Decomposition of the SU(2) gauge field in the maximal Abelian gauge

V. G. Bornyakov
NRC "Kurchatov Institute"—IHEP, Protvino 142281, Russia
and NRC "Kurchatov Institute"—ITEP, Moscow 117218, Russia

I. Kudrov
NRC "Kurchatov Institute"—ITEP, Moscow 117218, Russia

R. N. Rogalyov
NRC "Kurchatov Institute"—IHEP, Protvino 142281, Russia

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We study decomposition of SU(2) gauge field into monopole and monopoleless components. After fixing the maximal Abelian gauge in SU(2) lattice gauge theory with Wilson action we decompose the non-Abelian gauge field into the Abelian field created by monopoles and the modified non-Abelian field with monopoles removed. We then calculate respective static potentials in the fundamental and adjoint representations and confirm earlier findings that the sum of these potentials approximates the non-Abelian static potential with good precision at all distances considered. Repeating these computations at three lattice spacings we find that in both representations the approximation becomes better with decreasing lattice spacing. Our results thus suggest that this approximation becomes exact in the continuum limit. We further find the same relation (for one lattice spacing) to be valid also in the cases of improved lattice action and in the theory with quarks.

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I. INTRODUCTION

We study numerically the lattice SU(2) gluodynamics in the maximal Abelian gauge (MAG) and consider decomposition of the lattice gauge field $U_\mu(x)$ [1]

$$U_\mu(x) = U_\mu^{\mathrm{mod}}(x) U_\mu^{\mathrm{mon}}(x), \tag{1}$$

where $U_\mu^{\mathrm{mon}}(x)$ is the monopole component and $U_\mu^{\mathrm{mod}}(x)$ is, respectively, the monopoleless component which we also will call a modified gauge field. By modification we understand removal of monopoles.

The decomposition (1) was first considered in [1]. It was demonstrated for one value of lattice spacing a that the sum of the static potentials $V_{\mathrm{mod}}(r) + V_{\mathrm{mon}}(r)$ [computed respectively with use of $U_\mu^{\mathrm{mod}}(x)$ and $U_\mu^{\mathrm{mon}}(x)$] was a good approximation of the original non-Abelian static potential $V(r)$ at all distances while $V_{\mathrm{mod}}(r)$ could be well fitted by purely Coulomb fit function. Here we extend this study in a few directions as discussed below.

It is well known [2–6] that after performing the Abelian projection in the MAG [7,8], the Abelian string tension calculated from the Abelian static potential is very close to the non-Abelian string tension and the corresponding coefficient of the Coulomb term is about $1/3$ of that in the non-Abelian static potential. It was also observed that the Abelian component of the gauge field is responsible for the chiral symmetry breaking [9]. This observation, like many others, supports the concept of Abelian dominance (for a review see, e.g., [10]). It was further discovered [4,11,12] that the monopole static potential also has string tension close to the non-Abelian one and small coefficient of the Coulomb term. These observations are in agreement with conjecture that monopole degrees of freedom are responsible for confinement [13]. It is then interesting to see what kind of static potential one obtains if the monopole contribution into the gauge field is switched off, that is, if only off-diagonal gluons and the so called photon part of the Abelian gluon field are left interacting with static quarks.

Previously computations of this kind were made in [14–17], where it was shown that the topological charge, chiral condensate and effects of chiral symmetry breaking in quenched light hadron spectrum disappear after removal of the monopole contribution from the relevant operators. Similar computations were made within the scope of the Z_2

projection studies [18]. It was shown that modified gauge field with removed projected center vortices (P-vortices) produces Wilson loops without area law, i.e., devoid of the confinement property. We do a similar removal with monopoles. We consider three types of the static potential: $V_{\rm mod}(r)$ obtained from the Wilson loops of the modified gauge field $U_\mu^{\rm mod}(x)$, $V_{\rm mon}(r)$ obtained from the Wilson loops of the monopole gauge field $U_\mu^{\rm mon}(x)$ and the sum of these two static potentials.

