Second post-Newtonian motion in Reissner-Nordström spacetime

Bo Yang,[1] Chunhua Jiang,[1] and Wenbin Lin[1,2,*]
[1]*School of Mathematics and Physics, University of South China, Hengyang 421001, China*
[2]*School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China*

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We derive the second post-Newtonian solution for the quasi-Keplerian motion of a charged test particle in the Reissner-Nordström spacetime under harmonic coordinates. We formulate the solution in terms of the test particle's orbital energy and angular momentum, both of which are constants at the second post-Newtonian order. The charge effects on the test particle's motion including the orbital period and perihelion precession are displayed explicitly. Our results can be applied to the cases in which the test particle has small charge-to-mass ratio, or the test particle has arbitrary charge-to-mass ratio but the multiplication of the test particle and the gravitational source's charge-to-mass ratios is much smaller than 1.

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I. INTRODUCTION

The motion of bodies in gravitational fields is a classical problem in astronomy and cosmology. For the cases in which the gravitational fields are not extremely strong, the motion can be studied in post-Newtonian (PN) approximations. A large number of analytical PN solutions for the motion of the binary systems have been obtained, including the first and higher PN effects of the mass [1–11], the 1.5PN effects of the spin-orbit coupling effects [12–22] on the general motion, and the 2PN effects of the mass quadrupole on the circular motion [23]. When taking the limit of the extreme mass ratio, the solutions for the motion of the binary systems return to the solution of the motion of the test particle. Based on these studies, we have achieved the analytical solutions for the motion of the test particle under the most generally parametrized PN force [24,25] and those in the spacetime of the classical black holes, including the 2PN effects of the mass in the Wagoner-Will-Epstein-Haugan representation [26], the 2PN effects of the spin-induced quadrupole on the equatorial motion in Kerr spacetime [27], and the 2.5PN effects of the spin-orbit coupling on the general motion in the Kerr spacetime [28].

The Reissner-Nordström metric [29–31] is the unique spherically symmetric and asymptotically flat solution of the Einstein-Maxwell equations, which describes the exterior spacetime around an isolated spherical object of mass and electric charge. Although the charged astronomical bodies have not been discovered, the possibility of their existence in the Universe may not be ruled out. In fact, the charge effects of the astrophysical black holes and stars have been studied extensively [32–41]. For example, Ruffini and co-workers have investigated the circular motion of a charged test particle in the field of the Reissner-Nordström black hole [38,39]. In our previous work, we have shown that the black hole's electric charge can contribute to the orbital perihelion precession at 1PN order, but not to the orbital period at the same order [40]. It is interesting to further explore the effects of the black hole's charge on the particle's motion, including the orbital perihelion precession and period at higher PN orders. On the other hand, the electrically charged test particles are all around in astrophysics, so it is especially important to discuss the motion of the charged test particles in the field of the Reissner-Nordström black hole.

In this work, we derive the 2PN solution for the quasi-Keplerian motion of the charged test particle in the Reissner-Nordström spacetime and exhibit the charge effects on the orbital period and perihelion precession. The solution is formulated in terms of the test particle's charge-to-mass ratio, energy, and angular momentum, as well as the black hole's mass and charge-to-mass ratio.

The rest of this paper is organized as follows. Section II introduces the harmonic metric of the Reissner-Nordström black hole in the 2PN approximations, the geodesic equation with the Lorentz force, and the corresponding Lagrangian, as well as the orbital energy and angular momentum. In Sec. III we present a detailed derivation of the 2PN solution for the charged test particle's quasi-Keplerian motion. A summary is given in Sec. IV.

II. 2PN LAGRANGIAN, ENERGY, AND ANGULAR MOMENTUM IN REISSNER-NORDSTRÖM SPACETIME

The Reissner-Nordström black hole's mass and electric charge are denoted by M and Q. In the harmonic coordinates, the metric of the Reissner-Nordström black hole in the 2PN approximation can be written as [42]

*lwb@usc.edu.cn.

$$g_{00} = -1 + \frac{2M}{r} - \frac{2M^2}{r^2}\left(1 + \frac{1}{2}\epsilon_0^2\right) + \frac{2M^3}{r^3}(1+\epsilon_0^2), \quad (1)$$

$$g_{0i} = 0, \quad (2)$$

$$g_{ij} = \left(1 + \frac{2M}{r} + \frac{M^2}{r^2}\right)\delta_{ij} + \frac{M^2}{r^2}\left(1 - \epsilon_0^2\right)\frac{x^i x^j}{r^2}, \quad (3)$$

where $\epsilon_0 \equiv Q/M$ is the charge-to-mass ratio of the gravitational source. The non-naked singularity of the Reissner-Nordström spacetime requires $|\epsilon_0| \leq 1$. $r \equiv |\boldsymbol{x}|$ denotes the distance from the field position $\boldsymbol{x} \equiv (x, y, z)$ to the black hole located at the coordinate origin. The gravitational constant and the speed of light in vacuum are set as 1 ($G = 1$ and $c = 1$). The metric has a signature of $(- + ++)$. Latin indices i and j run from 1 to 3.

