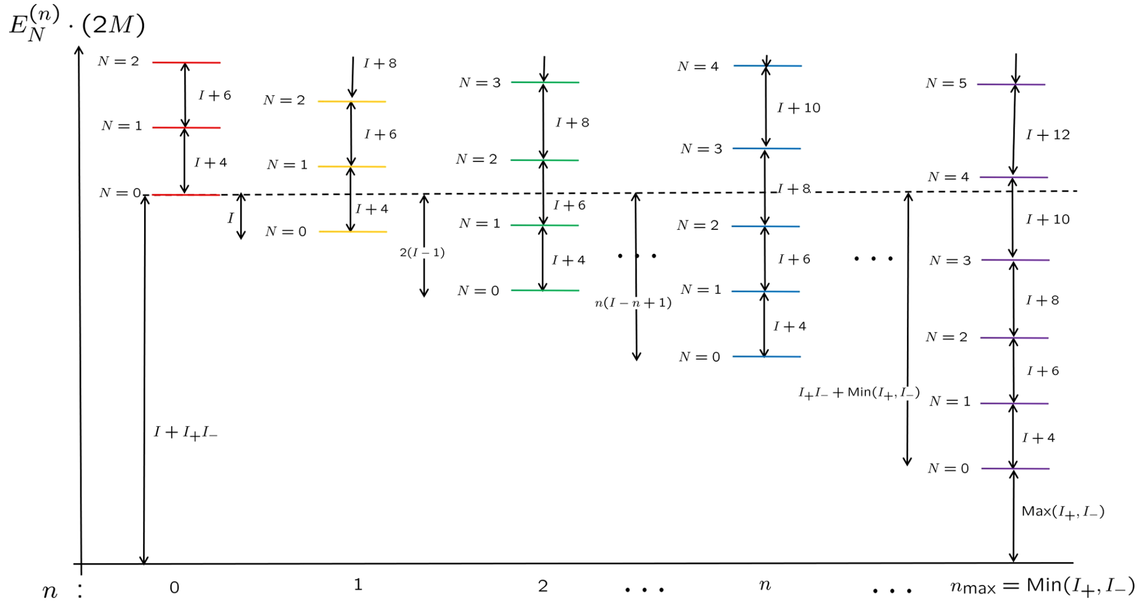

FIG. 7. For the $SO(4)$ monopole with $(\frac{I_+}{2},\frac{I_-}{2})_4$, there are $\mathrm{Min}(I_+,I_-)+1$ sectors, each of which exhibits the Landau levels.

$$N=0,1,2,\ldots \quad \text{and} \quad n=0,1,2,\ldots,\mathrm{Min}(I_+,I_-). \tag{140}$$

Since all possible $(p,q)_5$ are exploited by changing N and n in (131) for a given I, Eq. (139) exhausts all energy levels of the $SO(5)$ Landau Hamiltonian. Figure 7 schematically depicts the energy levels of (139). The corresponding degeneracy (31) is also derived as

$$D_N^{(n)}(I)=\frac{1}{6}(N+n+1)(I-2n+1)(I+N+2-n)\times(I+2N+3). \tag{141}$$

We here mentioned specific features of the energy levels. The original Landau levels in the $SU(2)$ monopole background correspond to the $n=0$ sector of the preset energy levels. Indeed, for $(I_+,I_-)=(0,I)$ and $n=0$, the above formulas exactly reproduce the results of Sec. III B. The Landau level spacing and the degeneracy depend only on the sum $I\equiv I_++I_-$ rather than both I_+ and I_-. Furthermore, the Landau level spacing does not depend on the sector index n and is common in all of the sectors:

$$E_{N+1}^{(n)}-E_N^{(n)}=\frac{1}{2M}(2N+I+4). \tag{142}$$

The Landau level energy monotonically lowers as n increases,

$$E_N^{(n+1)}-E_N^{(n)}=-\frac{1}{2M}(I-2n)\le 0, \tag{143}$$

and the minimum energy level is realized at the LLL of the $n_{\max}=\mathrm{Min}(I_+,I_-)$ sector,

$$E_{N=0}^{(n=n_{\max})}=\frac{1}{2M}\mathrm{Max}(I_+,I_-). \tag{144}$$

Recovering the radius R of the S^4 in (139), we take the thermodynamic limit, $I,R\to\infty$ with I/R^2 being fixed. From (142), we see that every Landau level spacing in all sectors becomes identical,

$$E_{N+1}^{(n)}-E_N^{(n)}\to\omega\equiv\frac{I}{2MR^2}, \tag{145}$$

which is the usual Landau level spacing on a (4D) plane.

VI. NONCOMMUTATIVE GEOMETRY AND MANY-BODY WAVE FUNCTION

Here, we investigate matrix geometries in the Landau levels by applying the Landau level projection [5,50,51]. With the Nth Landau level eigenstates in the n sector, we take matrix elements of the S^4 coordinates,

$$(X_a)_{\alpha\beta}=\int d\Omega_4\boldsymbol{\psi}_{N,\alpha}^{(n)\dagger}x_a\boldsymbol{\psi}_{N,\beta}^{(n)} \qquad (\alpha,\beta=1,2,\ldots,D_N^{(n)}). \tag{146}$$