Here we study this phenomenon at three lattice spacings using the Wilson lattice gauge field action and thus we can make conclusions about the continuum limit. We also present the results for one lattice spacing obtained with the tadpole improved lattice field action [19] thus checking the universality. Furthermore, we present results for the SU(2) theory with dynamical quarks, i.e., for $\mathrm{QC_2D}$.

The paper is organized as follows. In the next section we introduce relevant definitions and describe details of our computations. In Sec. III results for the static potential are presented. Section IV is devoted to discussion and conclusions.

II. DEFINITIONS AND SIMULATION DETAILS

We consider the SU(2) lattice gauge theory after fixing MAG. The Abelian projection means coset decomposition of the non-Abelian lattice gauge field $U_\mu(x)$ into the Abelian field $u_\mu(x)$ and the coset field $C_\mu(x)$ [7]:

$$U_\mu(x) = C_\mu(x) u_\mu(x). \tag{2}$$

The Abelian gauge field can be further decomposed into the monopole (singular) part $u_\mu^{\rm mon}(x)$ and the photon (regular) part $u_\mu^{\rm ph}(x)$ [20]:

$$u_\mu(x) = u_\mu^{\rm mon}(x) u_\mu^{\rm ph}(x). \tag{3}$$

In terms of the corresponding angles it has the form

$$\theta_\mu(x) = \theta_\mu^{\rm mon}(x) + \theta_\mu^{\rm ph}(x), \tag{4}$$

where $\theta_\mu(x) \in (-\pi, \pi]$ is defined by $u_\mu(x) = e^{i\theta_\mu(x)}$, and $\theta_\mu^{\rm mon,ph}(x)$ are defined analogously. $\theta_\mu^{\rm mon}(x)$ can be presented as follows:

$$\theta_\mu^{\rm mon}(x) = -2\pi \sum_y D(x-y) \partial'_\nu m_{\nu\mu}(y), \tag{5}$$

where $D(x)$ is lattice inverse Laplacian, ∂'_ν is lattice backward derivative, $m_{\nu\mu}(x)$ are Dirac plaquettes. This solution satisfies the Landau gauge condition $\partial'_\mu \theta_\mu^{\rm mon}(x) = 0$.

We calculate the usual Wilson loops

$$W(C) = \frac{1}{2} \mathrm{Tr} \mathcal{W}(C), \mathcal{W}(C) = \left(\prod_{l \in C} U(l) \right), \tag{6}$$

the monopole Wilson loops

$$W_{\rm mon}(C) = \frac{1}{2} \mathrm{Tr} \left(\prod_{l \in C} u^{\rm mon}(l) \right), \tag{7}$$

and the non-Abelian Wilson loops with removed monopole contribution

$$W_{\rm mod}(C) = \frac{1}{2} \mathrm{Tr} \mathcal{W}_{\rm mod}(C), \qquad \mathcal{W}_{\rm mod}(C) = \left(\prod_{l \in C} \tilde{U}(l) \right), \tag{8}$$

where the modified non-Abelian gauge field $\tilde{U}_\mu(x)$ is defined as

$$\tilde{U}_\mu(x) = C_\mu(x) u_\mu^{\rm ph}(x). \tag{9}$$

Note that $u_\mu^{\rm ph}(x)$ is the Abelian projection of $\tilde{U}_\mu(x)$ and involves no monopoles.

It is known that MAG fixing leaves $U(1)$ gauge symmetry unbroken. The general form of the $U(1)$ gauge transformation is given by

$$\theta'_\mu(x) = \theta_\mu(x) + \partial_\mu \omega(x) + 2\pi n_\mu(x), \tag{10}$$

where $\theta'_\mu(x), \omega(x) \in (-\pi, \pi]$, $n_\mu(x) = 0, \pm 1$. Thus there are "small" gauge transformations with $n_\mu(x) = 0$ and "large" gauge transformations with $n_\mu(x) = \pm 1$. The monopole Wilson loop $W_{\rm mon}(C)$ is invariant under these gauge transformations. This is not true for $W_{\rm mod}(C)$. It was shown in [1] that $W_{\rm mod}(C)$ is invariant only under "small" gauge transformations and it is necessary to remove "large" gauge transformations. To this end we fix the $U(1)$ Landau gauge using the gauge condition

$$\max_\omega \sum_{x,\mu} \cos(\theta'_\mu(x)). \tag{11}$$

Up to Gribov copies this condition fixes configuration of Dirac plaquettes $m_{\mu\nu}(x)$ completely. Fixing $U(1)$ Landau gauge is excessive for our purposes but is eligible for calculations of $W_{\rm mod}$.