The charged test particle has mass m and electric charge q. The particle's charge-to-mass ratio is $\epsilon_1 \equiv q/m$, and its motion is described by the geodesic equation with the Lorentz force

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \epsilon_1 F^\mu{}_\nu \frac{dx^\nu}{d\tau}, \quad (4)$$

where $\Gamma^\mu_{\nu\lambda}$ denotes Christoffel's symbols that, given by the derivatives of the chosen metric $g_{\mu\nu}$, τ is the proper time of the particle along its world line. The electromagnetic Faraday tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu, \quad (5)$$

where A_α is the associated electromagnetic potential vector

$$A_0 = -\frac{\epsilon_0 M}{r}\left(1 + \frac{M}{r}\right)^{-1}, \quad (6)$$

$$A_i = 0. \quad (7)$$

Substituting the 2PN metric into Eq. (4), we can obtain the 2PN equation of motion for the charged test particle as follows:

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} = &-\frac{M\boldsymbol{x}}{r^3}\left[(1-\epsilon_0\epsilon_1) - \frac{M}{r}(4 + \epsilon_0^2 - 5\epsilon_0\epsilon_1) + \boldsymbol{v}^2\left(1 + \frac{1}{2}\epsilon_0\epsilon_1\right) + \frac{M^2}{r^2}\left(9 + 6\epsilon_0^2 - \frac{27}{2}\epsilon_0\epsilon_1 - \frac{3}{2}\epsilon_0^3\epsilon_1\right)\right.\\ &\left.-\frac{2M(\boldsymbol{v}\cdot\boldsymbol{x})^2}{r^3}(1-\epsilon_0^2) - \frac{\boldsymbol{v}^2}{2}(\epsilon_0\epsilon_1 + \epsilon_0^2)\frac{M}{r} + \frac{1}{8}\boldsymbol{v}^4\epsilon_0\epsilon_1\right]\\ &+\frac{M(\boldsymbol{v}\cdot\boldsymbol{x})\boldsymbol{v}}{r^3}\left[(4-\epsilon_0\epsilon_1) - \frac{M}{r}(2 + 2\epsilon_0^2 - \epsilon_0\epsilon_1) + \frac{1}{2}\boldsymbol{v}^2\epsilon_0\epsilon_1\right],\end{aligned} \quad (8)$$

where $\boldsymbol{v}$ denotes the particle's velocity. When $\epsilon_1 = 0$ this equation reduces to the 2PN dynamics of a neutral particle in Reissner-Nordström spacetime.

From the equation of motion (8) and the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \boldsymbol{v}} = \frac{\partial \mathrm{L}}{\partial \boldsymbol{x}}, \quad (9)$$

we can obtain the corresponding 2PN Lagrangian of the charged test particle

$$\begin{aligned}\mathrm{L} = &\frac{1}{2}\boldsymbol{v}^2 + \frac{M}{r}(1-\epsilon_0\epsilon_1) + \frac{1}{8}\boldsymbol{v}^4 + \frac{3}{2}\frac{M}{r}\boldsymbol{v}^2 - \frac{1}{2}\frac{M^2}{r^2}(1 + \epsilon_0^2 - 2\epsilon_0\epsilon_1) + \frac{1}{2}\frac{M^3}{r^3}(1 + \epsilon_0^2 - 2\epsilon_0\epsilon_1)\\ &+\frac{1}{16}\boldsymbol{v}^6 + \frac{1}{4}\frac{M^2}{r^2}\boldsymbol{v}^2(7-\epsilon_0^2) + \frac{7}{8}\frac{M}{r}\boldsymbol{v}^4 + \frac{1}{2}\frac{M^2}{r^2}\frac{(\boldsymbol{v}\cdot\boldsymbol{x})^2}{r^2}(1-\epsilon_0^2).\end{aligned} \quad (10)$$

