New Chandrasekhar transformation in Kerr spacetime

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(Received 4 December 2021; accepted 17 February 2022; published 22 March 2022)

We construct a new type of Chandrasekhar transformation in Kerr spacetime using the different tortoise coordinate, which is useful for exact analysis to study the Teukolsky equation with arbitrary frequency. We also give the interpretation of our transformation using the formalism of the quantum Seiberg-Witten geometry.

DOI: 10.1103/PhysRevD.105.064036

I. INTRODUCTION

The direct observation of gravitational waves by LIGO and Virgo [1] has opened a new era of cosmology. The binary system such as the black hole merger is a good object to observe the gravitational waves, where the theoretical calculation is also possible. One of the methods for the calculation of the gravitational waves radiated from the binary system is the black hole perturbation theory [2], which is in particular useful for the case of the extreme mass-ratio inspiral (EMRI). In this formalism, we have to solve the linearized Einstein equation in the black hole spacetime. Fortunately, it was found that the separation of the variables is possible for particular gauges. For the Schwarzschild spacetime, the two equations for the radial variable have been obtained: the Regge-Wheeler equation [3] and the (radial) Teukolsky equation [4]. Those equations are derived from the same linearized Einstein equation but with the different gauges, and hence it is expected that there is a relation between the Regge-Wheeler and the Teukolsky equation, originated by the gauge transformation. The explicit one-to-one correspondence was found by Chandrasekhar [5], called the Chandrasekhar transformation. For the Kerr spacetime, it is not known how to obtain the Regge-Wheeler-type equation directly from the linearized Einstein equation, and only the Teukolsky equation is obtained. Hence some Regge-Wheeler-type equations [6–8] are proposed using the Chandrasekhar(-like) transformation from the Teukolsky equation. The Chandrasekhar transformation is also known as an example of the Darboux transformation [9–11].

Mathematically, the Regge-Wheeler and the Teukolsky equations belong to the confluent Heun's equation (CHE) [12], which has two regular singularities and one irregular singularity. CHE also has the so-called accessory parameter, which cannot be reproduced from the local leading behavior of the solution. Due to the existence of the accessory parameter, solving the equation globally is very difficult and so far some local behaviors of the solutions around the regular singularities are mainly studied. Moreover, the solution which is regular at the origin is recently implemented in *Mathematica*. For the global solution, the expression as the series of the hypergeometric functions [12], corresponding to the solution of the Teukolsky equation with the low-frequency expansion, have just been established [13,14].

Recently, CHE has also been found as the differential equation associated with the quantum Seiberg-Witten geometry [15–17] in supersymmetric gauge theories. Moreover, due to AGT (Alday-Gaiotto-Tachikawa) correspondence [18,19], the same equation can also be regarded as the BPZ (Belavin-Polyakov-Zamolodchikov) equation in two-dimensional conformal field theory [20]. This correspondence is helpful to study the solution to the Regge-Wheeler and the Teukolsky equations beyond the low-frequency approximation. It is also found that the Chandrasekhar(-like) transformation in the Schwarzschild spacetime can be interpreted as the exchange of the mass parameters [21,22]. Before that it has already been known that this exchange of the parameters is regarded as a particular integral transform [23,24]. Performing the integral transform needs to know the global behavior of the function. On the other hand, since the Chandrasekhar(-like) transformation just consists of the function itself and its derivative, one can easily find the local behavior (around the regular singularities, in particular) of the transformed function. In this paper, we will consider a new Chandrasekhar(-like) transformation in the Kerr spacetime, which can be interpreted as the transform of the mass parameters the quantum Seiberg-Witten geometry. Moreover, it would also help to study the problem with arbitrary frequency.[1]

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[1]If the frequency is high enough, the analysis using the geometrical optics is available.

The reminder of this work is organized as follows: in Sec. II, we review the Chandrasekhar transformation, which is extended from the original work by introducing a constant parameter for later convenience. In Sec. III, we consider the new Chandrasekhar(-like) transformation in the Kerr spacetime. In Sec. IV, we interpret our new transformation as the change of the parameters in CHE using the formalism of the quantum Seiberg-Witten geometry. Section V is devoted to summary and discussion.

