Brief note on Thurston geometries in 3D quadratic curvature theories

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We show that Thurston geometries are solutions to a large class of 3D quadratic curvature theories, where new massive gravity—which was studied by Flores-Alfonso *et al.* [Phys. Rev. Lett. **127,** 061102 (2021)]—is a special case.

Thurston's conjecture [1] states that a three-manifold with a given topology has a canonical decomposition into eight manifolds with high symmetry—the so-called Thurston geometries (TGs) [2–4]. Based on Hamilton's Ricci flow [5], which provides a way to deform a given metric in the form of a generalized heat equation, Perelman gave his celebrated proof [6,7]. Since many TGs are not Einstein spaces, one might wonder if a "master theory" that admits all TGs as solutions can be achieved by a modification of Einstein's equation.

In a recent paper [8], it was reported that TGs are solutions to new massive gravity (NMG) [9], where the authors stated that, to the best of their knowledge, NMG is the only 3D gravity theory with this property.[1] In a previous work [11], four of the TGs were shown to be solutions of the most general quadratic theory. In this paper we shall close a gap and show that the remaining four TGs are also solutions of the most general quadratic theory, whose action is displayed in Eq. (4). NMG is singled out by being the only ghost-free theory in this class.[2] However, ghost freedom is not essential in defining a flow where TGs are critical points.

In order to demonstrate this, let us present the logic of a uniformization theorem along the lines of Ref. [15]. Assuming that the manifold admits some metric, one needs to devise a mechanism to flow the initial metric into a highly symmetric one, which can be obtained by the following parabolic system:

$$\partial_t g_{\mu\nu} = \mathcal{O}(g_{\mu\nu}),\tag{1}$$

where $\mathcal{O}$ is an elliptic operator. The parabolicity of Eq. (1) ensures that the flow will end at fixed points as $t \to \infty$, for which $\mathcal{O}(g_{\mu\nu}) = 0$. However, since $\mathcal{O}(g_{\mu\nu})$ should transform as a tensor, it has to be constructed from curvature tensors, which makes it impossible to obtain a strictly parabolic system and leads to nonlinearity due to the inverse metric. Many issues regarding these complications should be resolved for a full proof. In his proof, Perelman made use of the operator

$$\mathcal{O}(g_{\mu\nu}) = -2(R_{\mu\nu} + \nabla_\mu \nabla_\nu f),\tag{2}$$

where f is a scalar field. It follows from the field equations of the action that

$$S[f, g] = \int \mathrm{d}^3 x \sqrt{|g|} e^{-f} [R + (\nabla f)^2],\tag{3}$$

and the action (3) is an increasing functional provided that $\partial_t f = \partial_t \ln \sqrt{|g|}$. Alternatively, one can also try to use the field equations of quadratic gravity theories as a candidate for such an operator, which can be defined through an action that is a functional of the metric tensor only. As long as one finds an operator that vanishes for all TGs, the ghost freedom of the theory plays no role in the construction.

Motivated by this, we consider the most general 3D quadratic gravity theory described by the action

[1]In Ref. [8], ghost-free theories were considered for simplicity [10].

[2]NMG is obtained after imposing the absence of the scalar mode, which is always a ghost, and the unitarity of the massive spin-2 mode around a maximally symmetric background spacetime [see Ref. [12] for an analysis on a flat background and Refs. [13,14] for constant-curvature backgrounds (dS/AdS)].

TABLE I.   Euclidean Thurston geometries as solutions to an arbitrary quadratic gravity theory. Parameters should satisfy the constraints i and ii, where examples of sufficient but not necessary conditions are given in Table III.