Based on this Lagrangian, we can calculate the 2PN energy $\mathcal{E}$ and angular momentum $\mathcal{J}$ of the charged test particle as follows:

$$\begin{aligned}\mathcal{E} = &\frac{1}{2}\boldsymbol{v}^2 - \frac{M}{r}(1-\epsilon_0\epsilon_1) + \frac{3}{8}\boldsymbol{v}^4 + \frac{3}{2}\frac{M}{r}\boldsymbol{v}^2 + \frac{1}{2}\frac{M^2}{r^2}(1 + \epsilon_0^2 - 2\epsilon_0\epsilon_1) - \frac{1}{2}\frac{M^3}{r^3}(1 + \epsilon_0^2 - 2\epsilon_0\epsilon_1)\\ &+\frac{5}{16}\boldsymbol{v}^6 + \frac{1}{4}\frac{M^2}{r^2}\boldsymbol{v}^2(7-\epsilon_0^2) + \frac{21}{8}\frac{M}{r}\boldsymbol{v}^4 + \frac{1}{2}\frac{M^2}{r^2}\frac{(\boldsymbol{v}\cdot\boldsymbol{x})^2}{r^2}(1-\epsilon_0^2),\end{aligned} \quad (11)$$

$$\mathcal{J} = |\boldsymbol{x}\times\boldsymbol{v}|\left[1 + \frac{1}{2}\boldsymbol{v}^2 + \frac{3M}{r} + \frac{3}{8}\boldsymbol{v}^4 + \frac{1}{2}\frac{M^2}{r^2}(7-\epsilon_0^2) + \frac{7}{2}\frac{M}{r}\boldsymbol{v}^2\right]. \quad (12)$$

Notice that the mass m of the test particle has been absorbed in the Lagrangian, the orbital energy, and angular momentum.

III. THE QUASI-KEPLERIAN MOTION IN THE 2PN APPROXIMATIONS

We follow the same procedure given by Soffel *et al.* [2] to derive the 2PN solution for the quasi-Keplerian motion of the charged test particle in the Reissner-Nordström spacetime.

Because of the spherical symmetry of the Reissner-Nordström spacetime, we only need to consider the motion of the charged test particle's motion in the equatorial plane, in which the particle's trajectory can be expressed as

$$\boldsymbol{x} = r(\cos\phi \boldsymbol{e}_x + \sin\phi \boldsymbol{e}_y), \tag{13}$$

where ϕ is the azimuthal angle. $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors of the x and y axes.

The expressions for the orbital energy and angular momentum in Eqs. (11) and (12) can be written as

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{M}{r}(1-\epsilon_0\epsilon_1) + \frac{3}{8}(\dot{r}^2 + r^2\dot{\phi}^2)^2 + \frac{3}{2}\frac{M}{r}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{M^2}{2r^2}(1+\epsilon_0^2 - 2\epsilon_0\epsilon_1)\left(1-\frac{M}{r}\right) \\ & + \frac{5}{16}(\dot{r}^2 + r^2\dot{\phi}^2)^3 + \frac{1}{4}\frac{M^2}{r^2}(\dot{r}^2 + r^2\dot{\phi}^2)(7-\epsilon_0^2) + \frac{21}{8}\frac{M}{r}(\dot{r}^2 + r^2\dot{\phi}^2)^2 + \frac{M^2}{2r^2}\dot{r}^2(1-\epsilon_0^2),\end{aligned} \tag{14}$$

$$\mathcal{J}^2 = r^4\dot{\phi}^2\left[1 + \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{3M}{r} + \frac{3}{8}(\dot{r}^2 + r^2\dot{\phi}^2)^2 + \frac{7}{2}\frac{M}{r}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{7}{2}\frac{M^2}{r^2}\left(1 - \frac{1}{7}\epsilon_0^2\right)\right]^2, \tag{15}$$

where the dot denotes the derivative with respect to the time.

From these two expressions, we can obtain

$$r^4\dot{\phi}^2 = \mathcal{J}^2\left\{1 - 2\mathcal{E} - \frac{M}{r}(8-2\epsilon_0\epsilon_1) + 3\mathcal{E}^2 + \frac{M^2}{r^2}[34 - 18\epsilon_0\epsilon_1 + \epsilon_0^2(2+3\epsilon_1^2)] + \mathcal{E}\frac{M}{r}(16-6\epsilon_0\epsilon_1)\right\}, \tag{16}$$

and

$$\dot{r}^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D}{r^3} + \frac{E}{r^4}, \tag{17}$$

with

$$A = 2\mathcal{E}\left(1 - \frac{3}{2}\mathcal{E} + 2\mathcal{E}^2\right), \tag{18}$$

$$B = 2M[(1-\epsilon_0\epsilon_1) - 3\mathcal{E}(2-\epsilon_0\epsilon_1) + 3\mathcal{E}^2(3-2\epsilon_0\epsilon_1)], \tag{19}$$

$$C = -\mathcal{J}^2\left\{1 - 2\mathcal{E} + \frac{M^2}{\mathcal{J}^2}[10 - 14\epsilon_0\epsilon_1 + \epsilon_0^2(1+3\epsilon_1^2)] - 6\frac{M^2\mathcal{E}}{\mathcal{J}^2}[6 - 7\epsilon_0\epsilon_1 + \epsilon_0^2(1+2\epsilon_1^2)] + 3\mathcal{E}^2\right\}, \tag{20}$$

$$D = M\mathcal{J}^2\left\{(8-2\epsilon_0\epsilon_1) - \mathcal{E}(16-6\epsilon_0\epsilon_1) + 2\frac{M^2}{\mathcal{J}^2}[13 - 25\epsilon_0\epsilon_1 + \epsilon_0^2(5+12\epsilon_1^2) - \epsilon_0^3\epsilon_1(3+2\epsilon_1^2)]\right\}, \tag{21}$$

$$E = -3M^2\mathcal{J}^2[11 - 6\epsilon_0\epsilon_1 + \epsilon_0^2(1+\epsilon_1^2)]. \tag{22}$$

