All solutions of Einstein-Maxwell equations with a cosmological constant in $2+1$ dimensions

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We present a *general* solution of the coupled Einstein-Maxwell field equations (without the source charges and currents) in three spacetime dimensions. We also admit any value of the cosmological constant. The whole family of such Λ-electrovacuum local solutions splits into two distinct subclasses, namely the nonexpanding Kundt class and the expanding Robinson-Trautman class. While the Kundt class only admits electromagnetic fields which are aligned along the geometrically privileged null congruence, the Robinson-Trautman class admits both aligned and also more complex nonaligned Maxwell fields. We derive all the metric and Maxwell field components, together with explicit constraints imposed by the field equations. We also identify the most important special spacetimes of this type, namely the coupled gravitational-electromagnetic waves and charged black holes.

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I. INTRODUCTION

Recently, in paper [1] we derived the most general solution of the Einstein equations with a cosmological constant Λ and also an aligned pure radiation matter field (possibly gyrating null dust/particles) in three spacetime dimensions. Here we extend this study to another important nonvacuum case, which is the presence of an electromagnetic field. In fact, we explicitly derive *all* solutions of the Einstein-Maxwell field equations with any value of Λ.

For many decades, the $2+1$-dimensional Einstein gravity has attracted a great deal of attention. The main reason is that such gravity theory is mathematically simpler than standard general relativity because the number of independent components of the curvature tensor is much lower. In fact, the Weyl tensor identically vanishes, and the Riemann and Ricci tensors have the same number of components. Consequently, there is *no classic dynamical degree of freedom in* $2+1$ *spacetimes*. The Ricci tensor—directly given by the Einstein field equations—fully determines the *local curvature* of the spacetime. This implies that a general *vacuum* solution of Einstein's equations is just the maximally symmetric Minkowski, de Sitter (dS), or anti–de Sitter (AdS) spacetime for $\Lambda = 0$, $\Lambda > 0$, or $\Lambda < 0$, respectively.

Despite such local simplicity/triviality of the $2+1$ gravity theory, it can serve as a very useful playground for various investigations, ranging from the black hole properties and cosmology to high-energy physics and quantum gravity. While the Einstein equations determine the spacetime locally, there can be *global topological degrees of freedom* reflected in the appropriate domains of the coordinates employed: It is possible to construct globally different geometries from locally identical spacetimes by various identifications. In the context of black holes, this has been successfully used for construction of famous Bañados-Teitelboim-Zanelli (BTZ)-type solutions with horizons when $\Lambda < 0$ by performing nontrivial identifications of the local AdS vacuum spacetime, pure radiation solutions, or spacetimes with electromagnetic field [2–4]. The corresponding topological degrees of freedom play a crucial role in quantum gravity models [5]. However, it is still not clear if they represent all possible nonvacuum spacetimes. It is thus desirable to obtain and investigate more general exact solutions in the presence of matter.

Many exact spacetimes in $2+1$-dimensional Einstein gravity have already been found. They are nicely summarized, classified, and described in a helpful comprehensive catalog [6]. Such solutions were found in a great number of works by making various specific assumptions on their symmetry, algebraic structure, or other geometrical or physical constraints. A general study of solutions of $2+1$-dimensional Einstein-Maxwell theory using the Rainich geometrization of the electromagnetic field was presented in [7]. Using a different approach, in this paper we solve the Einstein-Maxwell field equations generically, *without making any assumption*. In fact, we systematically derive all possible such spacetimes, extending and generalizing previously known exact electrovacuum solutions.

Specifically, in Sec. II we recall the key result of [1] that (virtually) all $2+1$ geometries belong either to the family

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of (nonexpanding) Kundt spacetimes or to the family of (expanding) Robinson-Trautman spacetimes. We also present the canonical metric form and the natural null triad. The related Appendix contains the corresponding Christoffel symbols and all components of the Riemann and Ricci tensors. In Sec. III we present the most general electromagnetic 2-form field in $2+1$ gravity, together with its dual 1-form, the equivalent Newman-Penrose scalars, and the energy-momentum tensor. In Sec. IV we formulate the (source-free) Einstein-Maxwell field equations with Λ, expressed in a simple form. Section V contains an explicit step-by-step integration of these field equations in the Kundt case, while Sec. VI contains an analogous procedure for the complementary Robinson-Trautman case. In both cases, the electromagnetic field is aligned with the privileged null direction of the gravitational field. The resulting complete families of such spacetimes are summarized in Secs. V H and VI H, respectively. The distinct family of Robinson-Trautman geometries with nonaligned electromagnetic fields is presented in Sec. VII, with a specific particular solution obtained in Sec. VII F. Final summary and further remarks can be found in concluding Sec. VIII.

II. ALL GEOMETRIES AND THEIR CANONICAL FORM IN 2+1 GRAVITY

In Sec. II of our previous work [1], we investigated general three-dimensional Lorentzian spacetimes $(\mathcal{M}, g_{ab})$ with the metric signature $(++-)$. We proved the uniqueness theorem, namely that the only possible such spacetimes are either *expanding geometries* of the *Robinson-Trautman type* (with $\Theta \neq 0$) or *nonexpanding geometries* of the *Kundt type* (with $\Theta = 0$). They are necessarily twist-free and shear-free; see Theorem 1 in [1] (this observation was already made in [8]).

In a C^1 spacetime there exists a *geodesic null* vector field $\mathbf{k}$ (defined as a tangent vector of null geodesics at any point), which in $D=3$ is equivalent to *hypersurface-orthogonality*; see Theorem 2 in [1]. Recall that the *expansion* Θ is the only nontrivial optical scalar,

$$\Theta = \rho \equiv k_{a;b} m^a m^b, \tag{1}$$

which characterizes the properties of a null congruence generated by $\mathbf{k}$, in a triad $\mathbf{e}_I \equiv \{\mathbf{k}, \mathbf{l}, \mathbf{m}\}$ of two null vectors $\mathbf{k}$, $\mathbf{l}$ and one spatial vector $\mathbf{m}$, normalized as

$$\mathbf{k}\cdot\mathbf{l} = -1, \qquad \mathbf{m}\cdot\mathbf{m} = 1. \tag{2}$$

In [1], we also introduced *canonical coordinates* $\{r, u, x\}$ for all Robinson-Trautman and Kundt metrics; see Theorem 3,

$$\mathrm{d}s^2 = g_{xx}(r,u,x)\mathrm{d}x^2 + 2g_{ux}(r,u,x)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2. \tag{3}$$

These coordinates are adapted to their unique geometry, namely r is an affine parameter along the null congruence generated by $\mathbf{k}$, the coordinate u labels null hypersurfaces (such that $k_a \propto u_{,a}$) which naturally foliate the spacetimes, and the spatial coordinate x spans the one-dimensional "transverse" subspace with constant u and r.

It is also convenient to recall that the nonvanishing contravariant metric components g^{ab} are

$$g^{xx} = 1/g_{xx}, \qquad g^{ur} = -1, \qquad g^{rx} = g_{ux}/g_{xx}, \\ g^{rr} = -g_{uu} + g_{ux}^2/g_{xx}, \tag{4}$$

equivalent to the inverse relations

$$g_{xx} = 1/g^{xx}, \qquad g_{ur} = -1, \qquad g_{ux} = g_{xx}g^{rx}, \\ g_{uu} = -g^{rr} + g_{xx}(g^{rx})^2. \tag{5}$$

The most natural choice of the null triad frame $\{\mathbf{k}, \mathbf{l}, \mathbf{m}\}$ satisfying (2) is

$$\mathbf{k} = \partial_r, \qquad \mathbf{l} = \frac{1}{2}g_{uu}\partial_r + \partial_u, \qquad \mathbf{m} = \frac{1}{\sqrt{g_{xx}}}(g_{ux}\partial_r + \partial_x). \tag{6}$$

A direct calculation for the metric (3) reveals that $k_{a;b} = \frac{1}{2}g_{ab,r}$. An explicit form of the expansion scalar (1) thus becomes $\Theta = k_{x;x}m^x m^x$, implying an important relation:

$$g_{xx,r} = 2\Theta g_{xx}, \tag{7}$$

For our next investigation it seems convenient to introduce a new function $G(r,u,x)$, which *fully* encodes the spatial metric function $g_{xx} > 0$ via the simple relation

$$G \equiv \frac{1}{\sqrt{g_{xx}}} \Leftrightarrow g_{xx} = G^{-2}. \tag{8}$$

The key relation (7) then takes the form

$$\Theta = -(\ln G)_{,r}. \tag{9}$$

Now it immediately follows that for vanishing expansion, $\Theta = 0$, the function G and thus also the spatial metric $g_{xx}(u,x)$ must be *independent of the coordinate* r. It yields the *Kundt class* of nonexpanding, twist-free, and shear-free geometries [9–13]. The complementary case $\Theta \neq 0$ gives the expanding *Robinson-Trautman class* of geometries [10,11,13–18], as summarized in Theorem 4 of our work [1].

The Christoffel symbols and all coordinate components of the Riemann and Ricci curvature tensors for the general metric (3), calculated using the relation (7), are listed in the Appendix.

III. GENERIC ELECTROMAGNETIC FIELD IN 2+1 GRAVITY

The aim of this work is to systematically investigate all possible gravitational and electromagnetic fields in $2+1$ dimensions, solving the coupled Einstein-Maxwell field equations.

Based on the results summarized in previous Sec. II, all such spacetimes can be conveniently written in the canonical coordinates $\{r, u, x\}$ for the general metric (3). Consequently, *generic electromagnetic field* takes the form of an antisymmetric 3×3 Maxwell tensor

$$F_{ab} = \begin{pmatrix} 0 & F_{ru} & F_{rx} \\ -F_{ru} & 0 & F_{ux} \\ -F_{rx} & -F_{ux} & 0 \end{pmatrix}, \tag{10}$$

which is equivalent to considering the 2-form $\mathbf{F} = \frac{1}{2}F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$, that is explicitly

$$\mathbf{F} = F_{ru}\mathrm{d}r \wedge \mathrm{d}u + F_{rx}\mathrm{d}r \wedge \mathrm{d}x + F_{ux}\mathrm{d}u \wedge \mathrm{d}x. \tag{11}$$

The field has *only three independent components*. These can be obtained from the electromagnetic potential 1-form $\mathbf{A} = A_a \mathrm{d}x^a$ by the standard relation

$$\mathbf{F} = \mathrm{d}\mathbf{A}. \tag{12}$$

Using (4), the corresponding contravariant components $F^{ab} \equiv g^{ac}g^{bd}F_{cd}$ read

$$F^{ru} = -\frac{F_x}{g_{xx}}, \qquad F^{rx} = \frac{F_u}{g_{xx}}, \qquad F^{ux} = -\frac{F_r}{g_{xx}}, \tag{13}$$

where the useful functions are

$$F_r \equiv F_{rx}, \tag{14}$$

$$F_x \equiv g_{xx}F_{ru} - g_{ux}F_{rx}, \tag{15}$$

$$F_u \equiv g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}. \tag{16}$$

In fact, these three functions are directly related to the components of the *dual Maxwell field 1-form* ${}^*\mathbf{F} = {}^*F_a \mathrm{d}x^a$ defined using the Hodge star operator,

$${}^*F^a \equiv \frac{1}{2}\omega^{abc}F_{bc}, \quad \text{where } \omega^{abc} = \frac{1}{\sqrt{-g}}\epsilon^{abc}. \tag{17}$$

Here g denotes the *determinant of the metric* g_{ab}, while ϵ^{abc} is the completely antisymmetric Levi-Civita symbol, for which we employ the convention that $\epsilon^{abc} = \epsilon_{abc} \equiv +1$ if abc is an even permutation of rux, it is -1 for odd permutation of rux, and 0 otherwise. For the metric (3) we immediately get

$$-g = g_{xx} \equiv G^{-2}, \tag{18}$$

and in view of (10) we obtain

$${}^*F^r = GF_{ux}, \qquad {}^*F^u = -GF_{rx}, \qquad {}^*F^x = GF_{ru}. \tag{19}$$

Using (14)–(16), the corresponding covariant components ${}^*F_a = g_{ab}{}^*F^b$ are

$${}^*F_a = GF_a, \tag{20}$$

so that the dual 1-form Maxwell field reads

$${}^*\mathbf{F} = G(F_r\mathrm{d}r + F_u\mathrm{d}u + F_x\mathrm{d}x). \tag{21}$$

For completeness let us also recall the inverse relation to (17),

$$F_{ab} = -\omega_{abc}{}^*F^c \quad \text{where } -\omega_{abc} = \sqrt{-g}\epsilon_{abc} = G^{-1}\epsilon_{abc}. \tag{22}$$

Next, it is necessary to evaluate the *electromagnetic invariants*

$$F^2 \equiv F_{ab}F^{ab}, \qquad {}^*F^2 \equiv {}^*F_a{}^*F^a. \tag{23}$$

A direct evaluation yields

$$F^2 = -2{}^*F^2 = -2G^2(g_{uu}F_{rx}^2 + 2F_{rx}(F_{ux} - g_{ux}F_{ru}) + g_{xx}F_{ru}^2). \tag{24}$$

Moreover, $F_{ab}{}^*F^a{}^*F^b = 0$ due to the symmetry reasons.

