Nonperturbative light-front effective potential for static sources in quenched scalar Yukawa theory

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We compute an effective potential between two fixed sources in light-front quantization of a quenched scalar Yukawa theory that models the interaction of complex scalar fields through the exchange of a neutral scalar. Despite the breaking of explicit rotational symmetry by the use of light-front coordinates, the effective potential is rotationally symmetric and matches the standard Yukawa potential for scalar exchange. The neutral scalar field is represented by a coherent state, which is obtained nonperturbatively as an eigenstate of our model Hamiltonian, with the eigenenergy determining the effective potential. The sources are represented by wave packets that are fixed with respect to ordinary time, but move in light-front coordinates. The theory is quenched, to remove pair-production processes that would otherwise cause the spectrum to be unbounded from below. For divergent contributions we consider both a simple momentum cutoff and a Pauli-Villars regularization, with the divergence absorbed by a renormalized mass for the sources.

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I. INTRODUCTION

A key quantity to obtain from quantum chromodynamics is the effective potential between a quark and an antiquark [1].[1] This has been studied quite carefully in the lattice formulation of QCD [3]. For comparison, it would be useful to be able to do the equivalent calculation in a nonperturbative light-front formulation.[2] In order to develop a method for doing so, we study static sources in a quenched scalar Yukawa theory, where the complications of gauge fields and intrinsic quantum numbers can be neglected. We focus on formulating an eigenvalue problem that yields the energy of a state with two static sources dressed by a cloud of scalar particles. While doing so, we accommodate the later necessity of associating dynamics with intrinsic quantum numbers of the sources. This is to allow for sources with spin and color charge that will change when interacting with the surrounding cloud. Here we restrict the model to complex source fields with ordinary charge; however, we accommodate the possibility of dynamical properties by placing the static sources in the quantum state, rather than using delta-function currents in the Lagrangian.

The analysis of static sources on a light front has been considered previously. In particular, Rozowsky and Thorn [10] studied the force between two sources on a light front by arranging a purely transverse separation, a limitation that we avoid. Burkardt and Klindworth [11] applied a transverse lattice approach [12] in $(2+1)$-dimensional QCD to the calculation of a $Q\bar{Q}$ potential which is nearly rotationally invariant. Blunden *et al.* [13] considered light-front models where a particle interacts with a static potential.

An important aspect of our work is that we obtain the effective potential through variation of the ordinary energy, built from light-front quantities, rather than the light-front energy alone. This is the physical definition of a potential that can be translated into a force between the sources. The necessity of considering the ordinary energy has been seen in other contexts [14–17]. Such a consideration is also important because the static sources prevent momentum conservation, leaving the ordinary energy as the only conserved part of the four-momentum.

We take the following as our definition of light-front coordinates [18]:

[1]For recent discussion of static sources, see, for example, [2].
[2]For reviews of light-front quantization and applications, see [4–9].

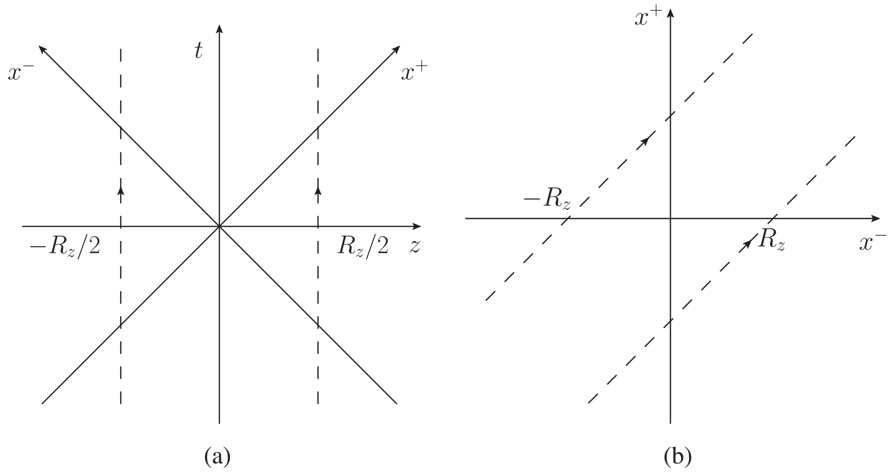

FIG. 1. Static source trajectories in the $z-t$ plane for (a) ordinary and (b) light-front coordinates.

$$x^{\pm} \equiv (t \pm z), \qquad \vec{x}_\perp = (x, y), \qquad \underline{x} = (x^-, \vec{x}_\perp), \tag{1.1}$$

with x^+ chosen as the light-front time. The inversion is obviously $t = \frac{1}{2}(x^+ + x^-)$ and $z = \frac{1}{2}(x^+ - x^-)$. These relationships are illustrated in Fig. 1(a).[3] The associated derivatives are

$$\frac{\partial}{\partial x^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial z}\right). \tag{1.2}$$

The light-front energy and momentum are

$$p^- \equiv E + p_z, \quad p^+ \equiv E - p_z, \quad \vec{p}_\perp = (p_x, p_y). \tag{1.3}$$

The dot product of four-vectors is $p \cdot x = \frac{1}{2} p^- x^+ + \underline{p} \cdot \underline{x}$ with $\underline{p} \cdot \underline{x} = \frac{1}{2} p^+ x^- - \vec{p}_\perp \cdot \vec{x}_\perp$ as the dot product of three-vectors. The mass-shell condition is then $p \cdot p = p^+ p^- - p_\perp^2 = m^2$, which implies $p^- = (m^2 + p_\perp^2)/p^+$.

In what follows, we first define quenched scalar Yukawa theory in light-front quantization, in Sec. II. The construction of a single static source dressed by neutral scalars is described in Sec. III; this includes mass renormalization such that the eigenenergy is equal to the physical mass of the complex scalar field. Two static sources are then constructed as a product of single sources placed with fixed separation in Sec. IV; here we obtain the key result that the energy of the two-source eigenstate is shifted from twice the physical mass of one by an amount equal to the Yukawa potential. We also compute the change in the average number of neutral scalars. To anticipate similar calculations in more complicated theories, particularly gauge theories, we rederive these results with use of Pauli-Villars regularization [9] in Sec. V to replace the simple momentum cutoff used in Secs. III and IV. A brief summary is given in Sec. VI. Details of the construction of wave packets on a light front are given in Appendix A, to properly represent a static source that moves in the x^- direction [13]. Some aspects of the calculations and an alternate approach to the single-source case are placed in additional Appendixes.

[3]The figure was drawn with JaxoDraw [19].

II. QUENCHED SCALAR YUKAWA THEORY

Scalar Yukawa theory couples a complex scalar field χ with bare mass m_0 to a real scalar field ϕ with mass μ; the physical mass of the complex scalar will be written as m. The Lagrangian is

$$\mathcal{L} = \partial_\mu \chi^* \partial^\mu \chi - m_0^2 |\chi|^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2 \phi^2 - g\phi|\chi|^2. \tag{2.1}$$

This model is also known as the (massive) Wick–Cutkosky model [20] and has received considerable attention in light-front quantization [21–31] in both two and four dimensions. The most recent work focuses on the construction of the eigenstate for a charged scalar dressed by a cloud of neutrals [32,33].

The quenched form of the theory excludes pair production. Without this restriction, the theory is ill defined, with a spectrum that is unbounded from below, as happens in any cubic scalar theory [34,35]. The quenching also means that the neutral scalar does not require mass renormalization, hence the use of the physical mass μ in the Lagrangian.