Making use of the relation

$$\dot{r}^2 = \left[\frac{d(1/r)}{d\phi}\right]^2(r^4\dot{\phi}^2), \tag{23}$$

and plugging Eqs. (16) and (17) into (23), we can write the radial equation in the form

$$\left[\frac{d(1/r)}{d\phi}\right]^2 = A' + \frac{B'}{r} + \frac{C'}{r^2} + \frac{D'}{r^3} + \frac{E'}{r^4}, \tag{24}$$

with

$$A' = \frac{2\mathcal{E}}{\mathcal{J}^2}\left(1 + \frac{1}{2}\mathcal{E}\right), \tag{25}$$

$$B' = \frac{2M}{\mathcal{J}^2}[(1-\epsilon_0\epsilon_1) + \mathcal{E}(4-\epsilon_0\epsilon_1) + 2\mathcal{E}^2], \tag{26}$$

$$C' = -\left\{1 - \frac{M^2}{\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] - 6(2-\epsilon_0\epsilon_1)\frac{M^2\mathcal{E}}{\mathcal{J}^2}\right\}, \tag{27}$$

$$D' = 2\frac{M^3}{\mathcal{J}^2}(3 - \epsilon_0^2 - 3\epsilon_0\epsilon_1 + \epsilon_0^2\epsilon_1^2), \tag{28}$$

$$E' = M^2(1-\epsilon_0^2). \tag{29}$$

Since the right-hand side of Eq. (24) is a fourth-order polynomial in r^{-1}, we can further rewrite it as

$$\left[\frac{d(1/r)}{d\phi}\right]^2 = \left[\frac{1}{r} - \frac{1}{a_r(1+e_r)}\right]\left[\frac{1}{a_r(1-e_r)} - \frac{1}{r}\right]\left(C_1 + \frac{C_2}{r} + \frac{C_3}{r^2}\right). \tag{30}$$

Comparing the coefficients between Eqs. (24) and (30), we have

$$a_r = \frac{M(1-\epsilon_0\epsilon_1)}{-2\mathcal{E}}\left\{1 + \frac{1}{2}\mathcal{E}\frac{7-\epsilon_0\epsilon_1}{1-\epsilon_0\epsilon_1} + \frac{1}{4}\mathcal{E}^2\frac{1+\epsilon_0\epsilon_1}{1-\epsilon_0\epsilon_1} + 2\frac{M^2\mathcal{E}}{\mathcal{J}^2}\left[(4-\epsilon_0^2) - \epsilon_0^2\frac{1-\epsilon_1^2}{1-\epsilon_0\epsilon_1}\right]\right\}, \tag{31}$$

$$\begin{aligned} e_r^2 = {} & 1 + \frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2} - \frac{\mathcal{E}}{(1-\epsilon_0\epsilon_1)^2}\left\{2[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] + \frac{3\mathcal{E}\mathcal{J}^2}{M^2}\frac{(5-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}\right\} \\ & + \frac{\mathcal{E}^2}{(1-\epsilon_0\epsilon_1)^2}\left\{\frac{30 - 36\epsilon_0\epsilon_1 - 3\epsilon_0^3\epsilon_1(3+\epsilon_1^2) + \epsilon_0^2(5+13\epsilon_1^2)}{(1-\epsilon_0\epsilon_1)} + \frac{2\mathcal{E}\mathcal{J}^2(40 - 15\epsilon_0\epsilon_1 + 2\epsilon_0^2\epsilon_1^2)}{M^2(1-\epsilon_0\epsilon_1)^2}\right. \\ & \left. - \frac{8M^2}{\mathcal{E}\mathcal{J}^2}[4 - 4\epsilon_0\epsilon_1 + \epsilon_0^3\epsilon_1 - \epsilon_0^2(2-\epsilon_1^2)](1-\epsilon_0\epsilon_1)\right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} C_1 = {} & 1 - \frac{M^2}{\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] - 2\frac{M^2\mathcal{E}}{\mathcal{J}^2}(7 - \epsilon_0^2 - 3\epsilon_0\epsilon_1) \\ & - \frac{4M^4}{\mathcal{J}^4}[4 - 4\epsilon_0\epsilon_1 + \epsilon_0^3\epsilon_1 - \epsilon_0^2(2-\epsilon_1^2)](1-\epsilon_0\epsilon_1), \end{aligned} \tag{33}$$

$$C_2 = -\frac{2M^3}{\mathcal{J}^2}[4 - 4\epsilon_0\epsilon_1 + \epsilon_0^3\epsilon_1 - \epsilon_0^2(2-\epsilon_1^2)], \tag{34}$$

$$C_3 = -M^2(1-\epsilon_0^2). \tag{35}$$

It can be seen from Eq. (30) that $r_\pm = a_r(1 \pm e_r)$ represent the maximal and minimal values for r. Hence, a_r and e_r can be regarded as the semimajor axis and the eccentricity of the quasi-Keplerian orbit.

