Maximum force for black holes and Buchdahl stars

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Black holes and Buchdahl stars are identified respectively by $\Phi(R) = 1/2, 4/9$, where $g_{tt} = 1-2\Phi(R)$ for a spherically-symmetric static metric. We investigate the maximum force for black holes and Buchdahl stars when one of the participating objects is charged and/or rotating while the other is neutral and nonrotating. It turns out that the maximum force between two Schwarzschild objects is universal, given in terms of the fundamental constant velocity of light and the gravitational constant in general relativity (GR) in the usual four-dimensional spacetime. In general this feature uniquely picks out the pure Lovelock gravity (having only one Nth order term in action which includes GR in the linear order $N = 1$) and the dimensional spectrum, $D = 3N + 1$, where N is degree of the Lovelock polynomial action.

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I. INTRODUCTION

In Newtonian gravity, there is no upper bound on force which goes on increasing as the distance between the objects decreases, and in the limit of the point particles it diverges. In general relativity (GR), on the other hand, as body becomes more and more compact it turns into a black hole—the black hole horizon forms that provide the lower bound on the separating distance. It could then define the maximum value that a force between any two objects can physically attain. It has been computed in various situations; the maximum force or tension between two equal mass static uncharged Schwarzschild black holes touching each other at the horizon [1,2] is given by

$$F_{\max} = c^4/4G, \tag{1}$$

where c is the velocity of light and G is the Newtonian gravitational constant. This would in turn define the maximum power attainable in any physical system as

$$P_{\max} = cF_{\max} = c^5/4G. \tag{2}$$

This is known as the so-called Dyson luminosity [3], or some multiple of it to account for geometrical factors $O(1)$. This limits maximum possible luminosity in gravitational or indeed any other forms of radiation that an isolated system may emit, [4,5]. Schiller has further proposed and surmised that the existence of a maximum force implies GR[1] just as maximum velocity characterizes special relativity. Recently Schiller [9] also proposed tests for the maximum force and power. There have been several computations of maximum force in different settings [10,11] and also of maximum entropy emission [12]. All this would have important implications on the cosmic censorship conjecture [13–15].

Further let us write the maximum force in Planck units in D spacetime dimensions which would read as

$$F_{\rm pl} = G_D^{2/(2-D)} c^{(4+D)/(D-2)} h^{(4-D)/(2-D)}, \tag{3}$$

which is free of the Planck's constant h if and only if $D = 4$. In four spacetime dimension, it is solely given in terms of the fundamental constant velocity of light and the gravitational constant, and is free of everything else.

Thus, the bound is universal except for the dimensionality of spacetime and it is expected to remain true even when quantum gravity effects are included or in full theory of quantum gravity. It also turns out that it remains unaltered when the cosmological constant Λ is included [2].

Further also note that the ratio of magnetic moment to angular momentum is also free of the Planck's constant h [16]. This indicates something fundamental and natural about these nonquantum universal units that is unique to

[1]This is not exactly true as it turns out that pure Lovelock theory as well as Moffat's gravity theory [6] do admit maximum force. What is true, however, is the fact [7] that the existence of maximum force bound in terms of only the fundamental constant velocity of light and the gravitational constant, does lead to pure Lovelock gravity in $D = 3N + 1$, which includes GR at the linear order $N = 1$, where N is degree of the Lovelock polynomial action [8].

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four dimensional spacetime [17]. This bound however does not exist in the Newtonian gravity [18], where point masses can approach arbitrarily close to each-other and the inverse-square gravitational force can then become arbitrarily large. It is the formation of an event horizon around these mass points in GR that is responsible for the maximum force bound. It is the inverse square law that makes it universal, free of black hole masses, and only given in terms of c and G. This is how the dimensionality of spacetime, $D = 4$ gets singled out in GR, and in general $D = 3N + 1$ in pure Lovelock theory which includes GR for $N = 1$ [19].

