

Top-down anatomy of flavor symmetry breakdownAlexander Baur^{1,2,*}, Hans Peter Nilles^{3,4,†}, Saúl Ramos-Sánchez^{2,‡},
Andreas Trautner^{5,§} and Patrick K. S. Vaudrevange^{1,||}¹*Physik Department, Technische Universität München,
James-Franck-Straße 1, 85748 Garching, Germany*²*Instituto de Física, Universidad Nacional Autónoma de México,
POB 20-364, Ciudad de México 01000, México*³*Bethe Center for Theoretical Physics and Physikalisches Institut der Universität Bonn,
Nussallee 12, 53115 Bonn, Germany*⁴*Arnold Sommerfeld Center for Theoretical Physics, Ludwigs-Maximilians-Universität München,
Theresienstraße 37, 80333 München, Germany*⁵*Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*

(Received 23 December 2021; accepted 10 February 2022; published 28 March 2022)

A top-down approach to the flavor puzzle leads to eclectic flavor groups which include modular and traditional flavor symmetries. Based on examples of semirealistic $\mathbb{T}^2/\mathbb{Z}_3$ orbifold compactifications of heterotic string theory, we discuss the breakdown patterns of the eclectic flavor group via the interplay of vacuum expectation values (vevs) of moduli and flavon fields. This leads to an attractive flavor scheme with various possibilities to obtain “flavor hierarchies” through the alignment of these vevs. Despite the fact that the top-down approach gives strong restrictions for bottom-up flavor model building, it seems to be well suited to provide a realistic flavor pattern for quarks and leptons.

DOI: [10.1103/PhysRevD.105.055018](https://doi.org/10.1103/PhysRevD.105.055018)**I. INTRODUCTION**

In his influential work [1], Feruglio considered finite modular groups for flavor model building. This proposal has triggered intense activity in the construction of bottom-up (BU) models of quark and lepton interactions based on various finite modular groups [2–7] (see e.g., the review [8] for a complete set of references and further details on the BU approach). One successful result of these constructions is that they can provide good fits to data by typically requiring a smaller number of free parameters than in models endowed with traditional flavor symmetries. However, the predictability of such BU constructions may be challenged through the appearance of uncontrollable terms in the Kähler potential [9]. In the BU approach, there are thus many working models with some degree of predictability, but we still lack a baseline theory coming from an underlying fundamental principle.

To address this problem, there has been extensive work devoted toward top-down (TD) derivations of modular flavor symmetries from ultraviolet complete models based on string theory [10,11]. Apart from heterotic orbifold compactifications [11,12], TD models include scenarios based on compactifications on toroidal orientifolds [13] and magnetized tori [14–17]. TD constructions typically give strong restrictions on the allowed symmetries and the particle spectrum and may also allow one to constrain (or even compute) the otherwise free terms of the Kähler potential [17–20]. Furthermore, TD models naturally include unification of flavor with CP -like transformations [11,12,21,22]. This unification becomes even more transparent in models where the modular group is extended to its metaplectic [17,23,24] or symplectic cover [25–29] as recently discussed both in the BU and TD approaches.

In the present paper, we concentrate on the TD approach. This approach to the flavor problem leads to a holistic view that necessarily encompasses all available kinds of discrete flavor symmetries. It thus has to include traditional flavor symmetries and R -symmetries as well as finite modular flavor symmetries and their associated CP transformations [11]. These symmetries reflect the symmetries of the underlying UV complete theory, which we consider here in the framework of heterotic strings with compactifications with elliptic fibrations. The discrete flavor symmetries can be derived in full generality as the outer automorphisms of the Narain space group [11,12]. This leads to the concept of

*alexander.baur@tum.de
†nilles@th.physik.uni-bonn.de
‡ramos@fisica.unam.mx
§trautner@mpi-hd.mpg.de
||patrick.vaudrevange@tum.de

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

the eclectic flavor group [18,20,30,31] as a multiplicative closure of all flavor symmetries. The eclectic flavor group is the maximal possible flavor group, but it is only partially linearly realized. The linearly realized flavor subgroup is nonuniversal in moduli space, a property that leads to the concept of “local flavor unification” [12]. This holistic picture teaches us some general lessons:

- (i) One cannot just consider a specific flavor group (e.g., modular flavor) without the others.
- (ii) There is always a traditional flavor group (universal in moduli space) that could give severe restrictions to the Kähler potential and superpotential of the theory.
- (iii) Apart from the finite discrete modular flavor group and its specific representations, one has to consider the modular weights of the matter fields as well, as they might lead to further R -symmetries that play the role of “shaping symmetries.”

Flavor symmetries have to be spontaneously broken, and thus the full eclectic picture requires several sources of breakdown. On the one hand, this leads to a serious complication of the picture. On the other hand, it is welcome once we want to obtain the hierarchical structure of masses and mixing angles of quarks *and* leptons. While the modular group can be broken via the moduli [with hierarchical patterns close to the fixed points of $SL(2, \mathbb{Z})$], we have to consider additional flavon fields to break the traditional flavor symmetries via nontrivial vacuum expectation values (vevs). These flavon vevs might then lead to a further breakdown of the discrete modular flavor symmetry as well. It is this subtle interplay of breakdown via moduli and flavons that is the main subject of the present paper. We shall see that there are various ways to obtain hierarchical patterns from the breakdown of the eclectic flavor group via moduli and flavons.

Flavor symmetries should, of course, also be discussed from a BU perspective. There, one has the free choice of the groups, the representations, and modular weights of matter fields to confront a model with existing data. Ideally, one might hope to find a specific model as a “best fit” to masses and mixing angles of quarks and leptons. This then might give useful hints toward a fundamental theory of flavor. Unfortunately, no one has yet been able to identify such a distinct model (or even a class of models). Good fits to the data can be achieved for various groups and representations. Still, even in the BU approach, we might try to find some theoretical guidelines. If we, for example, have a traditional flavor group G , we could design an eclectic scheme with a discrete modular flavor group that is included in the outer automorphisms of the group G [30]. This could be a first step to bridge the gap between the TD and BU approaches to flavor. Both approaches enjoy various desirable properties as e.g., the appearance of local flavor unification with enhanced symmetry at certain fixed points or regions in moduli space, eventually broken spontaneously via moduli vevs.

The TD approach is very restrictive and a challenge for realistic model building:

- (i) We first have to design models with the desired flavor groups explicitly in string compactifications.
- (ii) The explicit representations of the flavor group are then fixed and cannot be chosen by hand.
- (iii) Likewise, modular weights are fixed and determined as well.
- (iv) There are various restrictions from R -symmetries that appear in the six-dimensional compactification.

Given these restrictions, there remains still a wide gap between TD attempts and the models considered in the BU approach.

There have not been any attempts yet in TD model building, and in this paper, we want to make a first step in this direction. We are particularly interested in string models with elliptic fibrations, and these are classified according to two-dimensional $\mathbb{T}^2/\mathbb{Z}_k$ orbifold sectors with $k = 2, 3, 4, 6$. The traditional flavor symmetries of these scenarios were analyzed some time ago [32,33]. In those works, it is found that the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector (which in some sense also qualitatively covers the \mathbb{Z}_6 case as well) leads to the most promising class of string models that reproduce the matter spectrum of the minimal supersymmetric standard model (MSSM). These models are endowed with traditional flavor symmetry $\Delta(54)$ and twisted states that transform as (irreducible) triplet representations of this group. Therefore, we want to concentrate on this class of models with traditional flavor symmetry $\Delta(54)$ and modular flavor symmetry T' . Fortunately, there have been explicit semirealistic model constructions of heterotic string theory that exhibit elliptic fibrations of type \mathbb{Z}_3 [34]. A full classification of these models with the relevant field content is given in Table III of the present paper.

All of these models share the same eclectic flavor group but differ in the available representations of candidate flavon and matter fields. A first step toward phenomenological applications would then be the analysis of breakdown patterns of the eclectic flavor group, and this is the main goal of the present paper. We have to clarify which fields are needed for an efficient breakdown of the eclectic flavor group, which, in addition, allow a hierarchical pattern for a successful description of masses and mixing angles of quarks and leptons (originating partially through the proximity to local gauge group enhancements). Before proceeding to explicit model building, we would therefore like to know the qualitative breakdown pattern of $\Delta(54)$ and T' . In future work [35], we shall then use these results for the selection of suitable models (from the classes in Table III) for phenomenological applications.

The paper is structured as follows. In Sec. II, we describe a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector with traditional flavor group $\Delta(54)$ and eclectic flavor group¹ $\Omega(2) \cong [1944, 3448]$ [as the

¹We follow the notation of the SmallGroup library of GAP [36], where the first number in the square parentheses denotes the order of the group and the second number is the group identification. We also use the nomenclature conventions of Ref. [37].