We calculated $r \times t$ rectangular Wilson loops $W(r,t)$, $W_{\rm mon}(r,t)$ and $W_{\rm mod}(r,t)$. To extract respective static potentials the APE smearing [21] has been employed. Computations were done with the Wilson lattice action at $\beta = 2.4$, 2.5 [for SU(2) gauge group $\beta = 4/g^2$, where g is the bare coupling constant] on 24^4 lattices and at $\beta = 2.6$ on 32^4 lattices using 100 statistically independent configurations. With the tadpole improved action [19] the simulations were made at $\beta = 3.4$ on 24^4 lattices. The simulations in $\mathrm{QC_2D}$ were made on 32^4 lattices with small lattice spacing $a = 0.044$ fm [22]. To fix MAG, the simulated annealing algorithm [4] with one gauge copy was used.

III. STATIC POTENTIAL IN FUNDAMENTAL AND ADJOINT REPRESENTATIONS

We present our results for the sum $V_{\mathrm{mon}}(r) + V_{\mathrm{mod}}(r)$ and compare it with the non-Abelian potential $V(r)$ in Fig. 1 for lattice Wilson action and three lattice spacings. One can see that the non-Abelian static potential $V(r)$ is well approximated by this sum, i.e.,

$$V(r) \approx V_{\mathrm{mon}}(r) + V_{\mathrm{mod}}(r). \tag{12}$$

This observation can be formulated in the following way: potential $V(r)$ between static sources interacting with the non-Abelian gauge field $U_\mu(x)$ can be approximated by the sum of the potential $V_{\mathrm{mon}}(r)$ between the sources interacting only with the monopole field $U_\mu^{\mathrm{mon}}(x)$ and the potential $V_{\mathrm{mod}}(r)$ between the sources interacting only with the modified (monopoleless) field $U_\mu^{\mathrm{mod}}(x)$.

We fitted all static potentials to the fit function

$$V(r) = V_0 + \alpha/r + \sigma r. \tag{13}$$

The results for the Coulomb coefficient α, the string tension σa^2 in units of lattice spacing a as well as the values