The light-front Hamiltonian density is [9]

$$\mathcal{H} = |\vec{\partial}_\perp \chi|^2 + m_0^2 |\chi|^2 + \frac{1}{2}(\vec{\partial}_\perp \phi)^2 + \frac{1}{2}\mu^2 \phi^2 + g\phi|\chi|^2. \tag{2.2}$$

The mode expansions for the fields are

$$\phi(x) = \int \frac{dp^+ d^2 p_\perp}{\sqrt{16\pi^3 p^+}} [a(\underline{p}) e^{-ip\cdot x} + a^\dagger(\underline{p}) e^{ip\cdot x}] \tag{2.3}$$

and

$$\chi(x) = \int \frac{dp^+ d^2 p_\perp}{\sqrt{16\pi^3 p^+}} [c_+(\underline{p}) e^{-ip\cdot x} + c_-^\dagger(\underline{p}) e^{ip\cdot x}]. \tag{2.4}$$

The nonzero commutators are

$$[a(\underline{p}), a^\dagger(\underline{p}')] = \delta(\underline{p} - \underline{p}') \equiv \delta(p^+ - p'^+)\delta(\vec{p}_\perp - \vec{p}'_\perp),$$
$$[c_\pm(\underline{p}), c_\pm^\dagger(\underline{p}')] = \delta(\underline{p} - \underline{p}'). \tag{2.5}$$

The light-front Hamiltonian is $\mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}_{\rm int}^-$, with

$$\mathcal{P}_0^- = \int d\underline{p} \frac{m_0^2 + \vec{p}_\perp^2}{p^+} [c_+^\dagger(\underline{p}) c_+(\underline{p}) + c_-^\dagger(\underline{p}) c_-(\underline{p})] + \int d\underline{q} \frac{\mu^2 + \vec{q}_\perp^2}{q^+} a^\dagger(\underline{q}) a(\underline{q}), \tag{2.6}$$

and

$$\mathcal{P}_{\rm int}^- = g\int \frac{d\underline{p} d\underline{q}}{\sqrt{16\pi^3 p^+ q^+ (p^+ + q^+)}} [(c_+^\dagger(\underline{p}+\underline{q}) c_+(\underline{p}) + c_-^\dagger(\underline{p}+\underline{q}) c_-(\underline{p})) a(\underline{q}) + a^\dagger(\underline{q})(c_+^\dagger(\underline{p}) c_+(\underline{p}+\underline{q}) + c_-^\dagger(\underline{p}) c_-(\underline{p}+\underline{q}))]$$
$$+ g\int \frac{d\underline{p}_1 d\underline{p}_2}{\sqrt{16\pi^3 p_1^+ p_2^+ (p_1^+ + p_2^+)}} [c_+^\dagger(\underline{p}_1) c_-^\dagger(\underline{p}_2) a(\underline{p}_1 + \underline{p}_2) + a^\dagger(\underline{p}_1 + \underline{p}_2) c_+(\underline{p}_1) c_-(\underline{p}_2)]. \tag{2.7}$$

For the quenched theory, the second term in $\mathcal{P}_{\rm int}^-$ is dropped. The light-front momentum operator is

$$\mathcal{P}^+ = \int d\underline{q} q^+ a^\dagger(\underline{q}) a(\underline{q}) + \int d\underline{p} p^+ [c_+^\dagger(\underline{p}) c_+(\underline{p}) + c_-^\dagger(\underline{p}) c_-(\underline{p})]. \tag{2.8}$$

We then define an ordinary energy operator as

$$\mathcal{E} = \frac{1}{2}(\mathcal{P}^- + \mathcal{P}^+). \tag{2.9}$$

This will play a key role, because momentum is not conserved when static sources are present.

III. SINGLE SOURCE

A. Wave packet for the source

A static source is not at rest in light-front coordinates. It moves steadily in the positive x^- direction, as indicated in Fig. 1(b). For this we consider light-front wave packets for the sources, which are discussed in detail in Appendix A. We place a single source at $\pm\vec{R}/2$ with a state given by

$$|F^\pm\rangle = \int d\underline{p} \sqrt{p^+} F^\pm(\underline{p}) c_\pm^\dagger(\underline{p}) |0\rangle, \tag{3.1}$$

where $F^\pm(\underline{p})$ is a momentum-space envelope function peaked at a light-front momentum for an object of mass m at rest, $\underline{p} = (m, \vec{0}_\perp)$. The explicit factor of $\sqrt{p^+}$ is included to facilitate calculation of expectation values for the light-front energy and the probability density.

The model is intended to produce a spatial probability density that is sharply peaked at the source locations $\pm\vec{R}/2$. To impose this, we compute the expectation value of $|\chi|^2$ for the state $|F^\pm\rangle$

$$\langle F^\pm| : |\chi^2| : |F^\pm\rangle = \langle F^\pm| \int \frac{d\underline{p}'}{\sqrt{16\pi^3 p'^+}} \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} [c_+^\dagger(\underline{p}') c_+(\underline{p}) e^{i(p'-p)\cdot x} + c_-^\dagger(\underline{p}') c_-(\underline{p}) e^{i(p'-p)\cdot x} + \cdots] |F^\pm\rangle. \tag{3.2}$$

Here the extra dots indicate terms that do not contribute. Contractions of the various operators yield delta functions that resolve all but two sets of integrals. We evaluate at $x^+ = 0$:

$$\langle F^\pm| : |\chi^2| : |F^\pm\rangle|_{x^+=0} = \int \frac{d\underline{p}'}{\sqrt{16\pi^3}} F^{\pm *}(\underline{p}') e^{i\underline{p}'\cdot\underline{x}} \int \frac{d\underline{p}}{\sqrt{16\pi^3}} F^\pm(\underline{p}) e^{-i\underline{p}\cdot\underline{x}}. \tag{3.3}$$

With the definition

$$\psi^\pm(\underline{x}) = \int \frac{d\underline{p}}{\sqrt{16\pi^3}} F^\pm(\underline{p}) e^{-i\underline{p}\cdot\underline{x}}, \tag{3.4}$$

the expectation value reduces to

$$\langle F^\pm|:|\chi^2|:|F^\pm\rangle|_{x^+=0} = |\psi^\pm(\underline{x})|^2. \tag{3.5}$$

To represent a static source, we require that this become a delta function when the spatial packet becomes infinitesimally narrow[4]

$$\langle F^\pm|:|\chi^2|:|F^\pm\rangle|_{x^+=0} = N^2\delta(x^-\pm R_z)\delta(\vec{x}_\perp \mp \vec{R}_\perp/2), \tag{3.6}$$

with N^2 a normalization factor to be determined. Therefore, we identify

$$|\psi^\pm(\underline{x})|^2 \to N^2\delta(x^-\pm R_z)\delta(\vec{x}_\perp \mp \vec{R}_\perp/2). \tag{3.7}$$

The momentum-space wave-packet envelope is the Fourier transform

$$F^\pm(\underline{p}) = \int \frac{d\underline{x}}{\sqrt{16\pi^3}} e^{i\underline{p}\cdot\underline{x}}\psi^\pm(\underline{x}). \tag{3.8}$$

As transforms, $\psi^\pm$ and $F^\pm$ share the common normalization

$$N^2 = \int d\underline{x}|\psi^\pm(\underline{x})|^2 = \int d\underline{p}|F^\pm(\underline{p})|^2, \tag{3.9}$$

where the integral over p^+ can be extended to $-\infty$ because the envelope $F^\pm$ is sharply peaked about $p^+ = m > 0$.

The value of the normalization for $F^\pm$ is separately determined by the normalization of $|F^\pm\rangle$ as

$$1 = \langle F^\pm|F^\pm\rangle = \int d\underline{p}\, p^+|F^\pm(\underline{p})|^2 = m\int d\underline{p}|F^\pm(\underline{p})|^2 = mN^2, \tag{3.10}$$

where p^+ is replaced by the peak value of m and the p^+ integration is again extended to $-\infty$. Thus, we have $N = 1/\sqrt{m}$, which is dimensionally consistent with the Hamiltonian term $m^2|\chi|^2$ being an energy density.