Similarly as for general relativity in $D = 4$, it is convenient to define *Newman-Penrose scalars of the Maxwell field* by its three independent projections onto the frame (6),

$$\begin{aligned} \phi_0 &\equiv F_{ab}k^a m^b, \\ \phi_1 &\equiv F_{ab}k^a l^b, \\ \phi_2 &\equiv F_{ab}m^a l^b. \end{aligned} \tag{25}$$

Explicit calculation reveals that

$$\phi_0 = GF_{rx} = GF_r, \tag{26}$$

$$\phi_1 = F_{ru} = G^2(F_x + g_{ux}F_r), \tag{27}$$

$$\phi_2 = G\left(g_{ux}F_{ru} - F_{ux} - \frac{1}{2}g_{uu}F_{rx}\right) = G\left(F_u + \frac{1}{2}g_{uu}F_r\right), \tag{28}$$

so that the invariant can be expressed as

$$\frac{1}{2}F^2 = 2\phi_0\phi_2 - \phi_1^2. \tag{29}$$

These scalars have *distinct boost weights* $+1$, 0, -1, respectively, and can be used for invariant *algebraic classification* of the electromagnetic field [13], based on its (non-)alignment with the geometrically privileged null vector field $\mathbf{k} = \partial_r$ of the metric. By definition the field is *aligned* if its component with the highest boost weight vanishes. From (26) we immediately observe that

$$\text{electromagnetic field is aligned with } \mathbf{k}$$
$$\Leftrightarrow \phi_0 = 0 \Leftrightarrow F_{rx} = 0 \Leftrightarrow F_r = 0. \tag{30}$$

It can also be shown that this is equivalent to the special property of the field, namely

$$F_{ab}k^b = \mathcal{N}k_a. \tag{31}$$

Such an aligned field has just two components, namely $\phi_1 = F_{ru}$ and $\phi_2 = G(g_{ux}F_{ru} - F_{ux})$, and $F^2 = -2\phi_1^2$. When $\phi_1 = 0 \Leftrightarrow F_x = 0$, the field is *null*. When $\phi_2 = 0 \Leftrightarrow F_u = 0$, it is *non-null*.

In the case when the electromagnetic field is *both aligned and null*, the invariant vanishes, $F^2 = 0$. This describes *purely radiative field*, i.e., a propagating electromagnetic wave characterized by the only nonvanishing component F_{ux}.

There is a freedom in the choice of the frame normalized as (2), given by the local *Lorentz transformations*. It consists of a boost $\mathbf{k}' = B\mathbf{k}$, $\mathbf{l}' = B^{-1}\mathbf{l}$ which determines the distinct boost weights $+1$, 0, -1 of (25), respectively. The second Lorentz transformation is a null rotation with fixed $\mathbf{k}$ of the form

$$\mathbf{k}' = \mathbf{k}, \qquad \mathbf{l}' = \mathbf{l} + \sqrt{2}L\mathbf{m} + L^2\mathbf{k}, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}L\mathbf{k}. \tag{32}$$

There is also an analogous null rotation with fixed $\mathbf{l}$ which changes $\mathbf{k}$. However, in our case the direction of $\mathbf{k}$ is geometrically privileged (being twist-free and shear-free). Only (32) thus needs to be considered. It is easy to prove that the Maxwell scalars (25) transform as

$$\begin{aligned}
\phi_0' &= \phi_0, \\
\phi_1' &= \phi_1 + \sqrt{2}L\phi_0, \\
\phi_2' &= \phi_2 + \sqrt{2}L\phi_1 + L^2\phi_0.
\end{aligned} \tag{33}$$

Of course, the expression (29) is invariant since $2\phi_0'\phi_2' - \phi_1'^2 = 2\phi_0\phi_2 - \phi_1^2$.

Finally, we need to evaluate the *energy-momentum tensor* for a generic electromagnetic field which (in any dimension, including $D = 3$) is defined as

$$T_{ab} \equiv \frac{\kappa_0}{8\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2\right), \tag{34}$$

where $\kappa_0 > 0$ is a constant depending on the choice of the physical units. Interestingly, in arbitrary dimension $D \geq 3$ the Maxwell field *satisfies all the standard energy conditions*; see Proposition 21 in [19].

A straightforward (but somewhat lengthy) calculation reveals that

$$\begin{aligned}
\frac{8\pi}{\kappa_0}T_{rr} &= G^2F_{rx}^2, \\
\frac{8\pi}{\kappa_0}T_{rx} &= G^2F_{rx}(g_{xx}F_{ru} - g_{ux}F_{rx}), \\
\frac{8\pi}{\kappa_0}T_{ru} &= \frac{1}{2}G^2(g_{xx}F_{ru}^2 - g_{uu}F_{rx}^2), \\
\frac{8\pi}{\kappa_0}T_{xx} &= -F_{rx}(g_{ux}F_{ru} + F_{ux}) + \frac{1}{2}G^2(2g_{ux}^2 - g_{xx}g_{uu})F_{rx}^2 \\
&\quad + \frac{1}{2}g_{xx}F_{ru}^2, \\
\frac{8\pi}{\kappa_0}T_{ux} &= \frac{1}{2}G^2[g_{ux}g_{uu}F_{rx}^2 - 2g_{xx}g_{uu}F_{ru}F_{rx} \\
&\quad + g_{xx}F_{ru}(g_{ux}F_{ru} - 2F_{ux})], \\
\frac{8\pi}{\kappa_0}T_{uu} &= \frac{1}{2}G^2[2F_{ux}^2 + 2g_{uu}F_{rx}F_{ux} + g_{uu}^2F_{rx}^2 - 4g_{ux}F_{ru}F_{ux} \\
&\quad - 2g_{ux}g_{uu}F_{rx}F_{ru} + (2g_{ux}^2 - g_{xx}g_{uu})F_{ru}^2],
\end{aligned} \tag{35}$$

and the corresponding trace $T \equiv g^{ab}T_{ab}$ is

$$\frac{8\pi}{\kappa_0}T = G^2F_{rx}(g_{ux}F_{ru} - F_{ux}) - \frac{1}{2}G^2(g_{xx}F_{ru}^2 + g_{uu}F_{rx}^2). \tag{36}$$

Now, it is a nice fact that, by combining (35) with (36) as $T_{ab} - Tg_{ab}$, the result for *all components* can be written in a *simple factorized form* as

$$\frac{8\pi}{\kappa_0}(T_{ab} - Tg_{ab}) = G^2F_aF_b, \tag{37}$$

in terms of the functions F_a encoding the electromagnetic field, which we have introduced in (14)–(16).

IV. EINSTEIN-MAXWELL FIELD EQUATIONS WITH Λ

Having identified all three-dimensional Lorentzian geometries—which can be written in the canonical form (3)—and also the generic form of the electromagnetic field (10) with the energy-momentum tensor of the form (35) implying (37), we can now apply the field equations.

Einstein's equations are $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, in which we also admit a nonvanishing *cosmological constant* Λ. Their trace is $R = 2(3\Lambda - 8\pi T)$, so that the equations can be put into the form $R_{ab} = 2\Lambda g_{ab} + 8\pi(T_{ab} - T g_{ab})$. For the generic electromagnetic field F_{ab} we have derived the nice relation (37), and thus the *Einstein field equations* in $2+1$ gravity with Λ, coupled to an electromagnetic field, are simply

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{38}$$

where the functions F_a are defined by (14)–(16). Expressed in terms of the dual Maxwell field ${}^*\mathbf{F}$ 1-form components [see (21) and (20)] these are even simpler, namely

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 {}^*F_a {}^*F_b. \tag{39}$$

In addition to these equations for the gravitational field represented by the metric g_{ab}, we must also satisfy the *Maxwell equations* $\mathrm{d}{}^*\mathbf{F} = 4\pi {}^*\mathbf{J}$ and $\mathrm{d}\mathbf{F} = 0$ for the electromagnetic field F_{ab}. In the *absence of electric charges and currents*, in components these read $F^{ab}{}_{;b} = 0$, $F_{[ab;c]} = 0$. They are equivalent to

$$(\sqrt{-g} F^{ab})_{,b} = 0, \tag{40}$$

$$F_{[ab,c]} = 0, \tag{41}$$

where, using (18),

$$\sqrt{-g} = \sqrt{g_{xx}} = G^{-1}. \tag{42}$$

Recall also that the source-free Maxwell equation $\mathrm{d}{}^*\mathbf{F} = 0$, which is equivalent to (40), in components reads ${}^*F_{[a,b]} = 0$. In view of (20), it can be directly written as

$$(G F_a)_{,b} = (G F_b)_{,a}. \tag{43}$$

Our task is to completely integrate the coupled system of the field equations (38) and (40), (41) [or, equivalently, (43) instead of (40)] in $2+1$ dimensions for (3) and (10), both for the nonexpanding Kundt spacetimes (Sec. V) and the expanding Robinson-Trautman spacetimes (Sec. VI and Sec. VII). Explicit components of the Ricci tensor R_{ab}, which enter (38), for these twist-free and shear-free geometries are given by Eqs. (A24)–(A29) in the Appendix.

A. Einstein field equations with a massless scalar field

Let us also remark that in three dimensions there is a relation between the Einstein-Maxwell system (39) and the Einstein gravity equations (minimally) coupled to a *massless scalar field* Φ such that

$$g^{ab} \Phi_{;ab} = 0. \tag{44}$$

Indeed, the corresponding energy-momentum tensor reads

$$T_{ab} \equiv \Phi_{,a}\Phi_{,b} - \frac{1}{2} g_{ab} \Phi_{,c}\Phi^{,c}, \tag{45}$$

implying the trace $T = -\frac{1}{2}\Phi_{,c}\Phi^{,c}$, so that the Einstein equations $R_{ab} = 2\Lambda g_{ab} + 8\pi(T_{ab} - T g_{ab})$ become

$$R_{ab} = 2\Lambda\, g_{ab} + 8\pi\, \Phi_{,a}\Phi_{,b}. \tag{46}$$

With the identification

$$\Phi_{,a} \equiv \sqrt{\frac{\kappa_0}{8\pi}}\, {}^*F_a, \tag{47}$$

this system of equations is clearly equivalent to (39). The dual Maxwell field 1-form is thus

$${}^*\mathbf{F} = \sqrt{\frac{8\pi}{\kappa_0}}\, \mathrm{d}\Phi. \tag{48}$$

V. ALL KUNDT SOLUTIONS

In this section, we explicitly perform a step-by-step integration of the field equations in the nonexpanding case $\Theta = 0$, which defines the Kundt family of spacetimes. Recall a consequence of (8) and (9), namely that the function G is now r independent. It can be renamed as $G(u,x) \equiv P(u,x)$. Also, the one-dimensional spatial metric $g_{xx} = G^{-2}$ must be r independent, that is

$$g_{xx} \equiv P^{-2}(u,x). \tag{49}$$

Of course, $g^{xx} = P^2$. Now, we will employ the Einstein field equations (38), which for the Kundt spacetimes take the form

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 P^2 F_a F_b. \tag{50}$$

A. Integration of $R_{rr} = \kappa_0 P^2 F_r^2$

In view of Eq. (A24), $R_{rr} = 0$ for $\Theta = 0$. Therefore, this Einstein equation immediately requires $F_r = 0$, that is

$$F_{rx} = 0. \tag{51}$$

It means that, inevitably, *any electromagnetic field in the* $2+1$ *Kundt spacetimes must be aligned with* $\mathbf{k} = \partial_r$. Such fields are fully described by the functions

$$F_r = 0, \qquad F_x = P^{-2} F_{ru}, \qquad F_u = g_{ux} F_{ru} - F_{ux}. \tag{52}$$

There are only *two* possible components of the electromagnetic field, namely F_{ru} and F_{ux}.

In fact, this result is analogous to the situation in standard $3+1$ general relativity, for which it is well known that (due to the Mariot-Robinson theorem) any Einstein-Maxwell field (including a cosmological constant Λ) in the Kundt class of geometries must be aligned; see the introductions to Chapter 31 of [10] and Chapter 18 of [11].

B. Integration of $R_{rx} = \kappa_0 P^2 F_r F_x$

The Ricci tensor component (A25) for $\Theta = 0$ reduces to $R_{rx} = -\frac{1}{2} g_{ux,rr}$. Since $F_r = 0$, we obtain a general solution of this Einstein equation:

$$g_{ux} = e(u,x) + f(u,x)r, \tag{53}$$

where e and f are arbitrary functions of u and x. In view of Eqs. (4) and (49), the corresponding contravariant component of the Kundt metric is

$$g^{rx} = P^2[e(u,x) + f(u,x)r]. \tag{54}$$

C. Integration of $R_{ru} = -2\Lambda + \kappa_0 P^2 F_r F_u$

Using Eqs. (49) and (53), the Ricci tensor component (A26) is $R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} P^2 (f_{||x} + f^2)$, where

$$f_{||x} \equiv f_{,x} + \frac{P_{,x}}{P} f \Leftrightarrow P f_{||x} \equiv (Pf)_{,x}. \tag{55}$$

Actually, the symbol $_{||}$ denotes the covariant derivative (of a 1-form f) related to the spatial metric g_{xx} on the one-dimensional "transverse" subspace with constant u and r, namely $f_{||x} = f_{,x} - {}^S\Gamma^x_{xx} f$, where ${}^S\Gamma^x_{xx} \equiv \frac{1}{2} g^{xx} g_{xx,x}$ is the corresponding Christoffel symbol (see the Appendix). Although this notation seems to be superficial here, we employ it in order to see the relation to our previous studies [20–22] of Kundt and Robinson-Trautman spacetimes in any higher dimension $D \geq 4$ where this geometric notation plays a key role.

