Anomalous dispersion in gravity theories

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A wave pulse (be it a gravitational wave or a light wave) undergoes anomalous dispersion in a vacuum in flat spacetimes with an even number of spatial dimensions even if all the frequencies move at the same speed. Such an anomalous dispersion does not occur in spacetimes with an odd number of spatial dimensions. We study various gravity theories and show that dispersion-free propagation is possible in even number of spatial dimensions if the background is not the Minkowski but the de Sitter spacetime and the gravity theory is massive gravity with a tuned mass in terms of the cosmological constant. Mass and the cosmological constant conspire to get rid of the anomalous dispersion and restore Huygens's principle.

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I. INTRODUCTION

Recently [1] it was shown that the wave equation of a massive Klein-Gordon field with a tuned mass $m = \sqrt{\Lambda}$ in a $(2+1)$-dimensional de Sitter spacetime, with a positive cosmological Λ allows dispersionless propagation. Namely, an initial wave pulse does not broaden and change shape when it propagates in this background. This result, *a prirori*, is counterintuitive since it is well known that wave pulses in even spatial dimensions undergo anomalous (dimension-dependent) dispersion even if all modes propagate at the same speed [2]; and massive wave equations in all dimensions show (regular) dispersion as there is always propagation inside the light-cone due to the fact that the group velocity depends on the wave number of the individual waves constructing the pulse. But it turns out that these two effects help each other eliminate the anomalous dispersion in certain cases.

The result of [1] was inspired by two works: in [3] it was shown that adding one more *timelike* dimension to the $(2+1)$ flat spacetime, namely, considering a massless wave equation in a $(2+2)$-dimensional world, one has the possibility of dispersion-free propagation. This is rather surprising since we know that adding a spacelike dimension removes the anomalous dispersion, but even an extra timelike direction, albeit physically so removed from a spacelike direction, seems to do the job of removing the anomalous dispersion. Even though spacetimes with *two time* dimensions appear in theoretical physics [4], one would feel much pleased if an experimental effective model appears to have two time directions. This indeed happens in some hyperbolic metamaterials [5,6]: in a nondispersive, nonmagnetic, uniaxial anisotropic metamaterial (which can be constructed in a lab) the extraordinary (nontransverse) component of the electric field obeys a *massless* Klein-Gordon wave equation in a flat $(2+2)$-dimensional spacetime. In [3] this massless wave equation with constant coefficients in Cartesian coordinates was shown to be equivalent to a *modified* wave equation with time-dependent coefficients. Then this modified wave equation was shown to allow particular initial wave pulses to propagate without dispersion in vacuum. The modified wave equation introduced in [3] is somewhat *ad hoc* and the initial data chosen is rather specific. An explanation was given in [1]: it was shown that the modified wave equation exactly corresponds to a massive scalar field in a $(2+1)$-dimensional de Sitter spacetime with a tuned mass. Therefore the mentioned hyperbolic metamaterial acts like a $(2+1)$-dimensional de Sitter background and dispersion-free propagation is possible for a generic wave pulse if the mass is tuned to the cosmological constant.

In this work, building on these considerations, we give a detailed account of propagation of massless and massive gravity waves in generic $D = d + 1$ dimensions. These waves will be gravity waves defined in the weak field limit. It will turn out that in de Sitter backgrounds, massive fields with tuned masses allow dispersion-free propagation generalizing the results of [1].

The layout of the paper is as follows: In Sec. II, we study the D dimensional massless gravity (general relativity) in some detail to set up the formalism and to see the anomalous dispersion in the behavior of the spacetime Green's functions. In Sec. III, Fierz-Pauli massive gravity is studied in a flat spacetime background. In Sec. IV,

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D dimensional quadratic gravity is studied in a flat spacetime background; and in Sec. V, $2+1$ dimensional topologically massive gravity is studied. In Sec. VI, $2+1$ dimensional new massive gravity and massive Klein-Gordon fields in a D-dimensional de Sitter background are studied. The computations are straightforward but rather lengthy; we have provided some of the details of the computations in the Appendixes A and B.

II. MASSLESS GRAVITY IN $D=d+1$ DIMENSIONS

As it will be our guiding theory, we shall study the D dimensional massless gravity in some detail. Here we will give the background expansions of the relevant tensors that will also appear in various massive gravity theories studied in other sections. In $(d+1)$ dimensions, the Einstein-Hilbert action reads

$$I=\frac{1}{2\kappa}\int d^{d+1}x\sqrt{-g}R. \tag{1}$$

To compute the Green's function of the linearized theory around the flat spacetime, let us expand the action up to the second order in the metric fluctuations using

$$g_{\mu\nu}\coloneqq\bar{g}_{\mu\nu}+\tau h_{\mu\nu}, \tag{2}$$

where τ is a small expansion parameter, $\bar{g}_{\mu\nu}$ denotes the flat background spacetime metric in some coordinates. The inverse metric yields

$$g^{\mu\nu}=\bar{g}^{\mu\nu}-\tau h^{\mu\nu}+\tau^2 h^{\mu\sigma}h^\nu_\sigma+\mathcal{O}(\tau^3). \tag{3}$$

One also has the expansion of the square root of the determinant of the metric as

$$\sqrt{-g}=\sqrt{-\bar{g}}\left(1+\tau\frac{h}{2}+\tau^2\frac{1}{8}(h^2-2h^2_{\mu\nu})\right), \tag{4}$$

where $h^2_{\mu\nu}=h_{\mu\nu}h^{\mu\nu}$. Expansion of the metric yields an expansion of tensors that depends on the metric. In particular, the scalar curvature at the desired order becomes

$$R=\bar{R}+\tau(R)^{(1)}+\frac{\tau^2}{2}(R)^{(2)}, \tag{5}$$

where the first and the second order terms can be found to be

$$\begin{aligned}(R)^{(1)}&=\bar{g}^{\mu\nu}(R_{\mu\nu})^{(1)}-h^{\mu\nu}\bar{R}_{\mu\nu},\\ (R)^{(2)}&=\bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)}-2h^{\mu\nu}(R_{\mu\nu})^{(1)}+2h^{\mu\sigma}h^\nu_\sigma\bar{R}_{\mu\nu}.\end{aligned} \tag{6}$$