II. CHANDRASEKHAR TRANSFORMATION

We first review about the general procedure of the Chandrasekhar transformation [5]. Let the function $X(x)$ satisfy the differential equation

$$[\Lambda_-\Lambda_+ - V_X(x)]X = 0, \tag{2.1}$$

where the differential operators $\Lambda_\pm$ are defined by

$$\Lambda_\pm = \frac{d}{dx} \pm ip(x). \tag{2.2}$$

Then the Chandrasekhar transformation [5] is given by

$$Y = FX + G\Lambda_+ X, \tag{2.3}$$

where F is taken to be

$$F = \alpha^{-1} V_X, \tag{2.4}$$

with constant α, and G is some function of x to be determined later. Acting Λ_- to the both hand sides of (2.3) gives

$$\Lambda_- Y = AX + B\Lambda_+ X, \tag{2.5}$$

where A and B are defined by

$$A = F' - 2ipF + GV_X, \tag{2.6}$$

$$B = F + G', \tag{2.7}$$

and the prime denotes the derivative with respect to x. Again, acting Λ_+ to the both hand sides of (2.5) and eliminating X gives

$$\begin{aligned}\Lambda_+\Lambda_- Y - \frac{A'}{A}\Lambda_- Y - \alpha BY \\ = \left(A + B' + 2ipB - \alpha BG - \frac{B}{A}A'\right)\Lambda_+ X.\end{aligned} \tag{2.8}$$

In order to make the above be the closed equation for Y, we require that the right-hand side of (2.8) should vanish, namely

$$G = \alpha^{-1}\left(2ip + \frac{A}{B} - \frac{A'}{A} + \frac{B'}{B}\right). \tag{2.9}$$

Then Y satisfies

$$\Lambda_+\Lambda_- Y - \frac{A'}{A}\Lambda_- Y - \alpha BY = 0. \tag{2.10}$$

Finally, by multiplication transformation $Y = H\tilde{Y}$, we will obtain the desired form of the differential equation. For example, when we choose $H = A^{\frac{1}{2}}$, the differential equation for $\tilde{Y}$ becomes

$$[\Lambda_+\Lambda_- - V_Y]\tilde{Y} = 0, \qquad V_Y = B - \frac{H''}{H} - 2ip\frac{H'}{H} + 2\left(\frac{H'}{H}\right)^2, \tag{2.11}$$

which has a similar form as (2.1), but the order of Λ_+ and Λ_- is reversed.

Let us recapitulate the procedure of the transformation for the case of the Schwarzschild spacetime. In this background, X is assumed to satisfy the Regge-Wheeler equation [3] in the frequency domain

$$\left[\left(\frac{r-2M}{r}\frac{d}{dr}\right)^2 + \omega^2 - V_{\mathrm{RW}}(r)\right]X = 0. \tag{2.12}$$

$$V_{\mathrm{RW}}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right], \tag{2.13}$$

on the other hand the (radial) Teulkolsky equation [4] with spin $s = -2$ in the frequency domain is

$$\left[(r^2 - 2Mr)\frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U_{\mathrm{T}}(r)\right]R = 0, \tag{2.14}$$

$$U_{\mathrm{T}}(r) = \left(1 - \frac{2M}{r}\right)^{-1}[(\omega r)^2 - 4i\omega(r-3M)] - (l-1)(l+2). \tag{2.15}$$

Here r is the standard radial coordinate, ω is the frequency of the gravitational waves, M is the mass of the black hole, and l denotes the multipole which takes the value $l = 2, 3, \ldots$ By introducing the dimensionless coordinate $z = \frac{r}{2M}$, the Regge-Wheeler equation (2.12) and the Teukolsky equation (2.14) can be rewritten as

$$\left[\left(\frac{z-1}{z}\frac{d}{dz}\right)^2 + \epsilon^2 - \left(1 - \frac{1}{z}\right)\left(\frac{l^2+l}{z^2} - \frac{3}{z^3}\right)\right]X = 0. \tag{2.16}$$

$$\begin{aligned}z(z-1)\frac{d^2R}{dz^2} - (2z-1)\frac{dR}{dz} \\ + \left[\frac{z}{z-1}(\epsilon^2 z^2 - 2i\epsilon(2z-3)) - (l^2+l-2)\right]R = 0,\end{aligned} \tag{2.17}$$

where ϵ is the dimensionless frequency parameter defined by

$$\epsilon = 2M\omega. \tag{2.18}$$

We will use the tortoise coordinate

$$z^* = z + \ln(z-1), \tag{2.19}$$

as our variable x. The differential operators $\Lambda_\pm$ (2.2) are taken to be

$$\Lambda_\pm = \frac{d}{dz^*} \pm i\epsilon = \left(1-\frac{1}{z}\right)\frac{d}{dz} \pm i\epsilon. \tag{2.20}$$