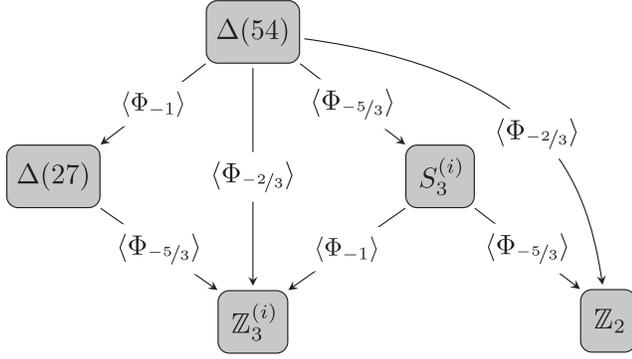


FIG. 1. Spontaneous breakdown patterns of the $\Delta(54)$ traditional flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector due to different flavon vevs. The index $i = 1, 2, 3, 4$ in $S_3^{(i)}$ and $\mathbb{Z}_3^{(i)}$ labels various (nonconjugate) subgroups built by different generators; see Table IV.

multiplicative closure of $\Delta(54)$, T' and \mathbb{Z}_9^R , derived as the two-dimensional elliptic fibration of a $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold. We identify the representations of the groups in the massless sector including the modular weights. At a generic point in moduli space, we have a flavor symmetry $\Delta(54) \cup \mathbb{Z}_9^R = \Delta'(54, 2, 1) \cong [162, 44]$. We discuss in detail the enhancements at the fixed points of the Kähler modulus $T = i, 1$, and $\omega := \exp(2\pi i/3)$ that lead to the groups $\Xi(2, 2) \cong [324, 111]$ (for $T = i$) or $[468, 125]$ (for $T = 1, \omega$). In Sec. III, we discuss the possible breakdown of $\Delta(54)$ via flavon vevs. Candidate flavons are identified via the inspection of the massless sector of the $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold under consideration. We show that triplets of $\Delta(54)$ are very efficient in the breakdown of the traditional flavor group. We extend this analysis to the breakdown of the eclectic flavor group in Sec. IV, with particular attention to the groups $[324, 111]$ and $[468, 125]$ at the fixed points $T = i$ and $T = 1, \omega$, respectively. The various breakdown patterns are summarized in the Figs. 1–3 with details given in Tables IV–VI. In Sec. V, we conclude by discussing the relevance of our analysis to flavor model building and give an outlook to future work.

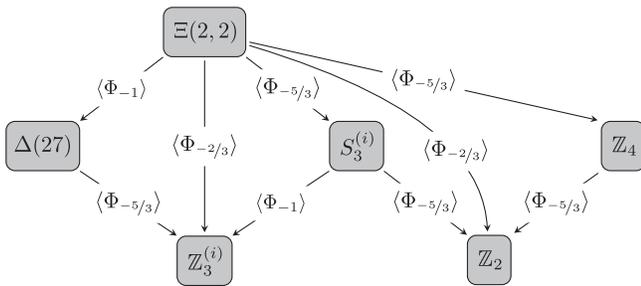


FIG. 2. Spontaneous breakdown patterns of the linearly realized unified flavor symmetry $\Xi(2, 2)$ at $(T) = i$ by $\mathbb{T}^2/\mathbb{Z}_3$ flavon vevs. The index $i = 1, 2$ in $S_3^{(i)}$ and $\mathbb{Z}_3^{(i)}$ labels various (nonconjugate) subgroups built by different generators; see Table V.

II. HOLISTIC PICTURE OF THE ECLECTIC FLAVOR SYMMETRY

Eclectic flavor groups [30] arise naturally in compactifications of string theory [16,18,20,31]. They consist of a nontrivial combination of the effective traditional flavor symmetries $G_{\text{traditional}}$ and finite modular symmetries G_{modular} under which string states are charged. Eclectic flavor groups can also include CP -like transformations and discrete R -symmetries. Let us here discuss the appearance of these symmetries in detail.

We focus on factorizable six-dimensional orbifolds, which contain three two-dimensional $\mathbb{T}^2/\mathbb{Z}_N$ orbifold sectors. Each of the \mathbb{T}^2 is endowed with two modular groups, $\text{SL}(2, \mathbb{Z})_U$ and $\text{SL}(2, \mathbb{Z})_T$, associated respectively with the complex structure modulus U and the (stringy) Kähler modulus T of the torus. For $N = 2$, the values of both T and U remain unrestricted. For $N > 2$, due to its geometric nature, U has to be fixed to a value that is compatible with the orbifold twist \mathbb{Z}_N . In these cases, $\text{SL}(2, \mathbb{Z})_U$ is broken down to a discrete remnant of the Lorentz group of discrete rotations in the compact dimensions, which is an Abelian group compatible with the orbifold. This subgroup appears as a discrete R -symmetry in the effective theory [31]. $\text{SL}(2, \mathbb{Z})_T$, however, remains a symmetry of the effective theory. This symmetry is nonlinearly realized, as can be seen from its action on the modulus T and matter fields Φ_n . An element $\gamma \in \text{SL}(2, \mathbb{Z})_T$ transforms these fields according to [38,39]

$$T \xrightarrow{\gamma} \frac{aT + b}{cT + d} \quad \text{and} \quad \Phi_n \xrightarrow{\gamma} (cT + d)^n \rho_s(\gamma) \Phi_n, \quad \text{with} \\ \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})_T. \quad (1)$$

Here, Φ_n denotes a multiplet of string matter states with identical quantum numbers, except for their location at different orbifold singularities. Furthermore, n denotes the modular weight of the matter multiplets Φ_n , $(cT + d)^n$ is known as automorphy factor, and $\rho_s(\gamma)$ is an s -dimensional matrix representation of γ in a (discrete) finite modular group G_{modular} that depends on the \mathbb{Z}_N twist of the orbifold sector.

On the other hand, the traditional flavor symmetries $G_{\text{traditional}}$ can be identified through the geometric features of toroidal orbifolds [32]. First, $G_{\text{traditional}}$ includes the permutations among the various equivalent orbifold singularities where matter states comprising the multiplets Φ_n are located. Since permutations are non-Abelian, so are these traditional flavor symmetries. Second, $G_{\text{traditional}}$ contains the discrete symmetries governing the admissible couplings among the states in Φ_n , which are known as string selection rules. The resulting traditional flavor group is obtained by multiplicative closure of these two types of symmetries. It is universal in moduli space, as all its elements only act on Φ_n .

In addition to traditional flavor and classical modular transformations, there are CP -like transformations. On the moduli T and U , these act with an element γ of

TABLE I. Eclectic flavor group $\Omega(2)$ for six-dimensional orbifolds that contain a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. In this case, $\text{SL}(2, \mathbb{Z})_U$ of the stabilized complex structure modulus $\langle U \rangle = \exp(2\pi i/3)$ is broken, resulting in a remnant \mathbb{Z}_9^R R -symmetry. Including \mathbb{Z}_9^R enhances the traditional flavor group $\Delta(54)$ to $\Delta'(54, 2, 1) \cong [162, 44]$, which together with T' finally leads to the eclectic group $\Omega(2) \cong [1944, 3448]$. Table from Ref. [31].

Nature of symmetry		Outer automorphism of Narain space group	Flavor groups				
Eclectic	Modular	Rotation $S \in \text{SL}(2, \mathbb{Z})_T$ Rotation $T \in \text{SL}(2, \mathbb{Z})_T$	\mathbb{Z}_4 \mathbb{Z}_3	T'			$\Omega(2)$
	Traditional flavor	Translation A Translation B	\mathbb{Z}_3 \mathbb{Z}_3	$\Delta(27)$	$\Delta(54)$	$\Delta'(54, 2, 1)$	
		Rotation $C = S^2 \in \text{SL}(2, \mathbb{Z})_T$	\mathbb{Z}_2^R				
		Rotation $R \in \text{SL}(2, \mathbb{Z})_U$	\mathbb{Z}_9^R				

challenge for flavor model building. One of the goals of this paper is precisely to address this question in an illustrative example of the eclectic picture based on string theory.

A. The Eclectic flavor symmetry of $\mathbb{T}^2/\mathbb{Z}_3$

We chose the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector as an example to illustrate the phenomenological potential of eclectic flavor symmetries. The outer automorphisms of the corresponding Narain space group yield the following symmetries [12,18,31]:

- (i) an $\text{SL}(2, \mathbb{Z})_T$ modular symmetry which acts as a $\Gamma'_3 \cong T'$ finite modular symmetry on matter fields and their couplings,
- (ii) a $\Delta(54)$ traditional flavor symmetry,
- (iii) a \mathbb{Z}_9^R discrete R -symmetry as a remnant of $\text{SL}(2, \mathbb{Z})_U$, and
- (iv) a \mathbb{Z}_2^{CP} CP -like transformation.

As explained in detail in Refs. [31,40] and summarized in Table I, the $\text{SL}(2, \mathbb{Z})_T$ modular generators S and T arise from rotational outer automorphisms. These generators can be represented by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (5)$$

and act on the modulus T and the matter fields Φ_n according to Eq. (1). Further, there is a reflectional outer automorphism which corresponds to a \mathbb{Z}_2^{CP} CP -like transformation. It can be chosen to be represented by

$$K_* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (6)$$

which acts on the modulus and matter fields as in Eq. (2).

The traditional flavor symmetry $\Delta(54)$ is generated by two translational outer automorphisms of the Narain space group A and B of order 3 together with the \mathbb{Z}_2 rotational outer automorphism $C := S^2$. The automatic identification of C with S^2 implies that $\Delta(54)$ and the modular symmetry have a nontrivial overlap and, furthermore, that $\Delta(54)$ is actually a non-Abelian R -symmetry.

Finally, in our example, the complex structure modulus is geometrically stabilized at $\langle U \rangle = \exp(2\pi i/3)$ in order for \mathbb{T}^2 to be compatible with the \mathbb{Z}_3 point group. If $\mathbb{T}^2/\mathbb{Z}_3$ is embedded in a six-dimensional orbifold, $\langle U \rangle$ breaks the original $\text{SL}(2, \mathbb{Z})_U$ of \mathbb{T}^2 to a discrete remnant generated by R that acts as a \mathbb{Z}_9^R symmetry on matter fields (normalizing their R -charges to be integers).

