Reduction of order and transseries structure of radiation reaction

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The Landau-Lifshitz equation is obtained from the Lorentz-Abraham-Dirac equation through "reduction of order." It is the first in a divergent series of approximations that, after resummation, eliminate runaway solutions. Using Borel plane and transseries analysis we explain why this is, and show that a nonperturbative formulation of reduction of order can retain runaway solutions. We also apply transseries analysis to solutions of the Lorentz-Abraham-Dirac equation, essentially treating them as expansions in both time and a coupling. Our results illustrate some aspects of such expansions under changes of variables and limits.

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I. INTRODUCTION

Radiation reaction (RR) continues to attract attention in classical and quantum electrodynamics, both experimentally [1,2] and theoretically [3–5] with a particular focus on intense laser fields where RR forces compare to or dominate the Lorentz force [6–8]. RR in strong fields is also relevant in gravitational physics, first clearly observed in the Hulse-Taylor binary pulsar [9], and studied theoretically in, e.g., Refs. [10–13].

Recently many authors have applied resummation [14–21] and resurgence and transseries concepts [22–24] in classical and quantum electrodynamics in strong backgrounds. (For introductions to and reviews of these concepts, see Refs. [25–31].) As a prominent example all-orders, resummed results [17,18] have been vital to progress on the Ritus-Narozhny conjecture [32,33] of the breakdown of Furry picture perturbation theory.

In this paper we use the Lorentz-Abraham-Dirac (LAD) equation of motion for radiation reaction [34–36] as a "test bed" for transseries analysis. It is a natural choice of a simple setting in which to explore transseries structures, essentially because we know that they must be there and their physical interpretation. They are the "unwanted" features of the LAD equation, preacceleration and runaway solutions, that are explicitly nonperturbative in τ_0, the time scale of radiation reaction. Indeed these are not seen in perturbative approaches, including reductive procedures which lead to e.g., the Landau-Lifshitz [37] (LL) equation, at any order [20]. We will see that the time-dependent nature of our problem means that even though the physics is quite simple, the formal structure can still be rich.

We extend our previous work [20], which iterated "reduction of order" *ad infinitum* in a constant crossed field (CCF) to obtain the all-orders (in τ_0) equation of motion LL_∞ by showing that this procedure eliminates nonperturbative transseries structure at the level of the equation of motion. We also show that the same holds in a circularly polarized monochromatic plane wave. The elimination of nonperturbative terms is, however, dependant on an "initial condition" matching to the Lorentz force at vanishing field. Other "initial conditions" keep nonperturbative terms and lead to runaway solutions of the order-reduced equation of motion. We then consider inserting a hard cutoff into a constant field; this is the simplest time dependence which allows us to unambiguously investigate preacceleration and its transseries structure.

This paper is organised as follows. We begin in Sec. II by reviewing reduction of order as applied to the LAD equation, and LL_∞. We show that nonperturbative contributions to LL_∞ are large as the coupling goes to zero, and lead to runaway solutions if kept. Next, in Sec. III we solve the two equations of motion in a step field profile, finding on the level of solutions to LAD instanton terms that are precisely the preaccelerating and runaway solutions. We conclude in Sec. IV.

II. LL_∞: REDUCTION OF ORDER AND TRANSSERIES

A. Conventions and notations

We will consider the momentum p^μ of a particle of charge e and mass m in a CCF given by

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$$f_{\mu\nu} := \frac{e}{m} F_{\mu\nu} = \mathcal{E} m n_{[\mu} \epsilon_{\nu]} \tag{1}$$

where $\mathcal{E}$ is the dimensionless field strength, n_μ is lightlike and $\epsilon^2 = -1$ with $n \cdot \epsilon = 0$. As we will only be concerned with one species of particle we henceforth use units where $m = 1$, although we will restore m in places for clarity. We will use light-front coordinates $p^\pm = p^0 \pm p^z$, $p^\perp = (p^1, p^2)$, with the z axis aligned such that $p^+ = n \cdot p$.

The LAD equation reads, using an overdot for a derivative with respect to proper time,

$$\dot{p}_\mu = f_{\mu\nu} p^\nu + \tau_0 P_{\mu\nu} \ddot{p}_\rho \tag{2}$$

where $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu$ is the projector orthogonal to p_μ and

$$\tau_0 := \frac{2\alpha}{3m}, \tag{3}$$

with α being the fine-structure constant; for an electron $\tau_0 \approx 6.2 \times 10^{-24}$ s. Nonperturbative effects in the solutions to the LAD equation occur on time scales of τ_0, but radiation reaction has observable effects over much longer time scales. The interaction is characterized by an energy parameter,

$$\delta^2 = \tau_0^2 p_\mu f^{\mu\nu} f_{\nu\rho} p^\rho = (\tau_0 \mathcal{E} p^+)^2. \tag{4}$$

When working at the level of the solution, the initial value δ_0 will play the role of a coupling. Note that $\delta = \tau_0 \chi$ where χ is the quantum nonlinearity parameter [5,38]; they are related in the same way that the classical electron radius and the Compton length are. This means that values $\delta \gtrsim 1$ are deeply in the quantum regime, and mainly relevant for classical electrodynamics as a formal theory.

B. Reduction of order and LL_∞

The LL equation [37] is obtained from the Lorentz-Abraham-Dirac equation by reduction of order: we apply $\mathrm{d}/\mathrm{d}\tau$ to both sides of Eq. (2), substitute for $\dot{u}$ according to Eq. (2) itself, and discard terms of order τ_0^2. This yields, in general,

$$\dot{p}_\mu = f_{\mu\nu} p^\nu + \tau_0 [(P f^2)_{\mu\nu} p^\nu + p^\rho \partial_\rho f_{\mu\nu} p^\nu], \tag{5}$$

although the final, gradient, term of course vanishes for a CCF.

The reduction of order procedure as just described reduces the order *in time*, but the procedure can be iterated any number of times to any order *in* τ_0 [39,40]. We will therefore refer to the first iteration (5) as LL_1. If reduction of order is iterated *ad infinitum*, i.e., to all orders in τ_0, it yields the equation of motion LL_∞,

$$\dot{p}^\mu = \mathcal{A}(\delta) f^{\mu\nu} p_\nu + \tau_0 \mathcal{B}(\delta) (P f^2)^{\mu\nu} p_\nu, \tag{6}$$

as discussed in a previous paper [20]. Here the functions $\mathcal{A}$ and $\mathcal{B}$ are solutions of the ordinary differential equations (ODEs)

$$\begin{cases} \delta^3 \mathcal{B} \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\delta} = & 1 - \mathcal{A} - 2\delta^2 \mathcal{A}\mathcal{B}, \\ \delta^3 \mathcal{B} \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\delta} = & -\mathcal{B} - 2\delta^2 \mathcal{B}^2 + \mathcal{A}^2, \end{cases} \tag{7}$$

and the initial conditions that recover the first-order Landau-Lifshitz equation are

$$\mathcal{A}(0) = \mathcal{B}(0) = 1. \tag{8}$$

The functions $\mathcal{A}$, $\mathcal{B}$ encode how the RR force varies with energy, vaguely analogous to a running coupling.

We emphasize here that when $\mathcal{A}$, $\mathcal{B}$ verify Eq. (7) the solution of LL_∞ is a solution of the LAD equation. Explicitly, differentiating Eq. (6) we obtain

$$\ddot{p}_\mu = \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\delta} \frac{\mathrm{d}\delta}{\mathrm{d}\tau} f_{\mu\nu} p^\nu + \mathcal{A} f_{\mu\nu} \dot{p}^\nu + \tau_0 \left[\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\delta} \frac{\mathrm{d}\delta}{\mathrm{d}\tau} (P f^2)_{\mu\nu} p^\nu + \mathcal{B}((P f^2)_{\mu\nu} \dot{p}^\nu - \dot{p}_\mu (p^+ \mathcal{E})^2 - p_\mu (\dot{p} f^2 p)) \right]. \tag{9}$$

Now by dotting n^μ into the LAD equation, it reads

$$n \cdot \dot{p} = \tau_0 (n \cdot \ddot{p} - p^+ p \cdot \ddot{p}) = -\tau_0^2 \left[\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\delta} \frac{\mathrm{d}\delta}{\mathrm{d}\tau} (p^+)^3 \mathcal{E}^2 + 2\mathcal{B} p^+ (\dot{p} f^2 p) + \mathcal{B} (n \cdot \dot{p}) (p^+ \mathcal{E})^2 \right] - \tau_0 \mathcal{A} p^+ (p f \dot{p}) - \tau_0^2 p^+ \mathcal{B} (\dot{p} f^2 p). \tag{10}$$