The Ricci tensor at the first order can be computed to be

$$(R_{\mu\nu})^{(1)}=\frac{1}{2}(\bar{\nabla}_\sigma\bar{\nabla}_\mu h^\sigma_\nu+\bar{\nabla}_\sigma\bar{\nabla}_\nu h^\sigma_\mu-\bar{\Box}h_{\mu\nu}-\bar{\nabla}_\mu\bar{\nabla}_\nu h), \tag{7}$$

where $\bar{\nabla}_\mu$ denotes the background metric compatible covariant derivative and $\bar{\Box}\coloneqq\bar{\nabla}_\mu\bar{\nabla}^\mu$. So the linearized scalar curvature becomes

$$(R)^{(1)}=\bar{\nabla}_\sigma\bar{\nabla}_\lambda h^{\sigma\lambda}-\bar{\Box}h-h^{\mu\nu}\bar{R}_{\mu\nu}, \tag{8}$$

while the second order Ricci tensor is more complicated:

$$\begin{aligned}(R_{\mu\nu})^{(2)}=&\bar{\nabla}_\sigma(\Gamma^\sigma_{\mu\nu})^{(2)}-\bar{\nabla}_\nu(\Gamma^\sigma_{\mu\sigma})^{(2)}+2(\Gamma^\sigma_{\sigma\lambda})^{(1)}(\Gamma^\lambda_{\mu\nu})^{(1)}\\ &-2(\Gamma^\sigma_{\nu\lambda})^{(1)}(\Gamma^\lambda_{\mu\sigma})^{(1)},\end{aligned} \tag{9}$$

where $(\Gamma^\lambda_{\mu\nu})^{(1)}$ denotes the first order Christoffel connection that reads as

$$(\Gamma^\lambda_{\mu\nu})^{(1)}=\frac{1}{2}\bar{g}^{\lambda\rho}(\bar{\nabla}_\mu h_{\nu\rho}+\bar{\nabla}_\nu h_{\mu\rho}-\bar{\nabla}_\rho h_{\mu\nu}). \tag{10}$$

$(\Gamma^\sigma_{\mu\nu})^{(2)}$ is the second order Christoffel connection of which the explicit form is not needed. Now we can expand the Einstein-Hilbert action (1) as

$$I=\bar{I}+\tau(I)^{(1)}+\frac{\tau^2}{2}(I)^{(2)}. \tag{11}$$

After making use of the above results, the second order term boils down to

$$\begin{aligned}(I)^{(2)}=&\frac{1}{2\kappa}\int d^{d+1}x\sqrt{-\bar{g}}(\bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)}-2h^{\mu\nu}(R_{\mu\nu})^{(1)}\\ &+2h^{\mu\sigma}h^\nu_\sigma\bar{R}_{\mu\nu}+h\left((R)^{(1)}+\frac{1}{4}\bar{R}(h^2-2h^2_{\mu\nu})\right).\end{aligned} \tag{12}$$

This expression is valid for a generic background metric; let us now consider the flat spacetime with Cartesian coordinates and take $\bar{g}_{\mu\nu}=\eta_{\mu\nu}$, $\bar{\nabla}_\mu=\partial_\mu$ and $\bar{R}_{\mu\nu}=0=\bar{R}$. Then (12) becomes

$$(I)^{(2)}=\frac{1}{2\kappa}\int d^{d+1}x(\bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)}-2h^{\mu\nu}(R_{\mu\nu})^{(1)}+h(R)^{(1)}), \tag{13}$$

which, making use of the linearized Einstein tensor

$$\begin{aligned}(G_{\mu\nu})^{(1)}&=(R_{\mu\nu})^{(1)}-\frac{1}{2}\bar{g}_{\mu\nu}(R)^{(1)}-\frac{1}{2}h_{\mu\nu}\bar{R}\\ &=(R_{\mu\nu})^{(1)}-\frac{1}{2}\bar{g}_{\mu\nu}(R)^{(1)},\end{aligned} \tag{14}$$

reduces to

$$(I)^{(2)} = \frac{1}{2\kappa}\int d^{d+1}x(\bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)} - 2h^{\mu\nu}(G_{\mu\nu})^{(1)}). \quad (15)$$

One can proceed in a gauge-invariant way, but here we impose the harmonic gauge to simplify the ensuing expressions. Then assuming

$$\partial_\mu h^\mu_\sigma = \frac{1}{2}\partial_\sigma h, \quad (16)$$

the linearized Einstein tensor becomes

$$(G_{\mu\nu})^{(1)} = \frac{1}{4}(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})\partial^2 h^{\alpha\beta}. \quad (17)$$

Dropping the boundary terms, in the harmonic gauge, one has $\bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)} = h^{\mu\nu}(G_{\mu\nu})^{(1)}$, and (15) reduces to

$$(I)^{(2)} = -\frac{1}{2\kappa}\int d^{d+1}x h^{\mu\nu}(G_{\mu\nu})^{(1)}. \quad (18)$$

This can be written as

$$(I)^{(2)} = \frac{1}{4\kappa}\int d^{d+1}x h^{\mu\nu}\mathcal{O}_{\mu\nu\alpha\beta}(x)h^{\alpha\beta}, \quad (19)$$

with the formally self-adjoint operator given as

$$\mathcal{O}_{\mu\nu\alpha\beta}(x) = -\frac{1}{2}(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})\partial^2. \quad (20)$$

Green's function is the inverse of the operator $\mathcal{O}_{\mu\nu\alpha\beta}$, under the assumed (sufficient decay at infinity) boundary conditions, hence one must solve the equation

$$\mathcal{O}_{\mu\nu\alpha\beta}(x)G^{\alpha\beta\lambda\tau}(x,x') = \frac{1}{2}(\delta^\lambda_\mu\delta^\tau_\nu + \delta^\lambda_\nu\delta^\tau_\mu)\delta^{(d+1)}(x-x'), \quad (21)$$

which, in the momentum space, reads as

$$\tilde{\mathcal{O}}_{\mu\nu\alpha\beta}(p)\tilde{G}^{\alpha\beta\lambda\tau}(p) = \frac{1}{2}(\delta^\lambda_\mu\delta^\tau_\nu + \delta^\lambda_\nu\delta^\tau_\mu). \quad (22)$$

$\tilde{\mathcal{O}}_{\mu\nu\alpha\beta}$ can be obtained from (20) by replacing ∂_μ with ip_μ to get

$$\tilde{\mathcal{O}}_{\mu\nu\alpha\beta}(p) = \frac{p^2}{2}(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}). \quad (23)$$

Then the solution satisfying (22) is

$$\tilde{G}^{\alpha\beta\lambda\tau}(p) = -\frac{1}{2p^2}\left(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda} - \frac{2\eta^{\alpha\beta}\eta^{\lambda\tau}}{d-1}\right). \quad (24)$$

The position space Green's function can be obtained from the Fourier transform

$$G^{\alpha\beta\lambda\tau}(x,x') = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} e^{-ip\cdot(x-x')}\tilde{G}^{\alpha\beta\lambda\tau}(p), \quad (25)$$

which reads

$$G^{\alpha\beta\lambda\tau}(x,x') = -\frac{1}{2}\left(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda} - \frac{2\eta^{\alpha\beta}\eta^{\lambda\tau}}{d-1}\right) \times \int \frac{d^{d+1}p}{(2\pi)^{d+1}} e^{-ip\cdot(x-x')}\frac{1}{p^2}. \quad (26)$$