The solution of Eq. (30) can be written as

$$r = \frac{a_r(1-e_r^2)}{1 + e_r\cos f}, \tag{36}$$

with f being the true anomaly for the quasi-Keplerian orbit and obeying

$$\left(\frac{df}{d\phi}\right)^2 = C_1 + \frac{C_2}{r} + \frac{C_3}{r^2}. \tag{37}$$

Substituting Eqs. (33)–(36) into Eq. (37), we have

$$\begin{aligned} \frac{df}{d\phi} = F\left\{1 - \frac{M^4}{\mathcal{J}^4}[5 - 8\epsilon_0\epsilon_1 - \epsilon_0^4\epsilon_1^2 - \epsilon_0^2(3-5\epsilon_1^2)\right. \\ \left. + \epsilon_0^3\epsilon_1(3-\epsilon_1^2)]e_r\cos f - \frac{M^4}{4\mathcal{J}^4}(1-\epsilon_0^2)e_r^2\cos 2f\right\}, \end{aligned} \tag{38}$$

with

$$F = 1 - \frac{M^2}{2\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] - \mathcal{E}\frac{M^2}{2\mathcal{J}^2}\left[\frac{1-\epsilon_0^2}{(1-\epsilon_0\epsilon_1)^2} + 2(7-\epsilon_0^2 - 3\epsilon_0\epsilon_1)\right]$$
$$- \frac{M^4}{8\mathcal{J}^4}[138 - 264\epsilon_0\epsilon_1 - 6\epsilon_0^2(11-28\epsilon_1^2) + \epsilon_0^3\epsilon_1(84-36\epsilon_1^2) + \epsilon_0^4(1-26\epsilon_1^2+\epsilon_1^4)]. \quad (39)$$

Integrating Eq. (38), we can obtain

$$\phi\left(\frac{2\pi}{\Phi}\right) = f + \frac{M^4}{\mathcal{J}^4}[5 - 8\epsilon_0\epsilon_1 - \epsilon_0^4\epsilon_1^2 - \epsilon_0^2(3-5\epsilon_1^2) + \epsilon_0^3\epsilon_1(3-\epsilon_1^2)]e_r \sin f + \frac{M^4}{8\mathcal{J}^4}(1-\epsilon_0^2)e_r^2 \sin 2f, \quad (40)$$

with

$$\Phi = 2\pi\left\{1 + \frac{M^2}{2\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] + \mathcal{E}\frac{M^2}{2\mathcal{J}^2}\left[\frac{1-\epsilon_0^2}{(1-\epsilon_0\epsilon_1)^2} + 2(7-\epsilon_0^2-3\epsilon_0\epsilon_1)\right]\right.$$
$$\left. + \frac{3M^4}{8\mathcal{J}^4}[70 - 136\epsilon_0\epsilon_1 - \epsilon_0^2(30-88\epsilon_1^2) + 4\epsilon_0^3\epsilon_1(9-5\epsilon_1^2) + \epsilon_0^4(1-10\epsilon_1^2+\epsilon_1^4)]\right\}. \quad (41)$$

Finally, we derive the time dependence of the quasi-Keplerian motion. Combining Eqs. (16), (38), and (39), we have

$$r^2\dot{f} = \mathcal{J}\left\{1 - \mathcal{E} - \frac{M}{r}(4-\epsilon_0\epsilon_1) - \frac{M^2}{2\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] + \mathcal{E}^2 + \frac{M^2}{r^2}[9 - 5\epsilon_0\epsilon_1 + \epsilon_0^2(1+\epsilon_1^2)]\right.$$
$$- \mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{9 - 16\epsilon_0\epsilon_1 - \epsilon_0^2(2-7\epsilon_1^2) + 2\epsilon_0^3\epsilon_1(1+\epsilon_1^2) - \epsilon_0^4\epsilon_1^2(1+\epsilon_1^2)}{2(1-\epsilon_0\epsilon_1)^2}$$
$$+ \frac{M}{r}\frac{M^2}{2\mathcal{J}^2}[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)](4-\epsilon_0\epsilon_1) + 2\mathcal{E}\frac{M}{r}(2-\epsilon_0\epsilon_1)$$
$$- \frac{M^4}{8\mathcal{J}^4}[138 - 264\epsilon_0\epsilon_1 - 6\epsilon_0^2(11-28\epsilon_1^2) + \epsilon_0^3\epsilon_1(84-36\epsilon_1^2) + \epsilon_0^4(1-26\epsilon_1^2+\epsilon_1^4)]$$
$$\left. - \frac{M^4}{\mathcal{J}^4}[5 - 8\epsilon_0\epsilon_1 - \epsilon_0^4\epsilon_1^2 - \epsilon_0^2(3-5\epsilon_1^2) + \epsilon_0^3\epsilon_1(3-\epsilon_1^2)]e_r\cos f - \frac{M^4}{4\mathcal{J}^4}(1-\epsilon_0^2)e_r^2\cos 2f\right\}. \quad (42)$$