The expectation value for the free part of the light-front energy for the complex field is given by

$$\langle \mathcal{P}^-_{0\chi}\rangle = \int d\underline{x}\langle F^\pm|:|\vec{\partial}_\perp\chi|^2 + m_0^2|\chi|^2:|F^\pm\rangle. \tag{3.11}$$

The first term can be evaluated from

$$\vec{\partial}_\perp\chi = \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} i\vec{p}_\perp[c_+(\underline{p})e^{-ip\cdot x} - c_-^\dagger(\underline{p})e^{ip\cdot x}]. \tag{3.12}$$

The presence of the $\vec{p}_\perp$ factor means that the first term in (3.11) is zero, because the wave packet is symmetrically peaked at $\vec{p}_\perp = 0$. The second term is readily obtained from (3.6), with the normalization $N^2 = 1/m$, as

$$\int d\underline{x}\langle F^\pm|:|m_0^2|\chi|^2:|F^\pm\rangle = \frac{m_0^2}{m}\int d\underline{x}\delta(x^-\pm R_z)\delta(\vec{x}_\perp \mp \vec{R}_\perp/2) = \frac{m_0^2}{m}. \tag{3.13}$$

The expectation value for the light-front longitudinal momentum is m, because the wave packets are sharply peaked at $p^+ = m$. Thus for $\mathcal{E} = (\mathcal{P}^- + \mathcal{P}^+)/2$ we have

$$\langle F^\pm|\mathcal{E}|F^\pm\rangle = m_0^2/2m + m/2, \tag{3.14}$$

which reduces to m when the coupling g is zero and the mass is not renormalized. The presence of interactions with the neutral scalar will renormalize the mass, as we show below.

B. Coherent state for the neutrals

On this source state $|F^\pm\rangle$, we build a coherent state of neutrals as an *ansatz* for the solution

$$|G_1^\pm F^\pm\rangle = \sqrt{Z_1^\pm}e^{\int d\underline{q}G_1^\pm(\underline{q})a^\dagger(\underline{q})}|F^\pm\rangle, \tag{3.15}$$

with $\sqrt{Z_1^\pm}$ a normalization factor given by

$$Z_1^\pm = e^{-\int d\underline{q}|G_1^\pm(\underline{q})|^2}. \tag{3.16}$$

It is an eigenstate of the annihilation operator

$$a(\underline{q})|G_1^\pm F^\pm\rangle = G_1^\pm(\underline{q})|G_1^\pm F^\pm\rangle, \tag{3.17}$$

and we require it also to be an eigenstate of the energy operator $\mathcal{E}$

$$\mathcal{E}|G_1^\pm F^\pm\rangle = E^\pm|G_1^\pm F^\pm\rangle. \tag{3.18}$$

This eigenvalue condition, projected onto $\langle F^\pm|$, reduces to

$$\left[\frac{m_0^2}{2m} + \frac{1}{2}m\right]|G_1^\pm\rangle + \int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+} + q^+\right]a^\dagger(\underline{q})G_1^\pm(\underline{q})|G_1^\pm\rangle + \frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\{e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}G_1^\pm(\underline{q}) + e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}a^\dagger(\underline{q})\}|G_1^\pm\rangle = E^\pm|G_1^\pm\rangle, \tag{3.19}$$

[4]The constraint for x^- is obtained by requiring $z = \frac{1}{2}(x^+ - x^-) = \pm R_z/2$ to hold at light-front time $x^+ = 0$.

where we have used (3.6) to replace $\langle F^\pm||\chi|^2|F^\pm\rangle$ and used the resulting delta functions to do the $\underline{x}$ integrals in $\mathcal{P}^-_{\rm int} = \int d\underline{x} g\phi|\chi|^2$.

For this eigenvalue equation to be satisfied, the terms that contain $a^\dagger$ must cancel, which means that

$$\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]G_1^\pm(\underline{q}) + \frac{g}{2m}\frac{1}{\sqrt{16\pi^3 q^+}}e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2} = 0. \tag{3.20}$$

The function $G_1^\pm$ must then be

$$G_1^\pm(\underline{q}) = -\frac{g}{m}\sqrt{\frac{q^+}{16\pi^3}}\frac{e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}}{(q^+)^2+q_\perp^2+\mu^2}. \tag{3.21}$$

The eigenenergy is then computed from the remaining terms as

$$\begin{aligned} E^\pm &= \frac{m_0^2}{2m}+\frac{1}{2}m+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}G_1^\pm(\underline{q}) \\ &= \frac{m_0^2}{2m}+\frac{1}{2}m-\frac{1}{2}\left(\frac{g}{m}\right)^2\int\frac{d\underline{q}}{16\pi^3}\frac{1}{(q^+)^2+q_\perp^2+\mu^2}. \end{aligned} \tag{3.22}$$

The integral in the last term is divergent. We introduce a cutoff Λ and define

$$I(\Lambda) = \int\frac{d\underline{q}}{16\pi^3\mu}\frac{\theta(\Lambda^2-(q^+)^2-q_\perp^2)}{(q^+)^2+q_\perp^2+\mu^2}, \tag{3.23}$$

and the eigenenergy becomes

$$E^\pm = \frac{m_0^2}{2m}+\frac{1}{2}m-\frac{1}{2}\left(\frac{g}{m}\right)^2\mu I(\Lambda). \tag{3.24}$$

We can arrange $E^\pm = m$, the physical mass, by choosing the bare mass such that

$$m_0^2 = m^2 + g^2\frac{\mu}{m}I(\Lambda). \tag{3.25}$$

The cutoff dependence is then removed.

We now have an exact solution for the single-source state that includes the source and a Fock-state expansion in the number of neutral scalars. The form is independent of the source location, except for a phase in the individual wave functions $G_1^\pm$. We also see that, in both cases, the normalization of the state is determined by

$$Z_1^\pm = e^{-\left(\frac{g}{m}\right)^2\int\frac{d\underline{q}}{16\pi^3}\frac{q^+}{((q^+)^2+q_\perp^2+\mu^2)^2}}. \tag{3.26}$$

The integral is divergent and requires a cutoff, to give meaning to the norm of the state.

Assuming that such a cutoff is in place, the single-source problem can be also formulated as a variational problem with $|G_1^\pm F^\pm\rangle$ as the trial state. The expectation value of the energy is

$$\begin{aligned} \langle G_1^\pm F^\pm|{:}\mathcal{E}{:}|G_1^\pm F^\pm\rangle &= \frac{m_0^2}{2m}+\frac{1}{2}m+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]G_1^{\pm*}(\underline{q})G_1^\pm(\underline{q}) \\ &\quad + \frac{g}{m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}[e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}G_1^\pm(\underline{q}) + e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}G_1^{\pm*}(\underline{q})]. \end{aligned} \tag{3.27}$$

Variation with respect to $G_1^{\pm*}$ yields (3.20) and the same results follow, including the value m of $\langle G_1^\pm F^\pm|{:}\mathcal{E}{:}|G_1^\pm F^\pm\rangle$ at this minimum, once the mass renormalization is taken into account.

For comparison with the double-source state, we also consider the average number $\langle n\rangle_\pm$ of neutral scalars in this single-source state. This is computed from the coherent state as

$$\langle n\rangle_\pm \equiv \int d\underline{q}\langle G_1^\pm F^\pm|a^\dagger(\underline{q})a(\underline{q})|G_1^\pm F^\pm\rangle = \int d\underline{q}|G_1^\pm(\underline{q})|^2. \tag{3.28}$$

However, on substitution of the form for $G_1^\pm$, the expectation value reduces to

$$\langle n\rangle_\pm = \left(\frac{g}{m}\right)^2\int\frac{d\underline{q}}{16\pi^3}\frac{(q^+)^2}{[(q^+)^2+q_\perp^2+\mu^2]^2}, \tag{3.29}$$

which contains the same divergent integral that defines the normalization. Thus, unless one chooses to fix the coupling g as a bare coupling by fixing the value of $\langle n\rangle_\pm$, the number of neutrals that dress a single source is effectively infinite. Instead of invoking a restriction on the coupling, we will accept this infinite value as part of the nature of the single-source state and investigate the change in the number of neutrals induced by the presence of a second source.