We are looking for the retarded Green's function, the poles should be displaced as such, and the result of the integral depends on the number of dimensions. For $d \geq 2$, defining $t \coloneqq t - t'$ and $r \coloneqq |\vec{x} - \vec{x}'|$, one arrives at (see Appendix A for details)

$$G^{\alpha\beta\lambda\tau}(t,r) = -\frac{1}{2}\left(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda} - \frac{2\eta^{\alpha\beta}\eta^{\lambda\tau}}{d-1}\right) \times \begin{cases} \frac{1}{4\pi}(-\frac{1}{2\pi r}\partial_r)^{\frac{d-3}{2}}\frac{\delta(t-r)}{r}: & \text{for odd } d, \\ \frac{\theta(t)}{2\pi}(-\frac{1}{2\pi r}\partial_r)^{\frac{d}{2}-1}\frac{\theta(t-r)}{\sqrt{t^2-r^2}}: & \text{for even } d. \end{cases} \quad (27)$$

For odd d, the Green's function is nonzero only for null separation and hence there is no tail inside the light cone. On the other hand, for even d, even though the Green's function is peaked around the null separation due to the appearance of the function $\frac{1}{\sqrt{t^2-r^2}}$, there is a tail inside the light cone. Hence, even a delta-function initial wave is dispersed and one has anomalous dispersion of gravitational waves. Note that, among the odd spatial dimensions, $d = 3$, our world is special since only for this dimension, there is no derivative on the delta function, hence the delta function pulse at $t = 0$ remains a delta function, only shifted to a new location, at all points and for all times.

III. FIERZ-PAULI MASSIVE GRAVITY IN $D = d + 1$ DIMENSIONS

The linearized Fierz-Pauli action, that has $\frac{(D+1)(D-2)}{2}$ degrees of freedom in a flat spacetime background, is

$$I = \frac{1}{2\kappa}\int d^{d+1}x\left(-\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} + \partial^\nu h^{\lambda\mu}\partial_\mu h_{\lambda\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{m^2}{2}(h^2_{\mu\nu} - h^2)\right). \quad (28)$$

The field equations coming from this action

$$\partial^2 h_{\mu\nu} - \partial_\lambda\partial_\nu h^\lambda_\mu - \partial_\lambda\partial_\mu h^\lambda_\nu + \partial_\nu\partial_\mu h + \eta_{\mu\nu}\partial_\sigma\partial_\lambda h^{\sigma\lambda} - \eta_{\mu\nu}\partial^2 h = m^2(h_{\mu\nu} - \eta_{\mu\nu}h) \quad (29)$$

can be recast as three equations

$$(\partial^2 - m^2)h_{\mu\nu} = 0, \qquad \partial_\mu h^{\mu\nu} = 0, \qquad h = 0. \tag{30}$$

The action (28) up to boundary terms reads

$$I = \frac{1}{4\kappa}\int d^{d+1}x h^{\mu\nu}\mathcal{O}_{\mu\nu\alpha\beta}(x)h^{\alpha\beta}, \tag{31}$$

with

$$\begin{aligned}\mathcal{O}_{\mu\nu\alpha\beta}(x) = &\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})(\partial^2 - m^2) \\ &+ \eta_{\alpha\beta}(\partial_\mu\partial_\nu - \eta_{\mu\nu}(\partial^2 - m^2)) + \eta_{\mu\nu}\partial_\alpha\partial_\beta \\ &- \frac{1}{2}(\eta_{\mu\beta}\partial_\alpha\partial_\nu + \eta_{\mu\alpha}\partial_\alpha\partial_\nu + \eta_{\nu\beta}\partial_\alpha\partial_\mu + \eta_{\nu\alpha}\partial_\alpha\partial_\mu).\end{aligned} \tag{32}$$

Following the steps in the previous section verbatim, one arrives at the momentum space Green's function

$$\tilde{G}^{\alpha\beta\sigma\lambda}(p) = -\frac{1}{2(p^2 + m^2)}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \frac{2}{d}\eta^{\alpha\beta}\eta^{\sigma\lambda}\right), \tag{33}$$

in which we dropped the terms proportional to p^α etc. as they do not contribute to any calculation for which the energy-momentum tensor is conserved ($p^\alpha T_{\alpha\beta} = 0$). Then we have the position space Green's function

$$\begin{aligned}G^{\alpha\beta\sigma\lambda}(x, x') = &-\frac{1}{2}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \frac{2}{d}\eta^{\alpha\beta}\eta^{\sigma\lambda}\right) \\ &\times \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2 + m^2}.\end{aligned} \tag{34}$$

Once again the results of this integral differ for odd and even d (see Appendix B for discussion). For odd d, one has

$$\begin{aligned}G^{\alpha\beta\sigma\lambda}_{\text{odd}\,d}(t, r) = &-\frac{1}{2}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \frac{2}{d}\eta^{\alpha\beta}\eta^{\sigma\lambda}\right) \\ &\times \frac{\Theta(t)}{2}\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{\frac{d-1}{2}} \\ &\times \left(J_0\left(m\sqrt{t^2 - r^2}\right)\Theta(t - r)\right),\end{aligned} \tag{35}$$

where J_0 is the Bessel function. For example for $d = 3$, one gets

$$\begin{aligned}G^{\alpha\beta\sigma\lambda}(t, r) = &-\frac{1}{2}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \frac{2}{3}\eta^{\alpha\beta}\eta^{\sigma\lambda}\right) \\ &\times \frac{\Theta(t)}{2}\left(-\frac{1}{2\pi r}\frac{d}{dr}\right) \\ &\times \left(J_0\left(m\sqrt{t^2 - r^2}\right)\Theta(t - r)\right).\end{aligned} \tag{36}$$

The derivative part yields

$$\begin{aligned}&\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)\left(J_0\left(m\sqrt{t^2 - r^2}\right)\Theta(t - r)\right) \\ &= \frac{\delta(t - r)J_0\left(m\sqrt{t^2 - r^2}\right)}{2\pi r} - \frac{m\theta(t - r)J_1\left(m\sqrt{t^2 - r^2}\right)}{2\pi\sqrt{t^2 - r^2}}.\end{aligned} \tag{37}$$

In the $m \to 0$ limit, this last equation gives the expected result $\frac{\delta(t-r)}{2\pi r}$; but (36) does not smoothly reduce to the corresponding $d = 3$ case of (27) due to the discrete difference in the third terms in the first brackets. This is the well-known van-Dam-Veltman-Zakharov discontinuity. Generically, as expected, in flat space for nonzero m, there is a tail inside the light cone and the retarded Green's function has support inside the light cone.