In astrophysics and cosmology, besides black holes which mark the limiting configuration, there are other compact objects of interest and consequence like massive compact stars or large-scale configurations. For a static object with perfect fluid interior there is well-known Buchdahl compactness bound [20] which is obtained for a fluid distribution under the very general conditions that density and isotropic pressure are positive and the former is decreasing outwards, and it is matched at the boundary with the Schwarzschild vacuum metric. The bound turns out to be

$$M/R \leq 4/9. \tag{4}$$

Let us identify with the limiting value, equality, the Buchdahl star; i.e., $M/R = 4/9$. Note that for black hole, the compactness limit is $M/R = 1/2$, which is absolute and unique. This is because its boundary is null, defining the event horizon and hence, is completely immune to other conditions. A black hole with a null boundary is naturally the most compact object. On the other hand a Buchdahl star is the most compact nonhorizon object with a timelike boundary. The Buchdahl bound can therefore be neither absolute nor unique. It can vary with the choice of equation of state, and anisotropy of pressure and energy conditions [21–24]. In a recent interesting and insightful paper [25], it has been shown that for realistic physical conditions of causality and stability of fluid interior, the Buchdahl bound is indeed always respected.

Since the Buchdahl star is the non–black hole's most compact object, it should be pertinent and interesting to investigate the maximum force between two such objects. That is what we wish to study in this paper. In addition we also investigate the maximum force between two objects, one of which is charged and/or rotating black hole or Buchdahl star while the other is neutral and nonrotating. Further, this investigation of maximum force for black holes and Buchdahl stars would be carried over to pure Lovelock gravity. In particular, it turns out that the maximum force for Buchdahl stars is $8/9$ th of the maximum force for black holes.

It is remarkable that existence of a maximum force between two black holes or Buchdahl stars, given entirely in terms of the fundamental constant velocity of light and the gravitational constant, uniquely singles out the pure Lovelock theory in the dimensional spectrum, $D = 3N + 1$, which includes GR for $N = 1$ [19]. We have considered the force between two equal mass Schwarzschild black holes with their horizons touching. If the masses are not equal then the force is bounded by the maximum bound obtained for equal masses [17].[2]

The paper is organized as follows: In the next section we shall characterize black holes and Buchdahl stars in terms of the potential, $\Phi(R)$, for radial motion, which in the static case is $g_{tt} = 1{-}2\Phi$. Then black holes and Buchdahl stars are always defined by $\Phi = 1/2, 4/9$ respectively. In Sec. III we study the maximum force for charged and/or rotating objects; black holes or Buchdahl stars. Section IV is concerned with pure Lovelock theory and investigation of the maximum force for pure Lovelock objects. We conclude with a discussion.

II. BLACK HOLES AND BUCHDAHL STARS

A static object is described by spherically symmetric metric,

$$ds^2 = f(R)dt^2 - dR^2/f(R) - R^2 d\Omega^2, \tag{5}$$

where $d\Omega^2$ is the metric on the unit sphere. Let us write $f(R) = 1{-}2\Phi(R)$; the above metric describes a neutral Shwarzschild object when $\Phi(R) = M/R$ and a charged Reissner-Nordström one for $\Phi(R) = (M - Q^2/2R)/R$.

Black holes and Buchdahl stars are identified by the condition $\Phi(R) = 1/2, 4/9$, respectively. There is also another insightful characterization [26]—the black hole horizon is defined when the gravitational field energy is equal to the nongravitational energy, and when it is half of nongravitational energy it defines a Buchdahl star.

A black hole is defined by its event horizon which would be given by $\Phi(R) = 1/2$. For a static Schwarzschild object, we know $\Phi(R) = M/R$ and $\Phi(R) = 1/2$ gives the black hole horizon, $R = 2M$. On the other hand a Buchdahl star is identified by $\Phi(R) = M/R = 4/9$. So black holes and Buchdahl stars are characterized by $\Phi(R) = 1/2, 4/9$ respectively, and this is universal and holds good, in general, for charged and rotating objects as well. Further, also note that $(M/R)_{\rm Buch}/(M/R)_{\rm BH} = 8/9$ is also universal [27]. We shall generally work in the units with $G = c = 1$ which would be restored in the expressions as and when required.