Using $\Lambda_\pm$, the Regge-Wheeler equation (2.16) can be rewritten in the form of (2.1) with

$$V_X = \left(1-\frac{1}{z}\right)\left(\frac{l^2+l}{z^2} - \frac{3}{z^3}\right). \tag{2.21}$$

In [5], Chandrasekhar chose $\alpha = 1$ and then the function F is just V_X, and chose G in (2.3) as

$$G = \frac{2z-3}{z^2} + 2i\epsilon. \tag{2.22}$$

Then A and B are computed from (2.6) and (2.7) as

$$A = \frac{3}{z^4}\left(1-\frac{1}{z}\right)^2,$$
$$B = \left(1-\frac{1}{z}\right)\left(\frac{l^2+l-2}{z^2} + \frac{3}{z^3}\right), \tag{2.23}$$

One can confirm that (2.9) is indeed satisfied. The differential equation (2.10) for Y becomes

$$\Lambda_+\Lambda_- Y + \frac{2(2z-3)}{z^2}\Lambda_- Y - \left(1-\frac{1}{z}\right)\left(\frac{l^2+l-2}{z^2} + \frac{3}{z^3}\right)Y = 0. \tag{2.24}$$

Finally (2.24) can be rewritten into the Teukolsky equation (2.17) by the multiplication transformation $Y = z^{-3}R$. Thus the Chandrasekhar transformation is

$$\begin{aligned} R &= z^3\left[V_X X + \left(\frac{2z-3}{z^2} + 2i\epsilon\right)\Lambda_+ X\right] \\ &= z^2 f \Lambda_+ f^{-1} \Lambda_+ z X, \end{aligned} \tag{2.25}$$

where $f = 1 - z^{-1}$.

III. NEW TRANSFORMATION IN KERR SPACETIME

Now we consider a similar Chandrasekhar-like transformation for the Teukolsky equation in Kerr spacetime. We use the conventional Boyer-Lindquist radial coordinate r. The outer and the inner horizons in this coordinate are located at

$$r = r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{3.1}$$

where a is the Kerr parameter. The limit $a \to 0$ corresponds to the Schwarzschild spacetime. The Teukolsky equation in Kerr spacetime with the spin $s = -2$ in the frequency domain [4] is given by

$$\Delta \frac{d^2R}{dr^2} - 2(r-M)\frac{dR}{dr} + U_\mathrm{T} R = 0, \tag{3.2}$$

$$U_\mathrm{T} = \frac{K^2 + 4i(r-M)K}{\Delta} - 8i\omega r - \lambda, \tag{3.3}$$

where λ is the eigenvalue of the equation, which approaches to $l^2 + l - 2$ in $a \to 0$ limit. Δ and K are defined by

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \tag{3.4}$$

$$K = (r^2 + a^2)\omega - am. \tag{3.5}$$

Here m can take the integer values with $-l \le m \le l$. In order to make the structure of the equation simpler, we introduce the dimensionless coordinate z by[2]

$$z = \frac{r - r_-}{r_+ - r_-}. \tag{3.6}$$

By this transformation the regular and the irregular singularities $r = r_-, r_+, \infty$ of the Teukolsky equation are mapped into $z = 0, 1, \infty$, respectively. The Teukolsky equation in terms of the z-coordinate becomes

$$\begin{aligned} &z(z-1)\frac{d^2R}{dz^2} - (2z-1)\frac{dR}{dz} \\ &\quad + \left[\frac{k^2 + 2i(2z-1)k}{z(z-1)} - 4i\tilde{k} - \lambda\right]R = 0, \end{aligned} \tag{3.7}$$

where k and $\tilde{k}$ are given by

$$k = \mathcal{A}\epsilon z^2 + \mathcal{B}\epsilon z + \mathcal{C}, \qquad \tilde{k} = \frac{dk}{dz} = 2\mathcal{A}\epsilon z + \mathcal{B}\epsilon, \tag{3.8}$$

$$\mathcal{A} = \frac{r_+ - r_-}{2M}, \qquad \mathcal{B} = \frac{r_-}{M}, \qquad \mathcal{C} = \frac{r_-\epsilon - am}{r_+ - r_-}. \tag{3.9}$$

[2] We assume that Kerr black hole is nonextremal, i.e., $a < M$. The extremal case ($a = M$) should be considered separately.