It follows from Eq. (6) that $p f \dot{p} = \mathcal{A} (p^+ \mathcal{E})^2$ and $\dot{p} f^2 p = -\tau_0 \mathcal{B} (p^+ \mathcal{E})^4$; we also have $\frac{\mathrm{d}\delta}{\mathrm{d}\tau} = \tau_0 \dot{p}^+ \mathcal{E} = -\tau_0^2 \mathcal{B} (p^+ \mathcal{E})^3$. Substituting these into the rhs of Eq. (10), writing out the lhs according to Eq. (6), and dividing by $\tau_0 (p^+)^3 \mathcal{E}^2$ it becomes

$$-\mathcal{B} = (\tau_0 p^+ \mathcal{E})^3 \mathcal{B} \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\delta} + 2 (\tau_0 p^+ \mathcal{E})^2 \mathcal{B}^2 - \mathcal{A}^2, \tag{11}$$

which is one of the ODEs (7). Hence the $^+$ component of the LAD equation will be satisfied if Eq. (6) holds, where $\mathcal{B}$ is a solution to Eq. (7). A similar calculation shows that the transverse components of the LAD equation will be satisfied if Eq. (6) holds and $\mathcal{A}$ is a solution to Eq. (7). The remaining component is fixed by the mass-shell condition.

As LL_∞ is obtained from reduction of order in a small parameter, it is essentially a resummed perturbative expansion. It is therefore entirely possible that the procedure could miss nonperturbatively small terms in the expansion parameter. We will here investigate the possible presence of such terms.

That is, Eqs. (7)(8) can be solved as perturbative series $\mathcal{A} \sim 1\text{–}2\delta^2 + \ldots, \mathcal{B} \sim 1 - 6\delta^2 + \cdots$ Although divergent, these series can be resummed with the Borel-Padé (see, e.g., Chapter 8 of Ref. [25]) or "educated match" [41] methods. As pointed out in Ref. [20], LL_∞ remains causal and free of runaways after such a resummation of perturbative terms. This is in contrast to the nonrelativistic case studied in Ref. [42], where these nonperturbative effects appear precisely after performing a Borel resummation. The question is thus raised whether nonperturbative effects appear from solutions of a more general transseries form

$$\begin{Bmatrix} \mathcal{A} \\ \mathcal{B} \end{Bmatrix} \sim \sum_{k,\ell \geq 0} \begin{Bmatrix} A_{k,\ell} \\ B_{k,\ell} \end{Bmatrix} \delta^{2k} e^{-\ell\kappa/\delta^\lambda}, \tag{12}$$

(for some κ, λ to be determined) which are not found by perturbative expansion or numerics. We use $\sim$ rather than an equality here and treat, for now, the expansion (12) formally: the space of such transseries is closed under algebraic operations and differentiation.

To determine the parameters κ, λ we linearize around $(\mathcal{A}, \mathcal{B}) = (1, 1)$ and $\delta = 0$; the general solution of the linearization is

$$\mathcal{A} = 1 - 2\delta^2 + c_1 \frac{1}{\delta^2} e^{1/2\delta^2} + \mathcal{O}(\delta^3), \tag{13a}$$

$$\mathcal{B} = 1 - 6\delta^2 + c_1 \frac{1}{\delta^4} e^{1/2\delta^2} + c_2 \frac{1}{\delta^2} e^{1/2\delta^2} + \mathcal{O}(\delta^3) \tag{13b}$$

for arbitrary constants c_1, c_2. We see that there are indeed nonperturbative terms depending exponentially on $1/\delta^2$, but these are *large* for real δ. The only solution that is finite as $\delta \searrow 0$ has $c_1 = c_2 = 0$, and hence lacks a nonperturbative part (its perturbative expansion is, we stress, divergent and must be resummed, though). We return to this at the end of this section.

We can strengthen our argument through the interpretation of $\kappa = -1/2$ as the location of the convergence-limiting singularity in the complex Borel plane. Borel singularities, and the overall transseries structure, are intimately related to the large-order growth of the perturbative coefficients [43]. In our case this can be determined to be, to leading order,

$$A_k, \qquad B_k \sim (-2)^k k! \tag{14}$$

by computing many coefficients using the recursion relations in Ref. [20]. We compute a normalized Borel transform

$$\mathrm{Borel}[\mathcal{A}](t) = \sum_k \frac{A_k}{2^k k!} t^k. \tag{15}$$

The transform cancels the factorial growth of the A_n, producing a series with finite radius of convergence, which can be analytically continued. With this normalization we expect the leading singularity to appear at $t = -1$.

The convergence-limiting singularity of the analytical continuation can now be probed using Padé approximants. The Padé method can struggle to identify multiple branch cuts, as it must accumulate poles along a cut to approximate it. This difficulty can be circumvented with a conformal map [30,31,44,45], making it also possible to identify singularities beyond the leading [24,43] and increase the accuracy of resummations [15,46,47]. Even without conformal mapping, though, there is a clear accumulation of Borel-Padé poles along the ray $t \leq -1$, seen in Fig. 1.

A fairly large number of terms are needed to see the structure in 1. The reason for this is that while

$$\frac{B_k}{kB_{k-1}} \overset{k\to\infty}{\to} \frac{1}{\kappa} = -2, \tag{16}$$

there are slowly decaying subleading corrections. Even after applying high-order Richardson extrapolation (see Chapter 8.1 of Ref. [25]), the slow convergence persists. Experimentally, this is because the subleading large-order behavior of the coefficients is *logarithmic*

$$\frac{B_k}{kB_{k-1}} \approx -2\left[1 + \frac{\Lambda}{k}(\log k)^2 + \mathcal{O}((\log k)^2/k^2)\right], \tag{17}$$

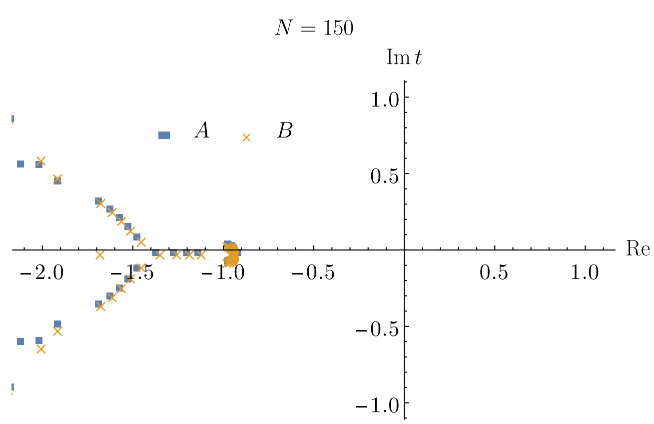

FIG. 1. Borel-Padé poles accumulating along the negative real axis, indicating the presence of a branch cut.

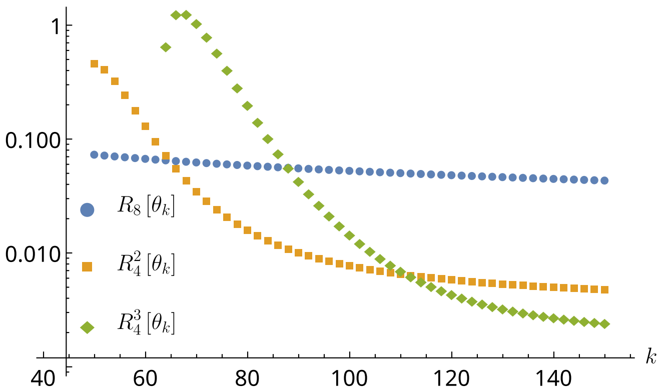

FIG. 2. Slow convergence of $\theta_k = -\frac{B_k}{kB_{k-1}} - 2$ as $k \to \infty$, even applying order-eight Richardson extrapolation (R_8), due to subleading logarithmic corrections. The modified extrapolations $R_K^{(2,3)}$ are accurate up to order $(\log k)^{(1,2)}/n^{-K}$.