In the case of a charged object, $\Phi(R) = \frac{M - Q^2/2R}{R}$ where the electric field energy $Q^2/2R$ lying exterior to radius R is subtracted out from mass M. Again $\Phi(R) = 1/2$ gives the familiar charged black hole horizon, $R_+ = M(1 + \sqrt{1 - \alpha^2}), \alpha^2 = Q^2/M^2$. On the other hand for the Buchdahl star, $\Phi(R) = 4/9$ would give [26,28,29],

[2]If masses are unequal, $F = (M'/M)F_{\rm max} < F_{\rm max}$ where $F_{\rm max} = c^4/4G$.

$$M/R = \frac{8/9}{1+\sqrt{1-(8/9)\alpha^2}}, \tag{6}$$

which is, $(M/R)_{\rm Buch} = (8/9)(M/R)_{\rm BH}(\alpha^2 \to (8/9)\alpha^2)$.

It was rather straightforward to define potential in the static case because of spherical symmetry while it is not so for the axially-symmetric rotating case. In this case we have to filter out the inherent frame-dragging effect by considering the potential for axial motion, which entirely lies in the t–r plane. It would then involve only gravitational contribution of rotation. It can be easily seen from the Kerr metric in the standard Boyer-Lindquist coordinates,

$$ds^2 = \frac{\Delta}{\rho^2}d\tau^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}[(R^2+a^2)d\phi - adt]^2, \tag{7}$$

where

$$d\tau = dt - a\sin^2\theta d\phi, \qquad \Delta = R^2 - 2MR + a^2, \\ \rho^2 = R^2 + a^2\cos^2\theta. \tag{8}$$

Then $\Phi(R)$ for the axial motion would be given by

$$\Phi(R) = \frac{MR}{(R^2+a^2)} = \frac{M/R}{1+\beta^2(M/R)^2}, \qquad \beta^2 = a^2/M^2. \tag{9}$$

Now $\Phi(R) = 1/2$ yields the familiar Kerr black hole horizon, $R_+ = M(1+\sqrt{1-\beta^2})$, while for the rotating Buchdahl star, $\Phi(R) = 4/9$ gives

$$M/R = \frac{8/9}{1+\sqrt{1-(8/9)^2\beta^2}} = (8/9)(M/R)_{\rm BH}(\beta^2 \to (8/9)^2\beta^2). \tag{10}$$

Generalizing to a Kerr-Newman charged and rotating object described by the above metric with $\Delta = R^2 - 2MR + a^2 + Q^2$, we shall similarly have

$$\Phi(R) = \frac{MR - Q^2/2}{(R^2+a^2)} = \frac{M/R - (\alpha^2/2)(M/R)^2}{1+\beta^2(M/R)^2}. \tag{11}$$

Then $\Phi(r) = 1/2$ gives the familiar Kerr-Newman black hole horizon, $R_+ = M(1+\sqrt{1-\alpha^2-\beta^2})$, while for the corresponding Buchdahl star where $\Phi(R) = 4/9$ we have

$$M/R = \frac{8/9}{1+\sqrt{1-(8/9)\alpha^2-(8/9)^2\beta^2}} = (8/9)(M/R)_{\rm BH}(\alpha^2 \to (8/9)\alpha^2, \beta^2 \to (8/9)^2\beta^2). \tag{12}$$

Note that a spherically-symmetric metric, Schwarzschild or Reissner-Nordström, describes a general static object which could be a black hole or a non–black hole object like a Buchdahl star. In contrast, strictly speaking the Kerr metric describes only a rotating black hole and not a rotating object in general. A rotating object would, in general, be deformed due to rotation and would have multipole moments. That is what the Kerr metric cannot accommodate and hence it can only describe a black hole which has no multipole moments—no hair. Despite vigorous attempts over decades, there still does not exist a proper metric describing a non–black hole rotating object; therefore, one has to resort to the Kerr metric for the description of a rotating Buchdahl star which should be valid only in the first approximation. Despite this, the result may however be indicative of a general behavior.