We introduce the $(I_++1)(I_-+1)\times D_N^{(n)}(I)$ matrix that represents the blue shaded region in Fig. 4,

$$\Psi_N^{(n)}\equiv(\boldsymbol{\psi}_{N,1}^{(n)}\quad\boldsymbol{\psi}_{N,2}^{(n)}\quad\cdots\quad\boldsymbol{\psi}_{N,D_N^{(n)}}^{(n)}), \tag{147}$$

which satisfies

$$\Psi_N^{(n)}\Psi_N^{(n)\dagger}=\mathbf{1}_{(I_++1)(I_-+1)}. \tag{148}$$

Using a $D_N^{(n)}(I)\times D_N^{(n)}(I)$ projection matrix $P_N^{(n)}$ made of (147)[17]

$$P_N^{(n)} \equiv \Psi_N^{(n)\dagger}\Psi_N^{(n)} \qquad (P_N^{(n)2} = P_N^{(n)}), \tag{149}$$

we can concisely represent the matrix coordinates (146) as

$$X_a = \int d\Omega_4 x_a P_N^{(n)}, \tag{150}$$

which obviously signifies the projection of the S^4 coordinates to the level.

A. The nonchiral LLL in the $n = I/2$ sector

Let us first consider the nonchiral LLL eigenstates of the $n_{\max} = I/2$ sector in the $SO(4)$ monopole background with $(I_+, I_-) = (I/2, I/2)$ (I: even). While the second Chern number vanishes, the zero-point energy $I/(4M)$ is finite and the LLL degeneracy is large as given by $D_{N=0}^{(n=\frac{I}{2})}(I) = \frac{1}{24}(I+2)(I+3)(I+4)$. Therefore, even though the second Chern number is zero, the nonchiral $SO(4)$ monopole system is not quite the same as a simple free system without the $SO(4)$ monopole. The LLL eigenstates constitute

$$(p,q)_5 = \left(\frac{I}{2},\frac{I}{2}\right)_5, \tag{151}$$

and x_a are

$$(p,q)_5 = (1,1)_5, \tag{152}$$

so the $SO(5)$ decomposition rule for $x_a\boldsymbol{\psi}$ in (146) signifies[18]

$$(1,1)_5 \otimes \left(\frac{I}{2},\frac{I}{2}\right)_5 = \left(\frac{I}{2}+1,\frac{I}{2}+1\right)_5 \oplus \left(\frac{I}{2}+1,\frac{I}{2}-1\right)_5 \oplus \left(\frac{I}{2}-1,\frac{I}{2}-1\right)_5. \tag{153}$$

The LLL irreducible representation (151) does not exist on the right-hand side of (153), and then

$$X_a = 0. \tag{154}$$

An intuitive explanation for this result is as follows. For nonchiral cases (see the right two of Fig. 6), the "center" of every probability distribution is at the origin, and hence the expectation values of the coordinates for such states are expected to be zeros as in the case of the spherical harmonics. Careful readers may derive the projection matrix (149) and explicitly check (154) by performing the integration (150).

For nonchiral LLL eigenstates, we explicitly computed the Fisher information metric

$$\begin{aligned} g_{\mu\nu} &= \mathrm{tr}\Bigg(\sum_{\alpha=1}^{D}(\partial_\mu\boldsymbol{\psi}_\alpha\partial_\nu\boldsymbol{\psi}_\alpha^\dagger + \partial_\nu\boldsymbol{\psi}_\alpha\partial_\mu\boldsymbol{\psi}_\alpha^\dagger) \\ &\quad - 2\sum_{\alpha,\beta=1}^{D}\partial_\mu\boldsymbol{\psi}_\alpha\boldsymbol{\psi}_\alpha^\dagger\boldsymbol{\psi}_\beta\partial_\nu\boldsymbol{\psi}_\beta^\dagger\Bigg) \\ &= \mathrm{tr}\Big(\partial_\mu\Psi_N^{(n)}\partial_\nu\Psi_N^{(n)\dagger} + \partial_\nu\Psi_N^{(n)}\partial_\mu\Psi_N^{(n)\dagger} \\ &\quad - 2\partial_\mu\Psi_N^{(n)}\Psi_N^{(n)\dagger}\Psi_N^{(n)}\partial_\nu\Psi_N^{(n)\dagger}\Big) \qquad (\mu,\nu = \xi,\chi,\theta,\phi) \end{aligned} \tag{155}$$

to have

$$g_{\mu\nu} \propto \mathrm{diag}(1,\ \sin^2\xi,\ \sin^2\xi\ \sin^2\chi,\ \sin^2\ \xi\sin^2\chi\ \sin^2\theta), \tag{156}$$

which is the polar coordinate metric on S^4. This is the same result as the $SU(2)$ monopole case [58] whose fuzzy geometry is the fuzzy four-sphere. The Fisher metric reflects the information of the manifold on which the wave functions are defined, while the matrix geometry reflects the shapes of the wave functions also.

B. The LLL in the $n = 0$ sector

Next, we proceed to the matrix geometry of the LLL in the $n = 0$ sector with degeneracy

$$D_{N=0}^{(n=0)}(I) = \frac{1}{6}(I+1)(I+2)(I+3) \qquad (I \equiv I_+ + I_-), \tag{157}$$

and X_a (146) are represented by $D_{N=0}^{(n=0)}(I)\times D_{N=0}^{(n=0)}(I)$ matrices.

For the $SU(2)$ monopole ($I_+ = I, I_- = 0$), the previous studies [5,58] showed the emergent matrix geometry is the fuzzy four-sphere,

$$X_a = \frac{1}{I+4}\Gamma_a^{(I)}, \tag{158}$$

[17] The $P_N^{(n)}$ holds two eigenvalues, 1 and 0, with degeneracies, $(I_+ + 1)(I_- + 1)$ and $D_N^{(n)}(I) - (I_+ + 1)(I_- + 1)$.