IV. DOUBLE SOURCE

The case of two static sources can also be solved with a coherent state. We take the variational approach, with a trial state built from a product of single-source states of opposite charge

$$|G_2G_1^+G_1^-F^+F^-\rangle = \sqrt{\frac{Z}{Z_1^+Z_1^-}}e^{\int d\underline{q}G_2(\underline{q})a^\dagger(\underline{q})}|G_1^+F^+\rangle|G_1^-F^-\rangle, \tag{4.1}$$

where $G_2(\underline{q})$ is a function to be determined and Z fixes the overall normalization as

$$Z = e^{-\int d\underline{q}|G_2(\underline{q})+G_1^+(\underline{q})+G_1^-(\underline{q})|^2}. \tag{4.2}$$

It will turn out that $G_2(\underline{q})$ is actually zero.

We choose the two sources to be of opposite charge, to have a simpler calculation without the cross terms that would arise for two identical sources with the same charge. However, this is not a serious restriction because one can argue that the overlap between spatial packets is effectively zero. This would allow the cross terms to be ignored.

When there are two sources, the expectation value of $:|\chi|^2:$ becomes

$$\begin{aligned}\langle F^+F^-|:|\chi^2|:|F^+F^-\rangle|_{x^+=0} = &\int\frac{d\underline{p}'}{\sqrt{16\pi^3p'^+}}F^{+*}(\underline{p}')e^{i\underline{p}'\cdot\underline{x}}\int\frac{d\underline{p}}{\sqrt{16\pi^3p^+}}F^+(\underline{p})e^{-i\underline{p}\cdot\underline{x}}\int d\underline{p}_2p_2^+|F^-(\underline{p}_2)|^2\\ &+\int\frac{d\underline{p}'}{\sqrt{16\pi^3p'^+}}F^{-*}(\underline{p}')e^{i\underline{p}'\cdot\underline{x}}\int\frac{d\underline{p}}{\sqrt{16\pi^3p^+}}F^-(\underline{p})e^{i\underline{p}\cdot\underline{x}}\int d\underline{p}_1p_1^+|F^+(\underline{p}_1)|^2.\end{aligned} \tag{4.3}$$

With the normalization requirement $\int d\underline{p}p^+|F^\pm(\underline{p})|^2 = 1$ and the limit (3.7), this reduces to

$$\langle F^+F^-|:|\chi^2|:|F^+F^-\rangle|_{x^+=0} = \frac{1}{m}[\delta(x^-+R_z)\delta(\vec{x}_\perp-\vec{R}_\perp/2)+\delta(x^--R_z)\delta(\vec{x}_\perp+\vec{R}_\perp/2)], \tag{4.4}$$

which places the sources appropriately at $\vec{R}/2$ and $-\vec{R}/2$. In terms of light-front coordinates, the peaks are at $x^- = \mp R_z$ and $\vec{x}_\perp = \pm\vec{R}_\perp/2$ when $x^+ = 0$.

The expectation value for $\mathcal{E}$ in the state is

$$\begin{aligned}\langle:\mathcal{E}:\rangle = &\frac{1}{2}\left[2\frac{m_0^2}{m}+2m\right]+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]|G_2(\underline{q})+G_1^+(\underline{q})+G_1^-(\underline{q})|^2\\ &+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\{e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+e^{-iq^+R_z/2-i\vec{q}_\perp\cdot\vec{R}_\perp/2}\}\\ &\times[G_2(\underline{q})+G_1^+(\underline{q})+G_1^-(\underline{q})+G_2^*(\underline{q})+G_1^{+*}(\underline{q})+G_1^{-*}(\underline{q})].\end{aligned} \tag{4.5}$$

Variation with respect to G_2^* results in

$$\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right][G_2(\underline{q})+G_1^+(\underline{q})+G_1^-(\underline{q})]+\frac{g}{2m}\frac{1}{\sqrt{16\pi^3q^+}}\{e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+\text{c.c.}\} = 0. \tag{4.6}$$

Substitution of the known expressions for $G_1^\pm(\underline{q})$ leaves $G_2(\underline{q}) = 0$. In other words, the effect of combining two single sources is included in the phase associated with each source location and the interference terms between G_1^+ and G_1^-.

On the use of $G_2 = 0$ and the expressions for $G_1^\pm$, the expectation value $\langle:\mathcal{E}:\rangle$ becomes, as discussed in Appendix C,

$$\langle:\mathcal{E}:\rangle = \frac{m_0^2}{m}+m-\left(\frac{g}{m}\right)^2\mu I(\Lambda)-\left(\frac{g}{2m}\right)^2\frac{e^{-\mu R}}{4\pi R}, \tag{4.7}$$

The first three terms of $\langle:\mathcal{E}:\rangle$ reduce to $2m$, with use of the constraint (3.25) for the bare mass. This brings us to our key result

$$\langle:\mathcal{E}:\rangle = 2m-\left(\frac{g}{2m}\right)^2\frac{e^{-\mu R}}{4\pi R}, \tag{4.8}$$

which shows that the eigenenergy of two static sources is their total mass plus a rotationally symmetric, attractive

Yukawa potential in the standard form for scalar exchange between scalars.[5]

Perhaps the most remarkable aspect is the rotational symmetry, despite the explicit breaking of rotational symmetry by light-front coordinates. This is achieved not by fine tuning but by staying close to the physics of the configuration, in that the effective potential between the sources is contained within the ordinary energy not the light-front energy and the sources are static with respect to ordinary time not light-front time.

The change in the number of neutral scalars, induced by the proximity of the two sources, is given by

$$\langle\delta n\rangle \equiv \int d\underline{q}\langle a^\dagger(\underline{q})a(\underline{q})\rangle - \langle n\rangle_+ - \langle n\rangle_- \tag{4.9}$$

$$= \int d\underline{q}|G_1^+(\underline{q}) + G_1^-(\underline{q})|^2 - \int d\underline{q}|G_1^+(\underline{q})|^2 - \int d\underline{q}|G_1^-(\underline{q})|^2. \tag{4.10}$$

Again, only the interference terms contribute

$$\langle\delta n\rangle = \int d\underline{q}[G_1^{+*}(\underline{q})G_1^-(\underline{q}) + G_1^{-*}(\underline{q})G_1^+(\underline{q})]. \tag{4.11}$$

Substitution of the form for $G_1^\pm$ and some additional calculus, shown in Appendix D, reduces this to

$$\langle\delta n\rangle = -\frac{1}{16\pi^2}\left(\frac{g}{m}\right)^2[e^{\mu R}\mathrm{Ei}(-\mu R) + e^{-\mu R}\mathrm{Ei}(\mu R)], \tag{4.12}$$

where Ei is the exponential integral function [36]. For large separations R, this simplifies to

$$\langle\delta n\rangle = -\frac{1}{8\pi^2}\left(\frac{g}{m}\right)^2\frac{1}{(\mu R)^2} + \mathcal{O}\left(\frac{1}{(\mu R)^3}\right), \tag{4.13}$$

which correctly goes to zero as the separation becomes infinite.

V. PAULI-VILLARS REGULARIZATION

A more sophisticated approach to regularization of the infinities encountered, instead of the momentum cutoff, is to use Pauli-Villars (PV) regularization by inserting a heavy neutral scalar with negative metric [9]. This particular formulation of PV regularization was developed in Yukawa theory [37] and successfully extended to QED [38]; a formulation for further extension to nonAbelian gauge theories also exists [9]. The addition to the Lagrangian is

$$\mathcal{L}_{\rm PV} = -\left[\frac{1}{2}(\partial_\mu\phi_{\rm PV})^2 - \frac{1}{2}\mu_{\rm PV}^2\phi_{\rm PV}^2\right] - g\phi_{\rm PV}|\chi|^2. \tag{5.1}$$

The mode expansion for the PV field is

$$\phi_{\rm PV}(x) = \int\frac{dp^+d^2p_\perp}{\sqrt{16\pi^3p^+}}[a_{\rm PV}(\underline{p})e^{-ip\cdot x} + a_{\rm PV}^\dagger(\underline{p})e^{ip\cdot x}], \tag{5.2}$$

with the nonzero commutator

$$[a_{\rm PV}(\underline{p}), a_{\rm PV}^\dagger(\underline{p}')] = -\delta(\underline{p} - \underline{p}'). \tag{5.3}$$

The minus sign is the negative metric that provides for cancellations between regular and PV contributions.