Because $F_r = 0$, the corresponding Einstein equation thus simplifies, and can be integrated to

$$g_{uu} = a(u,x) + b(u,x)r + c(u,x)r^2, \tag{56}$$

where $a(u,x)$ and $b(u,x)$ are *arbitrary* functions, while

$$c(u,x) \equiv 2\Lambda + \frac{1}{2} P^2 (f_{||x} + f^2). \tag{57}$$

D. Integration of the Maxwell equations

The crucial r dependence of all metric functions for the $2+1$ Kundt spacetimes is thus determined. In general, g_{uu} is quadratic, g_{ux} is linear, and $g_{xx} \equiv P^{-2}(u,x)$ is independent of r. Now, applying the Maxwell equations (40), (41) with $\sqrt{-g} = P^{-1}$, we will determine the r dependence of the electromagnetic field.

In the present setting, there are only four independent Maxwell equations, namely three components of $(\sqrt{-g}F^{ab})_{,b} = 0$ and just one component of $F_{[ab,c]} = 0$. Because (13) with (52) implies

$$F^{ru} = -F_{ru}, \qquad F^{rx} = P^2(g_{ux}F_{ru} - F_{ux}), \qquad F^{ux} = 0, \tag{58}$$

these four equations for the electromagnetic field have the form

$$F_{ru,r} = 0, \tag{59}$$

$$(g_{ux}F_{ru} - F_{ux})_{,r} = 0, \tag{60}$$

$$(P(g_{ux}F_{ru} - F_{ux}))_{,x} = \left(\frac{F_{ru}}{P}\right)_{,u}, \tag{61}$$

$$F_{ux,r} + F_{ru,x} = 0. \tag{62}$$

These equations can be completely solved for the two nontrivial components F_{ru} and F_{ux}. Starting with (59), we immediately obtain that

$$F_{ru} = Q(u,x), \tag{63}$$

where $Q(u,x)$ is an arbitrary function independent of r. By employing (62), we thus get

$$F_{ux} = -Q_{,x} r - \xi(u,x), \tag{64}$$

where $\xi(u,x)$ is another arbitrary function. Equation (60) gives the constraint

$$Q_{,x} = -fQ, \tag{65}$$

and (61) reduces to the equation

$$(P(eQ + \xi))_{,x} = \left(\frac{Q}{P}\right)_{,u}. \tag{66}$$

To summarize, by integrating all the Maxwell equations we obtained explicit components of the (necessarily aligned) *electromagnetic field in any* $2+1$ *Kundt spacetime*,

$$F_{rx} = 0, \qquad F_{ru} = Q, \qquad F_{ux} = fQr - \xi, \tag{67}$$

where the functions $Q(u,x)$ and $\xi(u,x)$ are constrained by Eqs. (65), (66). Consequently,

$$F_r = 0, \qquad F_x = P^{-2}Q, \qquad F_u = eQ + \xi, \tag{68}$$

and, due to (26)–(28),

$$\phi_0 = 0, \qquad \phi_1 = Q, \qquad \phi_2 = P(eQ + \xi). \tag{69}$$

When $\phi_1 = 0 \Leftrightarrow Q = 0$, the field is *null*, and then $\phi_2 = P\xi$. When $\phi_2 = 0 \Leftrightarrow eQ = -\xi$, it is *non-null*, and then $\phi_1 = Q$.

Now, we can integrate the remaining three Einstein equations, which impose the unique relation between the gravitational and electromagnetic field components.

E. Integration of $R_{xx} = 2\Lambda g_{xx} + \kappa_0 P^2 F_x^2$

For $\Theta = 0$, using Eqs. (A36) and (53), the Ricci tensor component (A27) reduces to $R_{xx} = -f_{xx} \equiv -(f_{\|x} + \frac{1}{2}f^2)$. The field equation $R_{xx} = 2\Lambda g_{xx} + \kappa_0 P^2 (P^{-2}Q)^2 = (2\Lambda + \kappa_0 Q^2)P^{-2}$ implies

$$\kappa_0 Q^2 = -\left[2\Lambda + P^2\left(f_{\|x} + \frac{1}{2}f^2\right)\right]. \tag{70}$$

The electromagnetic field component $F_{ru} \equiv \phi_1 = Q(u,x)$ is thus *explicitly determined by the cosmological constant* Λ *and by the metric functions* P, f [provided the right-hand side of (70) is non-negative]. It is now convenient to introduce a rescaled form of f entering the metric function $g_{ux} = e + fr$ [see (53)], namely

$$F \equiv P^2 f^2. \tag{71}$$

Then the field equation (70) can be rewritten as

$$P^2(f_{\|x} + f^2) = \frac{1}{2}F - 2\Lambda - \kappa_0 Q^2. \tag{72}$$

We can thus simplify the metric function g_{uu}, namely its coefficient c in (56) given by (57), to

$$c(u,x) = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2. \tag{73}$$

At this stage, the most general Kundt solution in $D = 3$ takes the form

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{\mathrm{d}x^2}{P^2} + 2(e + fr)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r \\ & + \left[a + br + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right]\mathrm{d}u^2, \end{aligned} \tag{74}$$

and the Einstein-Maxwell field equation (72) using (55) reads

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0 Q^2\right). \tag{75}$$

F. Integration of $R_{ux} = 2\Lambda g_{ux} + \kappa_0 P^2 F_u F_x$

Equation (A28) with $\Theta = 0$ for the metric (74) gives $R_{ux} = \frac{1}{2}[f_{,u} - b_{,x} - eP^2(f_{\|x} + f^2) - f(\ln P)_{,u}] - \frac{1}{4}[(F - 2\kappa_0 Q^2)_{,x} + 2fP^2(f_{\|x} + f^2)]r$. Applying (72) and (68), (53), the corresponding field equation $R_{ux} = 2\Lambda g_{ux} + \kappa_0 Q(eQ + \xi) = 2\Lambda e + \kappa_0(eQ^2 + Q\xi) + 2\Lambda fr$ splits into two conditions, resulting from the coefficients for the powers r^1 and r^0, namely

$$F_{,x} - 2\kappa_0(Q^2)_{,x} + (F - 4\Lambda - 2\kappa_0 Q^2)f = -8\Lambda f, \tag{76}$$

$$\begin{aligned} & f_{,u} - b_{,x} - \left(\frac{1}{2}F - 2\Lambda - \kappa_0 Q^2\right)e - f(\ln P)_{,u} \\ & \quad = 4\Lambda e + 2\kappa_0(eQ^2 + Q\xi). \end{aligned} \tag{77}$$

Using the field equation (75), Eq. (76) simplifies to $(Q^2)_{,x} = -2Q^2 f$ which is *identically satisfied* due to (65). Only the constraint (77) thus remains, which can be put into the form

$$b_{,x} = f_{,u} - f(\ln P)_{,u} - \frac{1}{2}(F + 4\Lambda + 2\kappa_0 Q^2)e - 2\kappa_0 Q\xi, \tag{78}$$

that is

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + Pe(Pf)_{,x} - 2\kappa_0 Q\xi. \tag{79}$$

This is an explicit expression *determining the metric function* $b(u,x)$.

G. Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0 P^2 F_u^2$

For $\Theta = 0$ and the Kundt metric (74), using the relation $e_{\|x} \equiv e_{,x} + eP_{,x}/P$ and similar for $f_{\|x}$, $e_{,u\|x}$, $f_{,u\|x}$, $a_{\|xx}$, $b_{\|xx}$ and $c_{\|xx}$ (see the Appendix), the last Ricci tensor component (A29) reads

$$R_{uu} = A + Br + Cr^2, \tag{80}$$

where

$$\begin{aligned} A = {} & a\left(c - \frac{1}{2}F\right) + P^2\left[-\frac{1}{2}a_{,xx} + \frac{1}{2}a_{,x}\left(f - \frac{P_{,x}}{P}\right)\right. \\ & -\frac{1}{2}b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) \\ & \left. + (f_{,u} - b_{,x} - ce)e + \left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right) + \frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right], \end{aligned} \tag{81}$$

$$B = b\left(c - \frac{1}{2}F - \frac{1}{2}P(Pf)_{,x}\right) + P^2\left[\left(f_{,u} - \frac{1}{2}b_{,x}\right)_{,x} + \left(f_{,u} - \frac{1}{2}b_{,x}\right)\left(f + \frac{P_{,x}}{P}\right) - c\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) - 2e(c_{,x} + fc)\right], \tag{82}$$

$$C = c(c - F) - P^2\left[\frac{1}{2}c_{,xx} + \frac{1}{2}c_{,x}\left(3f + \frac{P_{,x}}{P}\right) + c\left(f_{,x} + \frac{P_{,x}}{P}f + \frac{1}{2}f^2\right)\right]. \tag{83}$$

Due to (56), (68), the corresponding field equation is $R_{uu} = 2\Lambda(a + br + cr^2) + \kappa_0 P^2(eQ + \xi)^2$, which splits into the following three constraints:

$$A = 2\Lambda a + \kappa_0 P^2(eQ + \xi)^2, \tag{84}$$

$$B = 2\Lambda b, \tag{85}$$

$$C = 2\Lambda c. \tag{86}$$

From (73), (75), (65) we easily derive interesting identities for spatial derivatives of c,

$$c_{,x} = -fc, \qquad c_{,xx} = (f^2 - f_{,x})c. \tag{87}$$

By using (87), the expression (83) reduces to $C = c[c - \frac{1}{2}F - \frac{1}{2}P(Pf)_{,x}]$, and substituting from (73), (75) we obtain $C = 2\Lambda c$. Equation (86) *is thus identically satisfied.*

Surprisingly, Eq. (85) *is also identically satisfied.* Applying (75), the first term in (82) yields $2\Lambda b$, while the complicated combination of various terms in the square brackets vanishes by using the relations (87), (78), (73) and the field equations (65), (66). Therefore, $B = 2\Lambda b$, which is Eq. (85).

We are thus left with *only one equation*, namely (84). Using (70), (73), (75), and (78), it can be simplified to

$$a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + \frac{P_{,x}}{P}f\right) = -b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right) - Pe^2(Pf)_{,x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\xi^2. \tag{88}$$

This equation *determines the last metric function* $a(u, x)$.

Alternatively, it can be understood as an *explicit expression for the* $\xi(u, x)$ component of the Maxwell field, in terms of the metric functions P, e, f, a, b. Such an equation can be expressed in a covariant form as

$$2\kappa_0\xi^2 = -a_{\|xx} + (fa)_{\|x} - b\left(e_{\|x} + \frac{P_{,u}}{P^3}\right) + 2(e_{,u})_{\|x} - P^2e^2f_{\|x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right), \tag{89}$$

where $a_{\|xx} \equiv a_{,xx} + \frac{P_{,x}}{P}a_{,x}$ and $\psi_{\|x} \equiv \psi_{,x} + \psi P_{,x}/P$, for ψ representing $a_{,x}$, f, e, and $e_{,u}$.

H. Summary of the Kundt solutions

We have thus solved all the Einstein-Maxwell equations with a cosmological constant Λ in $2 + 1$ gravity for the complete Kundt family of nonexpanding spacetimes. The generic gravitational field of this type is

$$\begin{aligned} g_{xx} &= P^{-2}(u, x), \\ g_{ux} &= e(u, x) + f(u, x)r \\ g_{uu} &= a(u, x) + b(u, x)r + c(u, x)r^2, \end{aligned} \tag{90}$$

where

$$c = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2, \tag{91}$$

with

$$F \equiv P^2f^2, \tag{92}$$

cf. (73), (71),while the electromagnetic field (67) reads

$$\begin{aligned} F_{rx} &= 0, \\ F_{ru} &= Q(u, x) \\ F_{ux} &= f(u, x)Q(u, x)r - \xi(u, x). \end{aligned} \tag{93}$$

Written explicitly in a compact form,

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2(e + fr)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + \left(a + br + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right)\mathrm{d}u^2, \tag{94}$$

and

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u + (fQr - \xi)\mathrm{d}u \wedge \mathrm{d}x, \tag{95}$$

corresponding to the potential

$$\mathbf{A} = A_r\mathrm{d}r + A_x\mathrm{d}x, \tag{96}$$

where, considering (65),

$$A_r \equiv -\int Q\mathrm{d}u, \qquad A_x \equiv r\int fQ\mathrm{d}u - \int \xi\mathrm{d}u. \tag{97}$$

It is now important to recall the *Maxwell scalars* given by (69),

$$\begin{aligned}\phi_0 &= 0,\\ \phi_1 &= Q,\\ \phi_2 &= P(eQ+\xi).\end{aligned} \tag{98}$$

We have thus proved that *all electromagnetic fields in the Kundt spacetimes* in $2+1$ gravity *are necessarily aligned* ($\phi_0 = 0$). Moreover, they split into *two distinct subclasses*:

(i) The case $\phi_1 = 0 \Leftrightarrow Q = 0$: The field is *null*, in which case $\phi_2 = P\xi$ and $F_{ux} = -\xi$, so that

$$\mathbf{F} = -\xi\mathrm{d}u\wedge\mathrm{d}x. \tag{99}$$

(ii) The case $\phi_2 = 0 \Leftrightarrow \xi = -eQ$: The field is *non-null* with only $\phi_1 = Q$, corresponding to

$$\mathbf{F} = Q\mathrm{d}r\wedge\mathrm{d}u + Q(e+fr)\mathrm{d}u\wedge\mathrm{d}x. \tag{100}$$

Notice also that, applying the Lorentz null rotation (32) with fixed $\mathbf{k}$ and the uniquely chosen parameter $L = -\frac{1}{\sqrt{2}}eP$ in (33), the scalars (98) transform to

$$\begin{aligned}\phi'_0 &= 0,\\ \phi'_1 &= Q,\\ \phi'_2 &= P\xi.\end{aligned} \tag{101}$$

Therefore, with respect to the triad with $\mathbf{m}' = \mathbf{m} + \sqrt{2}L\mathbf{k} = P(\partial_x + fr\partial_r)$, the condition for the Maxwell field being non-null is $\phi'_2 = 0 \Leftrightarrow \xi = 0$.