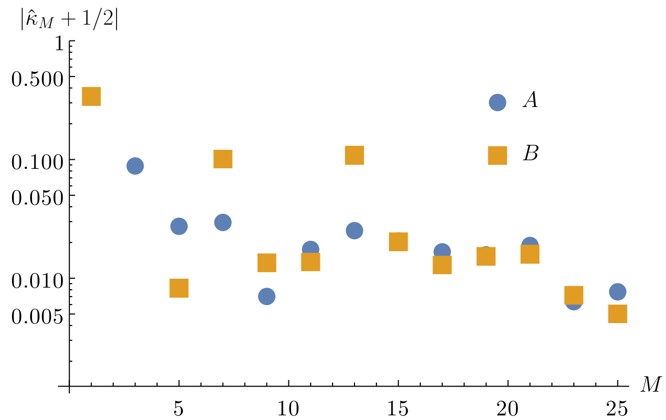

FIG. 4. Estimates of the branch point using a hypergeometric approximant based on perturbative coefficients up to order $2M + 1$.

and so not eliminated by standard Richardson extrapolation. Modifying Richardson extrapolation to account for logarithmic corrections (see Ref. [43] and Appendix B), the convergence is improved significantly, as shown in Fig. 2.

Instead of a Padé approximant, we can use a hypergeometric approximant in the Borel plane [48]. With perturbative data up to order $N = 2M + 1$ a hypergeometric ${}_{M+1}F_M(\cdots,\cdots;t/\hat{\kappa}_M)$ can be fitted; it has a built-in branch cut at $\hat{\kappa}_M$. Figure 3 shows an example ${}_2F_1$ approximant for Borel$[B](t)$, and Fig. 4 shows how $\hat{\kappa}_M$ converges to $\kappa = -\frac{1}{2}$.

We now return to the question of the sign of κ. While having exponentially *large* terms seems to be against the spirit of perturbation theory, in a purely formal treatment there is no "wrong sign" for κ, which may even be complex. An instructive example (discussed in detail in Sec. 2 of Ref. [26]) is the Airy functions, which have expansions

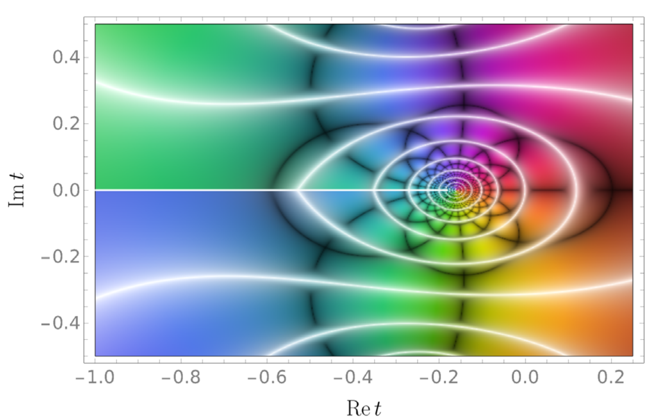

FIG. 3. Hypergeometric ${}_2F_1$ approximant to Borel$[B](t)$. The built-in branch cut along the negative real axis is evident as a discontinuity in the coloring. At this low order the estimate for the branch point is not very accurate, but this improves at higher order, cf. Fig. 4.

$$2\ \mathrm{Ai}(z), \qquad \mathrm{Bi}(z) \sim \frac{z^{-1/4}}{\sqrt{\pi}} e^{\mp\frac{2}{3}z^{3/2}}(1 + \mathcal{O}(z^{-3/2})) \tag{18}$$

as $z \to +\infty$ along the real axis. The exponentially large Bi is a valid solution to the Airy equation; it just does not match the boundary condition $f(+\infty) = 0$. As $z \to -\infty$ both Ai and Bi become oscillatory, corresponding to an imaginary κ. This is the eponymous phenomenon first studied by Stokes [49,50] in precisely the context of the Airy functions. (For a physical example with imaginary κ, see Ref. [24].)

For the LAD equation the initial acceleration is to be specified, while for LL_∞ it is determined by the initial momentum and $\mathcal{A}$, $\mathcal{B}$ at $\delta_0 = \tau_0 \mathcal{E} p_0^+$. Only the ODEs (7) need to hold for a solution of LL_∞ to be a solution to the LAD equation; hence the choice of initial condition for the ODEs (7) determines which, among all solutions of the LAD equation with a given initial momentum, is picked out by LL_∞.

By dotting n^μ into and squaring Eq. (6), respectively, we find that LL_∞ implies

$$\dot{p}^+ = -\tau_0 p^+ \mathcal{B}(\delta)\delta^2 \tag{19}$$

and

$$\tau_0^2 \dot{p}^2 = -\mathcal{A}(\delta)^2\delta^2 - \mathcal{B}(\delta)^2\delta^4. \tag{20}$$

With the Lorentz initial condition (8) the resummed perturbative $\mathcal{A}_{\mathrm{pert}}, \mathcal{B}_{\mathrm{pert}}$ are positive and approach 1 smoothly as $\delta \to 0$. This means that $\delta \to 0$ and thence $\dot{p}^2 \to 0$ as $\tau \to \infty$. The solution of LL_∞ is therefore the *physical, nonrunaway, solution* of the LAD equation, shown in Fig. 5. In other words, LL_∞ with the Lorentz initial condition (8) determines the *critical acceleration* (a concept first introduced in Ref. [51])

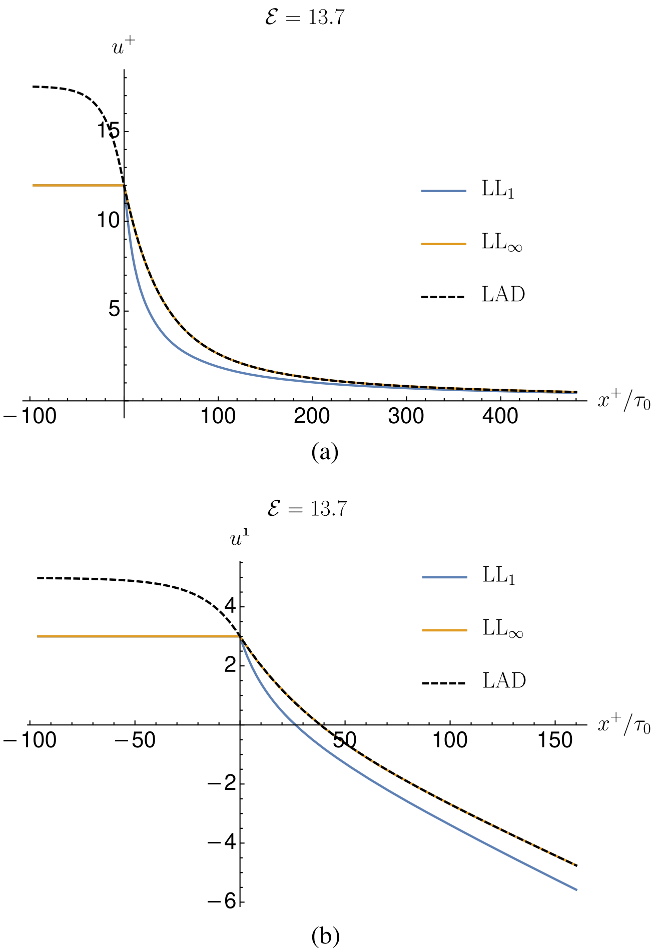

FIG. 5. Longitudinal (a) and transverse (b) momentum components across a step. LL$_\infty$ is seen to agree with the physical solution of the LAD equation after the step, while the latter exhibits preacceleration. The preacceleration occurs over a few τ_0 worth of proper time $\sim x^+/p^+$.

$$\dot{p}^\mu_{\text{crit}} \coloneqq \mathcal{A}_{\text{pert}}(\delta_0) f^\mu{}_\nu p_0^\nu + \mathcal{B}_{\text{pert}}(\delta_0)(Pf^2)^\mu{}_\nu p_0^\nu \tag{21}$$

that, for a given field strength and initial momentum, leads to the physical solution of the LAD equation.

As the purely perturbative solution of Eq. (7) leads to the physical solution to the LAD equation, the remaining, nonperturbative, solutions must lead to the runaways. We cannot find nonperturbative solutions with an initial condition at $\delta = 0$, but we can equally well set the initial condition at $\delta_0 = \tau_0 \mathcal{E} p_0^+$. Again using the Airy functions to illustrate, with the boundary condition $f(+\infty) = 0$ we discard Bi, but setting a condition at finite argument retains it. The solution of Eq. (7) satisfying

$$\mathcal{A}(\tau_0 \mathcal{E} p_0^+) = \mathcal{A}_{\text{pert}}(\tau_0 \mathcal{E} p_0^+) \tag{22a}$$

$$\mathcal{B}(\tau_0 \mathcal{E} p_0^+) = \mathcal{B}_{\text{pert}}(\tau_0 \mathcal{E} p_0^+) + \frac{\tau_0}{\delta_0^2}\varepsilon \tag{22b}$$

will give us a solution to the LAD equation with an initial longitudinal acceleration differing from the critical by ε; we expect this solution to be a runaway.