Note that under the limit $a \to 0$, the above quantities behave as

$$k \to \epsilon z^2, \quad \tilde{k} \to 2\epsilon z, \quad \mathcal{A} \to 1, \quad \mathcal{B}, \mathcal{C} \to 0. \tag{3.10}$$

Since it is not known how to obtain the Regge-Wheeler-type equation in Kerr spacetime directly from the linearized Einstein equation, we here consider the Chandrasekhar-like transformation from the Teukolsky equation. As the independent variable x we use the coordinate z^* (2.19) defined from (3.6). Note that in Kerr spacetime, z^* is different from the conventional tortoise coordinate z^{**} defined by

$$\begin{aligned} z^{**} &= z + \frac{2Mr_+}{(r_+ - r_-)^2}\ln(z-1) - \frac{2Mr_-}{(r_+ - r_-)^2}\ln z \\ &= z + \frac{1+\mathcal{A}}{2\mathcal{A}^2}\ln(z-1) - \frac{1-\mathcal{A}}{2\mathcal{A}^2}\ln z. \end{aligned} \tag{3.11}$$

In literature [6–8], z^{**} is used as the independent variable, since d/dz^{**} is the Killing vector field for the gravitational wave radiation. However, the differential equation using z^{**} has the apparent singularities at $z = (-1 + \mathcal{A} \pm i\sqrt{1-\mathcal{A}^2})/2\mathcal{A}$ (corresponding to $r = \pm ia$), which makes the analysis (in particular the discussion in the next section) complicated. Here we use the coordinate z^* since the apparent singularities do not appear and also the resulting equation is simpler. We take the differential operators $\Lambda_\pm$ as

$$\Lambda_\pm = \frac{d}{dz^*} \pm i\frac{k}{z^2} = \left(1 - \frac{1}{z}\right)\frac{d}{dz} \pm i\left(\mathcal{A}\epsilon + \frac{\mathcal{B}\epsilon}{z} + \frac{\mathcal{C}}{z^2}\right). \tag{3.12}$$

Then the Teukolsky equation (2.17) is rewritten as

$$\Lambda_+\Lambda_- R - \frac{2}{z}\Lambda_- R - \frac{z-1}{z^3}(3i\tilde{k} + \lambda)R = 0. \tag{3.13}$$

By the multiplication transformation $Y = z^{-3}R$, the differential equation for Y becomes

$$\begin{aligned} &\Lambda_+\Lambda_- Y + \frac{2(2z-3)}{z^2}\Lambda_- Y \\ &\quad - \left(1 - \frac{1}{z}\right)\left(\frac{\lambda - 3i\mathcal{B}\epsilon}{z^2} + \frac{3 - 6i\mathcal{C}}{z^3}\right)Y = 0. \end{aligned} \tag{3.14}$$

From the above, A and B can be read off as

$$\begin{aligned} A &= \frac{\alpha^{-1}c_0}{z^4}\left(1 - \frac{1}{z}\right)^2, \\ B &= \alpha^{-1}\left(1 - \frac{1}{z}\right)\left(\frac{\lambda - 3i\mathcal{B}\epsilon}{z^2} + \frac{3 - 6i\mathcal{C}}{z^3}\right), \end{aligned} \tag{3.15}$$

where c_0 is constant. G is computed from (2.9) as

$$\begin{aligned} \alpha G &= \frac{2ik}{z^2} + \frac{2z-3}{z^2} \\ &\quad + (c_0 - 3 + 6i\mathcal{C})\frac{z-1}{z^3}\left(\lambda - 3i\mathcal{B}\epsilon + \frac{3-6i\mathcal{C}}{z}\right)^{-1}. \end{aligned} \tag{3.16}$$

By choosing $c_0 = 3\text{–}6i\mathcal{C}$, the above can be simplified as

$$\begin{aligned} \alpha G &= \frac{2ik}{z^2} + \frac{2z-3}{z^2} \\ &= 2i\mathcal{A}\epsilon + \frac{2 + 2i\mathcal{B}\epsilon}{z} + \frac{-3 + 2i\mathcal{C}}{z^2}. \end{aligned} \tag{3.17}$$

We take the ansatz for F as

$$F = \alpha^{-1}V_X = \left(1 - \frac{1}{z}\right)\left(\frac{\lambda + \beta - 3i\mathcal{B}\epsilon}{z^2} - \frac{3 - 6i\mathcal{C}}{z^3}\right). \tag{3.18}$$