Introducing the post-Newtonian eccentric anomaly u by the relations

$$\sin f = \frac{(1-e_r^2)^{\frac{1}{2}}\sin u}{1-e_r\cos u}, \qquad \cos f = \frac{\cos u - e_r}{1-e_r\cos u},$$
$$f = 2\arctan\left(\sqrt{\frac{1+e_r}{1-e_r}}\tan\frac{u}{2}\right), \quad (43)$$

we have

$$\frac{df}{dt} = \frac{(1-e_r^2)^{1/2}}{1-e_r\cos u}\frac{du}{dt}, \quad (44)$$

and we can formulate the orbit given in Eq. (36) in terms of u as

$$r = a_r(1-e_r\cos u). \quad (45)$$

Integrating Eq. (42) and making use of Eqs. (43)–(45), we can achieve the final piece of the 2PN solution for the motion in Reissner-Nordström spacetime,

$$t\left(\frac{2\pi}{\mathrm{T}_u}\right) = u - e_t\sin u + \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1-\epsilon_0^2 + 2(1-\epsilon_0\epsilon_1)^2(7-\epsilon_0^2-3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}(f-u), \quad (46)$$

with T_u being the period for the eccentric anomaly u of the quasi-Keplerian motion

$$\mathrm{T}_u = \frac{2\pi M(1-\epsilon_0\epsilon_1)}{(-2\mathcal{E})^{\frac{3}{2}}}\left[1-\frac{3}{4}\mathcal{E}\frac{(5-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}-\frac{15}{32}\mathcal{E}^2\frac{(7+\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)} + \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1-\epsilon_0^2+2(1-\epsilon_0\epsilon_1)^2(7-\epsilon_0^2-3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}\right], \tag{47}$$

and e_t being the time eccentricity

$$e_t = e_r\left[1+2\mathcal{E}\frac{(4-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}+\mathcal{E}^2\frac{36-19\epsilon_0\epsilon_1+\epsilon_0^2\epsilon_1^2}{(1-\epsilon_0\epsilon_1)^2} + 2\mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{4(1-\epsilon_0\epsilon_1)-\epsilon_0^2(2-\epsilon_1^2-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)} - \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1-\epsilon_0^2+2(1-\epsilon_0\epsilon_1)^2(7-\epsilon_0^2-3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}\right]. \tag{48}$$

In the literatures, one usually uses another true anomaly υ to replace the true anomaly f in the formula of the quasi-Keplerian equation, requiring the $\sin\upsilon$ contribution in $\phi(\frac{2\pi}{\Phi})$ to vanish at each PN order [8,16,43]. Following the same method given in Ref. [43], we set

$$\upsilon = 2\arctan\left(\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right), \tag{49}$$

with

$$e_\phi = e_r(1+\epsilon c_1+\epsilon^2 c_2), \tag{50}$$

differing from the radial eccentricity e_r by some 1PN and 2PN level corrections c_1 and c_2. Here ϵ only denotes the PN order and does not have any value. Eliminating u in Eq. (43) with the help of Eq. (49), we have [43]

$$f = \upsilon+\epsilon c_1\frac{e_r}{e_r^2-1}\sin\upsilon+\epsilon^2\left[\left(c_2-c_1^2\frac{e_r^2}{e_r^2-1}\right)\frac{e_r}{e_r^2-1}\sin\upsilon + \frac{c_1^2}{4}\frac{e_r^2}{(e_r^2-1)^2}\sin 2\upsilon\right]. \tag{51}$$