[18] See [5] and references therein.

where $\Gamma_a^{(I)}$ are fully symmetric tensor products of I $SO(5)$ gamma matrices.[19] The basic properties of $\Gamma_a^{(I)}$ are given by

$$[\Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d] = 8(I+2)\epsilon_{abcde}\Gamma_e,$$
$$\Gamma_a^{(I)}\Gamma_a^{(I)} = I(I+4)\mathbf{1}_{D_{N=0}^{(n=0)}(I)}, \tag{160}$$

where the four bracket $[,,,]$ denotes the fully antisymmetric combinations of the four quantities inside the bracket,

$$[\Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d] \equiv \sum_\sigma \mathrm{sgn}(\sigma)\Gamma_{\sigma(a)}\Gamma_{\sigma(b)}\Gamma_{\sigma(c)}\Gamma_{\sigma(d)}. \tag{161}$$

For the $SO(4)$ monopole background with index $(\frac{I_+}{2}, \frac{I_-}{2})_4$, we explicitly evaluate X_a (150) using several low dimensional representations. From the obtained results, we deduce that the matrix geometry in the $SO(4)$ monopole background becomes

$$X_a = \frac{I_+ - I_-}{I(I+4)}\Gamma_a^{(I)}. \tag{162}$$

This naturally generalizes the original result (158). Notice that the matrix size of X_a depends only on the sum of the $SO(4)$ bi-spin indices while the overall coefficient depends on the difference of the $SO(4)$ bi-spin indices. The matrix coordinates (162) satisfy the quantum Nambu geometry of the fuzzy four-sphere,

$$[X_a, X_b, X_c, X_d] = (I+2)\left(\frac{2(I_+ - I_-)}{I(I+4)}\right)^3 \epsilon_{abcde}X_e, \tag{163}$$

and the radius is

$$X_aX_a = \frac{(I_+ - I_-)^2}{I(I+4)}\mathbf{1}_{D_{N=0}^{(n=0)}(I)}. \tag{164}$$

Equation (164) implies that the monopole and antimonopole oppositely contribute to the radius of the fuzzy four-sphere, and notably at the nonchiral case $I_+ = I_-$, the radius apparently vanishes. The Fisher metric is again given by the classical four-sphere metric (156).

C. 4D quantum Hall wave function

The noncommutative geometry is the underlying geometry of the quantum Hall effect and governs the LLL physics [59–61]. As the LLL geometry in the $n=0$ sector is given by the fuzzy four-sphere geometry the same as the original 4D quantum Hall effect, a Laughlin-like many-body wave function is expected to be realized in the present system. Recall that in the original 4D quantum Hall effect [4], the many-body wave function is constructed as the mth power of the Slater determinant,

$$\Psi^{(m)}(x_1, x_2, \ldots, x_D) = \Psi_{\mathrm{Slat}}(x_1, x_2, \ldots, x_{i_D})^m, \tag{165}$$

where

$$\Psi_{\mathrm{Slat}}(x_1, x_2, \ldots, x_D) = \epsilon_{i_1 i_2 \cdots i_D}\boldsymbol{\psi}_1(x_{i_1})\boldsymbol{\psi}_2(x_{i_2})\cdots\boldsymbol{\psi}_D(x_{i_D}). \tag{166}$$

The symbol m is taken to be an odd integer due to the Fermi statistics. The right-hand side of (166) is the tensor products of the Yang's LLL monopole harmonics with degeneracy $D = \frac{1}{6}(I+1)(I+2)(I+3)$. Since Yang's LLL monopole harmonics are given by the symmetric products of the $SO(5)$ fundamental spinors, it is legitimate to adopt $\Psi^{(m)}$ as a Laughlin-like many-body function [4]. We see that the power of each one-particle state is equally given by mI, which implies the corresponding $SU(2)$ monopole index to be $m\frac{I}{2}$.

In the same spirit, we construct a Laughlin-like many-body wave function for the LLL of the $n=0$ sector in the $SO(4)$ monopole background with indices,

$$\left(m\frac{I_+}{2}, m\frac{I_-}{2}\right)_4. \tag{167}$$

The filling factor is given by

$$\nu = \frac{D_{N=0}^{(n=0)}(I_+ + I_-)}{D_{N=0}^{(n=0)}(mI_+ + mI_-)} \xrightarrow{I_+ + I_- \to \infty} \frac{1}{m^3}. \tag{168}$$

It is straightforward to derive the Slater determinant wave function at filling $\nu = 1$ using the LLL monopole harmonics in the $n=0$ sector. The obtained Slater determinant is a singlet under the $SO(5)$ rotations and represents a uniformly distributed noninteracting many-body state on a four-sphere. However, in the construction of the Laughlin wave function, the situation is rather involved; powers of the Slater determinant are *not* generally confined in the LLL. This is because the LLL one-particle states in the $SO(4)$ monopole background are not simply given by homogeneous polynomials unlike the original $SU(2)$ case. Therefore, we have to implement the projection to the LLL,

$$\Psi^{(m)}(x_1, x_2, \ldots, x_D) = \mathrm{P}_{\mathrm{LLL}}\Psi_{\mathrm{Slat}}(x_1, x_2, \ldots, x_{i_D})^m, \tag{169}$$

where $\mathrm{P}_{\mathrm{LLL}}$ denotes the projection operator constructed by

[19] In particular,

$$\Gamma_m^{(I=1)} = \begin{pmatrix} 0 & \bar{q}_m \\ q_m & 0 \end{pmatrix} = \gamma_m, \qquad \Gamma_5^{(I=1)} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} = -\gamma_5. \tag{159}$$

$$\mathrm{P}_{\mathrm{LLL}} = \sum_{\mathrm{singlet}} |\mathrm{singlet}\rangle\langle\mathrm{singlet}|. \tag{170}$$