The *two electromagnetic components* Q, ξ and the *five metric functions* P, e, f, a, b describing the gravitational field are mutually constrained by the following Einstein-Maxwell field equations:

$$Q_{,x} = -fQ, \tag{102}$$

$$(QPe+P\xi)_{,x} = \left(\frac{Q}{P}\right)_{,u}, \tag{103}$$

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0 Q^2\right), \tag{104}$$

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + Pe(Pf)_{,x} - 2\kappa_0 Q\xi, \tag{105}$$

$$\begin{aligned}&a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + \frac{P_{,x}}{P}f\right)\\ &\quad= -b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right)\\ &\qquad - Pe^2(Pf)_{,x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\xi^2;\end{aligned} \tag{106}$$

see Eqs. (65), (66), (75), (79), and (88).

Interestingly, the form of the electromagnetic field (95) and also the same field equations (102)–(106) can *formally* be obtained by setting $D = 3$ in the corresponding equations for *higher-dimensional* Kundt spacetimes with an aligned Maxwell field [12].

Let us now separately discuss *two geometrically distinct subclasses*, namely $f = 0$ and $f \neq 0$.

1. *The subclass $f = 0$*

From (92) it follows that $f = 0 \Leftrightarrow F = 0$, so that Eqs. (102)–(106) considerably simplify to

$$Q_{,x} = 0, \tag{107}$$

$$(QPe+P\xi)_{,x} = \left(\frac{Q}{P}\right)_{,u}, \tag{108}$$

$$\kappa_0 Q^2 = -2\Lambda, \tag{109}$$

$$b_{,x} = -2\kappa_0 Q\xi, \tag{110}$$

$$\begin{aligned}(Pa_{,x})_{,x} &= -b\left((Pe)_{,x} + \frac{P_{,u}}{P^2}\right) + 2(Pe_{,u})_{,x}\\ &\quad + 2\left(\frac{P_{,u}}{P^2}\right)_{,u} - 2\kappa_0 P\xi^2.\end{aligned} \tag{111}$$

In this case, *Q is necessarily a constant*, and $\Lambda \leq 0$ because

$$2\Lambda = -\kappa_0 Q^2. \tag{112}$$

Therefore, the electromagnetic component ϕ_1 is also independent of u and x,

$$F_{ru} = \phi_1 = Q = \sqrt{-\frac{2}{\kappa_0}\Lambda}. \tag{113}$$

Keeping both the functions $P(u,x)$ and $\xi(u,x)$ *arbitrary*, Eq. (108) determines the metric function $e(u,x)$. Moreover, the function $b(u,x)$ is directly determined by the spatial integral of ξ via (110). Finally, integrating (111) we obtain $a(u,x)$.

Thus, we have obtained a complete and explicit family of such electrovacuum Kundt spacetimes in $2+1$ gravity, namely

$$ds^2 = \frac{dx^2}{P^2} + 2e\,du\,dx - 2du\,dr + (a + br + 2\Lambda r^2)du^2, \tag{114}$$

and

$$\mathbf{F} = Q\,dr \wedge du - \xi\,du \wedge dx. \tag{115}$$

It admits *four physically distinct subcases*:

(i) The case $Q = 0 = \xi$: The electromagnetic field **F** *vanishes*, and necessarily $\Lambda = 0$. The metric is

$$ds^2 = \frac{dx^2}{P^2} + 2e\,du\,dx - 2du\,dr + (a + br)du^2, \tag{116}$$

where $b(u)$ is independent of x. It is a vacuum solution without a cosmological constant, and thus in $2+1$ gravity it must be *flat Minkowski space*. We derived this metric in our previous work [1]; see Eq. (82) with $\mathcal{J} = 0 = \mathcal{N}$ therein.

(ii) The case $Q = 0$: Again, $\Lambda = 0$ and $b = b(u)$, so that the metric has the form (116), but there is now a *radiative (null) electromagnetic field*

$$\mathbf{F} = -\xi\,du \wedge dx. \tag{117}$$

The amplitude $\xi(u, x)$ must satisfy the field equation (108), which is $(P\xi)_{,x} = 0$. Therefore,

$$\xi(u, x) = \frac{\gamma(u)}{P(u, x)}, \tag{118}$$

where $\gamma(u)$ is *an arbitrary profile function* of the retarded time u. Finally, $a(u, x)$ is then obtained by integrating the remaining field equation (111).

(iii) The case $\xi = 0$: The *electromagnetic field is non-null*, and has the form

$$\mathbf{F} = Q\,dr \wedge du, \tag{119}$$

where Q is a *constant uniquely determined by negative cosmological constant* Λ via (113). The electromagnetic field is thus uniform, and positive (or zero) Λ is not allowed.

The metric is of the form (114). The field equation (110) implies that $b = b(u)$, while the remaining (108) and (111) reduce to

$$(Pe)_{,x} = -\frac{P_{,u}}{P^2}, \tag{120}$$

$$(Pa_{,x})_{,x} = 2(Pe_{,u})_{,x} - 2(Pe)_{,ux}. \tag{121}$$

The latter can be immediately integrated to

$$a_{,x} = 2e_{,u} - \frac{2}{P}(Pe)_{,u} + \frac{\delta(u)}{P}, \tag{122}$$

where $\delta(u)$ is any function of u. After prescribing an arbitrary metric function $P(u, x)$, we obtain $e(u, x)$ by integrating (120), and $a(u, x)$ by integrating (122).

(iv) The general case $Q \neq 0$, $\xi \neq 0$: In the generic case with both the non-null component of the electromagnetic field $Q = \text{const}$ and its null component $\xi(u, x)$, we obtain the superposition (115). The metric reads (114), with a cosmological constant $\Lambda < 0$ [notice that $\Lambda = 0$ implies $Q = 0$ due to (112), while $\Lambda > 0$ is forbidden]. The metric functions a and b are determined by the differential equations (110) and (111), respectively, and there is also the constraint (108) determining e.

This family of Kundt spacetimes in $2+1$ gravity can be interpreted as *mutually coupled exact gravitational and electromagnetic waves* [characterized by the functions $a(u, x)$ and $\xi(u, x)$, respectively] which propagate on the *background with* $\Lambda < 0$ *and uniform Maxwell field* (characterized by the constant Q). The simplest such background is

$$ds^2 = dx^2 - 2du\,dr + 2\Lambda r^2 du^2, \tag{123}$$

which is the $2+1$ analog of the exceptional electrovacuum type D metric with $\Lambda < 0$ found by Plebański and Hacyan [23]; see also Eq. (7.20) in [11]. Indeed, introducing $\mathcal{U} = 1/(2\Lambda u)$ and $\mathcal{V} = 2(u + 1/(\Lambda r))$, the metric (123) takes the form $ds^2 = dx^2 - 2d\mathcal{U}d\mathcal{V}/(1 - \Lambda\mathcal{U}\mathcal{V})^2$ which is clearly the direct-product $E^1 \times AdS_2$ spacetime.

2. The subclass $f \neq 0$

Recalling $F \equiv P^2 f^2$, cf. (92), in this case $F \neq 0$. The Kundt metric takes the general form (94), the aligned electromagnetic field is (95), and the corresponding Einstein-Maxwell field equations are (102)–(106).

By inspecting this system, it is seen that the first three differential equations (102), (103), (104) relate the metric functions P, e, f and the electromagnetic field components Q, ξ. Subsequently, the remaining two equations (105) and (106) can be used to evaluate the metric functions b and a, respectively.

Starting with (102), we immediately observe that there are *two distinct subcases*:

(i) The case $Q = 0$: The electromagnetic field is *null* (with $\phi_1 = 0$, $\phi_2 = P\xi$),

$$\mathbf{F} = -\xi\,du \wedge dx. \tag{124}$$

The field equation (102) is identically satisfied, putting no restriction on the function f, while (103), (104) reduce to

$$P\xi = \gamma(u), \tag{125}$$

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}(Pf)^2\right). \tag{126}$$

The first equation determines ξ, giving the same expression as (118), i.e., $\xi(u,x) = P^{-1}\gamma(u)$, while the second equation can be integrated for the variable (Pf) in terms of the integral of P^{-1}, yielding

$$f(u,x) = -2\sqrt{\Lambda}P^{-1}\tan\left[\sqrt{\Lambda}\int P^{-1}\mathrm{d}x\right] \quad \text{for } \Lambda > 0, \tag{127}$$

and the expression for $\Lambda < 0$ is analogous, replacing tan by tanh.

In the final step, the metric functions b and a are obtained by integrating the field equations (105) and (106), respectively.

(ii) The case $Q \neq 0$: In this generic case, the field equation (102) explicitly determines the metric function f in terms of the electromagnetic field component Q, which occurs in

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u + (fQr - \xi)\mathrm{d}u \wedge \mathrm{d}x, \tag{128}$$

as

$$f(u,x) = -(\ln Q)_{,x}. \tag{129}$$

However, there is a further constraint given the field equation (104),

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}(Pf)^2 + \kappa_0 Q^2\right). \tag{130}$$

Notice that it can also be rewritten as

$$F_{,x} = -f(F + 4\Lambda + 2\kappa_0 Q^2), \tag{131}$$

or, equivalently,

$$\kappa_0 Q^2 = -\frac{1}{2f}[F_{,x} + (F + 4\Lambda)f]. \tag{132}$$

It remains to be investigated what are the constraints resulting from the simultaneous solution of Eqs. (129) and (132).

VI. ALL ALIGNED ROBINSON-TRAUTMAN SOLUTIONS

After completing the derivation and preliminary description of the nonexpanding Kundt class, we will now concentrate on systematic integration of the field equations in the *expanding* case $\Theta \neq 0$, which defines the Robinson-Trautman family of spacetimes.

Recall that the field equations (38) take the form

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{133}$$

where F_a are defined by (14)–(16). In this section *we assume that the electromagnetic field is aligned* with $\mathbf{k} = \partial_r$ [see (31)], that is

$$F_{rx} = 0 \Leftrightarrow F_r = 0. \tag{134}$$

This considerably simplifies the field equations (133) whenever at least one of the index a, b is r.

A. Integration of $R_{rr} = 0$

From Eq. (A24) we immediately get the constraint

$$\Theta_{,r} + \Theta^2 = 0, \tag{135}$$

which determines the r dependence of the expansion scalar Θ. Its general solution can be written as $\Theta^{-1} = r + r_0(u,x)$. Because the metric (3) is invariant under the gauge transformation $r \to r - r_0(u,x)$, without loss of generality we can set the integration function $r_0(u,x)$ to zero. The expansion thus simplifies to

$$\Theta = \frac{1}{r}. \tag{136}$$

Integrating now the key relation (9) we obtain

$$G(r,u,x) = \frac{P(u,x)}{r}, \tag{137}$$

where $P(u,x)$ is any function independent of r. Using (8), we immediately get the generic spatial metric function $g_{xx} \equiv G^{-2}$ in the form

$$g_{xx} = \frac{r^2}{P^2(u,x)}. \tag{138}$$

Of course, by inversion $g^{xx} = P^2 r^{-2}$.

B. Integration of $R_{rx} = 0$

Using Eqs. (A25) and (135), which implies Eq. (136), the Ricci tensor component R_{rx} becomes

$$R_{rx} = -\frac{1}{2}(g_{ux,rr} - g_{ux,r}r^{-1}). \tag{139}$$

The corresponding field equation $R_{rx} = 0$ can be integrated, yielding a general solution

$$g_{ux} = e(u,x)r^2 + f(u,x), \tag{140}$$

where e and f are arbitrary functions of u and x. In view of Eqs. (5) and (138), the contravariant component of the Robinson-Trautman metric is

$$g^{rx} = P^2[e(u,x) + f(u,x)r^{-2}]. \tag{141}$$

C. Integration of the Maxwell equations

Now, applying the Maxwell equations (40), (41) with $\sqrt{-g} = G^{-1} = r/P$, we will determine the electromagnetic field. There are only four independent Maxwell equations, namely three components of $(\sqrt{-g}F^{ab})_{,b} = 0$ and just one component of $F_{[ab,c]} = 0$. Because (13) with (134) implies

$$F^{ru} = -F_{ru}, \qquad F^{rx} = \frac{P^2}{r^2}(g_{ux}F_{ru} - F_{ux}), \qquad F^{ux} = 0, \tag{142}$$

these four equations for the electromagnetic field take the form

$$(rF_{ru})_{,r} = 0, \tag{143}$$

$$(r^{-1}(g_{ux}F_{ru} - F_{ux}))_{,r} = 0, \tag{144}$$

$$r^2\left(\frac{F_{ru}}{P}\right)_{,u} = (P(g_{ux}F_{ru} - F_{ux}))_{,x}, \tag{145}$$

$$F_{ux,r} + F_{ru,x} = 0. \tag{146}$$

They can be solved for the nontrivial components F_{ru} and F_{ux}. From (143) we get

$$F_{ru} = \frac{Q(u,x)}{r}, \tag{147}$$

where $Q(u,x)$ is an arbitrary function of u and x. By employing (146), we thus obtain

$$F_{ux} = -Q_{,x}\ln|r| - \xi(u,x), \tag{148}$$

where $\xi(u,x)$ is another arbitrary function. Equation (144) with (140) then reduces to

$$\left(\frac{fQ}{r^2} + Q_{,x}\frac{\ln|r|}{r} + \frac{\xi}{r}\right)_{,r} = 0, \tag{149}$$

which gives the following three independent constraints:

$$fQ = 0, \qquad Q_{,x} = 0, \qquad \xi = Q_{,x}, \tag{150}$$

so that $\xi = 0$ and $Q = Q(u)$ is independent of x.