The conditions (2.6) and (2.7) fix the constants α and β as

$$\alpha = \frac{1 - \frac{2}{3}i\mathcal{C}}{1 - 2i\mathcal{C}}, \qquad \beta = 2(1 + i\mathcal{B}\epsilon)\alpha^{-1}. \tag{3.19}$$

V_X is obtained as

$$V_X = \left(1 - \frac{1}{z}\right)\left(\frac{\alpha\lambda + 2 + (2 - 3\alpha)i\mathcal{B}\epsilon}{z^2} - \frac{3 - 2i\mathcal{C}}{z^3}\right). \tag{3.20}$$

Then the resulting differential equation for X is of the form

$$\left[\left(\frac{z-1}{z}\frac{d}{dz}\right)^2 + p^2 - \left(1 - \frac{1}{z}\right)\left(\frac{\tilde{\lambda} + 2}{z^2} - \frac{3 - 4i\mathcal{C}}{z^3}\right)\right]X = 0, \tag{3.21}$$

where $p = k/z^2$ and $\tilde{\lambda}$ is defined by

$$\tilde{\lambda} = \alpha\lambda + 3i(1 - \alpha)\mathcal{B}\epsilon. \tag{3.22}$$

Note that (3.21) is reduced to the Regge-Wheeler equation under the limit $a \to 0$.

The behavior of the solution for the transformed equation at the boundary and how it is related to that for the Teukolsky equation can also be examined. The solution of the Teukolsky equation (3.7) at the boundary behaves as

$$R \sim \begin{cases} B_{\text{in}} z^{-1} e^{-i\mathcal{A}\epsilon z^*} + B_{\text{out}} z^3 e^{i\mathcal{A}\epsilon z^*} & \text{for } z \to \infty (z^* \to \infty), \\ \bar{B}_{\text{in}} z^2 (z-1)^2 e^{-i(\epsilon+\mathcal{C})z^*} + \bar{B}_{\text{out}} e^{i(\epsilon+\mathcal{C})z^*} & \text{for } z \to 1 (z^* \to -\infty), \end{cases} \tag{3.23}$$

where $B_{\rm in}$, $B_{\rm out}$, $\bar{B}_{\rm in}$, and $\bar{B}_{\rm out}$ are all constant. On the other hand, the solution of the transformed equation (3.21) at the boundary behaves as

$$X \sim \begin{cases} A_{\rm in} e^{-i\mathcal{A}\epsilon z^*} + A_{\rm out} e^{i\mathcal{A}\epsilon z^*} & \text{for } z \to \infty (z^* \to \infty), \\ \bar{A}_{\rm in} e^{-i(\epsilon+\mathcal{C})z^*} + \bar{A}_{\rm out} e^{i(\epsilon+\mathcal{C})z^*} & \text{for } z \to 1 (z^* \to -\infty), \end{cases} \tag{3.24}$$

where $A_{\rm in}$, $A_{\rm out}$, $\bar{A}_{\rm in}$, and $\bar{A}_{\rm out}$ are all constant. From the Chandrasekhar transformation (2.3) with $Y = z^{-3}R$, the relations between these coefficients can be found as

$$A_{\rm in} = -4\mathcal{A}^2\epsilon^2\alpha\zeta^{-1}B_{\rm in}, \tag{3.25}$$

$$A_{\rm out} = (-4\mathcal{A}^2\epsilon^2)^{-1}\alpha B_{\rm out}, \tag{3.26}$$

$$\bar{A}_{\rm in} = 2\alpha\zeta^{-1}[1-2i(\epsilon+\mathcal{C})][1-i(\epsilon+\mathcal{C})]\bar{B}_{\rm in}, \tag{3.27}$$

$$\bar{A}_{\rm out} = \frac{i}{2}(\epsilon+\mathcal{C})^{-1}[1-2i(\epsilon+\mathcal{C})]^{-1}\bar{B}_{\rm out}. \tag{3.28}$$

Here the constant ζ is given by

$$\zeta = \alpha^{-1}(\tilde{\lambda} - 3i\mathcal{B}\epsilon)(\tilde{\lambda} + 2 - i\mathcal{B}\epsilon) - 6i\mathcal{A}(1-2i\mathcal{C})\epsilon, \tag{3.29}$$

which is reduced to $(l-1)l(l+1)(l+2) - 6i\epsilon$ under the limit $a \to 0$. The relations (3.25)–(3.28) imply that at each boundary the "in" mode and the "out" mode are not mixed under the Chandrasekhar transformation. Then the boundary condition for no energy inflow in the Teukolsky equation $B_{\rm in} = \bar{B}_{\rm out} = 0$ is mapped to that in the transformed equation $A_{\rm in} = \bar{A}_{\rm out} = 0$ unless the frequency ϵ is equal to the zeroes or the poles of the coefficients in (3.25) and (3.28). Therefore, we can conclude that the spectra of the quasinormal modes in the Teukolsky equation and those in the transformed equation coincide generically.