Geometry	Metric	ℓ^2	Λ
E^3	$ds^2 = dx^2 + dy^2 + dz^2$	No length scale	0
S^3	$ds^2 = \ell^2(dx^2 + \sin^2 x dy^2 + \sin^2 x \sin^2 y dz^2)$	Arbitrary	$\frac{1}{\ell^2} - \sigma \frac{2(3\alpha+\beta)}{\ell^4}$
H^3	$ds^2 = \ell^2(dx^2 + \cosh^2 x dy^2 + \cosh^2 x \cosh^2 y dz^2)$	Arbitrary	$-\frac{1}{\ell^2} - \sigma \frac{2(3\alpha+\beta)}{\ell^4}$
$E^1 \times S^2$	$ds^2 = \ell^2(dx^2 + dy^2 + \sin^2 y dz^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{4(2\alpha+\beta)}$
$E^1 \times H^2$	$ds^2 = \ell^2(dx^2 + dy^2 + \cosh^2 y dz^2)$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{4(2\alpha+\beta)}$
Nil	$ds^2 = \frac{\ell^2}{4}[dx^2 + dy^2 + (xdy - dz)^2]$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
$SL(2,\mathbb{R})$	$ds^2 = \ell^2[dr^2 + \sinh^2 r \cosh^2 r d\theta^2 + (d\psi + \sinh^2 r d\theta)^2]$	$-\frac{25\alpha+23\beta}{10\Lambda(\alpha+\beta)}$	$-\sigma \frac{25\alpha+23\beta}{200(\alpha+\beta)^2}$
Sol	$ds^2 = \ell^2(e^{-2z}dx^2 + e^{2z}dy^2 + dz^2)$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+\beta)}$

$$S[g] = \int d^3x \sqrt{|g|}[\sigma(R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}], \tag{4}$$

where $\sigma = \pm 1$ is introduced to control the sign of the Einstein-Hilbert term and (α, β) are arbitrary constants. Field equations arising from the action (4) are given by

$$0 = \sigma(G_{\mu\nu} + \Lambda g_{\mu\nu}) + \alpha\left[2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 + 2g_{\mu\nu}\Box R - 2\nabla_\mu\nabla_\nu R\right]$$

$$+ \beta\left[\frac{3}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - 4R_\mu{}^\rho R_{\nu\rho} + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R + 3RR_{\mu\nu} - g_{\mu\nu}R^2\right]. \tag{5}$$

As we summarize in Tables I and II, TGs are solutions for different choices of (σ, α, β) that are subjected to two constraints:

(i) The length scale in the metric should satisfy $\ell^2 > 0$ in order to consider metrics with definite signature.

(ii) Denominators in the expressions for ℓ^2 and Λ should be nonzero.

As a simple example, to ensure that the theory admits Lorentzian TGs given in Table II as solutions, the following choice of parameters is enough:

$$\sigma = -1, \qquad \beta = \frac{1}{m^2} > 0, \qquad \alpha > -\frac{1}{2m^2}. \tag{6}$$

When one demands to obtain Lorentzian TGs as solutions to a pure gravity theory (no scalar mode) with unitary excitations around a maximally symmetric spacetime, NMG is the only choice and one has to fix the last parameter as $\alpha = -\frac{3}{8m^2}$ [9,13,14], for which the scalar mode decouples from the spectrum since it becomes infinitely heavy ($m_s^2 \to \infty$), and there remains only a unitary massive spin-2 graviton with $m_g^2 = m^2 + \frac{\Lambda}{2} > 0$.

For the Euclidean TGs, some possible choices of parameters are given in Table III, where NMG is again a special case. Note that $m^2 < 0$ corresponds to an imaginary mass. Analogous behavior was also observed in Refs. [15,16], where the Euclidean solutions were obtained from the

TABLE II.   Lorentzian Thurston geometries as solutions to an arbitrary quadratic gravity theory. Parameters should satisfy the constraints i and ii, where an example of sufficient but not necessary conditions is given in Eq. (6).