We thus conclude that the components of a generic *aligned electromagnetic field in any* $2+1$ *Robinson-Trautman spacetime* can be written as

$$F_{rx} = 0, \qquad F_{ru} = \frac{Q(u)}{r}, \qquad F_{ux} = 0, \tag{151}$$

with the constraint

$$fQ = 0, \tag{152}$$

and the Maxwell equation (145) which reduces to

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{153}$$

Consequently,

$$F_r = 0, \qquad F_x = P^{-2}Qr, \qquad F_u = eQr, \tag{154}$$

and, due to (26)–(28),

$$\phi_0 = 0, \qquad \phi_1 = \frac{Q}{r}, \qquad \phi_2 = ePQ. \tag{155}$$

When $\phi_1 = 0 \Leftrightarrow Q = 0$ then $\phi_2 = 0$. Therefore, there are *no null electromagnetic fields of this type*. When $\phi_2 = 0 \Leftrightarrow eQ = 0$, it is *non-null*, and then $\phi_1 = Q(u)/r$. Notice also, that due to (152), either we have a vacuum solution $(Q = 0)$ or a non-null electromagnetic field characterized by $Q(u)$ in the Robinson-Trautman spacetime *without the nondiagonal metric term* $(g_{ux} = 0)$.

Now, we will integrate the remaining Einstein's equations which couple the gravitational and electromagnetic fields. In view of (152), there are two cases to consider, namely $Q = 0$ and $f = 0$.

(i) The case $Q = 0$: The electromagnetic field completely vanishes, so that the spacetimes are *vacuum* (with any cosmological constant Λ). All such Robinson-Trautman solutions in $2+1$ gravity were found and described in our previous work [1]. Interestingly, for these vacuum spacetimes the function f remains nonvanishing (which is not true in $D \geq 4$).

(ii) The case $f = 0$: In this case, the metric component g_{ux} reduces to

$$g_{ux} = er^2 \Leftrightarrow g^{rx} = P^2e. \tag{156}$$

This simplifies the generic Ricci tensor components in the Appendix, which will now apply.

D. Integration of $R_{ru} = -2\Lambda$

Using (156), (136), and (138), the Ricci tensor component (A26) becomes

$$R_{ru} = -\frac{1}{2}(r g_{uu,r})_{,r} r^{-1} + \frac{1}{2} c r^{-1} + 2P^2 e^2, \tag{157}$$

where

$$c \equiv 2P^2 \left(e_{\|x} - \frac{1}{2} h_{xx,u} \right), \qquad e_{\|x} \equiv e_{,x} + eP_{,x}/P, \tag{158}$$

from which we obtain useful identities

$$P e_{\|x} = (Pe)_{,x}, \qquad eP^2 e_{\|x} = \frac{1}{2}(P^2 e^2)_{,x}, \tag{159}$$

and thus

$$c = 2[P(Pe)_{,x} + (\ln P)_{,u}]. \tag{160}$$

With Eq. (157), the Einstein equation $R_{ru} = -2\Lambda$ can now be easily integrated to give

$$g_{uu} = -a - b \ln|r| + cr + (\Lambda + P^2 e^2) r^2, \tag{161}$$

where $a(u,x)$ and $b(u,x)$ are arbitrary functions. The r dependence of all metric components is thus fully established.

E. Integration of $R_{xx} = 2\Lambda g_{xx} + \kappa_0 G^2 F_x^2$

Using Eqs. (135)–(138) and (156), the general Ricci tensor component (A27) becomes

$$R_{xx} = -cP^{-2} r - 2e^2 r^2 + P^{-2} r g_{uu,r}. \tag{162}$$

Substituting now the expression (161), we obtain $R_{xx} = 2\Lambda g_{xx} - b/P^2$. The corresponding Einstein equation with (154) reads $R_{xx} = 2\Lambda g_{xx} + \kappa_0 Q^2/P^2$. It is satisfied if, and only if,

$$b(u) = -\kappa_0 Q^2. \tag{163}$$

F. Integration of $R_{ux} = 2\Lambda g_{ux} + \kappa_0 G^2 F_u F_x$

Using Eqs. (136), (138), (156), and (161) with (163), the Ricci tensor component R_{ux} given by Eq. (A28) reads

$$R_{ux} = 2\Lambda g_{ux} + \kappa_0 e Q^2 - \frac{1}{2} a_{,x} r^{-1}. \tag{164}$$

The field equation with (154) is $R_{ux} = 2\Lambda g_{ux} + \kappa_0 e Q^2$, so that we obtain just one simple constraint:

$$a_{,x} = 0 \Leftrightarrow a = a(u). \tag{165}$$

The function a can depend only on the coordinate u, and the most general Robinson-Trautman aligned electrovacuum solution thus takes the form

$$\mathrm{d}s^2 = \frac{r^2}{P^2}\mathrm{d}x^2 + 2er^2\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + \left(-a(u) + \kappa_0 Q^2(u) \ln|r| + 2[P(Pe)_{,x} + (\ln P)_{,u}] r + (\Lambda + P^2 e^2) r^2 \right) \mathrm{d}u^2. \tag{166}$$

G. Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0 G^2 F_u^2$

The Ricci tensor component R_{uu} for the metric (166), given generally by Eq. (A29), becomes

$$R_{uu} = 2\Lambda g_{uu} + A + \frac{1}{2}\left[a_{,u} - \left(a - \frac{1}{2} b \right) c - \triangle c \right] \frac{1}{r} + \frac{1}{2}[b_{,u} - bc] \frac{\ln r}{r}, \tag{167}$$

where

$$A = -P^2 e^2 b + \frac{1}{4}c^2 + \frac{1}{2}P^2 e c_{,x} - \frac{1}{2}c_{,u} - \frac{1}{2}\triangle(P^2 e^2) + P(P e_{,u})_{,x} - 2\frac{P_{,u}^2}{P^2} + \frac{P_{,uu}}{P}, \tag{168}$$

c is given by Eq. (160), and

$$\triangle c \equiv h^{xx} c_{\|xx} = P(P c_{,x})_{,x} \tag{169}$$

is the covariant Laplace operator on the one-dimensional transverse Riemannian space spanned by x, applied on the function c. Remarkably, after substitution from (160) and evaluation, the expression for A enormously simplifies to

$$A = -P^2 e^2 b. \tag{170}$$

Moreover, using the Maxwell equation (153) which can be rewritten as

$$Q_{,u} = \frac{1}{2} c Q, \tag{171}$$

and the relation (163), that is $b = -\kappa_0 Q^2$, we easily prove that $b_{,u} = bc$. The last term in (167) thus always vanishes. To summarize, the last Ricci tensor component takes the form

$$R_{uu} = 2\Lambda g_{uu} + \kappa_0 e^2 P^2 Q^2 + \frac{1}{2}\left[a_{,u} - \left(a - \frac{1}{2} b\right) c - \triangle c\right]\frac{1}{r}. \tag{172}$$

Using (154), the corresponding field equation reads $R_{uu} = 2\Lambda g_{uu} + \kappa_0 e^2 P^2 Q^2$, so that we obtain *only one additional condition* determined by the term proportional to r^{-1}, namely

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c + \triangle c. \tag{173}$$

Let us observe that Eq. (171) implies

$$c(u) = 2(\ln Q)_{,u}, \tag{174}$$

i.e., the function c must necessarily be independent of the spatial coordinate x. Due to (169), $\triangle c = 0$, and the field equation (173) reduces to

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c. \tag{175}$$

Its general solution with (174) is

$$a(u) = Q^2 (\kappa_0 \ln |Q| - \mu), \tag{176}$$

where μ is any constant. The metric function $a(u)$ is thus directly related to the electromagnetic field $Q(u)$.

H. Summary of the aligned Robinson-Trautman solutions

We have solved all the Einstein-Maxwell equations with a cosmological constant Λ and *aligned* electromagnetic field in $2+1$ gravity for the Robinson-Trautman family of expanding spacetimes. In the canonical coordinates, the generic gravitational field of this type is

$$\begin{aligned} g_{xx} &= P^{-2}(u,x) r^2, \\ g_{ux} &= e(u,x) r^2, \\ g_{ur} &= -1, \\ g_{uu} &= \mu Q^2(u) - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + (\Lambda + P^2 e^2) r^2, \end{aligned} \tag{177}$$

where μ is a constant, $Q(u)$ is any function of u, and the metric functions P, e satisfy the field equation (153), that is

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{178}$$

The corresponding aligned electromagnetic field reads

$$\begin{aligned} F_{rx} &= 0, \\ F_{ru} &= \frac{Q(u)}{r}, \\ F_{ux} &= 0, \end{aligned} \tag{179}$$

see (151); i.e., it has *only one component* F_{ru}.

Written explicitly in the usual compact form, the solution is

$$\begin{aligned} ds^2 = {}& \frac{r^2}{P^2}(dx + eP^2 du)^2 - 2 du dr \\ & + \left(\mu Q^2 - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + \Lambda r^2\right) du^2, \end{aligned} \tag{180}$$

with

$$\mathbf{F} = \frac{Q}{r} dr \wedge du \quad \text{equivalent to} \quad {}^*\mathbf{F} = \frac{Q}{P} dx + ePQ du, \tag{181}$$

corresponding to the potential

$$\mathbf{A} = Q \ln\frac{r}{r_0} du, \tag{182}$$

and the Maxwell scalars (155)

$$\begin{aligned} \phi_0 &= 0, \\ \phi_1 &= \frac{Q}{r}, \\ \phi_2 &= ePQ. \end{aligned} \tag{183}$$

It follows that there are *no aligned (purely) null electromagnetic fields* in the Robinson-Trautman spacetimes in $2+1$ gravity because $\phi_1 = 0$ implies $\phi_2 = 0$. Moreover,

$\phi_2 = 0 \Leftrightarrow eQ = 0$. Either we have a vacuum solution ($Q = 0$) or a non-null electromagnetic field characterized by $Q(u)$ in the Robinson-Trautman spacetime without the nondiagonal metric term g_{ux} ($e = 0$).

The simplest $e \neq 0$ solution of the field equation (178), which can be rewritten as

$$(\ln P)_{,u} + P(eP)_{,x} = (\ln Q)_{,u}, \tag{184}$$

is

$$P = 1, \qquad e = x(\ln Q)_{,u} + \alpha(u), \tag{185}$$

where $\alpha(u)$ is an arbitrary function of u, yielding the metric

$$\mathrm{d}s^2 = r^2\Big(\mathrm{d}x + (\alpha + x(\ln Q)_{,u})\mathrm{d}u\Big)^2 - 2\mathrm{d}u\mathrm{d}r + \left(\mu Q^2 - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + \Lambda r^2\right)\mathrm{d}u^2. \tag{186}$$

Another interesting subclass of the Robinson–Trautman spacetimes (180) with aligned Maxwell field (181) arises when both sides of the field equation (178) vanish, $(Q/P)_{,u} = 0 \Leftrightarrow (eP)_{,x} = 0$. Then the metric functions P and e are *both factorized* in the coordinates u and x as

$$P = Q(u)\beta(x), \qquad e = \frac{\alpha(u)}{Q(u)\beta(x)}, \tag{187}$$

where $\alpha(u)$, $\beta(x)$ are arbitrary functions of the respective coordinates. Consequently, $eP = \alpha(u)$. [For $\beta = 1$ we obtain simply $P(u) = Q(u)$.] In such a case, the metric (180) takes the form

$$\mathrm{d}s^2 = \frac{r^2}{Q^2}\left(\frac{\mathrm{d}x}{\beta} + \alpha Q \mathrm{d}u\right)^2 - 2\mathrm{d}u\mathrm{d}r + \left(\mu Q^2 - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + \Lambda r^2\right)\mathrm{d}u^2, \tag{188}$$

and the Maxwell scalars are

$$\phi_0 = 0, \qquad \phi_1 = \frac{Q}{r}, \qquad \phi_2 = \alpha Q.$$

With respect to the natural triad (6), there are thus two components of the admitted Maxwell field, namely non-null component ϕ_1 and the electromagnetic radiation ϕ_2 ($\phi_2 \neq 0$ requires $\alpha \neq 0$). However, let us remark that, due to the freedom in the choice of the local null triad, under which the Maxwell scalars transform as (33), at a given point there exists a special triad in which $\phi_2' = 0$.