For even d, one has

$$\begin{aligned}G^{\alpha\beta\sigma\lambda}_{\text{even}\,d}(t, r) = &-\frac{1}{2}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \frac{2}{d}\eta^{\alpha\beta}\eta^{\sigma\lambda}\right) \\ &\times \frac{\Theta(t)}{2\pi}\left(\frac{-1}{2\pi r}\frac{d}{dr}\right)^{\frac{d-2}{2}}\left(\frac{\cos(m\sqrt{t^2 - r^2})\Theta(t - r)}{\sqrt{t^2 - r^2}}\right).\end{aligned} \tag{38}$$

For $d = 2$, this yields

$$\begin{aligned}G^{\alpha\beta\sigma\lambda}(t, r) = &-\frac{1}{2}(\eta^{\alpha\sigma}\eta^{\beta\lambda} + \eta^{\alpha\lambda}\eta^{\beta\sigma} - \eta^{\alpha\beta}\eta^{\sigma\lambda}) \\ &\times \frac{\Theta(t)}{2\pi}\cos\left(m\sqrt{t^2 - r^2}\right)\frac{\Theta(t - r)}{\sqrt{t^2 - r^2}}.\end{aligned} \tag{39}$$

For the even d case, there is a support inside the light cone and the Huygens's principle is violated. These results are expected in flat spacetime for massive fields.

IV. QUADRATIC CURVATURE GRAVITY IN $D = d + 1$ DIMENSIONS

We consider the following quadratic gravity action[1]

$$I_{\text{quad}} = \frac{1}{2\kappa}\int d^{d+1}x\sqrt{-g}(\sigma R + \alpha R^2 + \beta R_{\mu\nu}^2), \tag{40}$$

from which the second order action in the harmonic gauge can be found to be

$$(I_{\text{quad}})^{(2)} = \frac{1}{4\kappa}\int d^{d+1}x h^{\mu\nu}\mathcal{O}_{\mu\nu\alpha\beta}(x)h^{\alpha\beta}, \tag{41}$$

[1]We do not consider the $R^2_{\mu\nu\alpha\beta}$ term, since at the end we would like to study the particular $2 + 1$ dimensional gravity for which this term only shifts the parameters in the Lagrangian.

where the inverse propagator is a fourth order operator

$$\mathcal{O}_{\mu\nu\alpha\beta}(x) = \eta_{\mu\nu}\eta_{\alpha\beta}\left(\left(2\alpha + \frac{\beta}{2}\right)\partial^2 - \frac{\sigma}{2}\right)\partial^2 - (2\alpha+\beta)\eta_{\alpha\beta}\partial_\mu\partial_\nu\partial^2 + \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})(\sigma + \beta\partial^2)\partial^2. \tag{42}$$

From the Fourier transform of this operator, one can find the Green's function in the momentum space following the similar steps as in the second section. The Green's function satisfying (22) reads in the momentum space as

$$\begin{aligned}\tilde{G}^{\alpha\beta\lambda\tau}(p) = &\left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{\sigma(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{4\alpha+2\beta}{\sigma^2(d-1)}\eta^{\lambda\tau}p^\beta p^\alpha\right)\frac{1}{p^2}\\ &+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{\sigma d}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{\beta}{\sigma^2 d}\eta^{\lambda\tau}p^\beta p^\alpha\right)\frac{1}{p^2 - \sigma/\beta}\\ &+\left(-\frac{1}{\sigma d(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha d + \beta(d+1)}{\sigma^2 d(d-1)}\eta^{\lambda\tau}p^\beta p^\alpha\right)\frac{1}{p^2 + \frac{\sigma(d-1)}{4\alpha d+\beta(d+1)}}.\end{aligned} \tag{43}$$

From the pole structure, one can read the masses of the excitations: there is a massless spin 2 particle, there is a massive spin 2 particle with a mass $m_g^2 = -\frac{\sigma}{\beta}$, and there is a massive scalar mode with a mass $m_s^2 = \frac{\sigma(d-1)}{4\alpha d+\beta(d+1)}$. A complimentary study of these in (anti-)de Sitter spacetimes can be found in [7]: the masses get nontrivial contributions from the nonzero constant curvature background. In generic D dimensions, this theory has a massive ghost [8] which only disappears for $D = 3$ in a particular tuning of α and β which we shall study in the next section.

To get the retarded Green's function in the position space, we have to do the following integrals

$$\begin{aligned}G^{\alpha\beta\lambda\tau}(x,x') = &\left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{\sigma(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha+2\beta}{\sigma^2(d-1)}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2}\\ &+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{\sigma d}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{\beta}{\sigma^2 d}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2-\sigma/\beta}\\ &-\left(\frac{1}{\sigma d(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha d+\beta(d+1)}{\sigma^2 d(d-1)}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2 + \frac{\sigma(d-1)}{4\alpha d+\beta(d+1)}},\end{aligned} \tag{44}$$

which again should be studied in odd and even d separately.

i: Odd d case

$$\begin{aligned}G^{\alpha\beta\lambda\tau}(t,r) = &\left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{\sigma(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha+2\beta}{\sigma^2(d-1)}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{4\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-3)/2}\frac{\delta(t-r)}{r}\\ &+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{\sigma d}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{\beta}{\sigma^2 d}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2}J_0\left(m_g\sqrt{t^2-r^2}\right)\Theta(t-r)\\ &-\left(\frac{1}{\sigma d(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{1}{\sigma d m_s^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2}J_0\left(m_s\sqrt{t^2-r^2}\right)\Theta(t-r),\end{aligned} \tag{45}$$

where we have used the explicit forms of the masses m_g and m_s. In particular for $d = 3$, one arrives at

$$\begin{aligned}G^{\alpha\beta\lambda\tau}(t,r) = &\left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{2\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha+2\beta}{2\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{4\pi}\frac{\Theta(t)\delta(t-r)}{r}\\ &+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau} + \eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{3\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{\beta}{3\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\Theta(t)}{2}\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)J_0\left(m_g\sqrt{t^2-r^2}\right)\Theta(t-r)\\ &-\left(\frac{1}{6\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{12\alpha+4\beta}{6\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\Theta(t)}{2}\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)J_0\left(m_s\sqrt{t^2-r^2}\right)\Theta(t-r).\end{aligned} \tag{46}$$

Again, due to the massive parts, as expected, there is propagation inside the light cone. Hence the quadratic gravity violates the Huygens's principle in a flat spacetime.