The general solution (13) with $c_1 = 0, c_2 = -\tau_0 \varepsilon e^{-1/2\delta_0^2}$ verifies the initial condition (22). This is only the leading term at first nonperturbative order, but it will be sufficient. This gives us for the longitudinal acceleration

$$\begin{aligned} \frac{\mathrm{d}p^+}{\mathrm{d}x^+} &= -\frac{\delta^2}{\tau_0}\mathcal{B}(\delta) \\ &= -\frac{\delta^2}{\tau_0}\left[1 - 6\delta^2 + \ldots + \frac{\varepsilon}{\delta^2}\exp\left(\frac{1}{2\delta^2} - \frac{1}{2\delta_0^2}\right)\right]. \end{aligned} \tag{23}$$

After a short time $\delta(x^+) \approx \delta_0 + x^+ \mathcal{E}\tau_0 \frac{\mathrm{d}p^+}{\mathrm{d}x^+}$ and using this to expand we have

$$\frac{\mathrm{d}p^+}{\mathrm{d}x^+} = -\frac{\delta_0^2}{\tau_0}\left(\mathcal{B}_{\text{pert}}(\delta_0) + \frac{\tau_0 \varepsilon}{\delta_0^2} e^{\frac{x^+}{\tau_0 p_0^+}} + \ldots\right) \tag{24}$$

omitting some inessential terms. Clearly the second term inside the brackets is a runaway over a proper time τ_0, and it is only seen because we included nonperturbative terms in $\mathcal{B}$.

It is clear from Eq. (20) that if $\mathcal{B}(\delta_0) > 0$ and $\mathcal{B}$ remains positive as δ decreases we have a runaway in the $-\hat{z}$ direction. Likewise if $\mathcal{B}(\delta_0) < 0$ and keeps its sign as δ increases we have a runaway in the $+\hat{z}$ direction. It is less obvious what happens if $\mathcal{B}$ should cross 0. This is relevant because we can imagine perturbing $\dot{p}^\mu_{\text{crit}}$ in the $+\hat{z}$ direction, seeding a runaway instability. Initially, then, p^+ is decreasing, but after some time the instability begins to dominate and p^+ increases. If the dynamics are indeed described by LL$_\infty$, $\mathcal{B}$ must change sign when this happens.

Figure 6(a) shows the perturbative solution, as well as two solutions with initial conditions

$$\mathcal{A}_\pm(1) = \mathcal{A}_{\text{pert}}(1), \qquad \mathcal{B}_\pm(1) = (1 \pm \varepsilon)\mathcal{B}_{\text{pert}}(1). \tag{25}$$

Of these, $\mathcal{B}_-$ runs away to $+\infty$ as δ decreases, and so gives the $-\hat{z}$ runaway, while instead $\mathcal{B}_+$ approaches 0 at a finite argument $\tilde{\delta}$. Now, $\mathcal{B} = 0$ is a singular point of Eq. (7), but to leading order around it the system reads

$$\begin{cases} 0 = 1 - \mathcal{A}, \\ \delta^3 \mathcal{B}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\delta} = \mathcal{A}^2, \end{cases} \tag{26}$$

with the *two* solutions

$$\mathcal{B} \approx \pm |A(\tilde{\delta})| \sqrt{1/\tilde{\delta}^2 - 1/\delta^2}. \tag{27}$$

These "branches" are both shown in Fig. 6(b). What happens, then, for the $+\hat{z}$ runaway is that we start out on the upper branch, but as $\dot{p}^+$ becomes 0 and the runaway

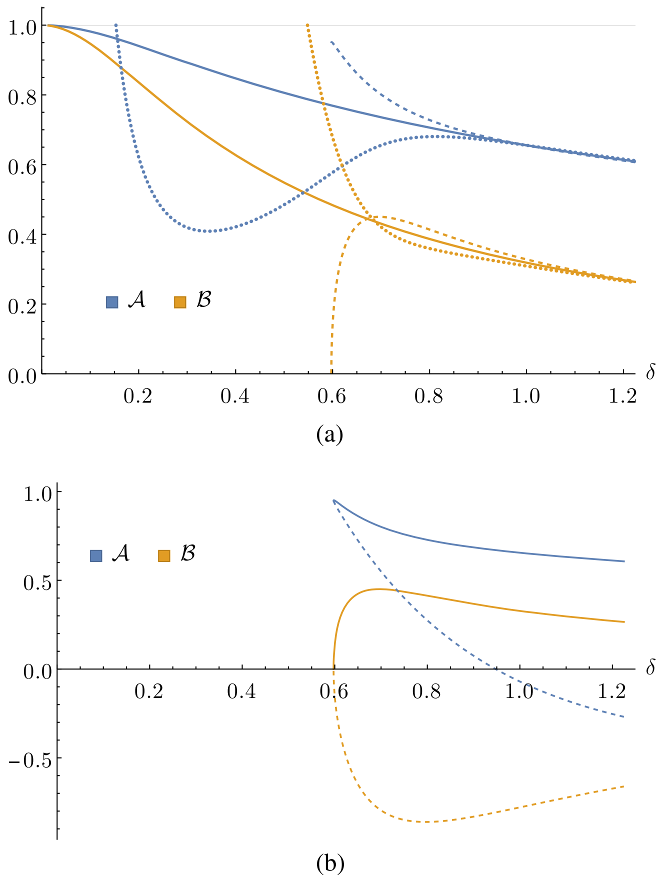

FIG. 6. (a) Three solutions to the ODEs (7). The solid curves represent the resummed perturbative solution smoothly approaching 1 as $\delta \to 0$; the dotted and dashed curves represent nonperturbative solutions, with the former blowing up exponentially at small δ, and the latter approaching the singular point $\mathcal{B} = 0$. (b) At the singular point $\mathcal{B} = 0$ the solution is nonunique. There is an "upper" (solid) and a "lower" branch (dashed), characterized by $\mathcal{B} \approx \pm|\mathcal{A}(\tilde{\delta})|\sqrt{1/\tilde{\delta}^2 - 1/\delta^2}$, respectively, near the singularity.

instability begins to dominate, the dynamics continue to be described by LL_∞, but now on the lower branch.

To verify this we solve the LAD equation numerically with the initial accelerations as implied by LL_∞, using $\mathcal{A}_\pm, \mathcal{B}_\pm$. Plotting p^z in Fig. 7 we find that the respective initial conditions indeed lead to the runaways we expect from Fig. 6(a). We also observe the switching between branches by plotting p^+ and its derivative for the $+\hat{z}$ runaway and comparing with LL_∞; see Fig. 8. We stress that in Figs. 7 and 8 we have solved the LAD equation *forward* in time: because the initial acceleration is either the critical or very close to it, the instability remains suppressed for several τ_0 worth of proper time.

We end this section by noting that the form of Eq. (6) is fully determined by there being only two possible tensor structures and one scalar invariant (δ) in the CCF geometry. Another highly restricted geometry is a circularly polarized monochromatic plane wave, and it is possible to derive equations similar to Eq. (7), and hence LL_∞ also in that

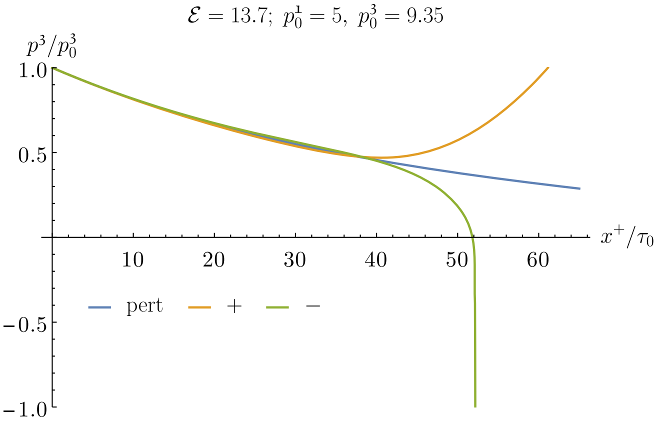

FIG. 7. z component of momentum for three solutions of the LAD equation with *initial* accelerations given by LL_∞, using either $\mathcal{A}_{\mathrm{pert}}, \mathcal{B}_{\mathrm{pert}}$ (blue) or $\mathcal{A}_\pm, \mathcal{B}_\pm$ (gold, green). The latter two are runaway solutions in the $\mp\hat{z}$ direction, respectively, while the first is the physical solution.