IV. COMPARISON WITH QUANTUM SEIBERG-WITTEN GEOMETRY

As well as in the Regge-Wheeler and the Teukolsky equations, the (confluent) Heun's equation also appears in the quantization of the Seiberg-Witten curves in supersymmetric gauge theories. For example in $\mathcal{N}=2$ supersymmetric SU(2) gauge theory coupled with three matter hypermultiplets in the fundamental representation of the gauge group, the quantum Seiberg-Witten geometry gives the following differential equation [17,19–22]

$$\left[\hbar^2 \frac{d^2}{dz^2} + \frac{q(z)}{z^2(z-1)^2}\right]\Psi(z) = 0, \tag{4.1}$$

where $\hbar$ is the quantization parameter (hereafter chosen as unity) and $q(z)$ is the quartic polynomial of z as

$$q(z) = \hat{A}_0 + \hat{A}_1 z + \hat{A}_2 z^2 + \hat{A}_3 z^3 + \hat{A}_4 z^4, \tag{4.2}$$

$$\begin{aligned} \hat{A}_0 &= -\frac{(m_1-m_2)^2}{4} + \frac{\hbar^2}{4}, \\ \hat{A}_1 &= -E - m_1 m_2 - \frac{m_3\Lambda_3}{8} - \frac{\hbar^2}{4}, \\ \hat{A}_2 &= E + \frac{3m_3\Lambda_3}{8} - \frac{\Lambda_3^2}{64} + \frac{\hbar^2}{4}, \\ \hat{A}_3 &= -\frac{m_3\Lambda_3}{4} + \frac{\Lambda_3^2}{32}, \\ \hat{A}_4 &= -\frac{\Lambda_3^2}{64}. \end{aligned} \tag{4.3}$$

Here m_1, m_2, and m_3 are the masses of the matter hypermultiplets, E is a moduli parameter and Λ_3 is the dynamical scale. For generic choice of the parameters, (4.1) is the form of CHE. Note that the symmetry under the exchange between m_1 and m_2 is manifest because $q(z)$ is unchanged, on the other hand the symmetry under the exchange between m_3 and another mass is not manifest[3] and the form of the differential equation is changed. However the origin of the parameters suggests that they describe the same physics, and hence there has to exist some correspondence between them. By this reason, we will use the above parametrization (4.3) instead of the standard parametrization [12] of CHE. For the relation between these parametrizations, see [20].

The Regge-Wheeler equation and the Teukolsky equation can be mapped as the form of (4.1) by the multiplication transformation. In the case of Schwarzschild spacetime, the correspondence of the parameters are given by[4]

$$\begin{aligned} &\hbar = 1, \qquad \Lambda_3 = 8i\epsilon, \qquad E = -l(l+1) + 2\epsilon^2 - \frac{1}{4}, \\ &m_1 = -2 + i\epsilon, \qquad m_2 = 2 + i\epsilon, \qquad m_3 = i\epsilon, \end{aligned} \tag{4.4}$$

for the Regge-Wheeler equation (2.16) and

$$\begin{aligned} &\hbar = 1, \qquad \Lambda_3 = 8i\epsilon, \qquad E = -l(l+1) + 2\epsilon^2 - \frac{1}{4}, \\ &m_1 = -2 + i\epsilon, \qquad m_2 = i\epsilon, \qquad m_3 = 2 + i\epsilon, \end{aligned} \tag{4.5}$$

for the Teukolsky equation (2.17). By comparing (4.4) and (4.5) one can find that the only difference is the exchange between m_2 and m_3. And as expected, the solutions to those two equations are related by the Chandrasekhar transformation (2.25). In the Teukolsky equation in Kerr spacetime, the correspondence of the parameters are computed as

[3] In the solution with low frequency expansion [13], this symmetry becomes manifest [14].