ii: Even d case

$$G^{\alpha\beta\lambda\tau}(t,r) = \left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{\sigma(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha+2\beta}{\sigma^2(d-1)}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}$$
$$+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{\sigma d}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{\beta}{\sigma^2 d}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\cos\left(m_g\sqrt{t^2-r^2}\right)\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}$$
$$-\left(\frac{1}{\sigma d(d-1)}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{1}{\sigma d m_s^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\cos\left(m_s\sqrt{t^2-r^2}\right)\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}. \tag{47}$$

In particular, for $d=2$, one has

$$G^{\alpha\beta\lambda\tau}(t,r) = \left(-\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda}) + \frac{1}{\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{4\alpha+2\beta}{\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{1}{2\pi}\Theta(t)\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}$$
$$+\left(\frac{1}{2\sigma}(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda}) - \frac{1}{2\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} - \frac{\beta}{2\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\Theta(t)}{2\pi}\cos\left(i\sqrt{\sigma/\beta}\sqrt{t^2-r^2}\right)\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}$$
$$-\left(\frac{1}{2\sigma}\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{8\alpha+3\beta}{2\sigma^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\Theta(t)}{2\pi}\cos\left(\sqrt{\frac{\sigma}{8\alpha+3\beta}}\sqrt{t^2-r^2}\right)\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}. \tag{48}$$

A particular $2+1$ dimensional model, the so-called new massive gravity (NMG) [9–11] is one of our main interests here. So let us consider this theory. Choosing $\beta = 1/m^2$ and $\alpha = -3/(8m^2)$ and $\sigma = -1$, (48) yields

$$G^{\alpha\beta\lambda\tau}_{\mathrm{NMG}}(t,r) = \frac{1}{4\pi}\left(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda} - 2\eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{1}{m^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\Theta(t)\Theta(t-r)}{\sqrt{t^2-r^2}}$$
$$-\frac{1}{4\pi}\left(\eta^{\alpha\lambda}\eta^{\beta\tau}+\eta^{\alpha\tau}\eta^{\beta\lambda} - \eta^{\alpha\beta}\eta^{\lambda\tau} + \frac{1}{m^2}\eta^{\lambda\tau}\partial^\beta\partial^\alpha\right)\frac{\cos(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\Theta(t)\Theta(t-r). \tag{49}$$

There is propagation inside the light cone and hence NMG in flat spacetime violates the Huygens's principle. We shall come back to the de Sitter version of this theory in Sec. VI.

V. TOPOLOGICALLY MASSIVE GRAVITY

The action for TMG is [12]

$$I_{\mathrm{TMG}} = \int d^3x\sqrt{-g}\left(\frac{1}{\kappa}R + \frac{1}{2\mu}\eta^{\mu\nu\alpha}\Gamma^\beta{}_{\mu\sigma}\times\left(\partial_\nu\Gamma^\sigma{}_{\alpha\beta} + \frac{2}{3}\Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\alpha\beta}\right)\right), \tag{50}$$

where $\eta^{\mu\nu\alpha}$ is the $3D$ antisymmetric tensor. The action yields the following field equations

$$\frac{1}{\kappa}G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{51}$$

where $C_{\mu\nu}$ denotes the Cotton tensor given as

$$C_{\mu\nu} = \eta_\mu{}^{\sigma\rho}\nabla_\sigma\left(R_{\rho\nu} - \frac{1}{4}g_{\rho\nu}R\right), \tag{52}$$

which is symmetric, divergence-free, and traceless. Linearization of the action around the flat spacetime yields the inverse propagator

$$\mathcal{O}_{\mu\nu\alpha\beta}(x) = -\frac{1}{2\kappa}(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})\partial^2 + \frac{1}{4\mu}(\eta_\mu{}^\sigma{}_\alpha\eta_{\nu\beta} + \eta_\mu{}^\sigma{}_\beta\eta_{\nu\alpha} + \eta_\nu{}^\sigma{}_\alpha\eta_{\mu\beta} + \eta_\nu{}^\sigma{}_\beta\eta_{\mu\alpha})\partial_\sigma\partial^2, \tag{53}$$

which in momentum space becomes

$$\tilde{\mathcal{O}}_{\mu\nu\alpha\beta}(p) = \frac{p^2}{2\kappa}(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}) - \frac{p^2}{4\mu}p^\lambda(\eta_{\mu\lambda\alpha}\eta_{\nu\beta} + \eta_{\mu\lambda\beta}\eta_{\nu\alpha} + \eta_{\nu\lambda\alpha}\eta_{\mu\beta} + \eta_{\nu\lambda\beta}\eta_{\mu\alpha}). \tag{54}$$

We obtain the momentum space propagator as

$$\tilde{G}^{\alpha\beta\rho\sigma}(p) = \frac{\kappa}{16\mu^4}(16\mu^4\eta^{\alpha\beta}\eta^{\rho\sigma} - 8\mu^4(\eta^{\alpha\sigma}\eta^{\beta\rho} + \eta^{\alpha\rho}\eta^{\beta\sigma}) - 6\mu^2\kappa^2(\eta^{\alpha\rho}p^\beta p^\sigma + \eta^{\beta\rho}p^\alpha p^\sigma + \eta^{\alpha\sigma}p^\beta p^\rho + \eta^{\beta\sigma}p^\alpha p^\rho)$$
$$+ 8\mu^2\kappa^2(\eta^{\alpha\beta}p^\rho p^\sigma + \eta^{\rho\sigma}p^\alpha p^\beta) - 6\kappa^4 p^\alpha p^\beta p^\rho p^\sigma - 4i\mu^3\kappa p_\kappa(\eta^{\kappa\alpha\rho}\eta^{\beta\sigma} + \eta^{\kappa\beta\rho}\eta^{\alpha\sigma} + \eta^{\kappa\alpha\sigma}\eta^{\beta\rho} + \eta^{\kappa\beta\sigma}\eta^{\alpha\rho})$$
$$- 3i\mu\kappa^3 p_\kappa(\eta^{\kappa\alpha\rho}p^\beta p^\sigma + \eta^{\kappa\beta\rho}p^\alpha p^\sigma + \eta^{\kappa\alpha\sigma}p^\beta p^\rho + \eta^{\kappa\beta\sigma}p^\alpha p^\rho))\frac{1}{p^2} + \frac{\kappa}{4\mu^4}(-2\mu^4\eta^{\alpha\beta}\eta^{\rho\sigma} + 2\mu^4(\eta^{\alpha\sigma}\eta^{\beta\rho} + \eta^{\alpha\rho}\eta^{\beta\sigma})$$
$$+ 2\mu^2\kappa^2(\eta^{\alpha\rho}p^\beta p^\sigma + \eta^{\beta\rho}p^\alpha p^\sigma + \eta^{\alpha\sigma}p^\beta p^\rho + \eta^{\beta\sigma}p^\alpha p^\rho) - 2\mu^2\kappa^2(\eta^{\alpha\beta}p^\rho p^\sigma + \eta^{\rho\sigma}p^\alpha p^\beta)$$
$$+ 2\kappa^4 p^\alpha p^\beta p^\rho p^\sigma + i\mu^3\kappa p_\kappa(\eta^{\kappa\alpha\rho}\eta^{\beta\sigma} + \eta^{\kappa\beta\rho}\eta^{\alpha\sigma} + \eta^{\kappa\alpha\sigma}\eta^{\beta\rho} + \eta^{\kappa\beta\sigma}\eta^{\alpha\rho})$$
$$+ i\mu\kappa^3 p_\kappa(\eta^{\kappa\alpha\rho}p^\beta p^\sigma + \eta^{\kappa\beta\rho}p^\alpha p^\sigma + \eta^{\kappa\alpha\sigma}p^\beta p^\rho + \eta^{\kappa\beta\sigma}p^\alpha p^\rho))\frac{1}{p^2 + \mu^2/\kappa^2}$$
$$- \frac{\kappa^3}{16\mu^4}(2\mu^2(\eta^{\alpha\rho}p^\beta p^\sigma + \eta^{\beta\rho}p^\alpha p^\sigma + \eta^{\alpha\sigma}p^\beta p^\rho + \eta^{\beta\sigma}p^\alpha p^\rho) + 2\kappa^2 p^\alpha p^\beta p^\rho p^\sigma$$
$$+ i\mu\kappa p_\kappa(\eta^{\kappa\alpha\rho}p^\beta p^\sigma + \eta^{\kappa\beta\rho}p^\alpha p^\sigma + \eta^{\kappa\alpha\sigma}p^\beta p^\rho + \eta^{\kappa\beta\sigma}p^\alpha p^\rho))\frac{1}{p^2 + 4\mu^2/\kappa^2}. \tag{55}$$