Thus, we have computed the compactness M/R expression for the various cases which we now use in the next section to compute the maximum force in these cases.

III. MAXIMUM FORCE

We compute the force between two equal mass objects, one of which is charged and rotating while the other is neutral and nonrotating. Their boundaries touch along the axis. The aim is to include only the gravitational contribution of charge and rotation and not the electromagnetic and spin-spin interactions. We shall find the contributions of charge and rotation to the maximum force.

Differentiating $\Phi(R)$ in Eq. (11) we obtain

$$F = \frac{M^2}{R^2}\frac{1-\alpha^2 M/R - \beta^2 M^2/R^2}{(1+\beta^2 M^2/R^2)^2}. \tag{13}$$

Now the maximum force between two equal mass black holes, one of which is Kerr-Newman and the other Schwarzschild, touching each other at the horizon on the axis, $\theta = 0$, for which $M/R = (1+\sqrt{1-\alpha^2-\beta^2})^{-1}$, would be given by

$$F_{\rm max}({\rm KN-BH}) = \frac{\sqrt{1-\alpha^2-\beta^2}}{2(1+\sqrt{1-\alpha^2-\beta^2})-\alpha^2} = 4\frac{\sqrt{1-\alpha^2-\beta^2}}{2(1+\sqrt{1-\alpha^2-\beta^2})-\alpha^2} \times F_{\rm max}({\rm Sch-BH}), \tag{14}$$

where $F_{\rm max}({\rm Sch-BH}) = c^4/4G$ when $\alpha^2 = \beta^2 = 0$.

For the Reissner-Nordström and Kerr black holes it will respectively read as follows:

$$F_{\max}(\mathrm{RN-BH}) = 4\frac{\sqrt{1-\alpha^2}}{(1+\sqrt{1-\alpha^2})^2}F_{\max}(\mathrm{Sch-BH}), \tag{15}$$

and

$$F_{\max}(\mathrm{Kerr-BH}) = 2\frac{\sqrt{1-\beta^2}}{1+\sqrt{1-\beta^2}}F_{\max}(\mathrm{Sch-BH}). \tag{16}$$

Their counterparts for the Buchdahl stars would be obtained by multiplying by $(8/9)^2$ and writing $\alpha^2 \to (8/9)\alpha^2$, $\beta^2 \to (8/9)^2\beta^2$ in the above expressions; i.e.,

$$\begin{aligned} F_{\max}(\mathrm{KN-Buch}) &= (8/9)^2 F_{\max}(\mathrm{KN-BH})(\alpha^2 \to (8/9)\alpha^2, \beta^2 \to (8/9)^2\beta^2) \\ &= 4F_{\max}(\mathrm{KN-BH})(\alpha^2 \to (8/9)\alpha^2, \beta^2 \to (8/9)^2\beta^2)F_{\max}(\mathrm{Buch}), \end{aligned} \tag{17}$$

where $F_{\max}(\mathrm{Buch})(\alpha^2=\beta^2=0)=(8/9)^2F_{\max}(\mathrm{Sch-BH})=(8/9)^2c^4/4G$.

In particular, the maximum force for charged and rotating Buchdahl stars would read, respectively, as follows:

$$F_{\max}(\mathrm{RN-Buch}) = 4\frac{\sqrt{1-(8/9)\alpha^2}}{2(1+\sqrt{1-(8/9)\alpha^2})-\alpha^2} \times F_{\max}(\mathrm{Buch}), \tag{18}$$

and

$$F_{\max}(\mathrm{Kerr-Buch}) = 4\frac{\sqrt{1-(8/9)^2\beta^2}}{2(1+\sqrt{1-(8/9)^2\beta^2})} \times F_{\max}(\mathrm{Buch}), \tag{19}$$

where $F_{\max}(\mathrm{Buch})=(8/9)^2F_{\max}(\mathrm{Sch-BH})=(8/9)^2c^4/G$.