The contribution to the free Hamiltonian is

$$\mathcal{P}_{0\rm PV}^- = -\int d\underline{q}\frac{\mu_{\rm PV}^2 + \vec{q}_\perp^{\,2}}{q^+}a_{\rm PV}^\dagger(\underline{q})a_{\rm PV}(\underline{q}), \tag{5.4}$$

and to the quenched interaction term

$$\mathcal{P}_{\rm int\,PV}^- = g\int\frac{d\underline{p}d\underline{q}}{\sqrt{16\pi^3p^+q^+(p^+ + q^+)}}[(c_+^\dagger(\underline{p}+\underline{q})c_+(\underline{p}) + c_-^\dagger(\underline{p}+\underline{q})c_-(\underline{p}))a_{\rm PV}(\underline{q}) + a_{\rm PV}^\dagger(\underline{q})(c_+^\dagger(\underline{p})c_+(\underline{p}+\underline{q}) + c_-^\dagger(\underline{p})c_-(\underline{p}+\underline{q}))]. \tag{5.5}$$

The contribution to the light-front momentum operator is

$$\mathcal{P}_{\rm PV}^+ = -\int d\underline{q}q^+a_{\rm PV}^\dagger(\underline{q})a_{\rm PV}(\underline{q}). \tag{5.6}$$

We can then define the PV contribution to the energy operator as $\mathcal{E}_{\rm PV} \equiv \frac{1}{2}(\mathcal{P}_{0\rm PV}^- + \mathcal{P}_{\rm int\,PV}^- + \mathcal{P}_{\rm PV}^+)$.

The *ansatz* for the solution to $(\mathcal{E} + \mathcal{E}_{\rm PV})|G_1^\pm G_{\rm PV}^\pm F^\pm\rangle = E^\pm|G_1^\pm G_{\rm PV}^\pm F^\pm\rangle$ for the single-source case is

$$|G_1^\pm G_{\rm PV}^\pm F^\pm\rangle = \sqrt{Z_1^\pm Z_{\rm PV}^\pm}e^{\int d\underline{q}G_1^\pm(\underline{q})a^\dagger(\underline{q})}e^{\int d\underline{q}G_{\rm PV}^\pm(\underline{q})a_{\rm PV}^\dagger(\underline{q})}|F^\pm\rangle, \tag{5.7}$$

with the normalization constants given by (3.16) and

$$Z_{\rm PV}^\pm = e^{\int d\underline{q}|G_{\rm PV}^\pm(\underline{q})|^2}. \tag{5.8}$$

[5] This differs slightly from the Yukawa potential between fermions, because in that case the interaction term in the Lagrangian is $g\phi\overline{\psi}\psi$ and g is dimensionless; here g has units of mass.

The action of the PV annihilation operator yields

$$a_{\rm PV}(\underline{q})|G_1^\pm G_{\rm PV}^\pm F^\pm\rangle = -G_{\rm PV}^\pm(\underline{q})|G_1^\pm G_{\rm PV}^\pm F^\pm\rangle, \tag{5.9}$$

where the minus sign is due to the negative PV metric.

Projection of the eigenvalue problem onto $\langle F^\pm|$ leaves

$$\left[\frac{m_0^2}{2m}+\frac{1}{2}m\right]|G_1^\pm G_{\rm PV}^\pm\rangle+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]a^\dagger(\underline{q})G_1^\pm(\underline{q})|G_1^\pm G_{\rm PV}^\pm\rangle+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu_{\rm PV}^2}{q^+}+q^+\right]a_{\rm PV}^\dagger(\underline{q})G_{\rm PV}^\pm(\underline{q})|G_1^\pm G_{\rm PV}^\pm\rangle$$
$$+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\{e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}[G_1^\pm(\underline{q})-G_{\rm PV}^\pm(\underline{q})]+e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}[a^\dagger(\underline{q})+a_{\rm PV}^\dagger(\underline{q})]\}|G_1^\pm G_{\rm PV}^\pm\rangle$$
$$=E^\pm|G_1^\pm G_{\rm PV}^\pm\rangle. \tag{5.10}$$

The terms that contain $a^\dagger$ and $a_{\rm PV}^\dagger$ must cancel separately. For $a^\dagger$ this yields the same equation as before, (3.20), and for $a_{\rm PV}^\dagger$ we have

$$\frac{1}{2}\left[\frac{q_\perp^2+\mu_{\rm PV}^2}{q^+}+q^+\right]G_{\rm PV}^\pm(\underline{q}) + \frac{g}{2m}\frac{1}{\sqrt{16\pi^3 q^+}}e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}=0. \tag{5.11}$$

The function $G_1^\pm$ is then unchanged from (3.21) and

$$G_{\rm PV}^\pm(\underline{q}) = -\frac{g}{m}\sqrt{\frac{q^+}{16\pi^3}}\frac{e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}}{(q^+)^2+q_\perp^2+\mu_{\rm PV}^2}. \tag{5.12}$$

The eigenenergy is computed from the remaining terms as

$$E^\pm = \frac{m_0^2}{2m}+\frac{1}{2}m+\frac{g}{2m}\times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}[G_1^\pm(\underline{q})-G_{\rm PV}^\pm(\underline{q})]. \tag{5.13}$$

Substitution of the forms of $G_1^\pm$ and $G_{\rm PV}^\pm$ and definition of the integral

$$I_{\rm PV}(\mu_{\rm PV}^2)=\int\frac{d\underline{q}}{16\pi^3\mu}\left[\frac{1}{(q^+)^2+q_\perp^2+\mu^2}-\frac{1}{(q^+)^2+q_\perp^2+\mu_{\rm PV}^2}\right], \tag{5.14}$$

provides for

$$E^\pm = \frac{m_0^2}{2m}+\frac{1}{2}m-\frac{1}{2}\left(\frac{g}{m}\right)^2\mu I_{\rm PV}(\mu_{\rm PV}^2). \tag{5.15}$$

The integral $I_{\rm PV}$ is, of course, the analog of the cutoff integral defined earlier in (3.23). It is finite, except in the limit of infinite PV mass, and is used to renormalize the source mass as before:

$$m_0^2 = m^2+g^2\frac{\mu}{m}I_{\rm PV}(\mu_{\rm PV}^2). \tag{5.16}$$

We then find that $E^\pm$ is just m. The insertion of the PV scalar also regulates the normalization:

$$Z_1^\pm Z_{\rm PV}^\pm = e^{-\int d\underline{q}[|G_1^\pm(\underline{q})|^2-|G_{\rm PV}^\pm(\underline{q})|^2]} = \exp\left\{-\left(\frac{g}{m}\right)^2\int\frac{d\underline{q}}{16\pi^3}\left(\frac{q^+}{(q^+)^2+q_\perp^2+\mu^2}-\frac{q^+}{(q^+)^2+q_\perp^2+\mu_{\rm PV}^2}\right)\right\}. \tag{5.17}$$

For the double-source case, the *ansatz* is a product of the single-source solutions

$$|G_1^+G_{\rm PV}^+G_1^-G_{\rm PV}^-F^+F^-\rangle = |G_1^+G_{\rm PV}^+F^+\rangle|G_1^-G_{\rm PV}^-F^-\rangle. \tag{5.18}$$