Applying the inverse Fourier transformation we obtain

$$G^{\alpha\beta\rho\sigma}(t,r) = \frac{\kappa}{16\mu^4}(16\mu^4\eta^{\alpha\beta}\eta^{\rho\sigma} - 8\mu^4(\eta^{\alpha\sigma}\eta^{\beta\rho} + \eta^{\alpha\rho}\eta^{\beta\sigma})$$
$$+ 4\mu^3\kappa(\eta^{\kappa\alpha\rho}\eta^{\beta\sigma} + \eta^{\kappa\beta\rho}\eta^{\alpha\sigma} + \eta^{\kappa\alpha\sigma}\eta^{\beta\rho} + \eta^{\kappa\beta\sigma}\eta^{\alpha\rho})\partial_\kappa)\frac{1}{2\pi}\Theta(t)\Theta(t-r)\frac{1}{\sqrt{t^2-r^2}}$$
$$+ \frac{\kappa}{4\mu^4}(-2\mu^4\eta^{\alpha\beta}\eta^{\rho\sigma} + 2\mu^4(\eta^{\alpha\sigma}\eta^{\beta\rho} + \eta^{\alpha\rho}\eta^{\beta\sigma})$$
$$- \mu^3\kappa(\eta^{\kappa\alpha\rho}\eta^{\beta\sigma} + \eta^{\kappa\beta\rho}\eta^{\alpha\sigma} + \eta^{\kappa\alpha\sigma}\eta^{\beta\rho} + \eta^{\kappa\beta\sigma}\eta^{\alpha\rho})\partial_\kappa)\frac{1}{2\pi}\Theta(t)\Theta(t-r)\frac{\cos(\frac{\mu}{\kappa}\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}, \tag{56}$$

where, in the last expression, we dropped the terms which vanish when the propagator is sandwiched between two conserved sources. There is a single massive spin-2 mode with only one degree of freedom since it is a parity noninvariant theory. Once again there is a tail inside the light cone and the Huygens's principle is violated.

VI. NEW MASSIVE GRAVITY IN DE SITTER SPACETIME

As mentioned in the Introduction, anomalous dispersion can disappear in the $2+1$ dimensional gravity in a curved background. To understand this, let us consider a generic quadratic gravity in a de Sitter background. This theory was studied in detail in [13]. Here, we shall only quote the pertaining details for our discussion. Generic three dimensional quadratic action is

$$I = \int d^3x\sqrt{-g}\left(\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2\right). \tag{57}$$

Consider the linearization of this theory in a de Sitter background given by the following metric

$$ds^2 = \frac{\ell^2}{t^2}(-dt^2 + dx^2 + dy^2), \tag{58}$$

where the effective cosmological constant is

$$\frac{1}{\ell^2} = \frac{1}{4\kappa(3\alpha+\beta)}\left(1 \pm \sqrt{1 - 8\kappa\Lambda_0(3\alpha+\beta)}\right). \tag{59}$$

Defining the perturbations as

$$g_{\mu\nu} = \frac{\ell^2}{t^2}\eta_{\mu\nu} + h_{\mu\nu}, \tag{60}$$

a rather long discussion given in [7,13] shows that for generic α, β, κ there are three propagating degrees of freedom. Two of these constitute the massive spin-2 field with the mass

$$m_g^2 = -\frac{1}{\kappa\beta} - \frac{12\alpha}{\ell^2\beta} - \frac{4}{\ell^2}, \tag{61}$$

and the third degree of freedom is a spin-0 mode with the mass

$$m_s^2 = \frac{1}{\kappa(8\alpha+3\beta)} - \frac{4}{\ell^2}\left(\frac{3\alpha+\beta}{8\alpha+3\beta}\right). \tag{62}$$

Let ϕ denote the spin-0 field, which arises as a gauge-invariant object then its action is given as

$$I_\phi = \frac{(8\alpha+3\beta)}{8}\int d^3x\left[\frac{t^3}{\ell^3}\dot{\phi}^2 - \frac{1}{(8\alpha+3\beta)}\frac{t}{\ell}\left(\frac{1}{\kappa} - \frac{12\alpha}{\ell^2} - \frac{4\beta}{\ell^2}\right)\phi^2\right]. \tag{63}$$

On the other hand, the two modes of the spin-2 field come with the following action

$$I_\sigma = \frac{1}{2}\int d^3x\left[\beta\frac{t^3}{\ell^3}(\dot{\sigma}^2 + \sigma\nabla^2\sigma) + \left(\frac{1}{\kappa} + \frac{12\alpha}{\ell^2} + \frac{4\beta}{\ell^2}\right)\frac{t}{\ell}\sigma^2\right]. \tag{64}$$