For $\alpha^2=1$, it reduces to $F_{\max}(\mathrm{RN-Buch}) = (4/5)F_{\max}(\mathrm{Buch})$. Similarly for rotating Buchdahl stars with $\beta^2=1$, we would have $F_{\max}(\mathrm{Kerr-Buch}) = 4\sqrt{17}/(9+\sqrt{17})F_{\max}(\mathrm{Buch}) \approx (8/13)F_{\max}(\mathrm{Buch})$. It is interesting to note that extremality for Buchdahl stars is over-extremality for black holes. The maximum force vanishes for extremal values which are $\alpha^2=1$, $\beta^2=1$ for Reissner-Nordström and Kerr black holes, respectively. These values are fine for Buchdahl stars and give a nonvanishing value for the maximum force. For Buchdhal stars, the corresponding extremal values are $\alpha^2=9/8$, $\beta^2=(9/8)^2$.

Note that $(M/R)_{\mathrm{Buch}} = (8/9)(M/R)_{\mathrm{BH}}(\alpha^2 \to (8/9)\alpha^2, \beta^2 \to (8/9)^2\beta^2)$, and the maximum force is given in terms of M/R—that is why the maximum force in the two cases is similarly transformed. The maximum force is independent of mass but it does depend upon the charge to mass, $\alpha^2 = Q^2/M^2$, and spin to mass, $\beta^2 = a^2/M^2$ ratios.

IV. LOVELOCK GRAVITY

In a D-dimensional spacetime, gravity can be described by an action functional involving arbitrary scalar functions of the metric and curvature, but not derivatives of curvature. In general, variation of such an arbitrary Lagrangian would lead to an equation having fourth-order derivatives of the metric. For them to be of second order, the gravitational Lagrangian, L, is constrained to be of the following Lovelock form [8],

$$L = \sum_N \lim \alpha_N L_N = \alpha_N \frac{1}{2^N}\delta^{a_1b_1a_2b_2\ldots a_nb_n}_{c_1d_1c_2d_2\ldots .c_nd_n}R^{c_1d_1}_{a_1b_1}R^{c_2d_2}_{a_2b_2}\ldots .R^{c_nd_n}_{a_nb_n}, \tag{20}$$

where $\delta^{pq.}_{rs\ldots}$ is the completely antisymmetric determinant tensor. Note that $N=1$, 2 respectively correspond to the familiar linear Einstein-Hilbert Lagrangian and the quadratic Gauss-Bonnet Lagrangian, which is given by

$$L_2 \equiv L_{\mathrm{GB}} = (1/2)(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2). \tag{21}$$

Lovelock's Lagrangian is a sum over N, where each term is a homogeneous polynomial in curvature and has an associated dimensionful coupling constant, α_N. Moreover, the complete antisymmetry of the δ tensor demands $D \geq 2N$, or it would vanish identically. Even for $D = 2N$ the Lagrangian reduces to a total derivative. Therefore, Lovelock's Lagrangian, L_N, is nontrivial only in dimension $D \geq 2N+1$.

Lovelock theory is the most natural and quintessential higher-dimensional generalization of GR with the remarkable property that the field equations continue to remain second order in the metric tensor despite the action being a homogeneous polynomial in the Riemann tensor.

A particular case of interest is that of the *pure Lovelock* which has only one Nth-order term in the Lagrangian without a sum over lower orders in the action and the equations of motion. It distinguishes itself by the property that gravity is kinematic in all critical odd $D = 2N + 1$ dimensions. It is well known that GR is kinematic in $D = 2 \times 1 + 1 = 3$ in the sense that Riemann is entirely given in terms of Ricci and hence a nontrivial vacuum solution cannot exist. Similarly for Lovelock theory, Lovelock-Riemann [30,31] is entirely determined by the corresponding Ricci in all $D = 2N + 1$, and hence therefore no vacuum solution can occur. Pure Lovelock theory thus universalizes the kinematic property to all critical odd $D = 2N + 1$ dimensions.