Since $\Delta(54)$ and \mathbb{Z}_9^R both act trivially on the modulus T , the traditional flavor symmetry is enhanced to $\Delta(54) \cup \mathbb{Z}_9^R \cong \Delta'(54, 2, 1) \cong [162, 44]$. Including the additional \mathbb{Z}_2^{CP} CP -like transformation (3), the full eclectic group according to Eq. (4) is a group of order 3888 given by

$$G_{\text{eclectic}} = \Omega(2) \rtimes \mathbb{Z}_2^{CP}, \quad \text{where } \Omega(2) \cong [1944, 3448]. \quad (7)$$

In semirealistic heterotic orbifold compactifications endowed with a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, the massless spectrum consists of (i) untwisted string states that transform as flavor singlet states Φ_0 or Φ_{-1} free to move in the bulk and (ii) twisted string states transforming as flavor triplets, attached to the three orbifold fixed points. The modular weights of twisted states depend on the twisted sector they belong to. As shown in Table II, in the θ sector, only $n \in \{-2/3, -5/3\}$ are possible, while only $n \in \{-1/3, 2/3\}$ appear in the θ^2 twisted sector. The transformations (1) of twisted triplets $\Phi_{-2/3}$ are governed by the three-dimensional matrix representations of the modular generators [12]

$$\rho(S) = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \quad \text{and} \quad \rho(T) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

which form the representation $s = 2' \oplus \mathbf{1}$ of the finite modular group $\Gamma'_3 \cong T' \cong [24, 3]$. Furthermore, the action of the $\Delta(54)$ generators on $\Phi_{-2/3}$ is given by

TABLE II. T' and $\Delta(54)$ irreducible representations and \mathbb{Z}_9^R R -charges of massless matter fields Φ_n with modular weights n in semirealistic heterotic orbifold compactifications with a $\mathbb{T}^2/\mathbb{Z}_3$ sector. T' , $\Delta(54)$, and \mathbb{Z}_9^R combine nontrivially to the eclectic flavor group $\Omega(2) \cong [1944, 3448]$, as described in Table I. Untwisted matter fields Φ_n (with integer modular weights n) form one-dimensional representations, while twisted matter fields Φ_n (with fractional modular weights n) build triplet representations. Table from Ref. [18].

		Eclectic flavor group $\Omega(2)$								\mathbb{Z}_9^R
Sector	Matter fields Φ_n	Modular T' subgroup				Traditional $\Delta(54)$ subgroup				R
		Irrep s	$\rho_s(S)$	$\rho_s(T)$	n	Irrep r	$\rho_r(A)$	$\rho_r(B)$	$\rho_r(C)$	
Bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	+1	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}' \oplus \mathbf{1}$	$\rho(S)$	$\rho(T)$	-2/3	$\mathbf{3}_2$	$\rho(A)$	$\rho(B)$	$+\rho(C)$	1
	$\Phi_{-5/3}$	$\mathbf{2}' \oplus \mathbf{1}$	$\rho(S)$	$\rho(T)$	-5/3	$\mathbf{3}_1$	$\rho(A)$	$\rho(B)$	$-\rho(C)$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}'' \oplus \mathbf{1}$	$(\rho(S))^*$	$(\rho(T))^*$	-1/3	$\bar{\mathbf{3}}_1$	$\rho(A)$	$(\rho(B))^*$	$-\rho(C)$	2
	$\Phi_{+2/3}$	$\mathbf{2}'' \oplus \mathbf{1}$	$(\rho(S))^*$	$(\rho(T))^*$	+2/3	$\bar{\mathbf{3}}_2$	$\rho(A)$	$(\rho(B))^*$	$+\rho(C)$	5
Superpotential	\mathcal{W}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

$$\rho(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and}$$

$$\rho(C) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \rho(S)^2, \quad (9)$$

and they generate the representation $r = \mathbf{3}_2$ of $\Delta(54)$. The representations associated with the other twisted triplets are expressed in terms of Eqs. (8) and (9), according to the prescription given in Table II. Finally, the integer \mathbb{Z}_9^R R -charges of matter fields can be uniquely determined by using their $SL(2, \mathbb{Z})_U$ properties and the fixed value $\langle U \rangle$; see Ref. [40], Sec. 4.2.

It is important to stress that the massless spectrum of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector of heterotic orbifold compactifications does not include all possible representations of $\Delta(54)$. In particular, note that there are no massless doublet representations. In fact, doublets arise as winding modes of strings around different singularities and, hence, correspond to massive states [42].

At different points $\langle T \rangle$ in moduli space, $SL(2, \mathbb{Z})_T$ and T' are broken down to the stabilizer subgroup, i.e., to the modular subgroup that leaves $\langle T \rangle$ invariant. Since the modulus is no longer transformed at $\langle T \rangle$, the surviving symmetry yields an enhancement of the traditional flavor symmetry $\Delta(54)$. This enhancement, associated with the specific location in moduli space, has been called *local flavor unification* [12]. Next, we study the most relevant scenarios of local flavor unification in the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector.

B. Flavor enhancement at $\langle T \rangle = i$

First, we discuss the point $\langle T \rangle = i$ in moduli space (see Sec. 6.2 in Ref. [40] for details of the derivation). In this case, $SL(2, \mathbb{Z})_T$ is broken to a \mathbb{Z}_4 subgroup generated by S . Twisted matter fields transform as

$$\Phi_{-2/3} \xrightarrow{S} \exp(2\pi i/6) \rho(S) \Phi_{-2/3}, \quad (10a)$$

$$\Phi_{-5/3} \xrightarrow{S} \exp(2\pi i 5/12) \rho(S) \Phi_{-5/3}. \quad (10b)$$

Here, we use $c = -1$, $d = 0$ for the automorphy factor $(c\langle T \rangle + d)^n = (-i)^n$ of $S \in SL(2, \mathbb{Z})_T$ at $\langle T \rangle = i$, which results in

$$(-i)^{-2/3} = \exp(2\pi i/6) \quad \text{for } n = -2/3, \quad (11a)$$

$$(-i)^{-5/3} = \exp(2\pi i 5/12) \quad \text{for } n = -5/3. \quad (11b)$$

The superpotential also transforms under S as $\mathcal{W} \xrightarrow{S} i\mathcal{W}$. This implies that also S generates a discrete R -symmetry, which altogether leads to a non-Abelian discrete R -symmetry [46].

At $\langle T \rangle = i$, the explicit representation matrices (which include the associated automorphy factors) of the unified flavor group of twisted matter fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$ are given by

$$\begin{aligned} \Phi_{-2/3}: \quad & \rho_{\mathbf{3}_2, i}(A) = \rho(A), \quad \rho_{\mathbf{3}_2, i}(B) = \rho(B), \\ & \rho_{\mathbf{3}_2, i}(C) = \rho(C), \\ & \rho_{\mathbf{3}_2, i}(R) = e^{2\pi i/9} \mathbb{1}_3, \quad \rho_{\mathbf{3}_2, i}(S) = e^{2\pi i/6} \rho(S), \quad \text{and} \end{aligned} \quad (12)$$

$$\begin{aligned} \Phi_{-5/3}: \quad & \rho_{\mathbf{3}_1, i}(A) = \rho(A), \quad \rho_{\mathbf{3}_1, i}(B) = \rho(B), \\ & \rho_{\mathbf{3}_1, i}(C) = -\rho(C), \\ & \rho_{\mathbf{3}_1, i}(R) = e^{-4\pi i/9} \mathbb{1}_3, \quad \rho_{\mathbf{3}_1, i}(S) = e^{2\pi i 5/12} \rho(S). \end{aligned} \quad (13)$$

The CP -like transformation generated by Eq. (6) acts on the modulus as $T \mapsto -\bar{T}$, which is conserved for $\langle T \rangle = i$. The corresponding automorphy factor and

representation matrix in Eqs. (2) and (3) are trivial, see Refs. [11,40], such that³ $\Phi_n \xrightarrow{CP} \bar{\Phi}_n$.

Altogether, the linearly realized unified flavor group at $\langle T \rangle = i$ is found to be

$$\Delta(54) \cup \mathbb{Z}_9^R \cup \mathbb{S} \cup \mathbb{Z}_2^{CP} = \Xi(2, 2) \rtimes \mathbb{Z}_2^{CP} \cong [324, 111] \rtimes \mathbb{Z}_2 \cong [648, 548]. \quad (14)$$

C. Flavor enhancement at $\langle T \rangle = \omega$

Next, following Sec. 6.3 in Ref. [40], we consider the point $\langle T \rangle = \omega$ in moduli space. There, $\text{SL}(2, \mathbb{Z})_T$ is broken to a \mathbb{Z}_3 subgroup generated by ST such that our twisted matter fields transform as

$$\Phi_{-2/3} \xrightarrow{ST} \exp(2\pi i 2/9) \rho(ST) \Phi_{-2/3}, \quad (15a)$$

$$\Phi_{-5/3} \xrightarrow{ST} \exp(2\pi i 5/9) \rho(ST) \Phi_{-5/3}. \quad (15b)$$

Here, we use $c = d = -1$ for $ST \in \text{SL}(2, \mathbb{Z})_T$, which yields $(c\langle T \rangle + d)^n = (-\omega - 1)^n = (\omega^2)^n$ and, hence, the automorphy factors

$$(\omega^2)^{-2/3} = \exp(2\pi i 2/9) \quad \text{for } n = -2/3, \quad (16a)$$

$$(\omega^2)^{-5/3} = \exp(2\pi i 5/9) \quad \text{for } n = -5/3. \quad (16b)$$

In addition, the superpotential transforms under ST as $\mathcal{W} \xrightarrow{ST} \omega \mathcal{W}$. Hence, also here the residual flavor symmetry is a non-Abelian discrete R -symmetry [46].