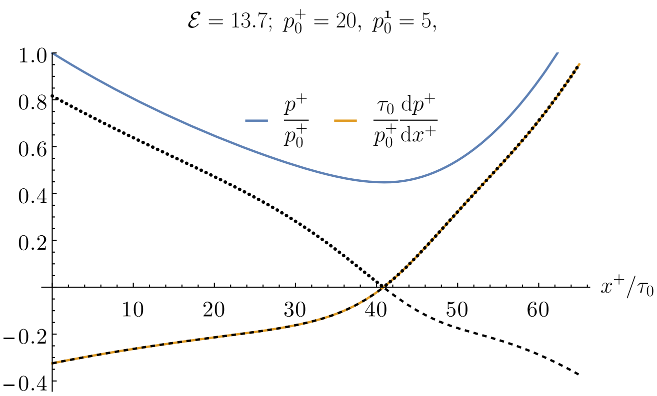

FIG. 8. Longitudinal momentum p^+ (blue) and acceleration (gold) for a runaway solution of the LAD equation in the $+\hat{z}$ direction. The dashed and dotted curves indicate the LL_∞ acceleration $\propto \mathcal{B}(\tau_0 \mathcal{E} p^+)$, using the upper and lower branches for $\mathcal{B}$, respectively; cf. Fig. 6(b). It is seen that as the longitudinal acceleration becomes zero it switches between the two branches.

case. It can be studied with the Borel plane methods we have applied to the CCF in this section; as the details are very similar, we defer them to Appendix A.

In either case, we obtain that reduction of order eliminates nonperturbative terms on the level of the *equation of motion* when a physical boundary condition—matching to the Lorentz force at vanishing field intensity—is imposed. We therefore now turn to how nonperturbative preaccelerating and runaway solutions arise on the level of *solutions* to the LAD equation.

III. THE LAD EQUATION AND LL_∞ IN A CROSSED STEP FIELD

We will now consider the LAD equation and LL_∞ in a field with a step profile, i.e.,

$$a'_\mu = \mathcal{E}\theta(x^+)\epsilon_\mu. \tag{28}$$

That the field is off for an interval of time will allow an unambiguous identification of preacceleration.

For the LAD equation we are faced with the problem of matching solutions before and after the step. The integro-differential form of the LAD equation [52–54], however, shows that the acceleration is continuous across a step, and so we should use the critical acceleration (21).

A. Exact solution to the free LAD equation

Before the step, with the field turned off, all the equations of motion can be solved exactly. For LL_1 and LL_∞ the solution is just uniform motion, while for the LAD equation we make an ansatz in terms of proper time τ and the rapidity ζ,

$$p^\mu(\tau) = \cosh(\zeta(\tau))p_0^\mu + \sinh(\zeta(\tau))\frac{\dot{p}_0^\mu}{\sqrt{-\dot{p}_0^2}}, \tag{29}$$

where the subscript 0 indicates values at $\tau = 0$. The LAD equation then implies an initial-value problem for ζ,

$$\tau_0\ddot{\zeta} = \dot{\zeta} \qquad \zeta(0) = 0 \qquad \dot{\zeta}(0) = \sqrt{-\dot{p}_0^2}, \tag{30}$$

with the solution

$$\zeta = \tau_0\sqrt{-\dot{p}_0^2}(e^{\tau/\tau_0} - 1). \tag{31}$$

We see that the pre-step solution is preaccelerating unless $\dot{p}_0^\mu = 0$. Viewed forwards in time this solution generalizes the well-known nonrelativistic runaway in that the exponential runaway is in the rapidity, rather than in the velocity. To the best of our knowledge, the covariant solution matched to the initial conditions, Eqs. (29)(31), has not previously appeared in the literature [55].

The solution has the form of a transseries in τ_0 with, expanding the hyperbolic functions, nonperturbative instanton terms of all orders. For $\tau > 0$ these become *large* as $\tau_0 \to 0$, corresponding to faster runaways; for $\tau < 0$ they become *small* in this limit, corresponding to the preacceleration occurring in a "boundary layer" of width $\approx 1/\tau_0$.

Note, though, that the solution is analytic in the proper time τ: the prefactor of each $e^{\ell\tau/\tau_0}$ term is some power series in τ_0, cf. Eq. (21). In fact τ only appears as τ/τ_0 and after a change of variables $(\tau, \tau_0) \to (\tilde{\tau}, \tau_0) = (\tau/\tau_0, \tau_0)$ the solution is analytic in both variables. This can be traced to that in the free LAD equation, or equivalently Eq. (30) the only scale is τ_0, which can be eliminated by rescaling. There is then no coupling in which to do perturbation theory, but the equation can be solved as a power series in rescaled time; τ_0 reenters when substituting for the initial condition (21).

This is a simple demonstration that the character of a transseries in *two* variables can change dramatically with a nonlinear change of variables. Such nonlinear transformations can in effect perform partial resummations in one of the variables, a point previously discussed in the contexts of a unitary matrix model [56] and radiation reaction [21]. As a consequence there can be subtleties in how (e.g., in which order) limits are taken; we will return to this shortly. (See also Refs. [57,58] for another example in strong-field physics where the manner of taking a limit matters.)

The above discussion has been in terms of proper time only while the rest of this paper uses light-front time. We therefore conclude this subsection with a short discussion of the solution of the free LAD equation in the light-front parametrization. In light-front time the equation for the rapidity retains factors of $\cosh\zeta, \sinh\zeta$ and cannot be solved analytically. Alternatively we can obtain the light-front time by quadrature,

$$x^+(\tau) = \int_0^\tau \mathrm{d}\sigma p^+(\sigma). \tag{32}$$

While this integral does have an analytic expression in terms of $Ei(\cdot)$, it gives only an implicit relation for $\tau(x^+)$. It can, though, be expanded to next-to-leading order (NLO) τ/τ_0 to find

$$\tau/\tau_0 = p_0^+x^+/\tau_0 - \frac{\tau_0}{2}\dot{p}_0^+(x^+/\tau_0)^2 + \mathcal{O}((x^+/\tau_0)^3). \tag{33}$$

Inserting this back into Eqs. (29)(31) yields another example of a nonlinear transformation strongly modifying the two-variable transseries structure.

B. Transseries solution of the LAD equation

We now come to the transseries structure of solutions to the LAD equation in a constant crossed field. This was briefly studied in Ref. [21] and we are mainly concerned with working out some implications of the results therein. Using notation slightly different from Ref. [21], we can formulate the LAD equation in a CCF as

$$g' = \delta[\partial_u(gg') + g^2P], \tag{34a}$$

$$h' = 1 + \delta[\partial_u(gh') + ghP],$$
$$P = (g')^2 - (h')^2 + 2g'\partial_u\frac{1 + h^2 - g^2}{2g}. \tag{34b}$$

Here g and h are normalized longitudinal and transverse components respectively [59],

$$g \coloneqq p^+/p_0^+, \tag{35a}$$

$$h \coloneqq gp_0^1 - p^1, \tag{35b}$$

the prime is a derivative with respect to a normalized light-front time $u \coloneqq \mathcal{E}x^+$, and $\delta^2 = \tau_0^2 p_0 f^2 p_0$, i.e., we drop the subscript on δ_0 from the previous section.

Reference [21] solved these equations iteratively by noting that if g, h have series expansions in δ, with the coefficients being functions of time,

$$\begin{Bmatrix} g \\ h \end{Bmatrix} \sim \sum_n \delta^n \begin{Bmatrix} g_n(u) \\ h_n(u) \end{Bmatrix} \tag{36}$$

the order n terms of the rhs are determined by terms of strictly lower order, so g_n, h_n can be found iteratively by simple integration. The zeroth order starting point is $g_0 = 1, h_0 = u$, corresponding to the Lorentz force. The coefficients are polynomials in u, with the first few being as follows:

$$g(u) = 1 - u\delta + u^2\delta^2 + (6u - u^3)\delta^3 + (-18u^2 + u^4)\delta^4 + \mathcal{O}(\delta^5), \tag{37a}$$

$$h(u) = u - \frac{1}{2}u^2\delta + \left(-2u + \frac{u^3}{2}\right)\delta^2 + \left(6u^2 - \frac{u^4}{2}\right)\delta^3 + \left(20u - \frac{41u^3}{3} + \frac{u^5}{2u}\right)\delta^4 + \mathcal{O}(\delta^5). \tag{37b}$$

Notably $g'(0), h'(0)$ have precisely the same perturbative expansion as one would find using LL_∞ for $\dot{p}_0^\mu$.