There is a *special case* $Q =$ const, for which the metric (188) simplifies to

$$\mathrm{d}s^2 = r^2\Big(\mathrm{d}\varphi + \alpha(u)\mathrm{d}u\Big)^2 - 2\mathrm{d}u\mathrm{d}r + \left(m - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + \Lambda r^2\right)\mathrm{d}u^2, \tag{189}$$

where the rescaled constant reads $m \equiv Q^2\mu$, and the new coordinate is

$$\varphi = \frac{1}{Q}\int \frac{\mathrm{d}x}{\beta(x)}. \tag{190}$$

For $\alpha(u) = 0$ (that is, without the electromagnetic radiation component), and for compact coordinate φ, this family of spacetimes represents *charged black holes* with any value of the cosmological constant Λ. Indeed, by introducing the time coordinate t via the transformation

$$\mathrm{d}u = \mathrm{d}t + \left(m - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + \Lambda r^2\right)^{-1}\mathrm{d}r, \tag{191}$$

we obtain the metric

$$\mathrm{d}s^2 = -\left(-m + \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{-m + \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2} + r^2\mathrm{d}\varphi^2, \tag{192}$$

with the electromagnetic field

$$\mathbf{F} = \frac{Q}{r}\mathrm{d}r \wedge \mathrm{d}t \quad \text{corresponding to} \quad \mathbf{A} = Q\ln\frac{r}{r_0}\mathrm{d}t. \tag{193}$$

This is the standard form of *cyclic symmetric, electrostatic solution with* Λ in polar "Schwarzschild" coordinates found by Peldan in 1993 [24], see Eq. (11.56) in [6], which extended previous solutions by Gott, Simon and Alpert [25,26], Deser and Mazur [27], and Melvin [28] to any cosmological constant; see also García [29]. A thorough review and discussion of this class of solutions is contained in [30] and also Sec. 11.2 of [6].

For $\alpha(u) \neq 0$ the spacetime (189) in general contains *additional* electromagnetic *radiation component* $\phi_2 \neq 0$. It remains to be analyzed in detail if such a situation can be physically interpreted as a charged black hole with a specific radiation, or if the function $\alpha(u)$ is just some kind of a kinematic parameter.

Similarly, the general Robinson-Trautman solution (180) with aligned electromagnetic field (181) needs to be understood and explicitly related to other known solutions summarized in Chapter 11 of [6], in particular the nonstatic ones. This seems to be in principle possible because, e.g., for $e \neq 0$ the transformation (191) introduces the metric component g_{tx} typical for stationary spacetimes.

VII. ALL NONALIGNED ROBINSON-TRAUTMAN SOLUTIONS

After completing the systematic derivation of all aligned electromagnetic fields in the family of expanding Robinson-Trautman geometries, we now investigate the *possible nonaligned fields*.

The Einstein–Maxwell equations are (133), in which the functions F_a are defined by (14)–(16). The generic non-aligned electromagnetic field has $\phi_0 \neq 0 \Leftrightarrow F_{rx} \neq 0 \Leftrightarrow F_r \neq 0$.

A. Integration of $R_{rr} = \kappa_0 G^2 F_r^2$

Using Eq. (A24) for the Ricci tensor component R_{rr}, we obtain the constraint

$$\kappa_0 F_r^2 = -g_{xx}(\Theta_{,r} + \Theta^2), \tag{194}$$

where $\Theta \neq 0$ is the optical scalar representing the *expansion* of the privileged null congruence generated by $\mathbf{k} = \partial_r$. Let us recall that it is directly related to the spatial metric function g_{xx} via the relations

$$g_{xx} = G^{-2} \quad \text{with} \quad \Theta = -(\ln G)_{,r} \equiv -\frac{G_{,r}}{G}; \tag{195}$$

see (8), (9). Therefore, the metric component g_{xx} must necessarily depend on the coordinate r, otherwise $\Theta = 0$.

It is possible to substitute from (195) into (194), but we found it more convenient to keep the expansion scalar Θ in (194). This equation *explicitly expresses the nonaligned Maxwell field component $F_{rx} \equiv F_r$ in terms of the metric component g_{xx}* (and its r derivatives via G). This relation can be rewritten as

$$\kappa_0 F_{rx}^2 = G^{-2}\Theta^2\Big((\Theta^{-1})_{,r} - 1\Big). \tag{196}$$

Notice that (in the Robinson-Trautman family) $F_{rx} = 0 \Leftrightarrow \Theta^{-1} = r + r_0(u,x)$. This fully corresponds to the previously studied aligned case, for which (136) applies.

B. Integration of $R_{rx} = \kappa_0 G^2 F_r F_x$

Using Eq. (A25) for the Ricci tensor component R_{rx} and (194), we get the relation

$$\frac{1}{2}(\Theta g_{ux,r} - g_{ux,rr}) = \kappa_0 G^2 F_r (F_x + g_{ux} F_r). \tag{197}$$

In view of (14), (15), this is equivalent to

$$\kappa_0 F_{ru} F_{rx} = \frac{1}{2}(\Theta g_{ux,r} - g_{ux,rr}). \tag{198}$$

Therefore, *by prescribing any metric function g_{ux}, the electromagnetic field component F_{ru} is explicitly determined.*

Notice that it *admits a special solution* $F_{ru} = 0 \Leftrightarrow \Theta g_{ux,r} = g_{ux,rr}$. This occurs *either* when g_{ux} is independent of the coordinate r,

$$g_{ux} = B(u,x), \tag{199}$$

or, using (195), when $\Theta = (\ln G^{-1})_{,r} = (\ln g_{ux,r})_{,r}$ which can be completely integrated as

$$g_{xx} = A(u,x)(g_{ux,r})^2, \tag{200}$$

where $A > 0$ is any function independent of r.

C. Integration of $R_{ru} = -2\Lambda + \kappa_0 G^2 F_r F_u$

The generic Ricci tensor component R_{ru} is given by (A26), so that the corresponding Einstein-Maxwell field equation becomes

$$\begin{aligned}
&-\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{rx}g_{ux,rr} + \frac{1}{2}g^{xx}(g_{ux,r\|x} + (g_{ux,r})^2)\\
&\quad - \Theta_{,u} - \frac{1}{2}\Theta(g^{xx}g_{xx,u} + g^{rx}g_{ux,r} + g_{uu,r})\\
&= -2\Lambda + \kappa_0 G^2 F_r F_u.
\end{aligned} \tag{201}$$

This uniquely *determines the third electromagnetic field component* (16) represented by F_u. Using (14)–(16) and then (194), (198), the last term on the right-hand side can be expressed as

$$\begin{aligned}
&\kappa_0 G^2 F_{rx}(g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx})\\
&= \kappa_0 g^{rx}F_{ru}F_{rx} - \kappa_0 g^{xx}F_{ux}F_{rx} - \kappa_0 g^{xx}g_{uu}F_{rx}^2\\
&= -\kappa_0 g^{xx}F_{ux}F_{rx} + \frac{1}{2}g^{rx}(\Theta g_{ux,r} - g_{ux,rr}) + g_{uu}(\Theta_{,r} + \Theta^2).
\end{aligned} \tag{202}$$

The field equation (201) thus reads

$$\begin{aligned}
\kappa_0 F_{ux}F_{rx} &= \frac{1}{2}g_{xx}(g_{uu,rr} + \Theta g_{uu,r} + 2(\Theta_{,r} + \Theta^2)g_{uu} - 4\Lambda)\\
&\quad + g_{ux}(\Theta g_{ux,r} - g_{ux,rr}) - \frac{1}{2}(g_{ux,r\|x} + (g_{ux,r})^2)\\
&\quad + \frac{1}{2}\Theta g_{xx,u} + g_{xx}\Theta_{,u}.
\end{aligned} \tag{203}$$

By prescribing any metric function g_{uu}, the third electromagnetic field component F_{ux} is thus explicitly determined.

To summarize, by employing three (out of six) independent components of the Einstein field equations, we have now derived explicit expressions (196), (198), and (203) which *determine all three components of the* *electromagnetic field*, namely F_{rx}, F_{ru}, and F_{ux}, respectively, in terms of the three (so far) independent metric components g_{xx}, g_{ux}, and g_{uu}.

These three expressions are equivalent to Eqs. (194), (197), (201) for the three dual electromagnetic functions $F_a \equiv {}^*F_a/G$. They can be written in a very compact form:

$$\kappa_0 F_r^2 = \alpha, \tag{204}$$

$$\kappa_0 F_r F_x = \beta - \alpha g_{ux}, \tag{205}$$

$$\kappa_0 F_r F_u = \gamma, \tag{206}$$

where the functions α, β, γ are useful shorthand for the combination of the three metric functions:

$$\alpha \equiv -g_{xx}(\Theta_{,r} + \Theta^2), \tag{207}$$

$$\beta \equiv \frac{1}{2} g_{xx}(\Theta g_{ux,r} - g_{ux,rr}), \tag{208}$$

$$\gamma \equiv \frac{1}{2}\Big[g_{xx}(4\Lambda - g_{uu,rr}) + g_{ux}g_{ux,rr} + g_{ux,r\|x} + (g_{ux,r})^2 - 2g_{xx}\Theta_{,u} - \Theta(g_{xx,u} + g_{ux}g_{ux,r} + g_{xx}g_{uu,r})\Big]. \tag{209}$$

Consequently,

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}}, \quad F_x = \left(\frac{\beta}{\alpha} - g_{ux}\right)F_r, \quad F_u = \frac{\gamma}{\alpha}F_r. \tag{210}$$

Let us recall that α is fully determined by g_{xx}, the function β is determined by g_{xx} and g_{ux}, while the third metric component g_{uu} enters only γ.

D. The Maxwell equations

As the next step, we apply the four independent Maxwell equations in the form (43) and (41), namely

$$(GF_a)_{,b} = (GF_b)_{,a} \quad \text{and} \quad F_{ux,r} + F_{ru,x} - F_{rx,u} = 0, \tag{211}$$

which restrict the possible electromagnetic field and its coupling to the gravitational field. For explicit evaluation of the partial derivatives with respect to $a, b = \{r, u, x\}$ we employ the expressions directly following from (195) and (204)–(206), implying (210), namely

$$G_{,a} = -\frac{1}{2}G^3 g_{xx,a}, \tag{212}$$

$$F_{r,a} = \frac{1}{\kappa_0 F_r}\left(\frac{1}{2}\alpha_{,a}\right), \tag{213}$$

$$F_{x,a} = \frac{1}{\kappa_0 F_r}\left((\beta - \alpha g_{ux})_{,a} - \frac{1}{2}(\beta - \alpha g_{ux})\frac{\alpha_{,a}}{\alpha}\right), \tag{214}$$

$$F_{u,a} = \frac{1}{\kappa_0 F_r}\left(\gamma_{,a} - \frac{1}{2}\gamma\frac{\alpha_{,a}}{\alpha}\right). \tag{215}$$

Using these relations in calculating $(GF_a)_{,b} = (GF_b)_{,a}$ for $ab = rx, ru, ux$ we obtain

$$\left(\alpha_{,x} + 2\alpha\frac{G_{,x}}{G}\right) - 2(\beta - \alpha g_{ux})_{,r} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,r}}{\alpha} + 2\Theta\right) = 0, \tag{216}$$

$$\left(\alpha_{,u} + 2\alpha\frac{G_{,u}}{G}\right) - 2\gamma_{,r} + \gamma\left(\frac{\alpha_{,r}}{\alpha} + 2\Theta\right) = 0, \tag{217}$$

$$\gamma_{,x} - \gamma\left(\frac{\alpha_{,x}}{2\alpha} - \frac{G_{,x}}{G}\right) - (\beta - \alpha g_{ux})_{,u} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,u}}{2\alpha} - \frac{G_{,u}}{G}\right) = 0, \tag{218}$$

respectively. Notice that the terms in the large brackets depend only on $g_{xx} \equiv G^{-2}$ and their derivatives. The last Maxwell equation (211), using the inversion of (14)–(16),

$$F_{rx} = F_r, \tag{219}$$

$$F_{ru} = G^2(F_x + g_{ux}F_r), \tag{220}$$

$$F_{ux} = g_{ux}G^2(F_x + g_{ux}F_r) - F_u - g_{uu}F_r, \tag{221}$$

reads

$$\beta_{,x} + \beta_{,r}g_{ux} + \beta\left[g_{ux,r} - g_{ux}\left(\frac{\alpha_{,r}}{2\alpha} + 2\Theta\right) - \left(\frac{\alpha_{,x}}{2\alpha} - 2\frac{G_{,x}}{G}\right)\right] - \frac{1}{2G^2}\left[2\gamma_{,r} - \alpha_{,r}\left(\frac{\gamma}{\alpha} - g_{uu}\right) + \alpha_{,u} + 2\alpha g_{uu,r}\right] = 0. \tag{222}$$

The four equations (216)–(218) and (222) put restrictions on the metric functions, encoded in G, α, β, γ.

E. Remaining Einstein equations $R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b$

Finally, it is necessary to solve the remaining three Einstein equations (38) for the components $ab = xx$, ux, uu. Using (210) we immediately derive their form:

$$R_{xx} = 2\Lambda g_{xx} + \frac{G^2}{\alpha}(\beta - \alpha g_{ux})^2, \tag{223}$$

$$R_{ux} = 2\Lambda g_{ux} + \frac{G^2}{\alpha}(\beta - \alpha g_{ux})\gamma, \tag{224}$$

$$R_{uu} = 2\Lambda g_{uu} + \frac{G^2}{\alpha}\gamma^2. \tag{225}$$

Substituting the explicit expressions for the corresponding Ricci tensor components (A27)–(A29) reveals a rather complicated system of partial differential equations for the metric functions which must be solved together with (216)–(218) and (222).