The explicit representation matrices of the unified flavor group of twisted matter fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$ are given by

$$\begin{aligned} \Phi_{-2/3}: \rho_{3_2, \omega}(A) &= \rho(A), \quad \rho_{3_2, \omega}(B) = \rho(B), \\ \rho_{3_2, \omega}(C) &= \rho(C), \\ \rho_{3_2, \omega}(R) &= e^{2\pi i/9} \mathbb{1}_3, \quad \rho_{3_2, \omega}(ST) = e^{2\pi i 2/9} \rho(ST), \text{ and} \end{aligned} \quad (17)$$

$$\begin{aligned} \Phi_{-5/3}: \rho_{3_1, \omega}(A) &= \rho(A), \quad \rho_{3_1, \omega}(B) = \rho(B), \\ \rho_{3_1, \omega}(C) &= -\rho(C), \\ \rho_{3_1, \omega}(R) &= e^{-4\pi i/9} \mathbb{1}_3, \quad \rho_{3_1, \omega}(ST) = e^{2\pi i 5/9} \rho(ST). \end{aligned} \quad (18)$$

A representative of the \mathbb{Z}_2^{CP} transformation is given by K_*T , which acts on the modulus and matter fields as

³For explicit computations, it is useful to combine $\Phi \oplus \bar{\Phi}$ and extend also the other generators to this $r \oplus \bar{r}$ -dimensional representation.

$$T \xrightarrow{CP} -\bar{T} - 1, \quad \text{and} \quad \Phi_n \xrightarrow{CP} \rho(T)^* \bar{\Phi}_n, \quad (19)$$

such that $\langle T \rangle = \omega$ is left invariant.

The linearly realized unified flavor group at $\langle T \rangle = \omega$ is thus

$$\Delta(54) \cup \mathbb{Z}_9^R \cup \text{ST} \cup \mathbb{Z}_2^{CP} = H(3, 2, 1) \rtimes \mathbb{Z}_2^{CP} \cong [486, 125] \rtimes \mathbb{Z}_2 \cong [972, 469]. \quad (20)$$

D. Flavor enhancement at $\langle T \rangle = 1$ and $\langle T \rangle = i\infty$

Since the point $\langle T \rangle = 1$ has not been investigated in the literature for residual symmetries, we give more details here. First, note that the discussion for $\langle T \rangle = 1$ also applies to the point $\langle T \rangle = i\infty$ because these points are dual via conjugation by the element ST^{-1} , i.e., working in the limit $\epsilon \rightarrow 0^+$,

$$\begin{aligned} ST^{-1} \circ \langle T \rangle &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \circ \langle T \rangle \\ &= \frac{0\langle T \rangle + 1}{-1\langle T \rangle + 1} \xrightarrow{\epsilon \rightarrow 0^+} i\infty \quad \text{for } \langle T \rangle = 1 + i\epsilon \xrightarrow{\epsilon \rightarrow 0^+} 1. \end{aligned} \quad (21)$$

To find the local enhancement of the traditional flavor symmetry by the stabilizer of the modulus, note that $\langle T \rangle = 1$ is left invariant by the $\gamma \in \text{SL}(2, \mathbb{Z})_T$ transformations satisfying

$$\begin{aligned} \gamma \circ \langle T \rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \langle T \rangle \\ &= \frac{a\langle T \rangle + b}{c\langle T \rangle + d} \stackrel{!}{=} \langle T \rangle \stackrel{\langle T \rangle = 1}{\Leftrightarrow} a + b \stackrel{!}{=} c + d. \end{aligned} \quad (22)$$

Combining this with the constraint $ad - bc = 1$, we obtain

$$(c + d - b)d - bc = 1 \Leftrightarrow (c + d)(d - b) = 1. \quad (23)$$

Recalling that all variables here are integers, we observe that

$$\begin{aligned} s := c + d &= d - b \quad \text{with} \quad s = \pm 1 \\ \Rightarrow \gamma &= \begin{pmatrix} s - b & b \\ -b & s + b \end{pmatrix}. \end{aligned} \quad (24)$$

This yields the stabilizer at $\langle T \rangle = 1$. All solutions to Eq. (22) then read

$$\begin{pmatrix} s - b & b \\ -b & s + b \end{pmatrix} = \begin{cases} (ST^{-2})^b & \text{for } s = +1, \\ S^2(ST^{-2})^{-b} & \text{for } s = -1, \end{cases} \quad (25)$$

for $b \in \mathbb{Z}$. Therefore, the stabilizer can be generated by the two elements

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad ST^{-2} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \quad (26)$$

The first generator corresponds to the $\Delta(54)$ element C, while the second generator *locally* (at $\langle T \rangle = 1$) enhances the traditional flavor symmetry $\Delta(54)$ to a larger group. Noting that $ST^{-2} = TS^{-1}TST^{-1} = (ST^{-1})^{-1}T(ST^{-1})$ shows that the second generator is dual to the T transformation $T \mapsto T + 1$ that leaves $\langle T \rangle = i\infty$ invariant. At the level of the finite modular group T' , the $SL(2, \mathbb{Z})_T$ transformation ST^{-2} is realized as the \mathbb{Z}_3 transformation $\rho(ST^{-2}) = \rho(ST)$. The automorphy factor of $(ST^{-2})^b$ is trivial at $\langle T \rangle = 1$ for any modular form of weight n ; i.e., using Eq. (25) for $s = 1$, we obtain

$$(c\langle T \rangle + d)^n = (-b + 1 + b)^n = 1. \quad (27)$$

Hence, for matter fields in $\Delta(54)$ representations $\mathbf{r} = \mathbf{3}_2$ (for $\Phi_{-2/3}$) or $\mathbf{r} = \mathbf{3}_1$ (for $\Phi_{-5/3}$), the explicit representation matrix of the generator that locally enhances the traditional flavor symmetry is given by

$$\rho_{\mathbf{r}, \langle T \rangle = 1}(ST) = \rho(ST). \quad (28)$$

We note that the $s = -1$ case would give rise to a nontrivial automorphy factor and, hence, different matrix generators for $\mathbf{3}_2$ and $\mathbf{3}_1$, but ultimately lead to exactly the same group.

Hence, altogether, the unified flavor group at $\langle T \rangle = 1$ can be generated by

$$\begin{aligned} \Phi_{-2/3}: \rho_{\mathbf{3}_{2,1}}(\mathbf{A}) &= \rho(\mathbf{A}), & \rho_{\mathbf{3}_{2,1}}(\mathbf{B}) &= \rho(\mathbf{B}), \\ \rho_{\mathbf{3}_{2,1}}(\mathbf{C}) &= \rho(\mathbf{C}), \\ \rho_{\mathbf{3}_{2,1}}(\mathbf{R}) &= e^{2\pi i/9} \mathbb{1}_3, & \rho_{\mathbf{3}_{2,1}}(ST) &= \rho(ST), \quad \text{and} \end{aligned} \quad (29)$$

$$\begin{aligned} \Phi_{-5/3}: \rho_{\mathbf{3}_{1,1}}(\mathbf{A}) &= \rho(\mathbf{A}), & \rho_{\mathbf{3}_{1,1}}(\mathbf{B}) &= \rho(\mathbf{B}), \\ \rho_{\mathbf{3}_{1,1}}(\mathbf{C}) &= -\rho(\mathbf{C}), \\ \rho_{\mathbf{3}_{1,1}}(\mathbf{R}) &= e^{-4\pi i/9} \mathbb{1}_3, & \rho_{\mathbf{3}_{1,1}}(ST) &= \rho(ST). \end{aligned} \quad (30)$$

One can show that the groups generated by Eqs. (17) and (29), as well as by Eqs. (18) and (30) are identical.

To identify the CP -like stabilizer at $\langle T \rangle = 1$, we solve

$$\gamma_{CP} \circ \langle T \rangle = \frac{a\langle \bar{T} \rangle + b}{c\langle \bar{T} \rangle + d} \stackrel{!}{=} \langle T \rangle \stackrel{\langle T \rangle = 1}{\Leftrightarrow} a + b \stackrel{!}{=} c + d, \quad (31)$$

for an element $\gamma_{CP} \in GL(2, \mathbb{Z})_T$ with $\det \gamma_{CP} = -1$. Hence, altogether,

$$(c + d)(d - b) \stackrel{!}{=} -1. \quad (32)$$

Once more, recalling that the matrix representation of γ_{CP} has integer entries, we see that

$$\begin{aligned} s &:= (c + d) = -(d - b) \quad \text{with} \\ s = \pm 1 &\Rightarrow \gamma_{CP} = \begin{pmatrix} s - b & b \\ 2s - b & -s + b \end{pmatrix}. \end{aligned} \quad (33)$$

This yields the CP -like stabilizer at $\langle T \rangle = 1$. The solutions to Eq. (31) can be written as

$$\begin{pmatrix} s - b & b \\ 2s - b & -s + b \end{pmatrix} = \begin{cases} (ST^{-2})^{-b+1}SK_* & \text{for } s = +1, \\ S^2(ST^{-2})^{b+1}SK_* & \text{for } s = -1, \end{cases} \quad (34)$$

for $b \in \mathbb{Z}$. Since all these transformations differ only by an element of the stabilizer group itself, it is clear that they give rise to equivalent resulting groups. For definiteness, we choose for the generator of the CP -like stabilizer the element with $s = 1$ and $b = 2$, i.e., $\rho((ST^{-2})^{-1}SK_*) = \rho(T^2K_*)$, such that the resulting \mathbb{Z}_2^{CP} acts on the modulus and matter fields as

$$T \stackrel{CP}{\mapsto} -\bar{T} + 2, \quad \text{and} \quad \Phi_n \stackrel{CP}{\mapsto} \rho(T^2)\bar{\Phi}_n. \quad (35)$$

Combining this with the unified flavor group, the full linearly realized unified flavor group at $\langle T \rangle = 1$ results as

$$\begin{aligned} \Delta(54) \cup \mathbb{Z}_9^R \cup ST \cup \mathbb{Z}_2^{CP} \\ = H(3, 2, 1) \rtimes \mathbb{Z}_2^{CP} \cong [486, 125] \rtimes \mathbb{Z}_2 \cong [972, 469]. \end{aligned} \quad (36)$$

We realize that this result at $\langle T \rangle = 1$ coincides with the linearly realized unified flavor group at $\langle T \rangle = \omega$, Eq. (20).