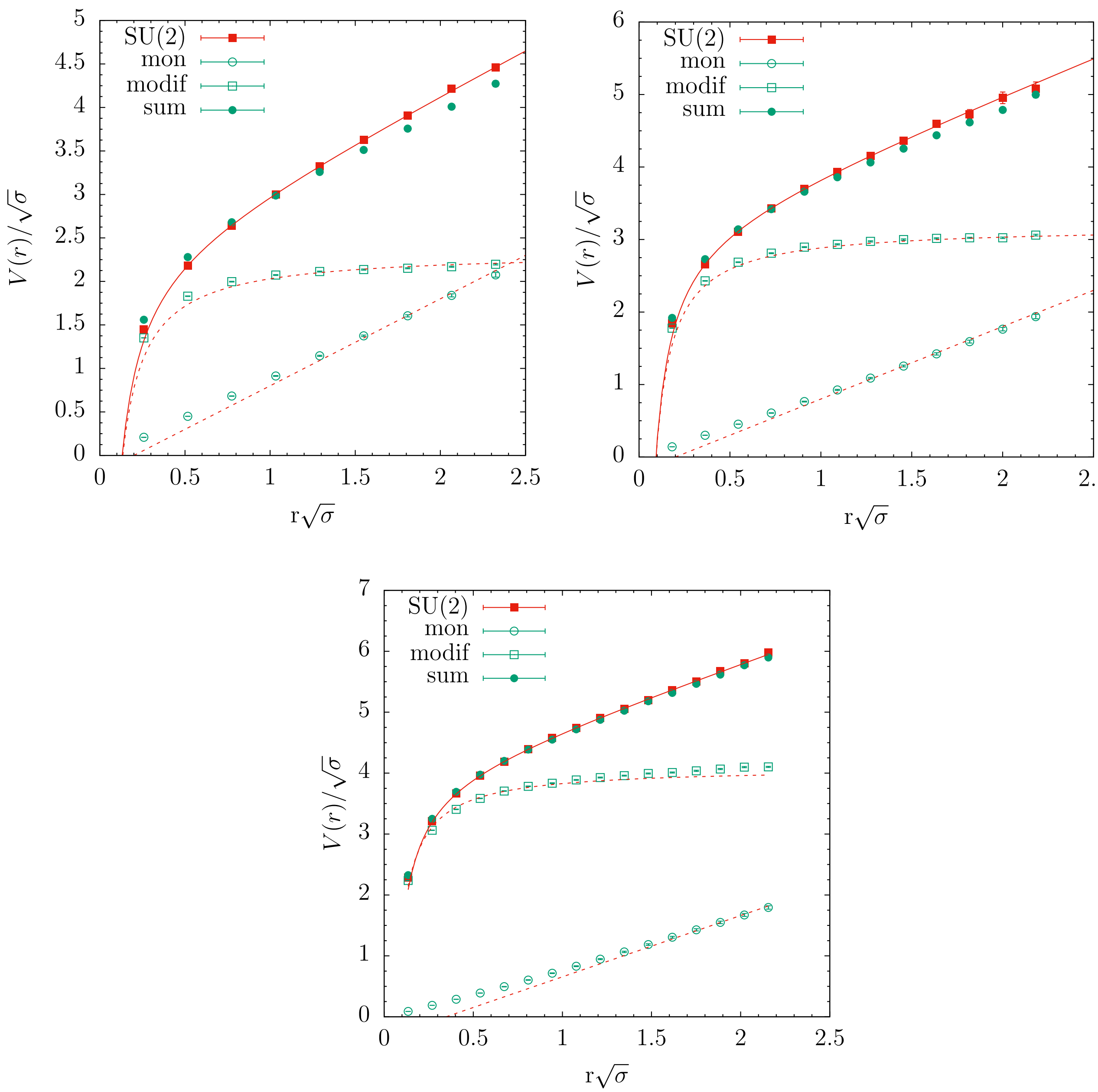

FIG. 1. Comparison of the non-Abelian potential $V(r)$ (filled squares) with the sum $V_{\mathrm{mod}}(r) + V_{\mathrm{mon}}(r)$ (filled circles) for $\beta = 2.4$ (left upper panel), $\beta = 2.5$ (right upper panel), $\beta = 2.6$ (lower panel). $V_{\mathrm{mod}}(r)$ (empty squares) and $V_{\mathrm{mon}}(r)$ (empty circles) are also depicted. The solid curve shows the fit to Eq. (13). Two dashed curves show its Coulomb and linear terms with adjusted constant terms.

TABLE I. Parameters of the potentials obtained by fits to the function (13).

Potential	$\beta=2.4,\ a=0.12$ fm σa^2	α	$\beta=2.5,\ a=0.09$ fm σa^2	α	$\beta=2.6,\ a=0.06$ fm σa^2	α	$\beta=3.4,\ a=0.09$ fm σa^2	α
V	0.067(1)	−0.31(1)	0.033(1)	−0.290(4)	0.0184(5)	−0.25(1)	0.032(1)	−0.26(1)
$V_{\mathrm{mon}}+V_{\mathrm{mod}}$	0.058(1)	−0.27(1)	0.030(1)	−0.27(1)	0.0175(4)	−0.25(1)	0.029(1)	−0.25(1)
V_{mon}	0.060(1)	−0.002(1)	0.030(1)	0.015(3)	0.0167(6)	0.06(2)	0.028(1)	0.034(5)
V_{mod}	⋯	−0.25(1)	⋯	−0.27(1)	0.002(1)	−0.27(1)	⋯	−0.30(1)