The solution (37) also illustrates the care needed in taking limits in a formal, divergent expansion. At each order in δ the leading behavior in u of g is $(-u\delta)^n$, the series has a finite radius of convergence, and can be resummed into $1/(1 + u\delta)$, which is the exact solution of LL_1 [39,60]. This has a single pole in the complex plane [61] and its Borel transform ($e^{-\delta t}$) is analytic everywhere. For any fixed u the linear term $\sim n!u$ will always win over u^n, though, meaning that the $u \to \infty$ limit must be taken *inside* the sum in Eq. (36).

If this iterative method is applied to the free LAD equation [which corresponds to striking the constant term on the rhs of Eq. (34b)], only the "trivial" solution of uniform motion is found. It is to be expected that solutions are lost as the method is only sensitive to initial conditions for the momentum, not the acceleration. In either case, the generated perturbative solution is the physical solution (but must be resummed), and we must introduce nonperturbative transseries terms to capture preacceleration and runaways.

To find all solutions, including preaccelerating and runaway solutions, instead of a simple series in δ, then, we should use a transseries ansatz [21],

$$g(u) \sim \sum_{n,\ell} \delta^n e^{\ell u/\delta} g_{n,\ell}(u). \tag{38}$$

We will refer to terms with $\ell \geq 1$ as *instanton* terms by analogy with quantum theory [64], even though their origin is different. Note again that the coefficients are functions of time; as in the previous subsection this is an expansion in two variables. The operator $\sim\delta \frac{\mathrm{d}^2}{\mathrm{d}u^2}$ on the rhs of Eq. (34) lowers by 1 the degree in δ of any term with $\ell \geq 1$. Thus we no longer have that $g'_{n,\ell}$ is determined by simply integrating lower-order coefficients, but rather by coupled first-order ODEs.

The expansion (38) is also lacking in that the initial conditions for the $g_{n,\ell}$ are grossly underdetermined, as we only have

$$g(0) = 1 \sim \sum_n \delta^n \sum_\ell g_{n,\ell}(0) \quad \text{and}$$
$$h(0) = 0 \sim \sum_n \delta^n \sum_\ell h_{n,\ell}(0) \tag{39}$$

and similar initial conditions for the acceleration,

$$g'(0) \sim \sum_n \delta^{n-1} \sum_\ell g'_{n-1,\ell}(0) + \ell g_{n,\ell}(0). \tag{40}$$

Since the zeroth and first derivatives of g, h at 0 determine all higher derivatives at 0 through the LAD equation (34), there is in principle an infinite hierarchy of constraints resolving the underdetermination. However, each rung of the ladder involves instanton terms of all orders, so we cannot proceed iteratively. (The system is not "triangular," so to speak.)

We are thus unable to iteratively determine fully self-consistently the precise transseries form of a specified runaway or preaccelerating solution. We can however truncate the system to one-instanton terms and assume that their initial amplitudes are $\mathcal{O}(\varepsilon)$, which will be accurate to $\mathcal{O}(\varepsilon^2 e^{2u/\delta})$. To make contact with the preceding section we will look for a solution such that

$$g'(0) = +\frac{\varepsilon}{\delta} - \delta\mathcal{B}_{\mathrm{pert}}(\delta), \tag{41a}$$

$$h'(0) = \mathcal{A}_{\mathrm{pert}}(\delta). \tag{41b}$$

This again corresponds to a runaway with initial acceleration ε different from the critical. (This is for concreteness only; the ODEs for the instanton coefficients are linear and other initial conditions pose no greater problems.)

At $n = 0, \ell = 0$, we have $g_{0,0} = 1 + \varepsilon, h_{0,0} = u$, keeping an integration constant that was implicitly dropped in the perturbative solution in order to account for instantonic contributions to the initial momentum. This implies order ε corrections to the following perturbative terms, beginning with $g_{1,0} = -2\varepsilon - u, h_{1,0} = +\varepsilon - \frac{u^2}{2}(\varepsilon + 1)$. The $n = 0, \ell = 1$ components [21] verify

$$\frac{\mathrm{d}}{\mathrm{d}u}\begin{pmatrix} g_{0,1} \\ h_{0,1} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{1,0}+u & -2 \\ 1+2u^2 & \tilde{g}_{1,0}-3u \end{pmatrix}\begin{pmatrix} g_{0,1} \\ h_{0,1} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} g_{0,1}(u) \\ h_{0,1}(u) \end{pmatrix} = -\varepsilon e^{u^2/2}\begin{pmatrix} \cos 2u \\ u\cos 2u - \sin 2u \end{pmatrix}. \tag{42}$$

For any initial acceleration other than the critical the instanton coefficients $g_{0,1}, h_{0,1}$ grow superexponentially, i.e., we have a runaway solution.

This procedure can in principle be iterated to any instanton order and any order in δ, although expanding the rhs of Eq. (34) becomes progressively costlier. At order $\delta^n e^{\ell u/\delta}$ the instanton coefficients take the form $\varepsilon^\ell \mathrm{Re}[P_{n,\ell}(u)e^{\ell(u^2/2-2iu)}]$ for some complex polynomial $P_{n,\ell}$ of degree n. We have calculated $P_{n,1}$ up to $n=16$, for which the constant terms and leading coefficients grow factorially and exponentially, respectively. Hence just like the perturbative series, the instanton series must also be resummed for small u, but are convergent when the limit $u\to\infty$ is taken inside the sum. We stress that this result, as well as Eqs. (37) and (42), agrees with Ref. [21].

The Gaussian form can be understood as the instanton coefficients reconstructing the nontrivial dependence $\tau(x^+)$. For the free solution,

$$\frac{\tau}{\tau_0} \approx \frac{x^+}{\tau_0 p_0^+} - \frac{(x^+)^2\dot{p}_0^+}{\tau_0(p_0^+)^3} = \frac{u}{\delta} + \frac{u^2}{2} + \mathcal{O}(u^3\delta) \tag{43}$$

when the initial acceleration is (close to) the critical. Because the quadratic term is independent of δ it appears separately at each order and the modification to the exponent can be read off directly. The next term in the exponent, going like $u^3\delta$, cannot be identified at a single order in δ, but would appear in an explicit resummation.

IV. CONCLUSIONS

We have used the Lorentz-Abraham-Dirac equation for radiation reaction as a "laboratory" setting in which to probe nonperturbative physics using transseries methods. Our choice of the LAD equation for this purpose is motivated both by a large current interest in radiation reaction [3–5,10–13,18–21], and the fact that the LAD equation features known, nonperturbative physics: preacceleration and runaway solutions. It is also a time-dependent problem, allowing us to study double expansions (in a time and a coupling), while most applications have looked at expansions in a coupling only [15,17,18,24,43,46] (but see Refs. [19,21,56]).

Extending our previous work on reduction of order and RR [20] we have shown that the nonperturbative runaway solutions are eliminated by reduction of order only when an essentially perturbative initial condition is applied. We illustrate this with the toy model (similar examples are found in several textbooks, e.g., Chapter 7 of Ref. [25])

$$z = 1 - \varepsilon z^2 \tag{44}$$

for a small parameter ε. The two solutions to this equation are

$$z_\pm = \frac{1}{2\varepsilon}\left(-1 \pm \sqrt{1+4\varepsilon}\right) = \begin{cases} 1-\varepsilon+2\varepsilon^2-5\varepsilon^3\cdots, \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+5\varepsilon^3\cdots. \end{cases} \tag{45}$$

If reduction of order is initiated with $z_0 = 1 + O(\varepsilon)$ only the purely perturbative solution z_+ is seen. If on the other hand an ansatz $z_0 = c_1/\varepsilon + c_2 + O(\varepsilon)$ including a possible nonperturbative term is made, one finds two branches $c_{i,+} = (0,1)$ and $c_{i,-} = (-1,-1)$. These generate z_+ and z_-, respectively. We see that it is not reduction of order itself that eliminates nonperturbative terms, but reduction of order combined with an initial condition on the purely perturbative branch. When nonperturbative terms are large, as is the case for the toy model (44) and the LAD equation, this is the only branch smoothly connected to vanishing expansion parameter. Thus we had to set an initial condition (22) at nonzero expansion parameter to keep nonperturbative runaway solutions with reduction of order.

We then considered the transseries structure of *solutions* to the LAD equation. We showed that to generate a solution of the LAD equation with a given initial (or final, for preaccelerating solutions) acceleration, instanton terms *of all orders* must, in general, be kept and their initial (final) coefficients must be chosen consistently with the LAD equation to the desired accuracy. The one exception to this is when the initial acceleration leads to the physical, nonrunaway solution; then all instanton terms vanish, and the solution is entirely perturbative.