The states $|\mathrm{singlet}\rangle$ signify the $SO(5)$ singlets made of the $D_{N=0}^{(n=0)}(I)$ tensor products of the LLL monopole harmonics in the $n=0$ sector with the $SO(4)$ background of indices (167). Applying the projection operator, we extract the LLL components of the mth power of the Slater determinant not ruining the $SO(5)$ symmetry. In this way, we can construct a Laughlin-like many-body ground state at filling (168).

VII. RELATIVISTIC $SO(5)$ LANDAU MODEL

We explore the relativistic version of the $SO(5)$ Landau model for a spinor particle and demonstrate the Atiyah-Singer index theorem for the $SO(4)$ monopole gauge field.

A. Synthetic gauge field and the relativistic Landau levels

With

$$\omega_{mn} = \frac{1}{1+x_5}(x_m dx_n - x_n dx_m), \tag{171}$$

the spin connection of S^4 is given by[20]

$$\omega = \frac{1}{2}\omega_{mn}(\sigma_{mn}^{(\frac{1}{2},0)_4} \oplus \sigma_{mn}^{(0,\frac{1}{2})_4}), \tag{173}$$

and the $SO(4)$ monopole gauge field (109) is

$$A = \frac{1}{2}\omega_{mn}\sigma_{mn}^{(\frac{I_+}{2},\frac{I_-}{2})_4}. \tag{174}$$

The relativistic $SO(5)$ Landau model describes a spinor particle on S^4, which interacts with the $SO(4)$ gauge field and the spin connection as well, and so their synthetic connection is the concern

$$\mathcal{A} \equiv \omega \otimes \mathbf{1}_{(I_++1)(I_-+1)} + 1_4 \otimes A. \tag{175}$$

The Dirac-Landau operator on S^4 is constructed as

$$-i\not{D} = -i\gamma^m e_m{}^\mu(\partial_\mu + i\mathcal{A}_\mu), \tag{176}$$

where μ denote the local coordinates on S^4, such as ξ, χ, θ, ϕ.

[20]The matrices of (173) are

$$\sigma_{mn}^{(\frac{1}{2},0)_4} = \frac{1}{2}\eta_{mn}^i\sigma_i, \qquad \sigma_{mn}^{(0,\frac{1}{2})_4} = \frac{1}{2}\bar{\eta}_{mn}^i\sigma_i. \tag{172}$$

Since the coordinate-dependent parts of ω and A are identical (171),[21] the synthetic gauge field is simply obtained by taking the tensor product of the $SO(4)$ matrices of (173) and (174). According to the $SO(4)$ decomposition rule[22]

$$\begin{aligned}&\left(\left(\frac{1}{2},0\right)_4 \oplus \left(0,\frac{1}{2}\right)_4\right) \otimes \left(\frac{I_+}{2},\frac{I_-}{2}\right)_4 \\ &\quad = \left(\frac{I_+}{2}+\frac{1}{2},\frac{I_-}{2}\right)_4 \oplus \left(\frac{I_+}{2},\frac{I_-}{2}+\frac{1}{2}\right)_4 \oplus \left(\frac{I_+}{2}-\frac{1}{2},\frac{I_-}{2}\right)_4 \\ &\qquad \oplus \left(\frac{I_+}{2},\frac{I_-}{2}-\frac{1}{2}\right)_4,\end{aligned}$$

we see that the synthetic connection consists of the four sectors:

$$\mathcal{A}^{(\frac{\mathcal{I}_+}{2},\frac{\mathcal{I}_-}{2})_4} = A^{(\frac{I_+}{2}+\frac{1}{2},\frac{I_-}{2})_4} \oplus A^{(\frac{I_+}{2},\frac{I_-}{2}+\frac{1}{2})_4} \oplus A^{(\frac{I_+}{2}-\frac{1}{2},\frac{I_-}{2})_4} \oplus A^{(\frac{I_+}{2},\frac{I_-}{2}-\frac{1}{2})_4}. \tag{179}$$

A standard way for deriving the spectra of the Dirac-Landau operator is to take its square and make use of the results of the corresponding nonrelativistic Landau problem. The formula is given by [6,62]

$$(-i\not{D})^2 = \sum_{a<b}\mathcal{L}_{ab}{}^2 - \sum_{a<b}F_{ab}{}^2 + \frac{1}{4}R_{S^4}. \tag{180}$$

The symbol $R_{S^4} = 6$ is the scalar curvature of S^4, and $\mathcal{L}_{ab}$ denote the angular momentum operators with the synthetic gauge field

$$\mathcal{L}_{ab} = -ix_a(\partial_b + i\mathcal{A}_b) + ix_b(\partial_a + i\mathcal{A}_b) + r^2\mathcal{F}_{ab}, \tag{181}$$

where $\mathcal{F}_{ab} = \partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a + i[\mathcal{A}_a, \mathcal{A}_b]$. The operators $\mathcal{L}_{ab}$ are just the familiar $SO(5)$ angular momentum operators with the $SO(4)$ monopole gauge field of the indices $(\frac{\mathcal{I}_+}{2}, \frac{\mathcal{I}_-}{2})$. We apply the results of Sec. V C to derive the spectra

[21]Recall that we have chosen the gauge group as the holonomy group of S^4.