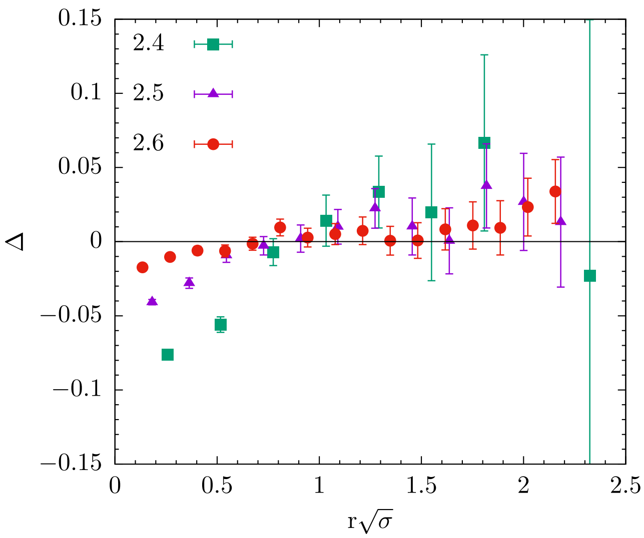

FIG. 2. The relative deviation $\Delta(r)$ defined in Eq. (14) vs distance r for three values of β.

of a in physical units are presented in Table I. The self energy V_0 is not important and is not shown. The values of a in physical units are obtained from the lattice data on σa^2 reported in Ref. [23] with the use of the phenomenological value $\sigma = 0.89$ GeV/fm. It should be noted that our data on σa^2 are in general consistent with those presented in [23].

One can see from Fig. 1 that the agreement between $V_{\mathrm{mon}}(r)+V_{\mathrm{mod}}(r)$ and $V(r)$ improves with decreasing lattice spacing. This is the main result of this paper. To make it more explicit we show in Fig. 2 the relative deviation determined as follows:

$$\Delta(r) = \frac{V(r) - (V_{\mathrm{mon}}(r) + V_{\mathrm{mod}}(r))}{V(r)}. \qquad (14)$$

More extended study with increased precision and an enlarged set of lattices is needed to make final conclusion about the continuum limit.

In Fig. 1 we also show the monopole $V_{\mathrm{mon}}(r)$ and the modified field $V_{\mathrm{mod}}(r)$ potentials separately. We find that $V_{\mathrm{mon}}(r)$ is linear at large distances and has small curvature