Projection of the eigenvalue problem

$$(\mathcal{E}+\mathcal{E}_{\rm PV})|G_1^+G_{\rm PV}^+G_1^-G_{\rm PV}^-F^+F^-\rangle = E|G_1^+G_{\rm PV}^+G_1^-G_{\rm PV}^-F^+F^-\rangle \tag{5.19}$$

onto $\langle F^+F^-|$ yields

$$\left\{2\left[\frac{m_0^2}{2m}+\frac{1}{2}m\right]+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]a^\dagger(\underline{q})[G_1^+(\underline{q})+G_1^-(\underline{q})]+\int d\underline{q}\frac{1}{2}\left[\frac{q_\perp^2+\mu_{\rm PV}^2}{q^+}+q^+\right]a_{\rm PV}^\dagger(\underline{q})[G_{\rm PV}^+(\underline{q})+G_{\rm PV}^-(\underline{q})]\right.$$
$$\left.+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}[e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+e^{-iq^+R_z/2-i\vec{q}_\perp\cdot\vec{R}_\perp/2}][G_1^+(\underline{q})-G_{\rm PV}^+(\underline{q})+G_1^-(\underline{q})-G_{\rm PV}^-(\underline{q})+a^\dagger(\underline{q})+a_{\rm PV}^\dagger(\underline{q})]\right\}$$
$$\times|G_1^+G_{\rm PV}^+G_1^-G_{\rm PV}^-\rangle=E|G_1^+G_{\rm PV}^+G_1^-G_{\rm PV}^-\rangle. \tag{5.20}$$

The previous determined forms of $G_1^\pm$ and $G_{\rm PV}^\pm$ are sufficient to render the coefficients of $a^\dagger$ and $a_{\rm PV}^\dagger$ as zero. Thus, the eigenvalue problem is solved as before with a simple overlap of single-source states, and the eigenvalue is

$$E=\frac{m_0^2}{m}+m+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}[e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+e^{-iq^+R_z/2-i\vec{q}_\perp\cdot\vec{R}_\perp/2}][G_1^+(\underline{q})-G_{\rm PV}^+(\underline{q})+G_1^-(\underline{q})-G_{\rm PV}^-(\underline{q})]. \tag{5.21}$$

Substitution of the expressions for $G_1^\pm$ and $G_{\rm PV}^\pm$ leaves

$$E=\frac{m_0^2}{m}+m-\frac{g^2}{m^2}\mu I_{\rm PV}(\mu_{\rm PV}^2)-\frac{g^2}{2m^2}[Y(R)-Y_{\rm PV}(R)], \tag{5.22}$$

where $I_{\rm PV}$ is given in (5.14), Y is defined in (C4), and

$$Y_{\rm PV}(R)\equiv\int_{q^+>0}\frac{d\underline{q}}{16\pi^3}\frac{e^{iq^+R_z+i\vec{q}_\perp\cdot\vec{R}_\perp}+e^{-iq^+R_z-i\vec{q}_\perp\cdot\vec{R}_\perp}}{(q^+)^2+q_\perp^2+\mu_{\rm PV}^2}. \tag{5.23}$$

Because $Y_{\rm PV}$ differs from Y only by the replacement of μ by $\mu_{\rm PV}$, we can immediately evaluate $Y_{\rm PV}$, as in Appendix C, and obtain, on use of the mass renormalization (5.16),

$$E=2m-\left(\frac{g}{2m}\right)^2\frac{1}{4\pi R}[e^{-\mu R}-e^{-\mu_{\rm PV}R}]. \tag{5.24}$$

In the limit of an infinite PV mass, we then recover our original result with an effective potential of $-(\frac{g}{2m})^2\frac{e^{-\mu R}}{4\pi R}$. The result for $\langle\delta n\rangle$ is also unchanged.

VI. SUMMARY

We have shown that, by considering the ordinary energy of static sources fixed with respect to ordinary time, a light-front calculation yields the correct Yukawa potential. Rotational symmetry is maintained despite the explicit breaking by the light-front coordinates themselves and without fine tuning of parameters. The effective potential arises from the overlap between the clouds of neutral scalars that dress the sources. It is essentially an interference term in the expectation value of the energy.

The success of the calculation is due to two factors. One is that we consider the ordinary energy E, not the light-front energy P^-. The other is that the sources are fixed with respect to ordinary time, not light-front time x^+. This is analogous to our work on the Casimir effect [17]. The primary observation is that changing coordinate systems does not and should not change the physics.

The calculation is nonperturbative, even though the resulting Yukawa potential is of order g^2. The eigensolution is obtained to all orders in g as a coherent state of neutral scalars. Such a solution is possible because the static sources remove the constraint of momentum conservation.

The approach can be extended to more complicated theories, although the solution of the eigenvalue problem will typically require numerical techniques. An obvious next application is to standard Yukawa theory with two fermions as sources static in position but dynamic with respect to spin. This can be done first as quenched but then also with fermion-pair contributions. Static-source potentials in QED and QCD are also clearly of interest; we suggest that the present work provides a starting point, with the implementation of Pauli-Villars regularization [9] as a key ingredient.

APPENDIX A: LIGHT-FRONT WAVE PACKETS

To be able to incorporate the steady movement in x^- of a static source, we consider the quantum mechanics of wave packets on a light front. We use such wave packets for the sources dressed by neutral scalars in a Fock-state expansion or a coherent state for the neutral scalars.

The ordinary time evolution of a particle with wave function Ψ is determined by the usual Schrödinger equation $i\frac{\partial\Psi}{\partial t}=\mathcal{P}^0\Psi$. The action of the momentum operator $\mathcal{P}^z$ is represented by $-i\frac{\partial}{\partial z}$. Therefore, the light-front time evolution is determined by

$$i\frac{\partial\Psi}{\partial x^+}=\frac{i}{2}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\Psi=\frac{1}{2}(\mathcal{P}^0-\mathcal{P}^z)\Psi=\frac{1}{2}\mathcal{P}^-\Psi. \tag{A1}$$

Separation of variables is then applied, with $\Psi(\underline{x},x^+)=\tau(x^+)\psi(\underline{x})$, to find

$$\frac{2i}{\tau}\frac{d\tau}{dx^+}=\frac{1}{\psi}\mathcal{P}^-\psi\equiv\frac{m^2+p_\perp^2}{p^+}, \tag{A2}$$

where the separation constant is written in the form of the on-shell light-front energy p^- for a particle of mass m with light-front momentum $\underline{p}=(p^+,\vec{p}_\perp)$. The light-front time evolution is then $\tau=\exp(-i\frac{m^2+p_\perp^2}{2p^+}x^+)$.

In momentum space, $\mathcal{P}^-$ for a free particle of momentum $\underline{p}$ is just the multiplicative operator $\frac{m^2+p_\perp^2}{p^+}$, and $\phi_{\underline{p}}(\underline{q})=N\delta(q^+-p^+)\delta(\vec{q}_\perp-\vec{p}_\perp)$ is the eigenfunction. A Fourier transform yields $\psi_{\underline{p}}(\underline{x})=\tilde{N}e^{i\underline{p}\cdot\underline{x}}$. A wave packet, with momentum envelope $\phi(\underline{p})$, is then given by

$$\Psi(\underline{x},x^+)=\int\frac{d\underline{p}}{\sqrt{16\pi^3}}\phi(\underline{p})\exp\left[i\left(\underline{p}\cdot\underline{x}-\frac{m^2+p_\perp^2}{2p^+}x^+\right)\right]. \tag{A3}$$

The normalization factor contains 16 rather than 8 because the dot product contains a factor of $1/2$ for the p^+x^- term.

For a static source, we have $\vec{p}_\perp=0$; however, p^+ is not zero. For a source at $z=\pm R_z/2$, we must have $\frac{1}{2}(x^+-x^-)=\pm R_z/2$ or $x^-=x^+\mp R_z$. Thus, x^- increases with light-front time [13]. For the wave packet, this corresponds to the factor $\exp[i(\frac{1}{2}p^+x^--\frac{m^2}{2p^+}x^+)]$, which, to be consistent with x^--x^+ being constant for the trajectory, must have $p^+=m$. This is, of course, the usual value for p^+ when the particle is at rest, but this simple analysis shows the connection with the associated wave packet and establishes that the envelope ϕ must be peaked at $\underline{p}=(m,\vec{0}_\perp)$.