[22]Since $SO(4) \simeq SU(2) \otimes SU(2)$, we can apply the $SU(2)$ decomposition rule to each of the $SU(2)$ s:

$$(j,k)_4 \otimes (j',k')_4 = \bigoplus_{J=|j-j'|}^{j+j'} \bigoplus_{K=|k-k'|}^{k+k'} (J,K)_4 \tag{177}$$

or

$$\begin{aligned}&\sigma_{mn}^{(j,k)_4} \otimes 1_{(2j'+1)(2k'+1)} + 1_{(2j+1)(2k+1)} \otimes \sigma_{mn}^{(j',k')_4} \\ &\quad = \sigma_{mn}^{(j\otimes j', k\otimes k')_4} = \bigoplus_{J=|j-j'|}^{j+j'} \bigoplus_{K=|k-k'|}^{k+k'} \sigma_{mn}^{(J,K)_4}.\end{aligned} \tag{178}$$

$$(-i\not{D})^2 = \lambda(p,q) - \frac{1}{2}(I_+(I_++2)+I_-(I_-+2)) + \frac{3}{2} \geq 0. \tag{182}$$

Similar to (131), the $SO(5)$ indices p and q are given by

$$p = N + \mathcal{I} - n, \qquad q = N + n, \tag{183}$$

where

$$\mathcal{I} \equiv \mathcal{I}_+ + \mathcal{I}_-, \quad n = 0,1,2,\ldots,\mathrm{Min}(\mathcal{I}_+,\mathcal{I}_-). \tag{184}$$

In the first two cases of (179), we have $\mathcal{I} = \mathcal{I}_+ + \mathcal{I}_- + 1 = I + 1$, and then

$$-i\not{D} = \pm\sqrt{N(N+3)+(I+1)(N-n)+n(n-1)+2I+I_+I_-+4}, \tag{185}$$

in which each of the positive and negative Landau levels holds the same degeneracy

$$D(N+I+1-n, N+n) = \frac{1}{6}(N+n+1)(I-2n+2)(I+N+3-n)(I+2N+4). \tag{186}$$

The minimum energy eigenvalue in magnitude is achieved at $N=0$, $n=\mathrm{Min}(\mathcal{I}_+,\mathcal{I}_-)$ to yield $|-i\not{D}| = \sqrt{2\mathrm{Max}(I_+,I_-)+4}$, and the spectra (185) do not realize zero modes. Meanwhile in the last two cases of (179), we have $\mathcal{I} = \mathcal{I}_+ + \mathcal{I}_- - 1 = I - 1$;

$$-i\not{D} = \pm\sqrt{N(N+3)+(I-1)(N-n)+n(n-1)+I_+I_-}, \tag{187}$$

in which each of the positive and negative Landau levels of (187) holds the same degeneracy

$$D(N+I-1-n, N+n) = \frac{1}{6}(N+n+1)(I-2n)(I+N+1-n)(I+2N+2). \tag{188}$$

For fixed N, n, I_+, and I_-, the eigenvalues of (187) are smaller than those of (185) in magnitude and realize zero modes at $N=0$, $n = n_{\mathrm{max}} = \mathrm{Min}(\mathcal{I}_+,\mathcal{I}_-)$.

B. Zero modes and the Atiyah-Singer index theorem

The Atiyah-Singer index theorem signifies equality between the zero-mode number and the Chern number.[23] For the present system, the Atiyah-Singer index theorem may be expressed as

$$\mathrm{ind}(-i\not{D}) \equiv \dim\mathrm{Ker}(-i\not{D}_+) - \dim\mathrm{Ker}(-i\not{D}_-) = c_2, \tag{189}$$

where $\not{D}_\pm$ are defined as

$$\not{D}_\pm = \frac{1}{2}(1\pm\gamma_5)\not{D} \tag{190}$$

and c_2 is the second Chern number of the $SO(4)$ monopole (121a). We evaluate the left-hand side of (189) to validate (189).

For $I_+ > I_-$, the zero modes are realized as those of $-i\not{D}_+$ in $(\frac{\mathcal{I}_+}{2},\frac{\mathcal{I}_-}{2}) = (\frac{I_+-1}{2},\frac{I_-}{2})$ at $N=0$ and $n=\mathrm{Min}(\mathcal{I}_+,\mathcal{I}_-)=I_-$. We then find $\dim\mathrm{Ker}(-i\not{D}_+) = D(I_+-1,I_-)$ and $\dim\mathrm{Ker}(-i\not{D}_-)=0$:

$$\mathrm{ind}(-i\not{D}) = D(I_+-1,I_-) = \frac{1}{6}(I_++1)(I_-+1)(I_++I_-+2)(I_+-I_-). \tag{191}$$