For completeness, we note that the generator of the CP -like stabilizer at the dual point $\langle T \rangle = i\infty$ can be represented by $S^2K_* = -K_*$, which acts as $T \mapsto -\bar{T}$ and clearly leaves $\langle T \rangle = i\infty$ invariant. The resulting linearized flavor group at $\langle T \rangle = i\infty$ is, as expected, the same as for $\langle T \rangle = 1$.

E. Heterotic configurations of $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ with $\Omega(2)$ eclectic symmetry

The $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector appears naturally in six-dimensional $\mathbb{T}^6/\mathbb{Z}_3$, $\mathbb{T}^6/\mathbb{Z}_6$ -II, $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_6)$ orbifolds. These orbifolds have been explored in the search of models arising from heterotic string compactifications with the MSSM spectrum plus vectorlike exotics [33,47,48]. The details of the matter spectra of these constructions are determined by the choices of the embedding of the six-dimensional orbifold into the gauge degrees of freedom, subject to consistency requirements, such as modular invariance; see e.g., Refs. [49–52].

TABLE III. Flavor symmetry representations of MSSM matter fields in $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sectors of five different types of six-dimensional $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ heterotic orbifold models (consistent string theory configurations; see text). For each different type of configuration, we display *all* possibilities for representations that multiplets of quark (q, \bar{u}, \bar{d}), lepton ($\ell, \bar{e}, \bar{\nu}$), and Higgs superfields can take on in the relevant $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. We use the field notation of Table II, where the subindices denote modular weights. Multiple subindices indicate that matter fields of all those modular weights appear in the respective model.

Model	ℓ	\bar{e}	$\bar{\nu}$	q	\bar{u}	\bar{d}	H_u	H_d	Flavons
A	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1}$
B	$\Phi_{-1/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-2/3}$	$\Phi_{-1/3}$	Φ_{-1}	Φ_0	$\Phi_{-2/3,-1}$
C	$\Phi_{-2/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-2/3}$	Φ_{-1}	Φ_{-1}	$\Phi_{-1/3,-1}$
D	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{\pm 2/3,0}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	$\Phi_{-1/3}$	Φ_0	$\Phi_{-1,0}$	$\Phi_{\pm 2/3,-1}$
E	$\Phi_{-2/3,-1/3}$	$\Phi_{-2/3,0}$	$\Phi_{0,-2/3,-1/3,-5/3}$	$\Phi_{-1,-2/3}$	$\Phi_{-2/3}$	$\Phi_{0,-2/3}$	Φ_0	Φ_0	$\Phi_{-2/3,-1/3,-5/3,-1}$

Inspecting the consistent semirealistic string models classified in Refs. [33,47,48] endowed with a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, we observe that the light MSSM matter superfields appear only in a reduced number of field configurations. For example, considering the $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ (1,1) orbifold geometry (see Ref. [53] for details on this geometry) with one and two vanishing Wilson lines, we identify five types of configurations of massless MSSM matter superfields, as summarized in Table III in terms of the field labels used in Table II. Given these configurations, one can arrive at the flavor phenomenology of the $\Omega(2)$ eclectic flavor symmetry by using the effective superpotential and Kähler potential given in Ref. [18], including the spontaneous breakdown of the eclectic flavor group triggered by the vevs of the indicated flavon representations and the Kähler modulus T . This shall be done explicitly for models of configuration type A in a companion paper [35].

To conclude this section, let us stress an important empirical TD observation. As we see in Table II (and use in Table III), the modular weights alone suffice to characterize the transformation behavior of fields under the flavor symmetries. That is, there is here a one-to-one relation between the modular weights and all flavor symmetry charges of the fields. This also holds for other known TD constructions; see e.g., Refs. [17,20,54–56]. If this feature turned out to be correct for generic TD models, it would suggest that consistent BU constructions should abide to the same rule. Namely, fields of the same modular weight should also transform in the very same representation of all modular flavor symmetries.

III. BREAKING OF THE TRADITIONAL FLAVOR SYMMETRY $\Delta(54)$

In order to understand the breakdown of the $\Omega(2)$ eclectic flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector, let us first discuss the breaking of the traditional flavor symmetry $\Delta(54) \subset \Omega(2)$. Since $\Delta(54)$ is universal in moduli space, it can only be broken via nontrivial vevs of flavon fields, but not by the moduli. Motivated by the spectra identified in Sec. II E, we consider the three matter multiplets Φ_{-1} , $\Phi_{-2/3}$, and $\Phi_{-5/3}$ as possible candidates for

flavon fields. Depending on their structure, the vevs can either preserve different subgroups of $\Delta(54)$ or break the traditional flavor symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector completely. The possible breaking patterns of individual vevs are schematically displayed in Fig. 1.

Let us consider a specific $\Delta(54) \rightarrow S_3$ example to understand how to arrive at the details of the breakdown patterns. The $\Delta(54)$ elements A and C generate a traditional flavor subgroup $S_3 \subset \Delta(54)$ that we denote as $S_3^{(1)}$. This group corresponds to the geometric permutation symmetry of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. Any representation of $\Delta(54)$ can be decomposed into irreducible representations of this subgroup. If the decomposition of a particular representation includes one or more trivial singlets of $S_3^{(1)}$, then the spontaneous breakdown of $\Delta(54)$ to $S_3^{(1)}$ occurs if a flavon field transforming in such a particular representation develops a vev along the trivial singlet direction(s). Using the characters of A and C, it is straightforward to determine how the $\Delta(54)$ representations of flavon candidates branch in $S_3^{(1)}$ (see e.g., Ref. [57] for a pedagogical introduction):

$$\langle \Phi_{-1} \rangle: \mathbf{1}' \rightarrow \mathbf{1}', \quad (37a)$$

$$\langle \Phi_{-2/3} \rangle: \mathbf{3}_2 \rightarrow \mathbf{1}' \oplus \mathbf{2}, \quad (37b)$$

$$\langle \Phi_{-5/3} \rangle: \mathbf{3}_1 \rightarrow \mathbf{1} \oplus \mathbf{2}. \quad (37c)$$

We see that only the representation $\mathbf{3}_1$ associated with the matter field $\Phi_{-5/3}$ branches into a trivial singlet of S_3 . Therefore, any nonzero vev of Φ_{-1} or $\Phi_{-2/3}$ breaks not only $\Delta(54)$ but also the $S_3^{(1)}$ subgroup. Only the vev of $\Phi_{-5/3}$ along the direction in field space associated with the trivial singlet can result in an unbroken $S_3^{(1)}$ symmetry generated by A and C. This direction can be identified in the $\mathbf{3}_1$ $\Delta(54)$ representation of $\Phi_{-5/3}$ as the simultaneous eigenvector of $\rho_{\mathbf{3}_1}(A) = \rho(A)$ and $\rho_{\mathbf{3}_1}(C) = -\rho(C)$ with eigenvalue 1, which is found to be $(1, 1, 1)^T$, up to an arbitrary overall factor. Note that there are physically

TABLE IV. Overview of subgroups of the $\Delta(54)$ traditional flavor group and the corresponding vevs for the breakdown. The second column shows the branching of the relevant representations into the subgroups. The 1 here denotes the trivial singlet, and nontrivial singlets are labeled by their eigenvalue under the Abelian generator. We provide examples for generators of each subgroup (up to conjugation) that are left unbroken by the different choices of vevs specified in the last two columns. The notation “ $\oplus \langle \Phi_{-1} \rangle$ ” means that a vev of a nontrivial $\Delta(54)$ singlet field has to be switched on in addition to the vev of a triplet in order to achieve the breaking to the respective subgroup. The provided generators and vevs are not unique since their conjugation can yield equivalent $\Delta(54)$ subgroups. All the subgroups are stated up to conjugation; i.e., the groups are distinct and not related by conjugation. We omit here an arbitrary global (normalization) factor for each of the vevs, α is an arbitrary complex number, while “...” denotes the absence of a suitable vev.

$\Delta(54)$ subgroup	Branchings			Subgroup generator(s)	Corresponding vevs	
	Φ_{-1}	$\Phi_{-2/3}$	$\Phi_{-5/3}$		$\langle \Phi_{-2/3} \rangle$	$\langle \Phi_{-5/3} \rangle$
$\Delta(27)$	1	3	3	A, B	$(0, 0, 0)^T \oplus \langle \Phi_{-1} \rangle$	$(0, 0, 0)^T \oplus \langle \Phi_{-1} \rangle$
$S_3^{(1)}$	1'	1' \oplus 2	1 \oplus 2	A, C	...	$(1, 1, 1)^T$
$Z_3^{(1)}$	1	1 \oplus 1$_{\omega}$ \oplus 1$_{\omega^2}$		A	$(1, 1, 1)^T$	$(1, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
$S_3^{(2)}$	1'	1' \oplus 2	1 \oplus 2	B, C	...	$(1, 0, 0)^T$
$Z_3^{(2)}$	1	1 \oplus 1$_{\omega}$ \oplus 1$_{\omega^2}$		B	$(1, 0, 0)^T$	$(1, 0, 0)^T \oplus \langle \Phi_{-1} \rangle$
$S_3^{(3)}$	1'	1' \oplus 2	1 \oplus 2	ABA, C	...	$(\omega, 1, 1)^T$
$Z_3^{(3)}$	1	1 \oplus 1$_{\omega}$ \oplus 1$_{\omega^2}$		ABA	$(\omega, 1, 1)^T$	$(\omega, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
$S_3^{(4)}$	1'	1' \oplus 2	1 \oplus 2	AB ² A, C	...	$(\omega^2, 1, 1)^T$
$Z_3^{(4)}$	1	1 \oplus 1$_{\omega}$ \oplus 1$_{\omega^2}$		AB ² A	$(\omega^2, 1, 1)^T$	$(\omega^2, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
Z_2	1$_{-1}$	1 \oplus 1$_{-1}$ \oplus 1$_{-1}$	1 \oplus 1 \oplus 1$_{-1}$	C	$(0, 1, -1)^T$	$(1, 0, 0)^T + \alpha(0, 1, 1)^T$

equivalent subgroups conjugate to the $S_3^{(1)}$ group generated by A and C. Each choice of conjugate generators naively yields a different vev structure, but all of them are related by conjugation with an element of $\Delta(54)$. By contrast, any pattern of $\langle \Phi_{-5/3} \rangle$ vevs that cannot be related to $(1, 1, 1)^T$ by conjugation will break $S_3^{(1)}$, too.