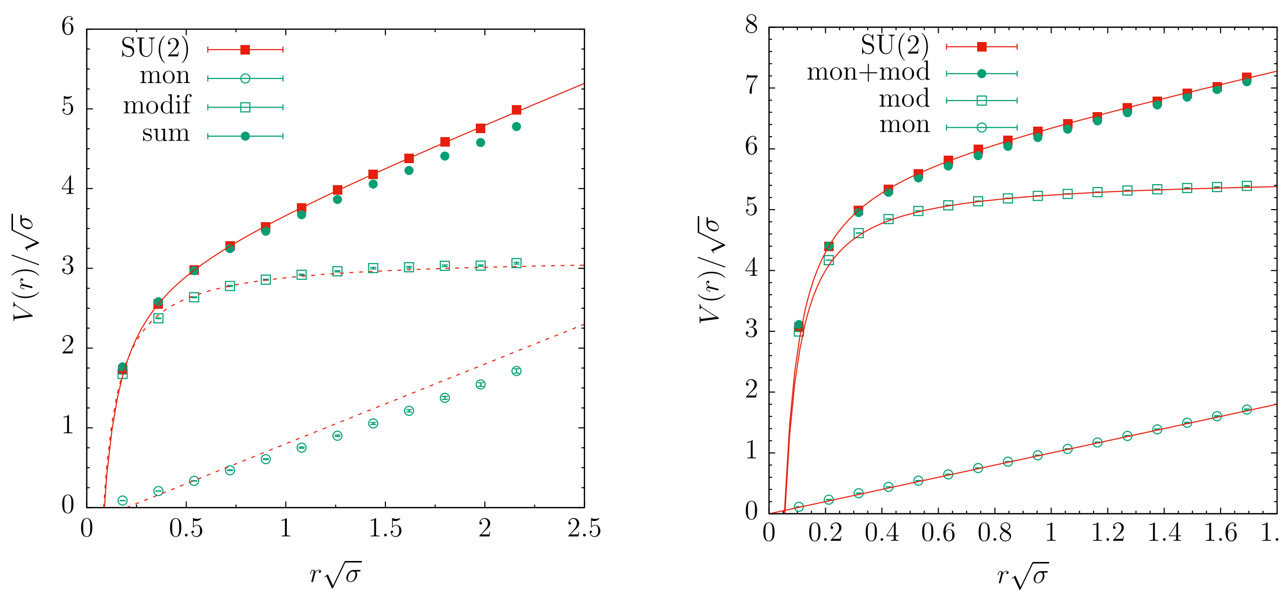

FIG. 3. Comparison of the non-Abelian potential $V(r)$ (filled squares) with the sum $V_{\mathrm{mod}}(r)+V_{\mathrm{mon}}(r)$ (filled circles) for improved action at $\beta=3.4$ (left) and for QC_2D (right). $V_{\mathrm{mod}}(r)$ (empty squares) and $V_{\mathrm{mon}}(r)$ (empty circles) are also shown. The solid curve and dashed curves carry the same meaning as in Fig. 1.

at small distances, which can be well fitted by the Coulomb behavior with small positive coefficient. The slope of $V_{\rm mon}(r)$ at large distances agrees better and better with that of $V(r)$ with decreasing lattice spacing. We shall note that increasing of the ratio $\sigma_{\rm mon}/\sigma$ with decreasing lattice spacing was reported before in [5].

It can be seen that $V_{\rm mod}(r)$ is of Coulombic form. Indeed it can be very well fitted by the fitting function $V_0^{\rm mod} - \alpha_{\rm mod}/r$ with $\alpha_{\rm mod} = 0.27(1)$ for $\beta = 2.5$ and similar values for $\beta = 2.4$, 2.6. One can see from Fig. 1 that $V_{\rm mod}(r)$ is in a very good agreement with the Coulombic part of $V(r)$. Thus removing the monopole contribution from the Wilson loop operator leaves Wilson loop which has no area law, i.e., the confinement property is lost. This result is similar to that obtained in [18] after removing P-vortices.

Apart from the approach to the continuum limit we studied the question of universality of the decomposition Eq. (12). The simulations were made with the tadpole improved action at $\beta = 3.4$. The lattice spacing at this β is approximately equal to that of the Wilson action at $\beta = 2.5$. The results are presented in Fig. 3 (left) and in Table I. One can see that agreement between $V(r)$ and $V_{\rm mod}(r) + V_{\rm mon}(r)$ is nearly as good as in Fig. 1 for $\beta = 2.5$.