Similarly for $I_+ < I_-$, the zero modes are realized as those of $-i\not{D}_-$ in $(\frac{\mathcal{I}_+}{2},\frac{\mathcal{I}_-}{2}) = (\frac{I_+}{2},\frac{I_--1}{2})$ at $N=0$ and $n=\mathrm{Min}(\mathcal{I}_+,\mathcal{I}_-)=I_+$. We then have $\dim\mathrm{Ker}(-i\not{D}_+)=0$ and $\dim\mathrm{Ker}(-i\not{D}_-) = D(I_--1,I_+) = -D(I_+-1,I_-)$, and so $\mathrm{ind}(-i\not{D}) = D(I_+-1,I_-)$, which yields (191) again. Finally, in the case $I_+ = I_- = \frac{I}{2}$ $(I=2,4,6,\ldots)$, the LLL of the $n_{\mathrm{max}} = \frac{I}{2}-1$ sector (187) does not realize the zero modes ($(-i\not{D}) = \pm 1 \neq 0$), i.e., $\dim\mathrm{Ker}(-i\not{D}) = 0$, which is also realized at $I_+ = I_-$ in (191). After all, for arbitrary $SO(4)$ indices, Eq. (191) generally holds and the most right-hand side is exactly equal to the second Chern number (121a). This obviously demonstrates the Atiyah-Sinder index theorem.

VIII. SUMMARY AND DISCUSSIONS

In this work, we fully solved the $SO(5)$ Landau problem in the $SO(4)$ monopole background and explored non-commutative geometry and 4D quantum Hall effect. For the

[23] Since the Dirac genus of sphere is trivial, we only need to take into account the Chern number in (189).

$SO(4)$ monopole with a bi-spin index, $(\frac{I_+}{2}, \frac{I_-}{2})$, we demonstrated that the $SO(5)$ Landau model is endowed with $\mathrm{Min}(I_+, I_-)$ sectors, each of which hosts the Landau levels whose level spacing is determined by the sum of the $SO(4)$ bi-spins (Fig. 7). It was shown that the Nth Landau level eigenstates in the n sector can be obtained as a block matrix of the nonlinear realization (the blue shaded block matrix in Fig. 4) with

$$(p, q)_5 = (N + I_+ + I_- - n, N + n)_5. \tag{192}$$

The matrix geometry of the LLL in the $n = 0$ sector was identified as the fuzzy four-sphere whose radius is determined by the difference between the $SO(4)$ bi-spin indices, while the matrix geometry of the nonchiral case is trivial. The classical S^4 geometry was recovered as the Fisher information metric in any cases. We constructed the Slater determinant from the newly obtained monopole harmonics and derive a Laughlin-like many-body wave function in the $SO(4)$ monopole background by applying the LLL projection. We also investigated the $SO(5)$ relativistic Landau model and derived the relativistic spectrum and the degeneracy. The number of the zero modes exactly coincides with the second Chern number of the $SO(4)$ monopole as anticipated by the Atiyah-Singer index theorem.

The $SO(4)$ monopole is quite unique for its gauge group being the only semisimple group among the $SO(n)$ groups, which endows the present system with a particular multi-sector structure of the Landau levels. It may be interesting to speculate experimental realizations of the present model in real condensed matter systems of synthetic dimensions. Of particular interest will be the nonchiral case $I_+ = I_-$, in which the second Chern number vanishes while the generalized Euler number does not and its physical implications have not been understood yet. There are many to be clarified in the present model itself, such as edge modes, effective field theory, and extended excitations. More explorations will be beneficial not only for further understanding of higher dimensional topological phases but also for noncommutative geometry and string theory.

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APPENDIX A: THE PONTYAGIN NUMBER AND THE EULER NUMBER

On the 4D manifold, the (first) Pontryagin number P_1 and the Euler number χ_4 are introduced as [57]

$$P_1(M^4) = \frac{1}{8\pi^2}\int_{M^4} R^{m_1m_2}R_{m_1m_2} = \frac{1}{32\pi^2}\int_{M^4} R^{m_1m_2}R^{m_3m_4}\mathrm{tr}(X_{m_1m_2}X_{m_3m_4}), \tag{A1a}$$

$$\chi_4(M^4) = \frac{1}{32\pi^2}\int_{M^4}\epsilon^{m_1m_2m_3m_4}R_{m_1m_2}R_{m_3m_4} = \frac{1}{128\pi^2}\int_{M^4}\epsilon^{m_3m_4m_5m_6}R^{m_1m_2}R_{m_3m_4}\mathrm{tr}(X_{m_1m_2}X_{m_5m_6}), \tag{A1b}$$

where $R^{m_1m_2}$ stand for the curvature two-form of the manifold and $X_{m_1m_2}$ denote the $SO(4)$ adjoint representation matrices:

$$(X_{m_1m_2})_{m_3m_4} \equiv -i\delta_{m_1m_3}\delta_{m_2m_4} + i\delta_{m_1m_4}\delta_{m_2m_3}. \tag{A2}$$

The topological quantities for the gauge field (116) are generalizations of (A1) by replacing the curvature two-form of the adjoint representation matrices with the field strength of arbitrary representation matrices.