[4] In the correspondence hereafter, there are three double signs appear in general [20]. We have fixed these signs in our convenience.

$$\hbar = 1, \qquad \Lambda_3 = 8i\mathcal{A}\epsilon, \qquad E = -\lambda - 2 + 2\epsilon^2 - \frac{am}{M}\epsilon - \frac{1}{4},$$
$$m_1 = -2 + i\epsilon, \qquad m_2 = i\epsilon + 2i\mathcal{C}, \qquad m_3 = 2 + i\epsilon. \quad (4.6)$$

On the other hand, the correspondence of the parameters for the differential equation (3.21) becomes

$$\hbar = 1, \qquad \Lambda_3 = 8i\mathcal{A}\epsilon, \qquad E = -\tilde{\lambda} - 2 + 2\epsilon^2 - \frac{am}{M}\epsilon - \frac{1}{4},$$
$$m_1 = -2 + i\epsilon + 2i\mathcal{C}, \qquad m_2 = 2 + i\epsilon, \qquad m_3 = i\epsilon. \quad (4.7)$$

By comparing (4.6) and (4.7), one can find that it is not only the exchange between m_2 and m_3, but also $2i\mathcal{C}$ is moved to m_1, and λ is replaced with $\tilde{\lambda}$. Note that the Regge-Wheeler-type equation which has the parameters in (4.6) with the exchange between m_2 and m_3 is given in [22]. However, that equation is found just by the exchange of the parameters, not by the Chandresekhar(-like) transformation.

V. SUMMARY AND DISCUSSION

In this paper, we have proposed the new kind of the Chandrasekhar transformation for the Teukolsky equation in Kerr spacetime, which reduces to the original work of Chandrasekhar under the limit $a \to 0$. The original Chandrasekhar transformation had been obtained from the point view of the gauge transformation for the linearized Einstein equation. We could expect that our transformation would have a similar origin, and it would be interesting to find it. One can also find that obviously there could be other transformation by different choices of the variable x, the differential operators $\Lambda_\pm$ and the multiplication transformation and the functions F and G. Well-known examples are of course the (Chandrasekhar-) Detweiler equation and the Sasaki-Nakamura equation [6–8], where the tortoise coordinate (3.11) is used. Instead, here we have used the coordinate z^* defined by (2.19) from (3.6), in order to keep the structure of the singularities. We have obtained the differential equation (3.21), which is also reduced to the Regge-Wheeler equation in the limit $a \to 0$ as the (Chandrasekhar-) Detweiler and the Sasaki-Nakamura equations are. The extension to the case of different spins (the scalar waves and the electromagnetic waves) [25–27] would also be interesting. We have also considered the behavior of the solution for the transformed equation at the boundary (far infinity and the outer horizon) and have shown that the spectra of the quasinormal modes in the transformed equation are generically the same as those in the Teukolsky equation. The explicit numerical evaluation of the quasinormal modes and comparison with the results from another method [20–22,28–31] would be interesting.

We have also given the interpretation of our transformation using the formalism which is recently proposed in the study of supersymmetric gauge theories. The Regge-Wheeler equation and the Teukolsky equation are examples of CHE, which also appears as the wave equation for the quantum Seiberg-Witten geometry. It turns out that our transformation is more nontrivial than the exchange of the mass parameters. Then at present there is no explanation why the spectra has to be the same in the side of the quantum Seiberg-Witten geometry and we may have to consider more nontrivial transformation as in [32,33]. A similar analysis for the (Chandrasekhar-)Detweiler equation and the Sasaki-Nakamura equation would also be useful.

Another possible generalization is to include the cosmological constant. The Regge-Wheeler and the Teukolsky equations in the background of the Kerr-de Sitter black hole are examples of the Heun's equation (HE) [34], which has four regular singularities. Since HE does not have the irregular singularity, it is slightly easier to handle it than CHE. The local solution of HE is included in *Mathematica*, as well as that of CHE. Some problems about evaporation and scattering are also discussed without approximation [35,36]. These exact analyses would help to study the problem with the arbitrary frequency, which is important for application of the scattering of the gravitational (electromagnetic or scalar) waves to more general cases.

ACKNOWLEDGMENTS

H. N. thanks Sung-Soo Kim, Xiaobin Li, and in particular Futoshi Yagi for valuable comments and discussions. This work was supported in part by the National Natural Science Foundation of China (Grant No. 11973025).

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