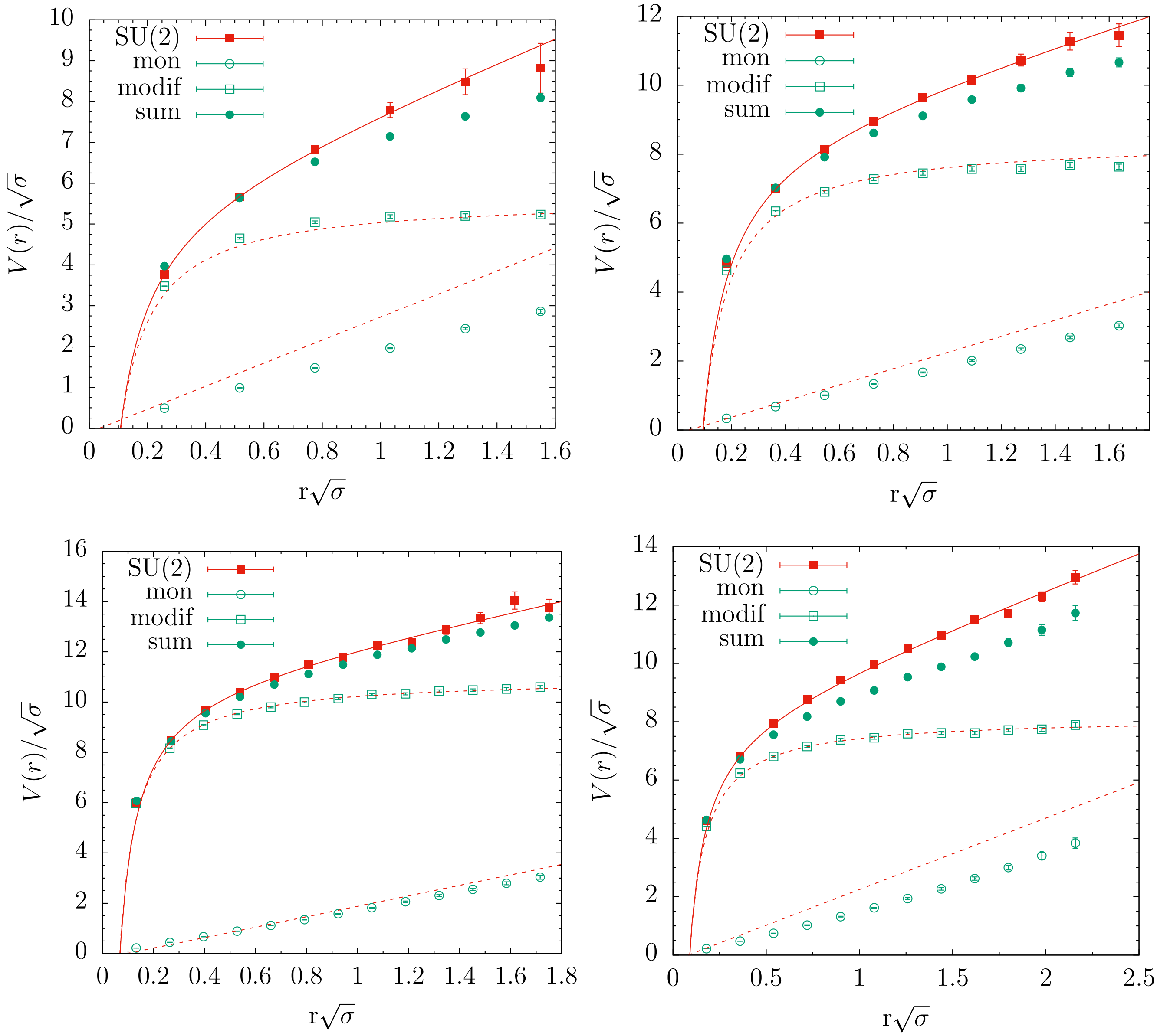

FIG. 4. Comparison of the adjoint non-Abelian potential $V_{\rm adj}(r)$ (filled squares) with the sum $V_{\rm mod,adj}(r) + V_{{\rm mon},q2}(r)$ (filled circles) for Wilson action at $\beta = 2.4$ (left, upper panel), $\beta = 2.5$ (right, upper panel), $\beta = 2.6$ (left, lower panel) and for improved action at $\beta = 3.4$ (right, lower panel). $V_{\rm mod,adj}(r)$ (empty squares) and $V_{{\rm mon},q2}(r)$ (empty circles) are also shown. The solid curve and dashed curves carry same meaning as in Fig. 1.

TABLE II. Parameters of the adjoint potentials obtained by fits to the function (13). In the columns for χ^2/N_{dof} the number in parenthesis shows N_{dof}.