To understand these modes, let us recall that a free scalar field in this background with mass m has the action

$$\begin{aligned} I &= -\frac{1}{2}\int d^3x\sqrt{-g}(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2) \\ &= -\frac{1}{2}\int d^3x\left\{\frac{\ell}{t}[-\dot{\Phi}^2 + (\partial_i\Phi)^2] + \frac{\ell^3}{t^3}m^2\Phi^2\right\}. \end{aligned}$$

Comparing this with (63) and (64), after scaling σ as $\sigma \to \frac{\ell^2}{t^2}\sigma$ and similarly $\phi \to \frac{\ell^2}{t^2}\phi$ one can read (62) from (63) and (61) from (64). In the NMG limit, that is $8\alpha + 3\beta = 0$ the ϕ field is infinitely massive and drops out of the spectrum. One is left with a massive spin-2 field with the mass

$$m_g^2 = -\frac{1}{\kappa\beta} + \frac{1}{2\ell^2}. \tag{65}$$

As shown in [1], a massive scalar field in de Sitter spacetime with the tuned mass $m = 1/\ell$ shows dispersionless propagation in $2+1$ dimensions. One can easily see this from the following construction: with the coordinate change $t = \ell e^{-\tau/\ell}$ and $a(\tau) = e^{\tau/\ell}$, de Sitter metric becomes

$$ds^2 = -d\tau^2 + a(\tau)^2(dx^2 + dy^2).$$

In these coordinates, the Fourier modes of the massive graviton has the dispersion relation

$$w_{\vec{k}}^2 = -\frac{1}{\ell^2} + m_g^2 + \frac{k^2}{a^2}. \tag{66}$$

For $m_g^2 = 1/\ell^2$, the group velocity is independent of $\vec{k}$. Hence anomalous dispersion disappears. This corresponds to the case $\kappa\beta = -2\ell^2$. This is possible for $\Lambda_0 = -27/\ell^2$. Note that, the bare cosmological constant is negative, but the effective cosmological constant is positive.

One can easily generalize the discussion of the previous section to generic D dimensions. Consider a massive scalar field living in the background spacetime with the metric

$$ds^2 = -d\tau^2 + a(\tau)^2\sum_{i=1}^{D-1}dx^i dx^i, \qquad a(\tau) = e^{H\tau},$$
$$H \coloneqq \sqrt{\frac{2\Lambda}{(D-1)(D-2)}}. \tag{67}$$

Then the wave equation $(\Box - m^2)\Phi = 0$ is solved by the Fourier modes

$$\Phi(\tau, x^i) \coloneqq \frac{f_{\vec{k}}(\tau)}{a(\tau)}e^{i\vec{k}\cdot\vec{x}}, \tag{68}$$

as long as the following equation is satisfied

$$\ddot{f}_{\vec{k}}(\tau) + \omega_{\vec{k}}^2 f_{\vec{k}}(\tau) = 0, \qquad \omega_{\vec{k}}^2 \coloneqq m^2 - \frac{\Lambda(D-1)}{2(D-2)} + \frac{k^2}{a(\tau)^2}. \tag{69}$$

Note that for the tuning $m^2 = \frac{\Lambda(D-1)}{2(D-2)}$, the group velocity $v_g^i = \frac{\partial\omega_k}{\partial k^i}$ is independent of k and hence there is no dispersion. For this case, the solution to (69) is given in terms of the Bessel function of the first and second kinds as

$$f_{\vec{k}}(\tau) = c_1 J_0\left(\frac{k}{a(\tau)H}\right) + c_2 Y_0\left(\frac{k}{a(\tau)H}\right), \tag{70}$$

and a generic wave pulse can be constructed from the superposition of these modes. Note that in [14][2] the same result was reached but to make the proper comparison the choice of $H = 1 \to \Lambda = \frac{(D-1)(D-2)}{2}$ choice should be made in our expressions.

VII. CONCLUSIONS

We studied the propagation of gravity waves in some detail in flat and de Sitter spacetimes for massless and massive gravity, quadratic gravity theories. It is quite well known that in flat backgrounds, with odd number of spatial dimensions (such as our Universe), there is no anomalous dispersion in a vacuum, while for all even spatial dimensions there is anomalous dispersion. So introducing one spacelike dimension changes the propagation dramatically. What has been a rather unexpected surprise was to see that

[2]We thank a conscientious referee for bringing this reference to our attention.

adding one timelike dimensions also removes anomalous dispersion which was demonstrated in [3] for a particular setting whose details have been given in [1]. In this work we have studied the extensions of these considerations to massive gravity theories; in particular we showed that for a particular tuning of the mass in terms of the cosmological constant, both scalar waves in D dimensional de Sitter spacetime and new massive gravity in $2+1$ dimensions allow dispersion-free propagation and hence the Huygens's principle survives.

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APPENDIX A: MASSLESS INTEGRAL

Here, for completeness, we give some of the details of the integrals that we used in the body of the text. We claim no originality in these two Appendixes as these results can be found in various forms in the literature [2,15–18]. We used [19] for various integrals and relations.

Consider the mass free integral

$$I_1 \coloneqq \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{-ip\cdot(x-x')}}{p^2}, \tag{A1}$$

which reads

$$I_1 = \int \frac{d^d p}{(2\pi)^d} e^{-i\vec{p}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{ip^0 t}}{\vec{p}^2 - (p^0)^2}, \tag{A2}$$

where we have defined $\vec{r} \coloneqq \vec{x} - \vec{x}'$ and $t \coloneqq x^0 - x'^0$. To obtain the retarded Green's function, in carrying out the p^0 integral, both poles are displaced in such a way that they are located in the upper-half plane and contribute to the integral. Hence the p^0 integral yields

$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{ip^0 t}}{\vec{p}^2 - (p^0)^2} = \frac{\sin(pt)}{p} \Theta(t), \tag{A3}$$

where $\Theta(t)$ denotes the Heaviside step function and $p \coloneqq |\vec{p}|$. Then

$$I_1 = \Theta(t) \int \frac{d^d p}{(2\pi)^d} e^{-i\vec{p}.\vec{r}} \frac{\sin(pt)}{p}. \tag{A4}$$

Assuming $\vec{p}\cdot\vec{r} = |\vec{p}||\vec{r}|\cos\theta_1$ and using

$$d^d p = p^{d-1}(\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\ldots \times \sin\theta_{d-2} d\theta_1 d\theta_2 \ldots . d\theta_{d-1} dp \tag{A5}$$

we obtain

$$I_1 = \Theta(t) \int_0^\infty \frac{dp}{(2\pi)^d} \sin(pt) p^{d-2} \times \int_0^\pi d\theta_1 e^{-i|\vec{p}||\vec{r}|\cos\theta_1} (\sin\theta_1)^{d-2} \times \int d\theta_2 \ldots . d\theta_{d-1} (\sin\theta_2)^{d-3} \ldots \sin\theta_{d-2}, \tag{A6}$$

where

$$\int d\theta_2 \ldots . d\theta_{d-1} (\sin\theta_2)^{d-3} \ldots \sin\theta_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \tag{A7}$$

is the solid angle in $d-1$ dimensions for both the even and odd dimensional cases. To evaluate the θ_1 integral we use the following formula

$$\int_0^\pi d\theta e^{ikr\cos\theta} (\sin\theta)^{m-2} = \sqrt{\pi} \left(\frac{2}{kr}\right)^{(m-2)/2} \Gamma\left(\frac{m-1}{2}\right) J_{m/2-1}(kr). \tag{A8}$$