Geometry	Metric	ℓ^2	Λ
Nil	$ds^2 = \frac{\ell^2}{4}[dx^2 + dy^2 - (xdy - dz)^2]$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
$SL(2,\mathbb{R})$	$ds^2 = \frac{\ell^2}{4}[-(d\Psi + \cos\Theta d\Phi)^2 + d\Theta^2 + \sin^2\Theta d\Phi^2]$	$\frac{25\alpha+23\beta}{10\Lambda(\alpha+\beta)}$	$-\sigma \frac{25\alpha+23\beta}{200(\alpha+\beta)^2}$
New Sol	$ds^2 = \ell^2(e^{-2z}dx^2 + e^{2z}dy^2 - dz^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+\beta)}$
Lorentz Sol	$ds^2 = \ell^2(2e^{-z}dxdz + e^{2z}dy^2)$	Arbitrary	$\sigma = 0$, no effect on the solution
Third Sol	$ds^2 = \ell^2(-e^{2z}dy^2 - 2dxdy + dz^2)$	$-4\sigma\beta$	0
Lorentz-Heisenberg	$ds^2 = \frac{\ell^2}{4}(-dx^2 + dy^2 + (xdy - dz)^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
AdS	$ds^2 = \ell^2[dr^2 + \sinh^2 r \cosh^2 r d\theta^2 - (d\psi + \sinh^2 r d\theta)^2]$	Arbitrary	$-\frac{1}{\ell^2} - \sigma \frac{2(3\alpha+\beta)}{\ell^4}$

TABLE III. Choice of parameters for Euclidean Thurston geometries. Note that these are sufficient but not necessary conditions.

Geometry	Parameters		
$E^1 \times S^2$	$\sigma = -1$	$\beta = \frac{1}{m^2} > 0$	$\alpha > -\frac{1}{2m^2}$
	$\sigma = +1$	$\beta = \frac{1}{m^2} < 0$	$\alpha < -\frac{1}{2m^2}$
Others	$\sigma = -1$	$\beta = \frac{1}{m^2} < 0$	$\alpha < -\frac{1}{2m^2}$
	$\sigma = +1$	$\beta = \frac{1}{m^2} > 0$	$\alpha > -\frac{1}{2m^2}$

low-energy limit of string theory supplemented by a one-form gauge field. For some of the solutions, gauge fields become imaginary, which is equivalent to having the kinetic terms with opposite signs in the action. However, this should not be considered as a problem since the aim here is to choose the parameters according to Table I and show that TGs correspond to fixed points.

TGs were also considered in another modification of Einstein's equations called minimal massive gravity (MMG) [17], whose field equations read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = 0, \qquad (7)$$

where the Cotton tensor $C_{\mu\nu}$ is related to the Schouten tensor $S_{\mu\nu}$ as

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha S_{\beta\nu}, \qquad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R, \qquad (8)$$

and the J tensor is given by

$$J_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} \epsilon_\nu{}^{\rho\sigma} S_{\alpha\rho} S_{\beta\sigma}. \qquad (9)$$

The field equations (7) do not arise from the variation of an action and are covariantly conserved only on shell. In Ref. [18], it was shown that all TGs except Sol geometry are solutions to Eq. (7) for certain (nonzero and finite) choices of parameters. Sol geometry requires the vanishing of the coefficient of the Cotton term ($\mu \to \infty$, $\gamma \to \infty$ such that $\frac{\gamma}{\mu^2} \to$ const), which in general ruins the physical consistency of the theory. However, such a solution is still acceptable if $\epsilon^{\mu\rho\sigma} S_\rho{}^\tau C_{\sigma\tau} = 0$, which is the case for the Sol geometry. Therefore, in addition to NMG, MMG defines an operator $\mathcal{O}(g_{\mu\nu})$ that vanishes for all TGs. What the field equations of MMG lack is the existence of an entropy functional along the flow that admits a gradient formulation (see Ref. [19] for an example in Cotton flow), which might be crucial in an attempt for a full proof of a uniformization theorem.

All in all, realizing TGs as vacuum solutions to higher-curvature modifications of Einstein's equations yields many possibilities where the ghost-free theories are not necessarily privileged. Due to the higher-derivative operators, it seems impossible to apply results from elliptic operator theory for rigorous results. However, the relevance of the unitarity of the theory might be checked through a detailed numerical analysis of the flow equation, as was done in Ref. [19] for the Cotton flow.

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