Substituting this result into Eq. (40) and requiring the $\sin\upsilon$ term to vanish in $\phi(\frac{2\pi}{\Phi})$, we can obtain

$$c_1 = 0, \tag{52}$$

$$c_2 = -2\mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{5-8\epsilon_0\epsilon_1-\epsilon_0^2(3-5\epsilon_1^2+\epsilon_0^2\epsilon_1^2)+\epsilon_0^3\epsilon_1(3-\epsilon_1^2)}{(1-\epsilon_0\epsilon_1)^2}, \tag{53}$$

which leads to

$$e_\phi = e_r\left[1-2\mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{5-8\epsilon_0\epsilon_1-\epsilon_0^2(3-5\epsilon_1^2+\epsilon_0^2\epsilon_1^2)+\epsilon_0^3\epsilon_1(3-\epsilon_1^2)}{(1-\epsilon_0\epsilon_1)^2}\right], \tag{54}$$

$$\phi\left(\frac{2\pi}{\Phi}\right) = \upsilon+\frac{M^4}{8\mathcal{J}^4}(1-\epsilon_0^2)\left[1+\frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2}\right]\sin 2\upsilon. \tag{55}$$

With the true anomaly υ, we can reexpress the time dependance of the quasi-Keplerian motion Eq. (46) in the form of

$$t\left(\frac{2\pi}{\mathrm{T}_u}\right) = u-e_t\sin u+\frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1-\epsilon_0^2+2(1-\epsilon_0\epsilon_1)^2(7-\epsilon_0^2-3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}(\upsilon-u). \tag{56}$$

Notice that $|\epsilon_0|\leq 1$ and $|\epsilon_0\epsilon_1|\ll 1$ are assumed in the above derivations, and all the formulas are valid up to 2PN accuracy.

IV. SUMMARY

Basing on the 2PN metric of the Reissner-Nordström black hole in the harmonic coordinates and the geodesic equation with the Lorentz force, we first calculate the corresponding Lagrangian, orbital energy, and angular momentum of the charged test particle. Then, through a function fitting method, we obtain the orbital parameters. Finally, we derive the quasi-Keplerian equation of the charged test particle in the Reissner-Nordström spacetime. We obtain two slightly different but equivalent formulations in the 2PN approximations. The results are summarized as follows.

The first formulation can be expressed as

$$\begin{aligned}
\boldsymbol{x} &= r(\cos\phi \boldsymbol{e}_x + \sin\phi \boldsymbol{e}_y),\\
r &= a_r(1 - e_r\cos u),\\
\phi\left(\frac{2\pi}{\Phi}\right) &= f + N_0\sin f + N_1\sin 2f,\\
f &= 2\arctan\left(\sqrt{\frac{1+e_r}{1-e_r}}\tan\frac{u}{2}\right),\\
t\left(\frac{2\pi}{\mathrm{T}_u}\right) &= u - e_t\sin u + N_2(f-u),
\end{aligned}$$

and the second formulation can be expressed as

$$\begin{aligned}
\boldsymbol{x} &= r(\cos\phi \boldsymbol{e}_x + \sin\phi \boldsymbol{e}_y),\\
r &= a_r(1 - e_r\cos u),\\
\phi\left(\frac{2\pi}{\Phi}\right) &= \upsilon + N_1\sin 2\upsilon,\\
\upsilon &= 2\arctan\left(\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right),\\
t\left(\frac{2\pi}{\mathrm{T}_u}\right) &= u - e_t\sin u + N_2(\upsilon-u),
\end{aligned}$$