As time-dependent quantities, solutions to the LAD equation exemplify that expansions in two variables can display strikingly different behavior in different regions of the variable plane and limits [56], and under nonlinear transformations. First, the solution to the free LAD equation contains nonperturbative terms of all instanton orders in one set of variables, but in another set these are transmuted into perturbative terms. Second, in a field, both the perturbative series and the instanton series are divergent and must be resummed at small times, but are convergent for large times.

The coupling parameter δ is smaller than the quantum nonlinearity parameter χ by a factor of α. Our results for $\delta \gtrsim 1$ should therefore be read as being about classical electrodynamics as a formal theory. It would however be interesting to consider, e.g., quantifying how much closer LL_∞, predicting less radiation reaction than LL_1, is to QED. Calculating to the necessary order in strong field QED remains extremely challenging, but recent progress on QED resummations and the Ritus-Narozhny conjecture [17–19,65] offers some encouragement.

Our results highlight that understanding the singularity structure of the Borel transform of a series is important for efficiently resumming it [30,47,48]. The series (37) is difficult to resum at large, finite, times because it "looks" convergent, with an analytic Borel transform, whereas the Padé approximant has poles. Reference [21] found that a nonlinear transformation effectively performed a partial resummation in one variable leading to an expansion divergent at all times, and therefore well-suited to Borel-Padé resummation. We take this and our results as a strong indication that a more thorough understanding of multi-variable divergent expansions, Borel transforms, and transseries would be highly useful to guide resummations in time-dependent problems and other expansions in multiple parameters.

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APPENDIX A: LL_∞ IN A MONOCHROMATIC PLANE WAVE

A circularly polarized monochromatic plane wave is characterized by the wave vector $k^\mu = \omega n^\mu$, and the invariants $\eta = k\cdot p/m^2$ and $\delta^2 = \tau_0^2 p f^2 p$. There are only three possible tensor structures that can enter into LL_∞, essentially $f, f^2, (p\cdot\partial)f$, Because of the circular polarization $\delta/\eta =: a_0$ is a constant and the form of LL_∞ must be

$$\dot{p}^\mu = \mathcal{A}_1(\delta) f_{\mu\nu} p^\nu + \tau_0 \mathcal{A}_2(\delta)(Pf^2)_{\mu\nu} p^\nu + \tau_0 \mathcal{A}_3(\delta) f_{\mu\nu,\rho} p^\nu p^\rho. \tag{A1}$$

Applying reduction of order leads to the fixed-point condition analogous to Eq. (7),

$$\begin{cases} -\delta^3 \mathcal{A}_2 \mathcal{A}_1' - 2\delta^2 \mathcal{A}_1 \mathcal{A}_2 - \frac{\delta^2}{a_0^2} \mathcal{A}_3 + 1 = \mathcal{A}_1, \\ -\delta^3 \mathcal{A}_2 \mathcal{A}_2' + \mathcal{A}_1^2 - 2\delta^2 \mathcal{A}_2^2 + \frac{\delta^2}{a_0^2} \mathcal{A}_3^2 = \mathcal{A}_2, \\ -\delta^3 \mathcal{A}_2 \mathcal{A}_3' - 2\delta^2 \mathcal{A}_2 \mathcal{A}_3 + \mathcal{A}_1 = \mathcal{A}_3. \end{cases} \tag{A2}$$

Note that a_0 enters as a parameter, but there are only derivatives with respect to δ, as a_0 is constant. If $a_0 \mapsto \infty, \mathcal{A}_3 \mapsto 0$ the first two equations form Eq. (7) after renaming. In this limit of a constant field, the third equation drops out: it is the coefficient in the equation of motion of the $f_{\mu\nu,\rho} p^\nu p^\rho$ term, which is then not present.

Starting with $\mathcal{A}_i = 1 + \mathcal{O}(\delta^2)$ it is straightforward to derive perturbative expansions in δ^2. The δ^{2n} coefficient is a degree n polynomial in $1/a_0^2$; for $\mathcal{A}_{1,2}$ the a_0^0 term recovers the factorially divergent perturbative expansions of $\mathcal{A}$, $\mathcal{B}$ from the main text. We therefore expect the same Borel singularity structure, and such is straightforwardly supported with the experimental, graphical methods in the main text.

To analytically determine the nonperturbative exponent we linearize the fixed-point equations (A2) around $\mathcal{A}_i = 1, 1/\delta^2 = z = \infty$, resulting in

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\tilde{\mathcal{A}}_1\\ \tilde{\mathcal{A}}_2\\ \tilde{\mathcal{A}}_3\end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}}_{\mathbb{M}}\begin{pmatrix}\tilde{\mathcal{A}}_1\\ \tilde{\mathcal{A}}_2\\ \tilde{\mathcal{A}}_3\end{pmatrix} + \frac{1}{2a_0^2 z}\underbrace{\begin{pmatrix}2a_0^2 & -1 & 1\\ 0 & 1+2a_0^2 & -2\\ 0 & 0 & 2a_0^2\end{pmatrix}}_{\mathbb{N}}\begin{pmatrix}\tilde{\mathcal{A}}_1\\ \tilde{\mathcal{A}}_2\\ \tilde{\mathcal{A}}_3\end{pmatrix} + \frac{1}{2a_0^2 z}\begin{pmatrix}1+2a_0^2\\ -1+2a_0^2\\ 2a_0^2\end{pmatrix}. \tag{A3}$$

The linearization is solved by writing $\mathcal{A}_i = \exp[\mathbb{M}z]_{ij}\mathcal{B}_j$ which leads to the equation for $\mathcal{B}_i$,

$$\frac{\mathrm{d}\mathcal{B}_i}{\mathrm{d}z} = \left(e^{-\mathbb{M}z}\frac{\mathbb{N}}{z}e^{\mathbb{M}z}\right)_{ij}\mathcal{B}_j + \frac{1}{2a_0^2}e^{-\mathbb{M}z}\begin{pmatrix}1+2a_0^2\\ -1+2a_0^2\\ 2a_0^2\end{pmatrix} = \frac{1}{z}\begin{pmatrix}\mathcal{B}_1\\ (1+\frac{1}{2a_0^2})\mathcal{B}_2\\ \mathcal{B}_3\end{pmatrix} + e^{-z/2}\begin{pmatrix}1 & 0 & 0\\ z/a_0^2 & 1 & 0\\ z/2a_0^2 & 0 & 1\end{pmatrix}\begin{pmatrix}1+2a_0^2\\ -1+2a_0^2\\ 2a_0^2\end{pmatrix}, \tag{A4}$$

and the general solution,

$$\begin{pmatrix}\tilde{\mathcal{A}}_1\\ \tilde{\mathcal{A}}_2\\ \tilde{\mathcal{A}}_3\end{pmatrix} = e^{\mathbb{M}z}\begin{pmatrix}c_1 z\\ c_2 z^{1+1/2a_0^2}\\ c_3 z\end{pmatrix} + \mathcal{A}_{i,\mathrm{p}} = e^{z/2}\begin{pmatrix}1 & 0 & 0\\ -z & 1 & 0\\ -z/2 & 0 & 1\end{pmatrix}\begin{pmatrix}c_1 z\\ c_2 z^{1+1/2a_0^2}\\ c_3 z\end{pmatrix} - \frac{1}{z}\begin{pmatrix}2+1/a_0^2\\ 6+1/a_0^2\\ 4+1/a_0^2\end{pmatrix}. \tag{A5}$$

We see that the same nonperturbatively *large* exponential $e^{1/2\delta^2}$ appears as for the CCF. One of the powers has an a_0 dependence not seen for the CCF; this corresponds to subleading, a_0-dependant large-order behavior of the perturbative coefficients.