The previous procedure can be repeated for different choices of generators, which yield different subgroups of $\Delta(54)$. All remnant subgroups (up to conjugation), a choice of their corresponding generators, and the flavon vev patterns associated with the spontaneous breakdown of $\Delta(54)$ to such subgroups are listed in Table IV. As in the case of $\Delta(54) \rightarrow S_3^{(1)}$, it mostly suffices to consider the vev of a single flavon to arrive at the different traditional flavor subgroups. To arrive at the subgroup $Z_3^{(1)}$, for example, it suffices that $\langle \Phi_{-2/3} \rangle$ acquires a nontrivial vev. However, if $\Phi_{-2/3}$ is not present in the spectrum, one may arrive at the same $Z_3^{(1)}$ subgroup by letting $\Phi_{-5/3}$ and Φ_{-1} develop vevs simultaneously.

Altogether, we see that vevs $\langle \Phi_{-1} \rangle$, $\langle \Phi_{-2/3} \rangle$, and $\langle \Phi_{-5/3} \rangle$ are quite efficient in breaking the traditional flavor symmetry. However, they can leave remnant symmetries unbroken, and this might happen by default, as potentials are known to be often minimized at symmetry enhanced points (i.e., by vevs with remnant subgroups). In order to break the flavor symmetry completely, vevs can be misaligned from their symmetry enhanced directions (thereby still allowing for *approximate* or softly broken symmetries

that can give rise to “flavor hierarchies”), or multiple vevs could be present simultaneously. From the stated patterns of single vevs, it is straightforward to work out remnant groups also if multiple vevs are present. The respective remnant groups would be given as the nontrivial intersection of the preserved symmetries of each individual vev. We will further explore the phenomenological consequences of these scenarios in our forthcoming paper [35].

IV. BREAKING OF THE COMPLETE ECLECTIC FLAVOR SYMMETRY $\Omega(2)$

After the $\Delta(54)$ example, let us consider the spontaneous breakdown patterns of the complete eclectic flavor symmetry $\Omega(2) \times Z_2^{CP}$. As discussed above, only parts of this group can be linearly realized on the spectrum of twisted matter fields, while other parts are necessarily broken by the vev $\langle T \rangle$ of the Kähler modulus. Compatibility with observations requires that, on top of the breaking induced by $\langle T \rangle$, the linearly realized unified flavor symmetry must be further broken by vevs of flavon fields.

At a generic point in moduli space, the traditional flavor symmetry is $\Delta(54)$, and the results of the previous section apply. In this section, we investigate analogous breaking patterns at the symmetry enhanced points in moduli space. In detail, we study the points $\langle T \rangle = i$ and $\langle T \rangle = \omega, 1, i\infty$, where the traditional flavor symmetry $\Delta(54)$ is enhanced, respectively, to $\Xi(2, 2) \cong [324, 111]$ and $H(3, 2, 1) \cong [486, 125]$ as discussed in Secs. II B to II D. As derived above, these flavor symmetries are additionally enhanced

TABLE V. Maximal subgroups (up to conjugation) of $\Xi(2, 2)$ that can be achieved from its breakdown by vevs of flavon fields at $\langle T \rangle = i$. We provide the branchings of the flavon representations under the resulting subgroups, followed by samples of the generators of such subgroups and the flavon vevs that yield the corresponding breakdown. Note that the \mathbb{Z}_2 generator satisfies $C = (S^3)^2$. We follow the same notation as in Table IV.

$\Xi(2, 2)$ subgroup	Branchings		Subgroup generator(s)	Corresponding vevs	
	$\Phi_{-2/3}$	$\Phi_{-5/3}$		$\langle \Phi_{-2/3} \rangle$	$\langle \Phi_{-5/3} \rangle$
$S_3^{(1)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	A, C	...	$(1, 1, 1)^T$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	A	$(1, 1, 1)^T$	$(1, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
$S_3^{(2)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	ABA, C	...	$(\omega, 1, 1)^T$
$\mathbb{Z}_3^{(2)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	ABA	$(\omega, 1, 1)^T$	$(\omega, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
\mathbb{Z}_4	$\mathbf{1}_{-1} \oplus \mathbf{1}_i \oplus \mathbf{1}_{-i}$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_i$	S^3	...	$(1 + \sqrt{3}, 1, 1)^T$
\mathbb{Z}_2	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{-1}$	C	$(0, 1, -1)^T$	$(1, 0, 0)^T + \alpha(0, 1, 1)^T$

by CP -like transformations. However, for practical reasons, we first focus on non- CP -like transformations and comment on the possible enhancements by CP -like transformations in the end.

We focus on the matter fields $\Phi_{-2/3}$ and $\Phi_{-5/3}$ from the θ sector of a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, as well as on the bulk field Φ_{-1} , because these fields arise as potential flavons in the models under consideration; see Sec. II E. The complete charge assignment of these fields together with the generators under the relevant modular and flavor transformations is summarized in Table II. As already stressed at the end of Sec. II E, the complete transformation behavior of a field can be inferred already from the respective modular weights which are fixed by the string theory construction.

Our results for the breakdown patterns induced by flavon vevs $\langle \Phi_{-2/3} \rangle$, $\langle \Phi_{-5/3} \rangle$ and/or $\langle \Phi_{-1} \rangle$ are depicted in Fig. 2 for the linearly realized unified flavor symmetry $\Xi(2, 2)$ at $\langle T \rangle = i$ and in Fig. 3 for the unified flavor symmetry $H(3, 2, 1)$ at $\langle T \rangle = \omega, 1, i\infty$. Details of our results are summarized in Table V (for $\langle T \rangle = i$) and Table VI (for $\langle T \rangle = \omega, 1, i\infty$). Just as in the previous section, we determine the branching of the respective irreducible representations into subgroups of the flavor symmetries. If at least one trivial singlet $\mathbf{1}$ is present in the branching, the associated flavon vev can break the original unified flavor group to the listed subgroup. We do not list subgroups that cannot be realized by any of the given vevs because they would actually preserve a larger subgroup. We give explicit

TABLE VI. Maximal subgroups (up to conjugation) of $H(3, 2, 1)$ that can be achieved from its breakdown by vevs of flavon fields at $\langle T \rangle = \omega$ (similar results hold for $\langle T \rangle = 1, i\infty$). We provide the branchings of the flavon representations under the resulting subgroups, followed by samples of the generators of such subgroups and the flavon vevs that yield the corresponding breakdown. In the first row of the top block, the subindices in the branching representations correspond to the charges with respect to the \mathbb{Z}_3 generated by $(B^2 A^2)^2 R(ST)^4$. We use the definitions $a := -1 + \eta - \eta^4 + \eta^5 - \eta^8$ and $b := -\eta + \eta^4 - \eta^8$ with $\eta := e^{2\pi i/18}$. Whenever the vev preserves not only the subgroup but a larger symmetry, the resulting symmetry is explicitly given in parentheses. In all other aspects, we follow the same notation as in Table IV.

$H(3, 2, 1)$ subgroup	Branchings		Subgroup generator(s)	Corresponding vevs	
	$\Phi_{-2/3}$	$\Phi_{-5/3}$		$\langle \Phi_{-2/3} \rangle$	$\langle \Phi_{-5/3} \rangle$
$S_3^{(2)} \times \mathbb{Z}_3$	$\mathbf{1}'_1 \oplus \mathbf{2}_\omega$	$\mathbf{1}_1 \oplus \mathbf{2}_\omega$	AC, $B^2 A^2$, $R(ST)^4$...	$(1, \omega^2, 1)^T$
$\mathbb{Z}_3^{(2)} \times \mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1}_{\omega^2, 1} \oplus \mathbf{1}_{\omega, \omega^2}$	$\mathbf{1} \oplus \mathbf{1}_{\omega^2, 1} \oplus \mathbf{1}_{\omega, \omega^2}$	$B^2 A^2$, $R(ST)^4$	$(1, \omega^2, 1)^T$	$(1, \omega^2, 1)^T$ (preserves $S_3^{(2)} \times \mathbb{Z}_3$)
$\mathbb{Z}_3^{(2)}$	$\mathbf{1} \oplus \mathbf{1}_{\omega^2} \oplus \mathbf{1}_\omega$	$\mathbf{1} \oplus \mathbf{1}_{\omega^2} \oplus \mathbf{1}_\omega$	$B^2 A^2$	$(1, \omega^2, 1)^T \oplus \langle \Phi_{-1} \rangle$	$(1, \omega^2, 1)^T \oplus \langle \Phi_{-1} \rangle$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	$R(ST)^4$	$(-\omega^2, 1, 0)^T + \alpha(-\omega^2, 0, 1)^T$	$(-\omega^2, 1, 0)^T + \alpha(-\omega^2, 0, 1)^T$
$S_3^{(1)}$	$\mathbf{1}' \oplus \mathbf{2}$	$\mathbf{1} \oplus \mathbf{2}$	C, A	...	$(1, 1, 1)^T$
$\mathbb{Z}_3^{(1)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	A	$(1, 1, 1)^T$	$(1, 1, 1)^T \oplus \langle \Phi_{-1} \rangle$
\mathbb{Z}_6	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-\omega}$	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_\omega$	$CR^2(ST)^8$	$(0, 1, -1)^T$	$(-2\omega^2, 1, 1)^T$
$\mathbb{Z}_3^{(3)}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{\omega^2}$	$R(ST)^4$	$(-\omega^2, 1, 0)^T + \alpha(-\omega^2, 0, 1)^T$	$(-\omega^2, 1, 0)^T + \alpha(-\omega^2, 0, 1)^T$
$\mathbb{Z}_3^{(4)}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$\mathbf{1} \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$	$BR(ST)^2$	$(b^*, \omega, a)^T$	$(1, a, b)^T$
\mathbb{Z}_2	$\mathbf{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}_{-1}$	C	$(0, 1, -1)^T$ (preserves \mathbb{Z}_6)	$(1, 0, 0)^T + \alpha(0, 1, 1)^T$

generators for all subgroups and the associated explicit vevs (up to conjugation). There exist multiple equivalent subgroups which can be obtained from the stated examples by conjugation.