where

$$\begin{aligned}
a_r &= \frac{M(1-\epsilon_0\epsilon_1)}{-2\mathcal{E}}\left\{1 + \frac{1}{2}\mathcal{E}\frac{7-\epsilon_0\epsilon_1}{1-\epsilon_0\epsilon_1} + \frac{1}{4}\mathcal{E}^2\frac{1+\epsilon_0\epsilon_1}{1-\epsilon_0\epsilon_1} + 2\frac{M^2\mathcal{E}}{\mathcal{J}^2}\left[(4-\epsilon_0^2) - \epsilon_0^2\frac{1-\epsilon_1^2}{1-\epsilon_0\epsilon_1}\right]\right\},\\
e_r^2 &= 1 + \frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2} - \frac{\mathcal{E}}{(1-\epsilon_0\epsilon_1)^2}\left\{2[6 - 6\epsilon_0\epsilon_1 - \epsilon_0^2(1-\epsilon_1^2)] + \frac{3\mathcal{E}\mathcal{J}^2}{M^2}\frac{(5-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}\right\}\\
&\quad + \frac{\mathcal{E}^2}{(1-\epsilon_0\epsilon_1)^2}\left\{\frac{30 + 5\epsilon_0^2 - 36\epsilon_0\epsilon_1 - 3\epsilon_0^3\epsilon_1(3+\epsilon_1^2) + 13\epsilon_0^2\epsilon_1^2}{(1-\epsilon_0\epsilon_1)} + \frac{2\mathcal{E}\mathcal{J}^2(40 - 15\epsilon_0\epsilon_1 + 2\epsilon_0^2\epsilon_1^2)}{M^2(1-\epsilon_0\epsilon_1)^2}\right.\\
&\quad \left. - \frac{8M^2}{\mathcal{E}\mathcal{J}^2}[4 - 4\epsilon_0\epsilon_1 + \epsilon_0^3\epsilon_1 - \epsilon_0^2(2-\epsilon_1^2)](1-\epsilon_0\epsilon_1)\right\},\\
e_t &= e_r\left[1 + 2\mathcal{E}\frac{(4-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)} + \mathcal{E}^2\frac{36 - 19\epsilon_0\epsilon_1 + \epsilon_0^2\epsilon_1^2}{(1-\epsilon_0\epsilon_1)^2} + 2\mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{4(1-\epsilon_0\epsilon_1) - \epsilon_0^2(2 - \epsilon_1^2 - \epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}\right.\\
&\quad \left. - \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1 - \epsilon_0^2 + 2(1-\epsilon_0\epsilon_1)^2(7 - \epsilon_0^2 - 3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}\right],\\
e_\phi &= e_r\left[1 - 2\mathcal{E}\frac{M^2}{\mathcal{J}^2}\frac{5 - 8\epsilon_0\epsilon_1 - \epsilon_0^2(3 - 5\epsilon_1^2 + \epsilon_0^2\epsilon_1^2) + \epsilon_0^3\epsilon_1(3-\epsilon_1^2)}{(1-\epsilon_0\epsilon_1)^2}\right],\\
\Phi &= 2\pi\left\{1 + \frac{M^2}{2\mathcal{J}^2}[6(1-\epsilon_0\epsilon_1) - \epsilon_0^2(1-\epsilon_1^2)] + \mathcal{E}\frac{M^2}{2\mathcal{J}^2}\left[\frac{1-\epsilon_0^2}{(1-\epsilon_0\epsilon_1)^2} + 2(7 - \epsilon_0^2 - 3\epsilon_0\epsilon_1)\right]\right.\\
&\quad \left. + \frac{3M^4}{8\mathcal{J}^4}[70 - 136\epsilon_0\epsilon_1 - \epsilon_0^2(30 - 88\epsilon_1^2) + 4\epsilon_0^3\epsilon_1(9 - 5\epsilon_1^2) + \epsilon_0^4(1 - 10\epsilon_1^2 + \epsilon_1^4)]\right\},\\
N_0 &= \frac{M^4}{\mathcal{J}^4}[5 - 8\epsilon_0\epsilon_1 - \epsilon_0^4\epsilon_1^2 - \epsilon_0^2(3 - 5\epsilon_1^2) + \epsilon_0^3\epsilon_1(3-\epsilon_1^2)]\left[1 + \frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2}\right]^{\frac{1}{2}},\\
N_1 &= \frac{M^4}{8\mathcal{J}^4}(1-\epsilon_0^2)\left[1 + \frac{2\mathcal{E}\mathcal{J}^2}{M^2(1-\epsilon_0\epsilon_1)^2}\right],\\
N_2 &= \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1 - \epsilon_0^2 + 2(1-\epsilon_0\epsilon_1)^2(7 - \epsilon_0^2 - 3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3},\\
\mathrm{T}_u &= \frac{2\pi M(1-\epsilon_0\epsilon_1)}{(-2\mathcal{E})^{\frac{3}{2}}}\left[1 - \frac{3}{4}\mathcal{E}\frac{(5-\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)} - \frac{15}{32}\mathcal{E}^2\frac{(7+\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)}\right.\\
&\quad \left. + \frac{2M\mathcal{E}^2}{\sqrt{-2\mathcal{E}\mathcal{J}^2}}\frac{1 - \epsilon_0^2 + 2(1-\epsilon_0\epsilon_1)^2(7 - \epsilon_0^2 - 3\epsilon_0\epsilon_1)}{(1-\epsilon_0\epsilon_1)^3}\right].
\end{aligned}$$

In the formulations, a_r, e_r, and u can be regarded as the semimajor axis, eccentricity, and eccentric anomaly of the quasi-Keplerian motion in the post-Newtonian approximations. f and v are two slightly different definitions of the true anomaly. T_u denotes the orbital period. The difference between Φ and 2π is the perihelion precession. The effects of the black hole's charge on the test particle's motion, including perihelion precession and orbital period, are characterized by the terms containing ϵ_0, and the effects of the test particle's charge are described by the terms containing ϵ_1. The achieved 2PN solution can be applied to the motion of the electrically charged test particles with small charge-to-mass ratio in the Reissner-Nordström spacetime, which has $|\epsilon_0| \leq 1$ for the non-naked singularity, e.g., the typically charged solar mass object in the field of the charged supermassive black hole. It can also be applied to the motion of the test particle with arbitrary charge-to-mass ratio in the field of the weakly charged black hole as long as $|\epsilon_0\epsilon_1| \ll 1$.

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