Pure Lovelock gravity is kinematic in all critical odd $D = 2N + 1$ and a nontrivial vacuum solution cannot exist unless $D \geq 2N + 2$. Finally, variation of the Lagrangian with respect to the metric, for pure Lovelock theories, leads to the following second-order equation,

$$-\frac{1}{2^{N+1}} \delta^{a_1 b_1 a_2 b_2 \ldots a_n b_n}_{c_1 d_1 c_2 d_2 \ldots . c_n d_n} R^{c_1 d_1}_{a_1 b_1} R^{c_2 d_2}_{a_2 b_2} \ldots . R^{c_n d_n}_{a_n b_n} = 8\pi G T_{ab}. \quad (22)$$

Since no derivatives of curvature appear, this equation is of second order in the derivatives of the metric tensor. Although not directly evident, the second-order derivatives also appear linearly and the equations are therefore quasi-linear, thereby ensuring unique evolution.

Another property that singles out pure Lovelock is the existence of *bound orbits* around a static object [32]. Note that, in GR, bound orbits exist around a static object only in $D = 4$. In view of these remarkable features, it has been argued that pure Lovelock is an attractive gravitational equation in higher dimensions [33].

As with the Schwarzschild solution for GR, there exists an exact solution for a pure Lovelock black hole [34], and it is given by Eq. (5), with

$$\Phi(R) = \frac{GM}{R^n}, \qquad n = \frac{(D - 2N - 1)}{N}, \quad (23)$$

where G is the gravitational constant appropriate for the corresponding dimension and Lovelock degree N. Clearly, $D \geq 2N + 2$ for nontrivial vacuum solutions, and the black hole horizon is given by

$$\Phi(R) = \frac{GM}{R^n} = \frac{1}{2}, \quad (24)$$

yielding the horizon, $R_H^n = 2GM/c^2$.

The force between two neutral static pure Lovelock black holes with their horizons touching would be given by

$$F = n \frac{GM^2}{R_H^{n+1}}. \quad (25)$$

This would not be independent of black hole masses unless $n = (D - 2N - 1)/N = 1$; i.e., $D = 3N + 1$. Then the maximum force takes the same value as for GR in $D = 4$; i.e.,

$$F_{\max}(\mathrm{BH}) = \frac{c^4}{4G}. \quad (26)$$

With $n = 1$ (equivalently $D = 3N + 1$), a Buchdahl star is identified by $\Phi(R) = 4/9$ and the maximum force between two equal mass Buchdahl stars is

$$F_{\max}(\mathrm{Buch}) = (8/9) F_{\max}(\mathrm{BH}), \quad (27)$$

where $F_{\max}(\mathrm{BH}) = c^4/4G$.

The pure Lovelock analog of force in Planck units [7] would read as

$$F_{\mathrm{pl}} = G^{2/(2-D)} c^{(4+D)/(D-2)} h^{(3N+1-D)/(2-D)}, \quad (28)$$

which would be free of h if and only if $D = 3N + 1$.

The property that the maximum force is universal and is equal to $c^4/4G$ uniquely picks out pure Lovelock theory in $D = 3N + 1$, which includes GR for $N = 1$ and $D = 4$.

V. DISCUSSION

Existence of maximum force depends upon occurrence of a black hole horizon that marks the lower bound on distance separation between the two black holes. Further, its independence on the mass of black holes critically hinges on the inverse square law which singles out four dimensions in GR. For the inverse square law, the potential should be $1/R$, which in GR could only happen for $D = 4$ because the potential is $1/R^{D-3}$. In contrast, in pure Lovelock gravity it is $1/R^n$ [34], where $n = (D - 2N - 1)/N$ which would be unity for $D = 3N + 1$ [19]. That is, the force would be the inverse square for $D = 4, 7, 10, \ldots$ respectively for linear GR, $N = 1$, quadratic Gauss-Bonnet, $N = 2$, cubic, $N = 3$, and so on. Therefore, the force would be the inverse square for the entire dimensional spectrum, $D = 3N + 1$.