As an example of these envelope functions, we consider a Gaussian form, parametrized by a width ϵ, that becomes a delta function in the appropriate limit.[6] The peak momentum value of $p^+=m$ is achieved by including a phase factor $e^{-imx^-/2}$

$$\psi^\pm(\underline{x})=\frac{1/\sqrt{m}}{(\epsilon\sqrt{\pi})^{3/2}}e^{-imx^-/2}e^{-(x^-\pm R_z)^2/2\epsilon^2}e^{-(\vec{x}_\perp\mp\vec{R}_\perp/2)^2/2\epsilon^2}. \tag{A4}$$

The Fourier transform is

$$\phi^\pm(\underline{p})=\frac{1}{\sqrt{2m}}\left(\frac{\epsilon}{\sqrt{\pi}}\right)^{3/2}e^{\mp i(p^+-m)R_z/2}e^{\pm i\vec{p}_\perp\cdot\vec{R}_\perp/2}\times e^{-\epsilon^2(p^+-m)^2/8}e^{-\epsilon^2p_\perp^2/2}. \tag{A5}$$

For a static source, the momentum and position distributions are both sharply defined. In our units, where $\hbar=1$, this is not mathematically obvious; the correct relationship is recovered in an $\hbar\to0$ limit.

APPENDIX B: FOCK-SPACE EXPANSION METHOD FOR THE SINGLE-SOURCE PROBLEM

We construct an eigenstate of $\mathcal{E}$ as a Fock-state expansion for the neutrals, built on a state for a single static source at $\pm\vec{R}/2$

$$|\psi F^\pm\rangle=\sum_n\left(\prod_i^n\int d\underline{p}_i\right)\psi_n(\underline{p}_1,\ldots,\underline{p}_n)\times\frac{1}{\sqrt{n!}}\prod_i^n a^\dagger(\underline{p}_i)|F^\pm\rangle. \tag{B1}$$

Because of the static source, (light-front) momentum is not conserved, and we do not seek simultaneous eigenstates for $\mathcal{P}^-$ and $\mathcal{P}^+$. Thus the Fock-state expansion does not contain a momentum conserving delta function to restrict the integrals over individual momenta. We simply require $\mathcal{E}|\psi F^\pm\rangle=E^\pm|\psi F^\pm\rangle$.

The action of individual parts of $\mathcal{E}$ [See Eqs. (2.6)–(2.8).] yield the following

$$\mathcal{P}_0^-|\psi F^\pm\rangle=\frac{m_0^2}{m}|\psi F^\pm\rangle+\sum_n\left(\prod_i^n\int d\underline{p}_i\right)\left(\sum_i^n\frac{p_{\perp i}^2+\mu^2}{p_i^+}\right)\psi_n(\underline{p}_1,\ldots,\underline{p}_n)\frac{1}{\sqrt{n!}}\prod_i^n a^\dagger(\underline{p}_i)|F^\pm\rangle, \tag{B2}$$

$$\mathcal{P}^+|\psi F^\pm\rangle=m|\psi F^\pm\rangle+\sum_n\left(\prod_i^n\int d\underline{p}_i\right)\left(\sum_i^n p_i^+\right)\psi_n(\underline{p}_1,\ldots,\underline{p}_n)\frac{1}{\sqrt{n!}}\prod_i^n a^\dagger(\underline{p}_i)|F^\pm\rangle, \tag{B3}$$

and

$$\mathcal{P}_{\rm int}^-|\psi F^\pm\rangle=g\sum_n\left(\prod_i^n\int d\underline{p}_i\right)\psi_n(\underline{p}_1,\ldots,\underline{p}_n)\int d\underline{x}:|\chi|^2:\times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}[a(\underline{q})e^{-iq^+x^-/2+i\vec{q}_\perp\cdot\vec{x}_\perp}+a^\dagger(\underline{q})e^{iq^+x^-/2-i\vec{q}_\perp\cdot\vec{x}_\perp}]\frac{1}{\sqrt{n!}}\prod_i^n a^\dagger(\underline{p}_i)|F^\pm\rangle. \tag{B4}$$

[6]The analysis presented in the main sections is independent of the specific form chosen.

Projection of $\mathcal{E}|\psi F^\pm\rangle = E^\pm|\psi F^\pm\rangle$ onto $\langle F^\pm|\frac{1}{\sqrt{n'!}}\prod_i^{n'} a(\underline{q}_i)$ yields

$$\left[\frac{m_0^2}{2m}+\frac{1}{2}m+\sum_i^{n'}\frac{1}{2}\left(\frac{q_{\perp i}^2+\mu^2}{q_i^+}+q_i^+\right)\right]\psi_{n'}(\underline{q}_1,\ldots,\underline{q}_{n'})+\frac{g}{2m}\sum_j^{n'}\frac{1}{\sqrt{16\pi^3q^+}}e^{\mp iq_j^+R_z/2\mp i\vec{q}_{\perp j}\cdot\vec{R}_\perp/2}$$
$$\times\frac{1}{\sqrt{n'}}\psi_{n'-1}(\underline{q}_1,\ldots,\underline{q}_{j-1},\underline{q}_{j+1},\ldots,\underline{q}_{n'})+\frac{g}{2m}\sum_j^{n'+1}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}\frac{1}{\sqrt{n'+1}}\psi_{n'+1}(\underline{q}_1,\ldots,\underline{q}_{j-1},\underline{q},\underline{q}_j,\ldots,\underline{q}_{n'})$$
$$=E^\pm\psi_{n'}(\underline{q}_1,\ldots,\underline{q}_{n'}). \tag{B5}$$

An analytic solution is obtained by writing ψ_n as a product of single-particle wave functions $G_1^\pm(\underline{q}_i)$

$$\psi_n(\underline{q}_1,\ldots,\underline{q}_n)=\frac{1}{\sqrt{n!}}\prod_i^n G_1^\pm(\underline{q}_i). \tag{B6}$$

Substitution of this product, and division by $\psi_{n'}$, leaves

$$\sum_i^{n'}\left[\frac{1}{2}\left(\frac{q_{\perp i}^2+\mu^2}{q_i^+}+q_i^+\right)+\frac{g}{2m}\frac{1}{\sqrt{16\pi^3q^+}}\frac{e^{\mp iq_i^+R_z/2\mp i\vec{q}_{\perp i}\cdot\vec{R}_\perp/2}}{G_1^\pm(\underline{q}_i)}\right]$$
$$+\frac{m_0^2}{2m}+\frac{1}{2}m+\frac{g}{2m}\frac{1}{n'+1}\sum_j^{n'+1}f^\pm(\vec{R})=E^\pm, \tag{B7}$$

where

$$f^\pm(\vec{R})\equiv\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}G_1^\pm(\underline{q})e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}. \tag{B8}$$

The equation is solved provided that the content of the square brackets is zero and that $E^\pm$ is given by

$$E^\pm=\frac{m_0^2}{2m}+\frac{1}{2}m+\frac{g}{2m}f^\pm(\vec{R}). \tag{B9}$$

We will find that for this single-source case any $\vec{R}$ dependence is actually absent; the function $f^\pm$ is simply constant.

For the square bracket in (B7) to be zero, we need

$$G_1^\pm(\underline{q})=-\frac{g}{m}\sqrt{\frac{q^+}{16\pi^3}}\frac{e^{\mp iq_i^+R_z/2\mp i\vec{q}_{\perp i}\cdot\vec{R}_\perp/2}}{(q^+)^2+q_\perp^2+\mu^2}. \tag{B10}$$

Substitution into the expression for $f^\pm$ and then into $E^\pm$ yields

$$E^\pm=\frac{m_0^2}{2m}+\frac{m}{2}-\frac{1}{2}\left(\frac{g}{m}\right)^2\mu I(\Lambda) \tag{B11}$$

where $I(\Lambda)$ is defined in (3.23). The mass renormalization presented in (3.25) then fixes $E^\pm=m$, the physical mass.

We see that, due to the factorization (B6), the neutral scalars form the coherent state used in Sec. III. The location of the source, at $\pm\vec{R}/2$, is encoded in the phase of the individual wave functions $G_1^\pm$, with the product in each Fock sector having a phase that corresponds to translation in light-front coordinates from the origin to $(\pm R_z,\pm\vec{R}_\perp/2)$, as generated by the total momentum.