At this stage, it does not seem possible to find a *general* solution of these equations. However, *we have achieved a separation of the variables representing the gravitational and the electromagnetic field*. Indeed, the system of seven equations (216)–(218), (222), and (223)–(225) with (A27)–(A29) *involves only the three metric functions* g_{xx}, g_{ux}, g_{uu}, encoded also in the functions G and α, β, γ defined in (195) and (207)–(209). After their solution is found, the corresponding three (dual) components of the electromagnetic field F_r, F_x, F_u are easily obtained by applying the relations (210). The components F_{rx}, F_{ru}, F_{ux} are then their simple combinations (219)–(221).

F. A simple particular solution

To demonstrate the usefulness of our formulation of the most general Einstein-Maxwell field equations and also *to show that the class of Robinson-Trautman* $2+1$ *spacetimes with nonaligned electromagnetic field is not empty*, we will now derive a special solution of the above system of equations.

Let us *assume* that *only the nonaligned component* F_r of the electromagnetic field is nontrivial, i.e.,

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}} \neq 0, \qquad F_x = 0, \qquad F_u = 0. \tag{226}$$

The field equations (204)–(206) then imply

$$\beta - \alpha g_{ux} = 0, \tag{227}$$

$$\gamma = 0. \tag{228}$$

Further simplification is achieved by *assuming*

$$g_{ux} = 0. \tag{229}$$

In such a case the condition (227) $\beta = 0$ is satisfied due to (208), while (228) gives

$$g_{uu,rr} - 4\Lambda + 2\Theta_{,u} + \Theta\left(g_{uu,r} - 2\frac{G_{,u}}{G}\right) = 0. \tag{230}$$

The Maxwell equations (216)–(218), (222) reduce to

$$\frac{\alpha_{,x}}{\alpha} + 2\frac{G_{,x}}{G} = 0, \tag{231}$$

$$\frac{\alpha_{,u}}{\alpha} + 2\frac{G_{,u}}{G} = 0, \tag{232}$$

$$\alpha_{,r} g_{uu} + \alpha_{,u} + 2\alpha g_{uu,r} = 0, \tag{233}$$

and the final three Einstein equations simplify as

$$R_{xx} = 2\Lambda g_{xx}, \tag{234}$$

$$R_{ux} = 0, \tag{235}$$

$$R_{uu} = 2\Lambda g_{uu}, \tag{236}$$

where

$$R_{xx} = g_{xx} g_{uu}(\Theta_{,r} + \Theta^2) + 2 g_{xx}\Theta_{,u} + \Theta(g_{xx} g_{uu,r} + g_{xx,u}), \tag{237}$$

$$R_{ux} = -\frac{1}{2} g_{uu,xr} + \frac{1}{2}\Theta g_{uu,x}, \tag{238}$$

$$R_{uu} = \frac{1}{2} g_{uu} g_{uu,rr} + \frac{1}{4} g^{xx} g_{xx,u} g_{uu,r} - \frac{1}{2} g^{xx} g_{xx,uu} - \frac{1}{2} g^{xx} g_{uu\|xx} + \frac{1}{4}(g^{xx} g_{xx,u})^2 + \frac{1}{2}\Theta(g_{uu} g_{uu,r} - g_{uu,u}). \tag{239}$$

Equations (231) and (232) can be easily integrated, yielding

$$\alpha = f(r) G^{-2} \equiv f(r) g_{xx}, \tag{240}$$

where $f(r)$ is any function of the coordinate r. Equation (233) gives the constraint

$$g_{uu,r} + \left(\frac{f'}{2f} + \Theta\right) g_{uu} - \frac{G_{,u}}{G} = 0, \tag{241}$$

in which f' is the derivative of f. It thus remains to solve (230), (241), and (234)–(236).

Now, combining (240) with the definition (207) we obtain

$$\Theta_{,r} + \Theta^2 = -f(r),$$

which is the Ricatti-type equation for the expansion Θ. Using the substitution $\Theta = z_{,r}/z$, it can be rewritten as the linear equation $z_{,rr} + f(r) z = 0$. Let us consider here only *the simplest case of a constant* f,

$$f \equiv C^2. \tag{242}$$

By applying (195) we obtain the explicit solution

$$\Theta(r) = C\cot(Cr), \tag{243}$$

$$G = \frac{P(u,x)}{\sin(Cr)}, \tag{244}$$

$$g_{xx} = \frac{\sin^2(Cr)}{P^2(u,x)}. \tag{245}$$

(We have applied the coordinate freedom, namely a trivial constant shift in the coordinate r, to simplify the expressions.) It is now easily seen that for the particular choice

$$P = 1, \tag{246}$$

$$g_{uu} = 0, \tag{247}$$

$$\Lambda = 0, \tag{248}$$

all the remaining field equations (230), (241), and (234)–(236) are satisfied because $R_{xx} = 0$, $R_{ux} = 0$, and $R_{uu} = 0$. We have thus obtained a *special Robinson-Trautman solution*,

$$\mathrm{d}s^2 = \sin^2(Cr)\mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}r, \tag{249}$$

with a nonaligned electromagnetic field:

$$F_r = \frac{C}{\sqrt{\kappa_0}G} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr), \qquad F_x = 0, \qquad F_u = 0, \tag{250}$$

that is,

$$^*\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\mathrm{d}r. \tag{251}$$

Using (219)–(221), this is equivalent to

$$\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr)\mathrm{d}r\wedge\mathrm{d}x, \tag{252}$$

corresponding to the potential

$$\mathbf{A} = -\frac{1}{\sqrt{\kappa_0}}\cos(Cr)\mathrm{d}x. \tag{253}$$

By rescaling the coordinates r and u the constant C can be set to $C = 1$, but we prefer to keep it free because it represents the value of the electromagnetic field and r is not dimensionless.

Actually, (249) is the metric 3) on page 133 of [31] for $q = 0$, which admits four Killing vectors [see also the metric (4.1) in [32]].

VIII. FINAL SUMMARY AND REMARKS

In this paper we systematically solved the Einstein-Maxwell equations with Λ, obtaining all electrovacuum $2+1$ spacetimes. We identified main geometrically distinct subclasses, and we explicitly derived the corresponding metrics and electromagnetic fields. In particular:

(1) The metric of *any* such spacetime can be written in canonical coordinates in the form (3)

$$\mathrm{d}s^2 = G^{-2}\mathrm{d}x^2 + 2g_{ux}\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + g_{uu}\mathrm{d}u^2. \tag{254}$$

(2) The *generic* electromagnetic Maxwell 2-form field and its dual 1-form have three independent components (11) and (21), namely

$$\mathbf{F} = F_{ru}\mathrm{d}r\wedge\mathrm{d}u + F_{rx}\mathrm{d}r\wedge\mathrm{d}x + F_{ux}\mathrm{d}u\wedge\mathrm{d}x, \tag{255}$$

$$^*\mathbf{F} = G(F_r\mathrm{d}r + F_u\mathrm{d}u + F_x\mathrm{d}x), \tag{256}$$

where $F_r = F_{rx}$, $F_x = g_{xx}F_{ru} - g_{ux}F_{rx}$, $F_u = g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}$.

(3) In terms of the Newman-Penrose scalars (25) of distinct boost weights $+1$, 0, -1, the Maxwell field *invariants* $F^2 \equiv F_{ab}F^{ab}$ and $^*F^2 \equiv {}^*F_a{}^*F^a$ are

$$\frac{1}{2}F^2 = -{}^*F^2 = 2\phi_0\phi_2 - \phi_1^2. \tag{257}$$

The electromagnetic field is *aligned with* $\mathbf{k} = \partial_r \Leftrightarrow \phi_0 = 0 \Leftrightarrow F_{rx} = 0 \Leftrightarrow F_r = 0$.

Such an aligned field has only two components, namely $\phi_2 = GF_u \equiv G(g_{ux}F_{ru} - F_{ux})$ and $\phi_1 = G^2F_x \equiv F_{ru}$. In the case when $\phi_2 = 0 \Leftrightarrow F_u = 0$, the electromagnetic field is *non-null*, characterized just by $\phi_1 = F_{ru}$. Contrarily, when $\phi_1 = 0 \Leftrightarrow F_x = 0$, it is *null (radiative)*, characterized just by $\phi_2 = -GF_{ux}$.

(4) Evaluating the energy-momentum tensor (34) we derived that, in terms of these quantities, the *Einstein-Maxwell field equations take a simple form* (38),

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{258}$$

(equivalent to $R_{ab} = 2\Lambda g_{ab} + \kappa_0{}^*F_a{}^*F_b$) and (43), (41),

$$(GF_a)_{,b} = (GF_b)_{,a}, \qquad F_{[ab,c]} = 0. \tag{259}$$

(5) In the triad (6) of the metric (254), all *optical scalars* of a congruence generated by the privileged null vector field $\mathbf{k} = \partial_r$ *vanish except*, possibly, *expansion:*

$$\Theta = -(\ln G)_{,r}. \tag{260}$$

There are thus *two geometrically distinct classes of spacetimes* to be investigated:

(a) $\Theta = 0$, defining the nonexpanding *Kundt class*, with the metric function

$$G \equiv P(u, x), \tag{261}$$

(b) $\Theta \neq 0$, defining the expanding *Robinson-Trautman class*, with the metric function

$$G \equiv G(r, u, x). \tag{262}$$

(6) Keeping the full generality, we explicitly integrated the coupled system of the field equations (258) and (259) *both* for the Kundt and the Robinson-Trautman spacetimes. It turned out that, as in standard $3 + 1$ general relativity, the *Kundt class only admits aligned electromagnetic fields* while the *Robinson-Trautman class admits both aligned and nonaligned electromagnetic fields*. Therefore, we treated these three distinct families of spacetimes in three separate sections of our paper, namely Sec. V, Sec. VI, and Sec. VII, respectively.

(7) All *Kundt* spacetimes (Sec. V) with *necessarily aligned electromagnetic fields* have the form

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{\mathrm{d}x^2}{P^2} + 2(e + fr)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r \\ & + \left(a + br + \left(\Lambda + \frac{1}{4}P^2 f^2 - \frac{\kappa_0}{2}Q^2\right) r^2\right)\mathrm{d}u^2, \end{aligned} \tag{263}$$

and

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u + (fQr - \xi)\mathrm{d}u \wedge \mathrm{d}x, \tag{264}$$

corresponding to the potential

$$\mathbf{A} = A_r\mathrm{d}r + A_x\mathrm{d}x, \tag{265}$$

where $A_r = -\int Q\mathrm{d}u$ and $A_x = r\int fQ\mathrm{d}u - \int \xi\mathrm{d}u$; see Eqs. (94)–(97). As summarized in Sec. V H, the function $Q(u, x)$ represents the *non-null* component, while the function $\xi(u, x)$ represents the *null* component of the Maxwell field. Their relation to the metric functions P, e, f and a, b is explicitly given by the Einstein-Maxwell equations (102)–(106). In Sec. V H we presented a basic description of these solutions, separately for *two geometrically distinct subclasses* $f = 0$ and $f \neq 0$.

This large family of nonexpanding Kundt spacetimes contains many interesting subclasses which represent electrovacuum universes and also waves on these cosmological backgrounds. The simplest of them are gravitational and electromagnetic *pp waves* with $\Lambda = 0$. These are defined by the condition $k_{a;b} = \frac{1}{2}g_{ab,r} = 0$ which requires $f = 0$, $b = 0$, $Q = 0$. The field equations (107)–(111) then yield the explicit metric in the Brinkmann form [33]:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2e\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + a\mathrm{d}u^2, \tag{266}$$

and the coupled electromagnetic wave:

$$\mathbf{F} = -\frac{\gamma(u)}{P(u, x)}\mathrm{d}u \wedge \mathrm{d}x, \tag{267}$$

corresponding to

$$\mathbf{A} = A_x\mathrm{d}x \quad \text{where} \quad A_x = -\int \frac{\gamma(u)}{P(u, x)}\mathrm{d}u. \tag{268}$$

Here $\gamma(u)$ is an *arbitrary profile function* of the retarded time u, while the metric function $a(u, x)$ is obtained by integrating the only remaining field equation (111).

(8) All *Robinson-Trautman* spacetimes (Sec. VI) with *aligned electromagnetic fields* [for which the metric function G simplifies to $G = P(u, x)/r$] can be written as

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{r^2}{P^2}(\mathrm{d}x + eP^2\mathrm{d}u)^2 - 2\mathrm{d}u\mathrm{d}r \\ & + \left(\mu Q^2 - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + \Lambda r^2\right)\mathrm{d}u^2, \end{aligned} \tag{269}$$

with

$$\begin{aligned} & \mathbf{F} = \frac{Q(u)}{r}\mathrm{d}r \wedge \mathrm{d}u \quad \text{corresponding to} \\ & \mathbf{A} = Q(u)\ln\frac{r}{r_0}\mathrm{d}u; \end{aligned} \tag{270}$$

see Eqs. (180)–(182). Here μ is a constant while the metric functions P and e satisfy the field equation (178), that is

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{271}$$

The dual 1-form Maxwell field reads

$$^{*}\mathbf{F} = \frac{Q}{P}\mathrm{d}x + ePQ\mathrm{d}u. \tag{272}$$

As summarized in Sec. VI H, the function $Q(u)$ gives the *non-null* component $\phi_1 = Q(u)/r$ of the

Maxwell field. Somewhat surprisingly, there is also an additional *null* (radiative) component $\phi_2 = ePQ$ when $e \neq 0$. However, such Maxwell fields *cannot be purely null* because $\phi_1 = 0$ implies $\phi_2 = 0$.