As noted already above, the flavor symmetry enhancement at $\langle T \rangle = \omega$, $\langle T \rangle = 1$ and at $\langle T \rangle = i\infty$ leads to isomorphic unified flavor groups. We have confirmed explicitly that also the respective matrix groups for the triplets generate identical groups (and irreducible representations). Hence, the analysis of the triplet vevs is exactly the same at all three points. For definiteness, we chose in Table VI the generator convention at $\langle T \rangle = \omega$ provided explicitly in Eqs. (17) and (18).

Here, we comment on the breakdown induced by the vev of the nontrivial singlet field $\langle \Phi_{-1} \rangle$. The transformation behavior of Φ_{-1} can be read off from Table II. Generators of the remnant modular group S and T furthermore have to be amended by the correct automorphy factors, stated in Eqs. (11) and (16) taken for modular weight $n = -1$. Altogether, we find that the single vev $\langle \Phi_{-1} \rangle$ induces the breakings

$$\begin{aligned} \langle \Phi_{-1} \rangle: \Xi(2, 2) &\rightarrow \Delta(27), & H(3, 2, 1)_{\langle T \rangle = \omega} &\rightarrow [81, 9], \\ H(3, 2, 1)_{\langle T \rangle = 1, i\infty} &\rightarrow [81, 7]. \end{aligned} \quad (38)$$

We see that, unlike the triplet winding states, the breaking induced by the vev of the nontrivial singlet bulk field $\langle \Phi_{-1} \rangle$ differs between points $\langle T \rangle = \omega$ and $\langle T \rangle = 1, i\infty$. However, the structure of the groups $[81, 9]$ and $[81, 7]$ coincides in both cases with $\Delta(27) \rtimes \mathbb{Z}_3$, which is the notation that we adopt in Fig. 3.

To conclude, let us discuss the stabilizers of the CP -like type, i.e., the possibility of enhanced CP -like transformations. Whether or not CP -like transformations are broken or preserved is often a model dependent statement. As we will see, this is because rephasing transformations of fields are important in this context, and whether or not those rephasings are physical or unphysical depends on the specifics of the model. This is why in our previous discussion we have focused on the non- CP -like flavor symmetries, where model independent statements are possible. However, independently of the model, one can answer the question of whether or not the vev of a specific field would automatically break the associated CP -like transformation (i.e., whether or not the respective vev has a CP -like stabilizer). The transformations we investigate here are described by Eq. (3); i.e., they transform $\Phi \mapsto U\bar{\Phi}$ in short. In addition, the transformation might include an element of the initial (non- CP -like) eclectic flavor symmetry that is preserved at the specific location $\langle T \rangle$, i.e., a transformation of the type $\Phi \mapsto \rho(g)U\bar{\Phi}$ with $g \in G_{\text{traditional}} \cup G_{\text{modular}} \cup G_{\text{R}}$. Note that there is a crucial difference here with respect to generic bottom-up constructions: from the BU perspective, U has to be a representation matrix of a suitable outer automorphism

transformation of the flavor symmetry that maps the representations to their complex conjugates. Such matrices are only defined up to a global phase by construction.

Therefore, in solving the equation $U\bar{\Phi} \stackrel{!}{=} \Phi$, a phase can always be absorbed in U . This is to be contrasted with the TD construction. Here, first of all, not all outer automorphisms of the flavor symmetry are possible, but only those which are part of the eclectic group and preserved by the specific vev $\langle T \rangle$ of the modulus. Second, as the matrix U itself is a representation matrix within the eclectic group, arbitrary global phases for U are not admissible.

In order to clarify whether or not the vevs listed in Tables V and VI have a CP -like stabilizer, we take the representative U of the respective CP -like transformation, stated in Eqs. (35) and (19), and before Eq. (14). Furthermore, we allow to amend this generator by multiplying it by any element $\rho(g)$ of the unified flavor symmetries at the respective moduli location $\langle T \rangle$. Then, we check whether the flavon vev can solve the equation

$$\rho(g)U\langle \bar{\Phi} \rangle \stackrel{!}{=} \langle \Phi \rangle. \quad (39)$$

We find that all of the flavon vevs listed in Tables V and VI exhibit CP -like stabilizers, with one exception. The sole exception is the vev $(1, 1, 1)^T$ for $H(3, 2, 1)$ (as always irrespective of whether $\langle T \rangle$ is ω , 1 or $i\infty$). This vev does not allow for a solution of Eq. (39) *unless* a rephasing by a global phase of i to $(i, i, i)^T$ is admitted. Note that also the omission of the global prefactors for all other vevs is, of course, an explicit choice of a global phase, in the sense that Eq. (39) would be spoiled also for these vevs upon a “wrong” choice of global phase. Altogether, we see that all of the vevs have CP -like stabilizers (modulo the global choice of phase). In other words, none of the discussed breaking patterns leads itself to CP violation in a model independent way. Nonetheless, CP will certainly be broken (just as the other residual symmetries) once the flavon and/or modulus vevs are deflected away from their symmetry enhanced points or, alternatively, if multiple vevs with incommensurable stabilizers are switched on at the same time.

V. CONCLUSIONS AND OUTLOOK

We have analyzed in detail the breakdown of the eclectic flavor group as it appears in the top-down approach based on string theory. In a realistic setup, the eclectic flavor symmetry has to be broken. This breaking is induced by two mechanisms. First, the vev of the modulus breaks the finite modular symmetry, at least partially. Second, the traditional flavor symmetry is universal in moduli space and, hence, unbroken by the modulus vev. It can be enhanced by elements of the finite modular symmetry at specific, symmetry enhanced points in moduli space. Thus, in the top-down approach, one cannot just consider the

finite modular flavor symmetry and ignore the traditional flavor group. The latter enhances the predictive power of the scheme as it gives severe restrictions on the Kähler potential and superpotential of the theory (see Ref. [18], Sec. III, for a detailed discussion). Of course, the (possibly enhanced) traditional flavor symmetry has to be broken as well, and this requires the introduction of flavon fields. This increases the number of parameters, but there is no alternative. The flavon fields that break the traditional flavor symmetry might break the modular symmetries as well. This leads to an attractive flavor structure due to the subtle interplay of the symmetry breakdown via flavons and moduli as it allows the incorporation of various possibilities for flavor hierarchies through the alignment of vevs.

We illustrate the scheme in detail for an example based on the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector with traditional flavor group $\Delta(54)$, modular flavor group T' , and eclectic flavor group $\Omega(2)$ as displayed in Table I. For the top-down approach, we consider a representation content as it appears in orbifold compactifications of the heterotic string (here, the $\mathbb{T}^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold, as discussed in Ref. [34]). All possible massless representations are given in Table II. As usual in the top-down approach, the spectrum is very selective, and only a few representations of $\Delta(54)$ and T' appear as massless modes. In the present example, we also observe that the automorphy factors of the fields are strictly correlated with the corresponding representations of the discrete modular group T' . Thus, there is here no freedom

to choose modular weights by hand; they are fixed by the underlying string theory construction.

As a warm-up example, we discuss the breakdown of $\Delta(54)$ via flavon fields in Sec. III. The breakdown pattern is shown in Fig. 1 and Table IV. The main result of the paper concerns the breakdown patterns of the eclectic flavor group $\Omega(2)$, which is derived in Sec. IV. We specifically consider the flavor groups $\Xi(2,2) = [324, 111]$ and $H(3,2,1) = [486, 125]$, which appear as unbroken subgroups of $\Omega(2)$ at the fixed points $T = i$ and $T = 1, \omega$ (as well as at $T = i\infty$ which is dual to $T = 1$), respectively. The qualitative breakdown patterns via flavon fields are summarized in Figs. 2 and 3. The specific form of the corresponding flavon vevs is given in Tables V and VI. This shows that even a simple system like the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector exhibits a rich web of breakdown patterns via flavon and modulus vevs that might be suitable to be applied to discuss the flavor structure of quarks and leptons in the standard model of particle physics. In a companion paper [35], we shall show that a successful fit of the masses and mixing angles of quarks and leptons can be achieved.

ACKNOWLEDGMENTS

P. V. and A. B. were supported by the Deutsche Forschungsgemeinschaft (SFB1258). H. P. N. thanks the ASC-LMU and in particular Dieter Lüst for hospitality and support.