APPENDIX C: DOUBLE-SOURCE EXPECTATION VALUE FOR $\mathcal{E}$

On use of $G_2=0$ and the expression (3.21) for $G_1^\pm$, the double-source expectation value $\langle:\mathcal{E}:\rangle$ given in (4.5) becomes

$$\langle:\mathcal{E}:\rangle=\frac{m_0^2}{m}+m+\frac{1}{2}\left(\frac{g}{m}\right)^2\int d\underline{q}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]\frac{q^+}{16\pi^3}\frac{|e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+e^{-iq^+R_z/2-i\vec{q}_\perp\cdot\vec{R}_\perp/2}|^2}{[(q^+)^2+q_\perp^2+\mu^2]^2}$$
$$+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left(-2\frac{g}{m}\right)\sqrt{\frac{q^+}{16\pi^3}}\frac{|e^{iq^+R_z/2+i\vec{q}_\perp\cdot\vec{R}_\perp/2}+e^{-iq^+R_z/2-i\vec{q}_\perp\cdot\vec{R}_\perp/2}|^2}{(q^+)^2+q_\perp^2+\mu^2}. \tag{C1}$$

The last two terms differ only in their sign, and a factor of 2, and reduce to

$$\langle:\mathcal{E}:\rangle=\frac{m_0^2}{m}+m-\frac{1}{2}\left(\frac{g}{m}\right)^2\int\frac{d\underline{q}}{16\pi^3}\frac{2+e^{iq^+R_z+i\vec{q}_\perp\cdot\vec{R}_\perp}+e^{-iq^+R_z-i\vec{q}_\perp\cdot\vec{R}_\perp}}{(q^+)^2+q_\perp^2+\mu^2}. \tag{C2}$$

The 2 in the numerator corresponds to a divergent integral that, with a cutoff Λ, is proportional to the integral (3.23) previously encountered in the single-source case. From that definition, we have

$$\langle :\mathcal{E}: \rangle = \frac{m_0^2}{m} + m - \left(\frac{g}{m}\right)^2 \mu I(\Lambda) - \frac{1}{2}\left(\frac{g}{m}\right)^2 Y(R), \quad \text{(C3)}$$

with

$$Y(R) \equiv \int_{q^+>0} \frac{d\underline{q}}{16\pi^3} \frac{e^{iq^+R_z + i\vec{q}_\perp\cdot\vec{R}_\perp} + e^{-iq^+R_z - i\vec{q}_\perp\cdot\vec{R}_\perp}}{(q^+)^2 + q_\perp^2 + \mu^2}$$
$$= \frac{1}{2}\int \frac{d\underline{q}}{16\pi^3} \frac{e^{iq^+R_z + i\vec{q}_\perp\cdot\vec{R}_\perp} + e^{-iq^+R_z - i\vec{q}_\perp\cdot\vec{R}_\perp}}{(q^+)^2 + q_\perp^2 + \mu^2}. \quad \text{(C4)}$$

In the second integral there is no restriction on the range of q^+. This unrestricted integral is easily evaluated in spherical coordinates $\vec{q} = (q_x, q_y, q^+) = (q, \theta, \phi)$, relative to an axis parallel to $\vec{R}$. The ϕ integral is trivial, leaving

$$Y(R) = \frac{1}{16\pi^2}\int_0^\infty q^2 dq \int_{-1}^1 d\cos\theta \frac{e^{iqR\cos\theta} + e^{-iqR\cos\theta}}{q^2+\mu^2}. \quad \text{(C5)}$$

The $\cos\theta$ integral reduces this to

$$Y(R) = \frac{1}{16\pi^2}\int_0^\infty \frac{q^2 dq}{q^2+\mu^2}\left[\frac{e^{iqR} - e^{-iqR}}{iqR} + \frac{e^{-iqR} - e^{iqR}}{-iqR}\right]$$
$$= \frac{1}{4\pi^2 R}\int_0^\infty \frac{q^2 dq}{q^2+\mu^2}\sin(qR) = \frac{1}{4\pi^2 R}\frac{\pi}{2}e^{-\mu R}. \quad \text{(C6)}$$

Substitution into the expression (C3) for $\langle :\mathcal{E}: \rangle$ brings us to a nearly final form

$$\langle :\mathcal{E}: \rangle = \frac{m_0^2}{m} + m - \left(\frac{g}{m}\right)^2 \mu I(\Lambda) - \left(\frac{g}{2m}\right)^2 \frac{e^{-\mu R}}{4\pi R}, \quad \text{(C7)}$$

All that remains is to invoke mass renormalization.

APPENDIX D: CHANGE IN NUMBER OF SCALARS

The change induced in the number of scalars by the combination of two static sources is

$$\langle \delta n \rangle = \int d\underline{q}[G_1^{+*}(\underline{q})G_1^-(\underline{q}) + G_1^{-*}(\underline{q})G_1^+(\underline{q})]. \quad \text{(D1)}$$

Substitution of the form (3.21) for $G_1^\pm$ leaves

$$\langle \delta n \rangle = \left(\frac{g}{m}\right)^2 \int_{q^+>0} d\underline{q} \frac{q^+}{16\pi^3} \frac{2 + e^{iq^+R_z + i\vec{q}_\perp\cdot\vec{R}_\perp} + e^{-iq^+R_z - i\vec{q}_\perp\cdot\vec{R}_\perp}}{[(q^+)^2 + q_\perp^2 + \mu^2]^2}. \quad \text{(D2)}$$

Use of the same spherical coordinates as in Appendix C reduces this to

$$\langle \delta n \rangle = \left(\frac{g}{m}\right)^2 \frac{1}{16\pi^2}\int_0^\infty \frac{p^3 dp}{(p^2+\mu^2)^2}$$
$$\times \int_{-1}^1 d\cos\theta \cos\theta [e^{iqR\cos\theta} + e^{-iqR\cos\theta}]. \quad \text{(D3)}$$

Performance of the $\cos\theta$ integration, and some algebraic rearrangement, yields

$$\langle \delta n \rangle = \left(\frac{g}{m}\right)^2 \frac{1}{4\pi^2 R^2}\int_0^\infty \frac{dp}{(p^2+\mu^2)^2}$$
$$\times [Rp^2 \sin(pR) + p\cos(pR) - p]. \quad \text{(D4)}$$

The individual p integrations can be computed as follows

$$\int_0^\infty \frac{p\,dp}{(p^2+\mu^2)^2} = \frac{1}{2\mu^2}, \quad \text{(D5)}$$

$$\int_0^\infty \frac{p\cos(pR)dp}{(p^2+\mu^2)^2} = \frac{d}{dR}\int_0^\infty \frac{\sin(pR)dp}{(p^2+\mu^2)^2}, \quad \text{(D6)}$$

$$\int_0^\infty \frac{p^2\sin(pR)dp}{(p^2+\mu^2)^2} = -\frac{d}{dR}\int_0^\infty \frac{p\cos(pR)dp}{(p^2+\mu^2)^2}, \quad \text{(D7)}$$

$$\int_0^\infty \frac{\sin(pR)dp}{(p^2+\mu^2)^2} = -\frac{1}{2\mu}\frac{d}{d\mu}\int_0^\infty \frac{\sin(pR)dp}{p^2+\mu^2}, \quad \text{(D8)}$$

and [39]

$$\int_0^\infty \frac{\sin(pR)dp}{p^2+\mu^2} = \frac{1}{2\mu}[e^{-\mu R}\mathrm{Ei}(\mu R) - e^{\mu R}\mathrm{Ei}(-\mu R)], \quad \text{(D9)}$$

where Ei is the exponential integral function [36]. The combination of the various integrals yields

$$\langle \delta n \rangle = -\frac{1}{16\pi^2}\left(\frac{g}{m}\right)^2 [e^{\mu R}\mathrm{Ei}(-\mu R) + e^{-\mu R}\mathrm{Ei}(\mu R)]. \quad \text{(D10)}$$

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