The simplest $e \neq 0$ solution of the field equation (271) is $P = 1$, $e = x(\ln Q)_{,u} + \alpha(u)$, which yields the metric (186).

Another interesting subclass (188) arises for factorized P such that $P = Q(u)\beta(x)$ and $eP = \alpha(u)$. The special case $\alpha = 0$ and $Q = \text{const}$ of these expanding Robinson-Trautman spacetimes is equivalent to the solution (192), (193),

$$\mathrm{d}s^2 = -\Phi(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi(r)} + r^2\mathrm{d}\varphi^2,$$
$$\Phi(r) = -m + \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2, \tag{273}$$

which is the family of *cyclic symmetric, electrostatic black holes with* Λ found in [24] and discussed in Sec. 11.2 of [6].

(9) The complementary class of *Robinson-Trautman* spacetimes with *nonaligned electromagnetic fields* is presented in Sec. VII. In this more complex case, the metric has the form (254) with a general function $G(r,u,x)$; cf. (262). Moreover, the electromagnetic field now has a nontrivial component $\phi_0 \neq 0 \Leftrightarrow F_{rx} \neq 0 \Leftrightarrow F_r \neq 0$, which considerably complicates the solution of the Einstein-Maxwell equations.

Nevertheless, we were able to explicitly express the generic three components of the Maxwell field *separately* in terms (of the combination) of the *metric functions* as

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}}, \qquad F_x = \left(\frac{\beta}{\alpha} - g_{ux}\right)F_r, \qquad F_u = \frac{\gamma}{\alpha}F_r, \tag{274}$$

where the functions α, β, γ are defined in (207)–(209). Interestingly, α is determined only by g_{xx}, β is determined by g_{xx} and g_{ux}, while the third metric component g_{uu} enters only γ.

We also derived a fully explicit form (216)–(218), (222) of all four Maxwell equations (259). Finally, there are three remaining Einstein equations (223)–(225). This system of seven equations involves only three *metric* functions. After their solution is found, all components F_r, F_x, F_u of the corresponding electromagnetic field are easily obtained using (274). In this sense, *we have achieved a separation of the variables representing the gravitational and the electromagnetic field*.

Although at present it is not possible for us to find a *general* solution to these seven equations, the formulation of the problem presented here seems to be useful. This fact has been demonstrated in Sec. VII F, where we have explicitly identified a *particular solution with nonaligned electromagnetic field*

$$\mathrm{d}s^2 = \sin^2(Cr)\mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}r, \tag{275}$$

with

$$\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr)\mathrm{d}r \wedge \mathrm{d}x \quad \text{corresponding to}$$
$$\mathbf{A} = -\frac{1}{\sqrt{\kappa_0}}\cos(Cr)\mathrm{d}x. \tag{276}$$

This special exact Robinson-Trautman spacetime contains electromagnetic field which has *only* the nonaligned component $F_r = (C/\sqrt{\kappa_0})\sin(Cr)$. It admits four Killing vectors [31,32,34].

Of course, many questions have remained open. First of all, it is necessary to find explicit relations to already known solutions summarized in [6]. Some basic identifications have already been presented here, namely:

(i) Maximally symmetric backgrounds (Minkowski, de Sitter, AdS) are contained both in the Kundt and Robinson-Trautman class of spacetimes (263) and (269), respectively.
(ii) There are electrovacuum backgrounds in the form of direct-product geometries, such as the $2+1$ analog of the exceptional Plebański-Hacyan metric with $\Lambda < 0$ and uniform Maxwell field (123).
(iii) We identified the complete family of *pp* waves in flat space, which are spacetimes admitting a covariantly constant null vecor field. In the Brinkmann form (266) they include the off-diagonal metric terms.
(iv) Within the Robinson-Trautman class with aligned fields we explicitly identified the cyclic symmetric charged black holes with any cosmological constant and electrostatic field (273).

Our main problem now is to identify all other known classes of solutions in $2+1$ dimensions by using specific invariant geometrical characterizations (such as an algebraic structure, symmetries, identification of rotation, and acceleration of the sources, etc.). Subsequently, explicit coordinate transformation must be found to relate our form of the solutions to those derived previously.

After identification of new spacetimes, their geometrical and physical analysis should be performed. Also, a systematic integration of the field equations for nonaligned Maxwell fields in the Robinson-Trautman class is desirable. However, these tasks are left for future works.

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APPENDIX: CONNECTIONS AND CURVATURE COMPONENTS IN CANONICAL COORDINATES

The Christoffel symbols for the general nontwisting spacetime (3) after applying the condition (7) are

$$\Gamma^r_{rr} = 0, \tag{A1}$$

$$\Gamma^r_{ru} = -\frac{1}{2} g_{uu,r} + \frac{1}{2} g^{rx} g_{ux,r}, \tag{A2}$$

$$\Gamma^r_{rx} = -\frac{1}{2} g_{ux,r} + \Theta g_{ux}, \tag{A3}$$

$$\Gamma^r_{uu} = \frac{1}{2}\left[-g^{rr} g_{uu,r} - g_{uu,u} + g^{rx}(2g_{ux,u} - g_{uu,x})\right], \tag{A4}$$

$$\Gamma^r_{ux} = \frac{1}{2}\left[-g^{rr} g_{ux,r} - g_{uu,x} + g^{rx} g_{xx,u}\right], \tag{A5}$$

$$\Gamma^r_{xx} = -\Theta g^{rr} g_{xx} - g_{ux||x} + \frac{1}{2} g_{xx,u}, \tag{A6}$$

$$\Gamma^u_{rr} = \Gamma^u_{ru} = \Gamma^u_{rx} = 0, \tag{A7}$$

$$\Gamma^u_{uu} = \frac{1}{2} g_{uu,r}, \tag{A8}$$

$$\Gamma^u_{ux} = \frac{1}{2} g_{ux,r}, \tag{A9}$$

$$\Gamma^u_{xx} = \Theta g_{xx}, \tag{A10}$$

$$\Gamma^x_{rr} = 0, \tag{A11}$$

$$\Gamma^x_{ru} = \frac{1}{2} g^{xx} g_{ux,r}, \tag{A12}$$

$$\Gamma^x_{rx} = \Theta, \tag{A13}$$

$$\Gamma^x_{uu} = \frac{1}{2}\left[-g^{rx} g_{uu,r} + g^{xx}(2g_{ux,u} - g_{uu,x})\right], \tag{A14}$$

$$\Gamma^x_{ux} = \frac{1}{2}\left[-g^{rx} g_{ux,r} + g^{xx} g_{xx,u}\right], \tag{A15}$$

$$\Gamma^x_{xx} = -\Theta g^{rx} g_{xx} + {}^S\Gamma^x_{xx}, \tag{A16}$$

where

$${}^S\Gamma^x_{xx} \equiv \frac{1}{2} g^{xx} g_{xx,x} = -\frac{G_{,x}}{G} \tag{A17}$$

is the Christoffel symbol with respect to the only spatial coordinate x, i.e., coefficient of the covariant derivative on the transverse one-dimensional space spanned by x.

The nonvanishing Riemann curvature tensor components are then

$$R_{rxrx} = -(\Theta_{,r} + \Theta^2) g_{xx}, \tag{A18}$$

$$R_{rxru} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r}, \tag{A19}$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} g^{xx}(g_{ux,r})^2, \tag{A20}$$

$$R_{rxux} = \frac{1}{2} g_{ux,r||x} + \frac{1}{4}(g_{ux,r})^2 - g_{xx}\Theta_{,u} - \frac{1}{2}\Theta(g_{xx,u} + g_{xx} g_{uu,r}), \tag{A21}$$

$$R_{ruux} = g_{u[u,x],r} + \frac{1}{4} g^{rx}(g_{ux,r})^2 - \frac{1}{4} g^{xx} g_{xx,u} g_{ux,r} + \Theta\left(g_{ux,u} - \frac{1}{2} g_{uu,x} - \frac{1}{2} g_{ux} g_{uu,r}\right), \tag{A22}$$

$$\begin{aligned} R_{uxux} = {} & -\frac{1}{2}(g_{uu})_{||xx} + g_{ux,u||x} - \frac{1}{2} g_{xx,uu} + \frac{1}{4} g^{rr}(g_{ux,r})^2 - \frac{1}{2} g_{uu,r} e_{xx} + \frac{1}{2} g_{uu,x} g_{ux,r} - \frac{1}{2} g^{rx} g_{xx,u} g_{ux,r} + \frac{1}{4} g^{xx}(g_{xx,u})^2 \\ & - \frac{1}{2}\Theta g_{xx}[g^{rr} g_{uu,r} + g_{uu,u} - g^{rx}(2g_{ux,u} - g_{uu,x})]. \end{aligned} \tag{A23}$$

Finally, the components of the Ricci tensor are

$$R_{rr} = -(\Theta_{,r} + \Theta^2), \tag{A24}$$

$$R_{rx} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r} + (\Theta_{,r} + \Theta^2) g_{ux}, \tag{A25}$$

$$R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} g^{rx} g_{ux,rr} + \frac{1}{2} g^{xx}(g_{ux,r||x} + (g_{ux,r})^2) - \Theta_{,u} - \frac{1}{2}\Theta(g^{xx} g_{xx,u} + g^{rx} g_{ux,r} + g_{uu,r}), \tag{A26}$$

$$R_{xx} = -g_{xx}g^{rr}(\Theta_{,r} + \Theta^2) + 2g_{xx}(\Theta_{,u} - g^{rx}\Theta_{,x}) + 2g_{ux}\Theta_{,x} - f_{xx} + \Theta[2g_{ux\|x} + 2g_{ux,r}g_{ux} + g_{xx}(g_{uu,r} - 2g^{rx}g_{ux,r}) - 2e_{xx}], \tag{A27}$$

$$\begin{aligned} R_{ux} = & -\frac{1}{2}g^{rr}g_{ux,rr} - \frac{1}{2}g_{uu,rx} + \frac{1}{2}g_{ux,ru} - \frac{1}{2}g^{rx}\left[g_{ux,r\|x} + (g_{ux,r})^2\right] + g^{xx}\left(\frac{1}{2}g_{ux,r}g_{ux\|x} - \frac{1}{2}e_{xx}g_{ux,r}\right) + g_{ux}\Theta_{,u} \\ & + \Theta\left[g_{ux}g_{uu,r} - \frac{1}{2}(g_{uu}g_{ux,r} - g_{uu,x}) - g_{ux,u} + \frac{1}{2}g^{rx}g_{ux,r}g_{ux} + \frac{1}{2}g^{rx}g_{xx,u}\right], \end{aligned} \tag{A28}$$

$$\begin{aligned} R_{uu} = & -\frac{1}{2}g^{rr}g_{uu,rr} - g^{rx}g_{uu,rx} - \frac{1}{2}g^{xx}e_{xx}g_{uu,r} + g^{rx}g_{ux,ru} - \frac{1}{2}g^{xx}g_{xx,uu} \\ & + g^{xx}\left(g_{ux,u\|x} - \frac{1}{2}g_{uu\|xx}\right) + \frac{1}{2}(g^{rr}g^{xx} - g^{rx}g^{rx})(g_{ux,r})^2 + \frac{1}{2}g^{xx}g_{ux,r}g_{uu,x} + \frac{1}{4}(g^{xx}g_{xx,u})^2 \\ & + \frac{1}{2}\Theta\left[-g^{rx}(2g_{ux,u} - g_{uu,x} - g_{ux}g_{uu,r}) + g_{uu}g_{uu,r} - g_{uu,u}\right], \end{aligned} \tag{A29}$$

and the Ricci scalar is

$$R = g_{uu,rr} - 2g^{rx}g_{ux,rr} - 2g^{xx}g_{ux,r\|x} - \frac{3}{2}g^{xx}(g_{ux,r})^2 + 2\Theta_{,r}g_{uu} + 4\Theta_{,u} + 2\Theta^2 g_{uu} + \Theta(2g_{uu,r} + 2g^{rx}g_{ux,r} + 2g^{xx}g_{xx,u}). \tag{A30}$$

The symbol $\|$ denotes the covariant derivative with respect to g_{xx}:

$$g_{ux\|x} = g_{ux,x} - g_{ux}{}^S\Gamma^x_{xx}, \tag{A31}$$

$$g_{ux,r\|x} = g_{ux,rx} - g_{ux,r}{}^S\Gamma^x_{xx}, \tag{A32}$$

$$g_{ux,u\|x} = g_{ux,ux} - g_{ux,u}{}^S\Gamma^x_{xx}, \tag{A33}$$

$$(g_{uu})_{\|xx} = g_{uu,xx} - g_{uu,x}{}^S\Gamma^x_{xx}, \tag{A34}$$

where e_{xx} and f_{xx} are convenient shorthand defined as

$$e_{xx} \equiv g_{ux\|x} - \frac{1}{2}g_{xx,u}, \tag{A35}$$

$$f_{xx} \equiv g_{ux,r\|x} + \frac{1}{2}(g_{ux,r})^2. \tag{A36}$$

The expressions (A24)–(A29) of the Ricci tensor enable us to write explicitly the gravitational field equations for any $D = 3$ Kundt or Robinson-Trautman spacetime.

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