For spheres S^d, we have

$$R_{mn} = e_m \wedge e_n, \tag{A3}$$

and (A1) becomes

$$P_1(S^4) = 0, \qquad \chi_4(S^4) = 2. \tag{A4}$$

Equation (A4) is realized as a special case of (121) for the $SO(4)$ vector representation $(\frac{I_+}{2}, \frac{I_-}{2}) = (\frac{1}{2}, \frac{1}{2})$. This is because $X_{m_1m_2}$ (A2) are unitarily equivalent to $\sigma^{(\frac{1}{2},\frac{1}{2})_4}_{m_1m_2}$ (70):

$$\sigma^{(\frac{1}{2},\frac{1}{2})_4}_{m_1m_2} = U^\dagger X_{m_1m_2} U, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}. \tag{A5}$$

Consequently,

$$P_1(S^4) = c_2^{(\frac{1}{2},\frac{1}{2})}, \qquad \chi_4(S^4) = \frac{1}{2}\tilde{c}_2^{(\frac{1}{2},\frac{1}{2})}. \tag{A6}$$

APPENDIX B: NONCHIRAL $SO(5)$ MONOPOLE HARMONICS

For a better understanding, we derive several $SO(5)$ monopole harmonics. We represent the nonlinear realization matrix (104) as

$$\Psi^{(1,0)_5} = \begin{pmatrix} \psi_1^{[0,\frac{1}{2}]} & \psi_2^{[0,\frac{1}{2}]} & \psi_3^{[0,\frac{1}{2}]} & \psi_4^{[0,\frac{1}{2}]} \\ \psi_1^{[0,-\frac{1}{2}]} & \psi_2^{[0,-\frac{1}{2}]} & \psi_3^{[0,-\frac{1}{2}]} & \psi_4^{[0,-\frac{1}{2}]} \end{pmatrix}. \tag{B1}$$

The upper column quantities, $\psi_1^{[0,\frac{1}{2}]}, \psi_2^{[0,\frac{1}{2}]}, \psi_3^{[0,\frac{1}{2}]}, \psi_4^{[0,\frac{1}{2}]}$, denote the fourfold degenerate LLL eigenstates in the $SU(2)$ monopole background $(\frac{I_+}{2}, \frac{I_-}{2}) = (\frac{1}{2}, 0)$:

$$\begin{aligned}
\psi_1^{[0,\frac{1}{2}]} &= \sqrt{\frac{1+x_5}{2}}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\xi}{2}\\0\end{pmatrix},\\
\psi_2^{[0,\frac{1}{2}]} &= \sqrt{\frac{1+x_5}{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\cos\frac{\xi}{2}\end{pmatrix},\\
\psi_3^{[0,\frac{1}{2}]} &= \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix}x_4+ix_3\\-x_2+ix_1\end{pmatrix}\\
&= \begin{pmatrix}\sin\frac{\xi}{2}(\cos\chi + i\sin\chi\cos\theta)\\ i\sin\frac{\xi}{2}\sin\chi\sin\theta e^{i\phi}\end{pmatrix},\\
\psi_4^{[0,\frac{1}{2}]} &= \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix}x_2+ix_1\\x_4-ix_3\end{pmatrix}\\
&= \begin{pmatrix}i\sin\frac{\xi}{2}\sin\chi\sin\theta e^{-i\phi}\\ \sin\frac{\xi}{2}(\cos\chi - i\sin\chi\cos\theta)\end{pmatrix},
\end{aligned} \tag{B2}$$

while the lower column quantities, $\psi_1^{[0,-\frac{1}{2}]}, \psi_2^{[0,-\frac{1}{2}]}, \psi_3^{[0,-\frac{1}{2}]}$, $\psi_4^{[0,-\frac{1}{2}]}$, represent the fourfold degenerate LLL eigenstates in the $SU(2)$ antimonopole background $(\frac{I_+}{2}, \frac{I_-}{2}) = (0, \frac{1}{2})$:

$$\begin{aligned}
\psi_1^{[0,-\frac{1}{2}]} &= \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix}-x_4+ix_3\\-x_2+ix_1\end{pmatrix}\\
&= \begin{pmatrix}-\sin\frac{\xi}{2}(\cos\chi - i\sin\chi\cos\theta)\\ i\sin\frac{\xi}{2}\sin\chi\sin\theta e^{i\phi}\end{pmatrix},\\
\psi_2^{[0,-\frac{1}{2}]} &= \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix}x_2+ix_1\\-x_4-ix_3\end{pmatrix}\\
&= \begin{pmatrix}i\sin\frac{\xi}{2}\sin\chi\sin\theta e^{-i\phi}\\ -\sin\frac{\xi}{2}(\cos\chi + i\sin\chi\cos\theta)\end{pmatrix},\\
\psi_3^{[0,-\frac{1}{2}]} &= \sqrt{\frac{1+x_5}{2}}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\xi}{2}\\0\end{pmatrix},\\
\psi_4^{[0,-\frac{1}{2}]} &= \sqrt{\frac{1+x_5}{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\cos\frac{\xi}{2}\end{pmatrix}.
\end{aligned} \tag{B3}$$

Following the prescription in the main text, we can derive the tenfold degenerate LLL $SO(5)$ eigenstates in the $n=0$ sector of the $SO(4)$ background $(\frac{I_+}{2}, \frac{I_-}{2})_4 = (\frac{1}{2}, \frac{1}{2})_4$. From the nonlinear realization matrix of $(p,q)_5 = (2,0)_5$, we have