	$\beta = 2.4$, $a = 0.12$ fm			$\beta = 2.5$, $a = 0.09$ fm			$\beta = 2.6$, $a = 0.06$ fm		
Potential	$\sigma_{\mathrm{adj}}/\sigma$	α_{adj}	$\chi^2/N_{\mathrm{dof}}(N_{\mathrm{dof}})$	$\sigma_{\mathrm{adj}}/\sigma$	α_{adj}	$\chi^2/N_{\mathrm{dof}}(N_{\mathrm{dof}})$	$\sigma_{\mathrm{adj}}/\sigma$	α_{adj}	$\chi^2/N_{\mathrm{dof}}(N_{\mathrm{dof}})$
V_{adj}	2.8(2)	−0.61(4)	0.28 (5)	2.3(2)	−0.80(5)	0.20 (6)	2.1(2)	−0.74(4)	0.66 (10)
$V_{\mathrm{mon},q2} + V_{\mathrm{adj,mod}}$	1.6(1)	−0.66(3)	0.56 (4)	1.8(1)	−0.67(5)	0.34 (7)	1.85(4)	−0.68(2)	0.30 (10)
$V_{\mathrm{mon},q2}$	1.9(1)	−0.010(4)	0.72 (5)	1.85(5)	0.006(4)	0.06 (7)	1.67(3)	0.001(1)	1.2 (11)
$V_{\mathrm{adj,mod}}$	⋯	−0.49(8)	2.2 (5)	⋯	−0.65(1)	1.1 (8)	⋯	−0.74(1)	0.86 (11)

Furthermore, we did the same study in QC$_2$D on 32^4 lattices with small lattice spacing $a = 0.044$ fm. The lattice field configurations were generated at pion mass $m_\pi = 740$ MeV with the rooted staggered fermion action corresponding in the continuum limit to $N_f = 2$ quark flavors and Symanzik improved gauge action (for details of simulations see, e.g., [22]). The results are presented in Fig. 3 (right). One can see clearly that approximate decomposition is fulfilled with rather high precision in this case as well.

Next we come to the static potential in the adjoint representation. It is well known that in the adjoint case the static quarks are screened at large distances by the gluons [24]. Still, the overlap of the adjoint Wilson loop computed here with the broken string state is known to be so tiny that the string breaking cannot be observed without adding explicitly the glue-lump state to the space of trial states [25]. Moreover, the adjoint string breaking distance [25] is beyond the range of the distances considered here. The Abelian projection for the adjoint representation was studied in [26]. In the adjoint representation case we check the validity of the relation

$$V_{\mathrm{adj}}(r) \approx V_{\mathrm{adj,mod}}(r) + V_{\mathrm{mon},q2}(r). \tag{15}$$

Here $V_{\mathrm{mon},q2}(r)$ is the static potential computed with the use of the lattice gauge field $u_\mu^{\mathrm{mon},q2}(x) = (u_\mu^{\mathrm{mon}}(x))^2$. Our numerical results for three lattice spacings for the Wilson action and for one lattice spacing for the improved action are presented in Fig. 4. The results of the fits to Eq. (13) are presented in Table II where notations σ_{adj} and α_{adj} were used for the linear and Coulomb terms coefficients. We show χ^2/N_{dof} in this Table to demonstrate the fit quality.

One can see that in the adjoint case the precision of our results is lower than for the fundamental representation case. Still it is seen that the relation (15) is satisfied quite well. The signature of improving agreement between lhs and rhs in (15) with decreasing lattice spacing is also seen although this should be checked in more precise measurements.

IV. CONCLUSIONS

We studied the decomposition of the static potential in the fundamental and adjoint representations into the linear term produced by the monopole (Abelian) gauge field $U_{\mathrm{mon}}(x)$ and the Coulomb term produced by the monopoleless non-Abelian gauge field $U_{\mathrm{mod}}(x)$. We confirm the results of Ref. [1] and improve them in a few respects. First, we made computations with varying lattice spacing and found that in both representations the agreement becomes better with decreasing lattice spacing. Our results suggest that the relations (12) and (15) become exact in the continuum limit. Further work is needed to provide more evidence for this conclusion. Second, we checked that the decomposition is valid also in the case of improved lattice action and in the theory with quarks. These results make it even more interesting to check this decomposition in the case of $SU(3)$ gauge group.

There are few conclusions to be drawn from the decomposition (12). It suggests that the monopole part $U_{\mathrm{mon}}(x)$ is responsible for the classical part of the hadronic string energy while the monopoleless part $U_{\mathrm{mod}}(x)$ produces the fluctuating part of that energy, i.e., while at small distances $U_{\mathrm{mod}}(x)$ should reproduce the perturbative results at large distances it contributes to the nonperturbative physics.

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