-
- [1] F. Feruglio, Are neutrino masses modular forms?, in *From My Vast Repertoire ...: Guido Altarelli's Legacy*, edited by A. Levy, S. Forte, and G. Ridolfi (World Scientific, Singapore, 2019), [10.1142/9789813238053_0012](https://arxiv.org/abs/10.1142/9789813238053_0012).
 - [2] J. C. Criado and F. Feruglio, Modular invariance faces precision neutrino data, *SciPost Phys.* **5**, 042 (2018).
 - [3] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Neutrino mixing from finite modular groups, *Phys. Rev. D* **98**, 016004 (2018).
 - [4] F. J. de Anda, S. F. King, and E. Perdomo, $SU(5)$ grand unified theory with A_4 modular symmetry, *Phys. Rev. D* **101**, 015028 (2020).
 - [5] J. T. Penedo and S. T. Petcov, Lepton masses and mixing from modular S_4 symmetry, *Nucl. Phys.* **B939**, 292 (2019).
 - [6] G.-J. Ding, S. F. King, and X.-G. Liu, Modular A_4 symmetry models of neutrinos and charged leptons, *J. High Energy Phys.* **09** (2019) 074.
 - [7] G.-J. Ding and F. Feruglio, Testing moduli and flavon dynamics with neutrino oscillations, *J. High Energy Phys.* **06** (2020) 134.
 - [8] F. Feruglio and A. Romanino, Lepton flavor symmetries, *Rev. Mod. Phys.* **93**, 015007 (2021).
 - [9] M.-C. Chen, S. Ramos-Sánchez, and M. Ratz, A note on the predictions of models with modular flavor symmetries, *Phys. Lett. B* **801**, 135153 (2020).
 - [10] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, Modular symmetry and non-Abelian discrete flavor symmetries in string compactification, *Phys. Rev. D* **97**, 116002 (2018).
 - [11] A. Baur, H. P. Nilles, A. Trautner, and P. K. Vaudrevange, Unification of flavor, CP , and modular symmetries, *Phys. Lett. B* **795**, 7 (2019).
 - [12] A. Baur, H. P. Nilles, A. Trautner, and P. K. Vaudrevange, A string theory of flavor and CP , *Nucl. Phys.* **B947**, 114737 (2019).
 - [13] T. Kobayashi and H. Otsuka, Classification of discrete modular symmetries in Type IIB flux vacua, *Phys. Rev. D* **101**, 106017 (2020).
 - [14] T. Kobayashi and S. Tamba, Modular forms of finite modular subgroups from magnetized D-brane models, *Phys. Rev. D* **99**, 046001 (2019).
 - [15] Y. Kariyazono, T. Kobayashi, S. Takada, S. Tamba, and H. Uchida, Modular symmetry anomaly in magnetic flux compactification, *Phys. Rev. D* **100**, 045014 (2019).

- [16] H. Ohki, S. Uemura, and R. Watanabe, Modular flavor symmetry on magnetized torus, *Phys. Rev. D* **102**, 085008 (2020).
- [17] Y. Almumin, M.-C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, and S. Shukla, Metaplectic flavor symmetries from magnetized tori, *J. High Energy Phys.* **05** (2021) 078.
- [18] H. P. Nilles, S. Ramos-Sánchez, and P. K. Vaudrevange, Lessons from eclectic flavor symmetries, *Nucl. Phys.* **B957**, 115098 (2020).
- [19] M.-C. Chen, V. Knapp-Perez, M. Ramos-Hamud, S. Ramos-Sánchez, M. Ratz, and S. Shukla, Quasi-eclectic modular flavor symmetries, *Phys. Lett. B* **824**, 136843 (2022).
- [20] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, Completing the eclectic flavor scheme of the \mathbb{Z}_2 orbifold, *J. High Energy Phys.* **06** (2021) 110.
- [21] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, Generalised CP symmetry in modular-invariant models of flavour, *J. High Energy Phys.* **07** (2019) 165.
- [22] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, CP violation in modular invariant flavor models, *Phys. Rev. D* **101**, 055046 (2020).
- [23] X.-G. Liu, C.-Y. Yao, B.-Y. Qu, and G.-J. Ding, Half-integral weight modular forms and application to neutrino mass models, *Phys. Rev. D* **102**, 115035 (2020).
- [24] C.-Y. Yao, X.-G. Liu, and G.-J. Ding, Fermion masses and mixing from the double cover and metaplectic cover of the A_5 modular group, *Phys. Rev. D* **103**, 095013 (2021).
- [25] G.-J. Ding, F. Feruglio, and X.-G. Liu, Automorphic forms and fermion masses, *J. High Energy Phys.* **01** (2021) 037.
- [26] G.-J. Ding, F. Feruglio, and X.-G. Liu, CP symmetry and symplectic modular invariance, *SciPost Phys.* **10**, 133 (2021).
- [27] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sánchez, and P. K. Vaudrevange, Siegel modular flavor group and CP from string theory, *Phys. Lett. B* **816**, 136176 (2021).
- [28] H. P. Nilles, S. Ramos-Sánchez, A. Trautner, and P. K. Vaudrevange, Orbifolds from $Sp(4, \mathbb{Z})$ and their modular symmetries, *Nucl. Phys.* **B971**, 115534 (2021).
- [29] K. Ishiguro, T. Kobayashi, and H. Otsuka, Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications, *Nucl. Phys.* **B973**, 115598 (2021).
- [30] H. P. Nilles, S. Ramos-Sánchez, and P. K. Vaudrevange, Eclectic flavor groups, *J. High Energy Phys.* **02** (2020) 045.
- [31] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, Eclectic flavor scheme from ten-dimensional string theory—I. Basic results, *Phys. Lett. B* **808**, 135615 (2020).
- [32] T. Kobayashi, H. P. Nilles, F. Plöger, S. Raby, and M. Ratz, Stringy origin of non-Abelian discrete flavor symmetries, *Nucl. Phys.* **B768**, 135 (2007).
- [33] Y. Olguín-Trejo, R. Pérez-Martínez, and S. Ramos-Sánchez, Charting the flavor landscape of MSSM-like Abelian heterotic orbifolds, *Phys. Rev. D* **98**, 106020 (2018).
- [34] B. Carballo-Pérez, E. Peinado, and S. Ramos-Sánchez, $\Delta(54)$ flavor phenomenology and strings, *J. High Energy Phys.* **12** (2016) 131.
- [35] A. Baur, H. P. Nilles, S. Ramos-Sánchez, A. Trautner, and P. K. S. Vaudrevange (to be published).
- [36] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.11.1, 2021, <https://www.gap-system.org>.
- [37] D. Jurčiukonis and L. Lavoura, GAP listing of the finite subgroups of $U(3)$ of order smaller than 2000, *Prog. Theor. Exp. Phys.* **2017**, 053A03 (2017).
- [38] J. Lauer, J. Mas, and H. P. Nilles, Duality and the role of nonperturbative effects on the world sheet, *Phys. Lett. B* **226**, 251 (1989).
- [39] J. Lauer, J. Mas, and H. P. Nilles, Twisted sector representations of discrete background symmetries for two-dimensional orbifolds, *Nucl. Phys.* **B351**, 353 (1991).
- [40] H. P. Nilles, S. Ramos-Sánchez, Saúl, and P. K. S. Vaudrevange, Eclectic flavor scheme from ten-dimensional string theory—II. Detailed technical analysis, *Nucl. Phys.* **B966**, 115367 (2021).
- [41] M.-C. Chen, M. Fallbacher, K. T. Mahanthappa, M. Ratz, and A. Trautner, CP violation from finite groups, *Nucl. Phys.* **B883**, 267 (2014).
- [42] H. P. Nilles, M. Ratz, A. Trautner, and P. K. S. Vaudrevange, CP violation from string theory, *Phys. Lett. B* **786**, 283 (2018).
- [43] K. S. Narain, New heterotic string theories in uncompactified dimensions < 10 , *Phys. Lett. B* **169**, 41 (1986).
- [44] K. S. Narain, M. H. Sarmadi, and E. Witten, A note on toroidal compactification of heterotic string theory, *Nucl. Phys.* **B279**, 369 (1987).
- [45] S. G. Nibbelink and P. K. S. Vaudrevange, T-duality orbifolds of heterotic Narain compactifications, *J. High Energy Phys.* **04** (2017) 030.
- [46] M.-C. Chen, M. Ratz, and A. Trautner, Non-Abelian discrete R symmetries, *J. High Energy Phys.* **09** (2013) 096.
- [47] H. P. Nilles and P. K. S. Vaudrevange, Geography of fields in extra dimensions: String theory lessons for particle physics, *Mod. Phys. Lett. A* **30**, 1530008 (2015).
- [48] E. Parr, P. K. S. Vaudrevange, and M. Wimmer, Predicting the orbifold origin of the MSSM, *Fortschr. Phys.* **68**, 2000032 (2020).
- [49] D. Bailin and A. Love, Orbifold compactifications of string theory, *Phys. Rep.* **315**, 285 (1999).
- [50] F. Plöger, S. Ramos-Sánchez, M. Ratz, and P. K. S. Vaudrevange, Mirage torsion, *J. High Energy Phys.* **04** (2007) 063.
- [51] P. K. S. Vaudrevange, Grand unification in the heterotic brane world, [arXiv:0812.3503](https://arxiv.org/abs/0812.3503).
- [52] S. Ramos-Sánchez, Towards low energy physics from the heterotic string, *Fortschr. Phys.* **57**, 907 (2009).
- [53] M. Fischer, M. Ratz, J. Torrado, and P. K. S. Vaudrevange, Classification of symmetric toroidal orbifolds, *J. High Energy Phys.* **01** (2013) 084.
- [54] S. Kikuchi, T. Kobayashi, and H. Uchida, Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds, *Phys. Rev. D* **104**, 065008 (2021).
- [55] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, The eclectic flavor symmetry of the \mathbb{Z}_2 orbifold, *J. High Energy Phys.* **02** (2021) 018.
- [56] K. Ishiguro, T. Kobayashi, and H. Otsuka, Symplectic modular symmetry in heterotic string vacua: flavor, CP , and R -symmetries, *J. High Energy Phys.* **01** (2022) 020.
- [57] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Non-Abelian discrete symmetries in particle physics, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010).