We could paraphrase this general result along the lines of Bertrand's theorem of classical mechanics as follows:

"The property that maximum force for uncharged static black hole and Buchdahl star is entirely given in terms of the fundamental constant velocity of light and the gravitational constant, uniquely picks out pure Lovelock gravity in the dimensional spectrum $D = 3N + 1$."

In pure Lovelock gravity it is universal and also free of the Planck's constant in $D = 3N + 1$ as we have seen in Eqs. (3) and (28). This indicates that this is a purely classical result which would remain true even when quantum gravity effects are included or in the full theory of quantum gravity. It indicates something fundamental and natural about these nonquantum universal units which is unique to $D = 4$ in GR, and in general to $D = 3N + 1$ in pure Lovelock gravity.

It is well known that GR is kinematic in three dimension because Riemann is entirely given in terms of Ricci, and that no nontrivial vacuum solution exists. Pure Lovelock gravity universalizes this property which is kinematic

(Lovelock-Riemann is given in terms of Lovelock-Ricci and no nontrivial pure Lovelock vacuum solution could occur) in all critical odd $D = 2N + 1$ dimensions [30,31]. The action Lagrangian contains only one Nth-order term without summing over lower orders. There are several interesting features of pure Lovelock gravity; for example, bound orbits, which exist only for $D = 4$ in GR, could exist in the dimensional window $3N + 1 \leq D \leq 4N$ [32,33]. The existence of a maximum force given entirely in terms of c and G adds yet another distinguishing property for pure Lovelock gravity. We have argued elsewhere [33] that the pure Lovelock equation is perhaps the right gravitational equation in higher dimensions.

We have evaluated the maximum force for black holes and Buchdahl stars having charge and/or rotation. It is interesting to note how the compactness ratio, M/R, and the maximum force transform in going from one to the other. That is, the compactness ratio, $(M/R)_{\mathrm{Buch}} = (8/9)(M/R)_{\mathrm{BH}}$, and maximum force, $F_{\mathrm{max}}(\mathrm{Buch}) = (8/9)^2 F_{\mathrm{max}}(\mathrm{BH})$ with $\alpha^2 \to (8/9)\alpha^2$ and $\beta^2 \to (8/9)^2\beta^2$. In particular, for the neutral static Buchdahl star, $(M/R)_{\mathrm{Buch}} = (8/9)(M/R)_{\mathrm{BH}} = 8/9 \times 1/2 = 4/9$ and $F_{\mathrm{max}}(\mathrm{Buch}) = (8/9)^2 F_{\mathrm{max}}(\mathrm{BH})$ where $F_{\mathrm{max}}(\mathrm{BH}) = c^4/4G$. For an uncharged static Buchdahl star, the compactness ratio is scaled relative to a black hole by the factor $8/9$, while the maximum force by its square, and α^2 and β^2 are similarly scaled, respectively.

Also note that the same factor $(8/9)$ is the ratio between square of their escape velocities; i.e., $(v_{\mathrm{Buch}}/v_{\mathrm{BH}})^2 = 8/9$ and this is universal [27] as it is the same for charged and/or rotating objects as well. Further, it turns out that like the black hole [35], nonextremal charged Buchdahl stars also cannot be extremalized. This is because the parameter window for particles reaching the star pinches off as the extremality is approached. This is exactly parallel to what happens for a black hole. Thus black holes and Buchdahl stars share many similar features.

Maximum force or maximum luminosity could be computed for non–black hole object (Buchdahl star), because it has, like a black hole, a bound on M/R. Observationally, luminosity of any phenomenon involving compact stars would always be bounded by $(8/9)^2$ of black hole luminosity. The most remarkable feature is that the bound is given in terms of the fundamental constant velocity of light and the gravitational constant.

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