Then we get

$$I_1 = \Theta(t)(2\pi)^{-d/2} \int_0^\infty dp \sin(pt) p^{d-2} \frac{J_{d/2-1}(-pr)}{(-pr)^{d/2-1}}. \tag{A9}$$

To complete the calculation, we need to compute the term $J_{d/2-1}(-pr)/(-pr)^{d/2-1}$. We use the identity [15]

$$\frac{J_{v+n}(x)}{x^{v+n}} = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{J_v(x)}{x^v}. \tag{A10}$$

Now, let us consider the even and odd dimensional cases separately.

1. Odd d case

We have $x = -pr$ in (A10). Let $v = -1/2$, then one has $n = (d-1)/2$ and we use

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \tag{A11}$$

to arrive at

$$\frac{J_{d/2-1}(-pr)}{(-pr)^{d/2-1}} = 2^{d/2} \pi^{(d-2)/2} p^{1-d} \left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2} \cos(pr). \tag{A12}$$

Then I_1 integral reduces to

$$I_1 = \frac{1}{\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2}\int_0^\infty dp\,\frac{\sin(pt)\cos(pr)}{p}. \tag{A13}$$

In order to take the p integral one needs [19]

$$\int_0^\infty dx\frac{\sin(ax)\cos(bx)}{x} = \begin{cases} \pi/2, & a > b \geq 0 \\ \pi/4, & a = b > 0 \\ 0, & b > a \geq 0. \end{cases} \tag{A14}$$

Using these one ends up with

$$I_1 = \frac{1}{4\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-3)/2}\frac{\delta(t-r)}{r}. \tag{A15}$$

2. Even d case

We have $x = -pr$ in (A10). Let $v = 0$, then one has $n = (d-2)/2$ and (A10) yields

$$\frac{J_{d/2-1}(-pr)}{(-pr)^{d/2-1}} = p^{(2-d)/2}\left(\frac{1}{r}\frac{d}{d(-pr)}\right)^{(d-2)/2}J_0(-pr), \tag{A16}$$

where

$$J_0(z) = \sum_{k=0}^\infty (-1)^k\frac{z^{2k}}{2^{2k}(k!)^2}. \tag{A17}$$

One has $J_0(-z) = J_0(z)$, then we obtain

$$\frac{J_{d/2-1}(-pr)}{(-pr)^{d/2-1}} = (2\pi)^{(d-2)/2}p^{2-d}\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}J_0(-pr). \tag{A18}$$

Substituting this in (A9) we get

$$I_1 = \frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\int_0^\infty dp\sin(pt)J_0(-pr), \tag{A19}$$

and we end up with

$$I_1 = \frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}. \tag{A20}$$

We can summarize the results as follows [2,16,17]

$$I_1 = \int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2} = \begin{cases} \frac{1}{4\pi}\Theta(t)(-\frac{1}{2\pi r}\frac{d}{dr})^{(d-3)/2}\frac{\delta(t-r)}{r}: & \text{for odd } d, \\ \frac{1}{2\pi}\Theta(t)(-\frac{1}{2\pi r}\frac{d}{dr})^{(d-2)/2}\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}: & \text{for even } d. \end{cases} \tag{A21}$$

APPENDIX B: MASSIVE INTEGRAL

Let us consider the following integral

$$I_2 \coloneqq \int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip\cdot(x-x')}}{p^2+m^2}. \tag{B1}$$

Similar steps in the previous section yields

$$I_2 = \Theta(t)(2\pi)^{-d/2}\int_0^\infty dp\,p^{d-1}\frac{\sin(t\sqrt{\vec{p}^2+m^2})}{\sqrt{\vec{p}^2+m^2}} \times\frac{J_{d/2-1}(-pr)}{(-pr)^{d/2-1}}. \tag{B2}$$

Now we need to consider the odd and even d separately.

1. Odd d case

Using the identity (A12) the I_2 integral reduces to

$$I_2 = \frac{1}{\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2} \times\int_0^\infty dp\frac{\sin(t\sqrt{\vec{p}^2+m^2})}{\sqrt{\vec{p}^2+m^2}}\cos(pr). \tag{B3}$$

In order to take the p integral we use

$$\int_0^\infty dx\frac{\sin(p\sqrt{x^2+a^2})\cos(bx)}{x^2+a^2} = \begin{cases} \pi J_0(a\sqrt{p^2-b^2})/2, & 0 < b < p,\ a > 0 \\ 0, & b > p > 0,\ a > 0 \end{cases} \tag{B4}$$

and we arrive at

$$I_2 = \frac{1}{2}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-1)/2}\Theta(t-r)J_0\left(m\sqrt{t^2-r^2}\right). \tag{B5}$$

2. Even d case

Using (A18) we get

$$I_2 = \frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\int_0^\infty dp \frac{p\sin\left(t\sqrt{\vec{p}^2+m^2}\right)}{\sqrt{\vec{p}^2+m^2}} J_0(pr), \tag{B6}$$

where

$$\int_0^\infty dp J_0(pr)\frac{p\sin\left(t\sqrt{\vec{p}^2+m^2}\right)}{\sqrt{\vec{p}^2+m^2}} = \frac{\cos\left(m\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}}\Theta(t-r). \tag{B7}$$

Then one obtains

$$I_2 = \frac{1}{2\pi}\Theta(t)\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{(d-2)/2}\frac{\cos\left(m\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}}\Theta(t-r). \tag{B8}$$

We can summarize the results as follows [18]

$$I_2 = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{-ip(x-x')}}{p^2+m^2} = \begin{cases} \frac{1}{2}\Theta(t)(-\frac{1}{2\pi r}\frac{d}{dr})^{(d-1)/2}\Theta(t-r)J_0\left(m\sqrt{t^2-r^2}\right): & \text{for odd}\, d, \\ \frac{1}{2\pi}\Theta(t)(-\frac{1}{2\pi r}\frac{d}{dr})^{(d-2)/2}\frac{\cos(m\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\Theta(t-r): & \text{for even}\, d. \end{cases} \tag{B9}$$

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