APPENDIX B: MODIFIED RICHARDSON EXTRAPOLATION FOR LOGARITHMIC CORRECTIONS

Richardson extrapolation (see Chapter 8.1 of Ref. [25]) can be used to accelerate the convergence of a quantity

$$f_n \sim \sum_{k\geq 0} a_k n^{-k} \overset{n\to\infty}{\longrightarrow} a_0 \tag{B1}$$

from $\mathcal{O}(1/n)$ to $\mathcal{O}(n^{-K-1})$ for large n. Specifically letting $(\Delta_n f) \coloneqq f_{n+1} - f_n$ it holds that (in this Appendix all asymptotic statements are as $n \to \infty$)

$$R_K[f_n] = \frac{1}{K!}(\Delta_n^K n^K f_n) = a_0 + \mathcal{O}(n^{-K-1}). \tag{B2}$$

However as discussed in Ref. [43], if the quantity f_k has logarithmic corrections, viz.,

$$f_k \sim \sum_{k\geq 0} a_k n^{-k} + \log n \sum_{k\geq 1} b_k n^{-k} \tag{B3}$$

the acceleration is spoiled. This was solved in Ref. [43] by applying R_K twice such that

$$R_K[R_K[f_n]] = a_0 + \mathcal{O}(n^{-K-1}\log n). \tag{B4}$$

There is an intuitive explanation for this. The operator Δ_n acts like a derivative: it lowers the degree of polynomials by 1, annihilates constants, and satisfies a quasi-Leibniz rule. Furthermore $\Delta_n \log n = \log(1 + 1/n) = \mathcal{O}(1/n)$. Hence in the leading term with a logarithm in Eq. (B3), $n^{K-1}\log n$, the "derivative" Δ_n^K has to act on $\log n$ at least once to produce something nonzero. Acting another $K-1$ times produces something that goes like n^{-1}. This is why logarithmic corrections spoil accelerated convergence, but $R_K[f_k]$ is itself free of logarithms. Thus simply applying R_K again kills subleading terms to order n^{-K-1}, as desired.

Suppose now that there are subleading terms with powers of $\log n$, viz.,

$$f_k \sim \sum_{k\geq 0} a_k n^{-k} + \sum_{\ell=1}^{p} (\log n)^{\ell} \sum_{k\geq 1} b_{\ell,k} n^{-k}. \tag{B5}$$

Similarly to before, Δ_n^K has to act on at least one logarithmic factor in $n^{-K-1}(\log n)^p$ to contribute, and consequently $\Delta_n^K n^K f_k$ goes like $n^{-1}(\log n)^{p-1}$. Iterating we realize that $R_k^p[f_n]$ is free of logarithms, and

$$R_k^{p+1}[f_n] = a_0 + \mathcal{O}(n^{-K-1}(\log n)^p). \tag{B6}$$

[1] J. M. Cole *et al.*, Phys. Rev. X **8**, 011020 (2018).
[2] K. Poder *et al.*, Phys. Rev. X **8**, 031004 (2018).
[3] D. A. Burton and A. Noble, Contemp. Phys. **55**, 110 (2014).
[4] T. G. Blackburn, Rev. Mod. Plasma Phys. **4**, 5 (2020).
[5] A. Gonoskov, T. G. Blackburn, M. Marklund, and S. S. Bulanov, arXiv:2107.02161.
[6] C. N. Danson *et al.*, High Power Laser Sci. Eng. **7**, e54 (2019).
[7] H. Abramowicz *et al.*, Eur. Phys. J. Special Topics **230**, 2445 (2021).
[8] S. Meuren *et al.*, arXiv:2002.10051.
[9] J. H. Taylor, L. A. Fowler, and P. M. McCulloch, Nature (London) **277**, 437 (1979).
[10] T. Damour, Phys. Rev. D **102**, 124008 (2020).
[11] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, J. High Energy Phys. 07 (2021) 169.
[12] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, J. High Energy Phys. 10 (2021) 148.
[13] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, Phys. Rev. D **104**, 026009 (2021).
[14] G. Torgrimsson, Phys. Rev. D **102**, 116008 (2020).
[15] G. Torgrimsson, Phys. Rev. D **102**, 096008 (2020).
[16] F. Karbstein, Phys. Rev. Lett. **122**, 211602 (2019).
[17] A. A. Mironov, S. Meuren, and A. M. Fedotov, Phys. Rev. D **102**, 053005 (2020).
[18] T. Heinzl, A. Ilderton, and B. King, Phys. Rev. Lett. **127**, 061601 (2021).
[19] G. Torgrimsson, Phys. Rev. Lett. **127**, 111602 (2021).
[20] R. Ekman, T. Heinzl, and A. Ilderton, Phys. Rev. D **104**, 036002 (2021).
[21] G. Torgrimsson, Phys. Rev. D **104**, 056016 (2021).
[22] H. Taya, T. Fujimori, T. Misumi, M. Nitta, and N. Sakai, J. High Energy Phys. 03 (2021) 082.
[23] H. Taya, M. Hongo, and T. N. Ikeda, Phys. Rev. B **104**, L140305 (2021).
[24] G. V. Dunne and Z. Harris, Phys. Rev. D **103**, 065015 (2021).
[25] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, 1999).
[26] M. Mariño, Fortschr. Phys. **62**, 455 (2014).
[27] D. Dorigoni, Ann. Phys. (Amsterdam) **409**, 167914 (2019).
[28] G. V. Dunne and M. Ünsal, Proc. Sci., LATTICE2015 (**2016**) 010 [arXiv:1511.05977].

[29] I. Aniceto, G. Basar, and R. Schiappa, Phys. Rep. **809**, 1 (2019).
[30] O. Costin and G. V. Dunne, J. Phys. A **52**, 445205 (2019).
[31] O. Costin and G. V. Dunne, Phys. Lett. B **808**, 135627 (2020).
[32] N. B. Narozhny, Phys. Rev. D **21**, 1176 (1980).
[33] A. M. Fedotov, J. Phys. Conf. Ser. **826**, 012027 (2017).
[34] M. Abraham, *Theorie der Elektrizität* (Teubner, Leipzig, 1905).
[35] H. A. Lorentz, *The Theory of Electrons* (Teubner, Leipzig, 1909).
[36] P. A. M. Dirac, Proc. R. Soc. A **167**, 148 (1938).
[37] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, London, 1975).
[38] A. Di Piazza, C. Muller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. **84**, 1177 (2012).
[39] A. Di Piazza, Lett. Math. Phys. **83**, 305 (2008).
[40] R. Ekman, T. Heinzl, and A. Ilderton, New J. Phys. **23**, 055001 (2021).
[41] G. Álvarez and H. J. Silverstone, J. Phys. Commun. **1**, 025005 (2017).
[42] S. Zhang, Prog. Theor. Exp. Phys. **2013**, 123A01 (2013).
[43] M. Borinsky, G. V. Dunne, and M. Meynig, SIGMA **17**, 087 (2021).
[44] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4-Theories* (World Scientific, Singapore, 2001).
[45] *Large-Order Behaviour of Perturbation Theory*, edited by J.-C. Le Guillou and J. Zinn-Justin (Elsevier, New York, 2012).
[46] A. Florio, Phys. Rev. D **101**, 013007 (2020).
[47] O. Costin and G. V. Dunne, Eur. Phys. J. Special Topics **230**, 2679 (2021).
[48] H. Mera, T. G. Pedersen, and B. K. Nikolić, Phys. Rev. D **97**, 105027 (2018).
[49] G. G. Stokes, Trans. Cambridge Philos. Soc. **9**, 166 (1851).
[50] G. G. Stokes, Trans. Cambridge Philos. Soc. **10**, 105 (1864).
[51] H. Spohn, Europhys. Lett. **50**, 287 (2000).
[52] F. Rohrlich, Ann. Phys. (N.Y.) **13**, 93 (1961).
[53] G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).
[54] G. G. Alcaine and F. J. Llanes-Estrada, Phys. Rev. E **88**, 033203 (2013).
[55] An expression involving sinh appears in Ref. [53] but the approach there is not explicitly covariant and only seeks to prove that the nonrunaway solution is uniform motion.
[56] A. Ahmed and G. V. Dunne, J. High Energy Phys. 11 (2017) 054.
[57] T. Podszus and A. Di Piazza, Phys. Rev. D **99**, 076004 (2019).
[58] A. Ilderton, Phys. Rev. D **99**, 085002 (2019).
[59] It is not necessary to normalize to specifically p_0; any on-shell momentum will do. This choice is the most convenient for the present purpose, though.
[60] H. Heintzmann and M. Grewing, Z. Phys. **251**, 77 (1972).
[61] The pole indicates that there is a minimum light-front time in the past at which the particle was at the speed of light, cf. Refs. [40,62,63].
[62] T. N. Tomaras, N. C. Tsamis, and R. P. Woodard, Phys. Rev. D **62**, 125005 (2000).
[63] R. P. Woodard, Nucl. Phys. B, Proc. Suppl. **108**, 165 (2002).
[64] L. N. Lipatov, JETP Lett. **25**, 104 (1977).
[65] A. A. Mironov and A. M. Fedotov, arXiv:2109.00634.