$$\begin{aligned}
\psi_{(1,1)}^{[0,0]} &= \frac{1}{\sqrt{2}}\begin{pmatrix}-\sin\xi(\cos\chi - i\sin\chi\cos\theta)\\ i\sin\xi\sin\chi\sin\theta e^{i\phi}\\ 0\\ 0\end{pmatrix},\\
\psi_{(1,2)}^{[0,0]} &= \frac{1}{2}\begin{pmatrix}i\sin\xi\sin\chi\sin\theta e^{-i\phi}\\ -\sin\xi(\cos\chi + i\sin\chi\cos\theta)\\ -\sin\xi(\cos\chi - i\sin\chi\cos\theta)\\ i\sin\xi\sin\chi\sin\theta e^{i\phi}\end{pmatrix},\\
\psi_{(3,3)}^{[0,0]} &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\ i\sin\xi\sin\chi\sin\theta e^{-i\phi}\\ -\sin\xi(\cos\chi + i\sin\chi\cos\theta)\end{pmatrix},\\
\psi_{(1,3)}^{[0,0]} &= \begin{pmatrix}\cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}(\cos^2\chi + \sin^2\chi\cos^2\theta)\\ i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi + i\sin\chi\cos\theta)e^{i\phi}\\ -i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi - i\sin\chi\cos\theta)e^{i\phi}\\ -\sin^2\frac{\xi}{2}\sin^2\chi\sin^2\theta e^{2i\phi}\end{pmatrix},\\
\psi_{(1,4)}^{[0,0]} &= \begin{pmatrix}-i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi - i\sin\chi\cos\theta)e^{-i\phi}\\ \cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}\sin^2\chi\sin^2\theta\\ -\sin^2\frac{\xi}{2}(\cos\chi - i\sin\chi\cos\theta)^2\\ i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi - i\sin\chi\cos\theta)e^{i\phi}\end{pmatrix},\\
\psi_{(2,3)}^{[0,0]} &= \begin{pmatrix}i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi + i\sin\chi\cos\theta)e^{-i\phi}\\ -\sin^2\frac{\xi}{2}(\cos\chi + i\sin\chi\cos\theta)^2\\ \cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}\sin^2\chi\sin^2\theta\\ -i\sin^2\frac{\xi}{2}(\cos\chi + i\sin\chi\cos\theta)\sin\chi\sin\theta\end{pmatrix},\\
\psi_{(2,4)}^{[0,0]} &= \begin{pmatrix}-\sin^2\frac{\xi}{2}\sin^2\chi\sin^2\theta e^{-2i\phi}\\ -i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi + i\sin\chi\cos\theta)e^{-i\phi}\\ i\sin^2\frac{\xi}{2}\sin\chi\sin\theta(\cos\chi - i\sin\chi\cos\theta)e^{-i\phi}\\ \cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}(\cos^2\chi + \sin^2\chi\cos^2\theta)\end{pmatrix},\\
\psi_{(3,3)}^{[0,0]} &= \frac{1}{\sqrt{2}}\begin{pmatrix}\sin\xi(\cos\chi + i\sin\chi\cos\theta)\\ 0\\ i\sin\xi\sin\chi\sin\theta e^{i\phi}\\ 0\end{pmatrix},\\
\psi_{(3,4)}^{[0,0]} &= \frac{1}{2}\begin{pmatrix}i\sin\xi\sin\chi\sin\theta e^{-i\phi}\\ \sin\xi(\cos\chi + i\sin\chi\cos\theta)\\ \sin\xi(\cos\chi - i\sin\chi\cos\theta)\\ -i\sin\xi\sin\chi\sin\theta e^{i\phi}\end{pmatrix},\\
\psi_{(4,4)}^{[0,0]} &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ i\sin\xi\sin\chi\sin\theta e^{-i\phi}\\ 0\\ \sin\xi(\cos\chi - i\sin\chi\cos\theta)\end{pmatrix}.
\end{aligned} \tag{B4}$$

Equation (B4) is realized as a symmetric combination of the direct products of the monopole harmonics (B2) and the antimonopole harmonics (B3):

$$\boldsymbol{\psi}^{[0,0]}_{(\alpha,\beta)} = \left(\frac{1}{\sqrt{2}}\right)^{\delta_{\alpha\beta}} \left(\boldsymbol{\psi}_\alpha^{[0,\frac{1}{2}]} \otimes \boldsymbol{\psi}_\beta^{[0,-\frac{1}{2}]} + \boldsymbol{\psi}_\beta^{[0,\frac{1}{2}]} \otimes \boldsymbol{\psi}_\alpha^{[0,-\frac{1}{2}]}\right) \quad (\alpha,\beta = 1,2,3,4). \tag{B5}$$

With the $SO(5)$ charge conjugation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{B6}$$

we see that (B5) is equivalent to $(C\Sigma_{ab}^{(1,0)_5})_{\alpha\beta}\boldsymbol{\psi}_\alpha^{[\frac{1}{2},0]} \otimes \boldsymbol{\psi}_\beta^{[0,\frac{1}{2}]}$. In (B5), the monopole and antimonopole harmonics equivalently contribute to the nonchiral monopole harmonics. In the group theory point of view, Eq. (B5) corresponds to the symmetric $(2,0)_5$ representation made of two $(1,0)_5$ representations. Since the monopole harmonics and antimonopole harmonics, respectively, have the $SU(2)$ gauge symmetry, their tensor products (B5) enjoy the $SU(2) \otimes SU(2) \simeq SO(4)$ gauge symmetry. In general, the LLL nonchiral monopole harmonics in the $n = 0$ sector of the $SO(4)$ monopole background $(\frac{I}{4}, \frac{I}{4})_4$ (I: even integers) can be obtained as the symmetric representation of the tensor product of two LLL monopole harmonics of the $SU(2)$ monopole background $(\frac{I}{4}, 0)_4$ and the antimonopole background $(0, \frac{I}{4})_4$:

$$\left(\frac{I}{2}, 0\right)_5 \otimes \left(\frac{I}{2}, 0\right)_5 \to (I, 0)_5. \tag{B7}$$

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