Bosonic algebraic approach applied to the $[QQ][\bar{Q}\bar{Q}]$ tetraquarks

A. J. Majarshin,[1,*] Yan-An Luo,[1,†] Feng Pan,[2,3,‡] and J. Segovia[4,§]

[1]*School of Physics, Nankai University, Tianjin 300071, People's Republic of China*
[2]*Department of Physics, Liaoning Normal University, Dalian 116029, People's Republic of China*
[3]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA*
[4]*Departamento Sistemas Físicos, Químicos y Naturales, Universidad Pablo de Olavide, 41013 Sevilla, Spain*

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The exact eigenenergies of the $T_{4c} = [cc][\bar{c}\bar{c}]$, $T_{4b} = [bb][\bar{b}\bar{b}]$, and $T_{2[bc]} = [bc][\bar{b}\bar{c}]$ tetraquarks are calculated within the extended transitional Hamiltonian approach, in which the so-called Bethe *ansatz* within an infinite-dimensional Lie algebra is used. We fit the parameters appearing in the transitional region from phenomenology associated with potential candidates of tetraquarks. The rotation and vibration transitional theory seems to provide a better description of heavy tetraquarks than other attempts within the same formalism. Our results indicate that the pairing strengths are large enough to provide binding; an extended comparison with the current literature is also performed.

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I. INTRODUCTION

A system of interacting bosons is a well-studied problem. Having its roots in the Bose-Einstein condensates [1–3], the framework has been applied to studies of nuclear and molecular structure [4–6], and examples of algebraic methods applied to hadron physics can be found in Refs. [7–11]. We have recently applied the interacting boson approximation proposed by Arima and Iachello [12], which includes two types of bosons (s and d-bosons), to the computation of wave functions in an interacting sl many-body boson system [13]. Note herein that, in general, the building blocks of the boson system are associated with both s and l bosons for single and quadrupole angular momentum. Typically, as an extension, the largest dynamical symmetry group generated by s and l operators is $U(2l+2)$ where the total number of bosons is a conserved quantity. The Hamiltonian of the model can be defined based on a linear combination of first- and second-order Casimir operators when only one- and two-body interactions are present.

Quarks can combine to form hadrons such as mesons (quark-antiquark pair) and baryons (three-quarks). Within an algebraic framework, the spectrum of hadrons began to be studied with the seminal work of Iachello in 1989 [7]. He was also able to elucidate some features about the structure of mesons and baryons, and the emergence of general patterns. An extension of the interacting boson approximation for studying eigenenergies of mesons in the $U(4)$ model was proposed by Pan *et al.* in 2006 [14]. The mass spectra of $Q\bar{Q}$ mesons, with Q either c or b quark, has been recently discussed in the framework of the $U(3) \to O(4)$ transitional theory [7,8,14]. Herein, we want to extend this model to multiquark states [15], in particular, tetraquarks with only heavy-quark content. In an extension of the sl boson system, the largest dynamical symmetry group is generated by s and l $(l = Q, \bar{Q}, \ldots)$ boson operators. We examine a similar Hamiltonian, based on the $SU(1,1)$ algebraic technique [13,16–18] and in a sl boson system to describe the masses of the $[QQ][\bar{Q}\bar{Q}]$ tetraquarks. Our predictions will provide a new solvable model in hadron physics. We shall show that the masses of the $[QQ][\bar{Q}\bar{Q}]$ tetraquarks are sensible to the vector quark pairing strengths.

Fully heavy tetraquarks have recently received considerable attention, both experimentally and theoretically. On the experimental side, it is thought that all-heavy tetraquark states will be very easy to spot because their masses should be far away from the typical mass regions populated by both conventional heavy mesons and the XYZ states discovered until now [19]. A search for deeply bound $bb\bar{b}\bar{b}$ tetraquark states at the LHC was motivated by Eichten *et al.* in Ref. [20], and it was carried out by the LHCb collaboration [21] determining that no significant excess is found in the $\mu^+\mu^-\Upsilon(1S)$ invariant-mass

*jalili@nankai.edu.cn
†luoya@nankai.edu.cn
‡daipan@dlut.edu.cn
§jsegovia@upo.es

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distribution. Note, however, that a search for exotic mesons at the CMS experiment reported a potential candidate of a fully bottom tetraquark $T_{4b} = [bb][\bar{b}\,\bar{b}]$ around 18–19 GeV [22]. On the other hand, the LHCb collaboration has recently released a study of the J/ψ-pair invariant mass spectrum finding a narrow peak and a broad structure which could originate from hadron states consisting of four charm quarks [23].

From the theoretical side, we find fully-heavy tetraquark computations based on phenomenological mass formulae [24–26], QCD sum rules [27–30], QCD motivated bag models [31], NR effective field theories [32,33], potential models [34–47], phenomenological approaches [26,27,48–52], nonperturbative functional methods [53], and even some exploratory lattice-QCD calculations [54]. Some works predict the existence of stable $QQ\bar{Q}\bar{Q}$ ($Q = c$ or b) bound states with masses slightly lower than the respective thresholds of quarkonium pairs (see, for instance, Refs. [24,25,27–29,32,33,41]. In contrast, there are other studies that predict no stable $cc\bar{c}\,\bar{c}$ and $bb\bar{b}\bar{b}$ tetraquark bound states because their masses are larger than two-quarkonium thresholds (see, e.g., Refs. [26,34,36,38,54]). To some extent, a better understanding of the mass locations of fully heavy tetraquark states would be desirable, if not crucial, for our comprehension of their underlying dynamics and their experimental hunting.

II. THEORETICAL METHOD

Within this framework, diquark clusters must be assumed in order to describe a tetraquark system. According to this, a tetraquark

$$T = Q_1 Q_2 \bar{Q}_3 \bar{Q}_4, \tag{1}$$

contains two point-like diquarks and we extend the interacting boson model to multilevel pairing considering algebraic solutions of an sl-boson system [13]. Note that the dynamical symmetry group is generated by s and l operators, where l can be the configuration of the multiquark states. In the Vibron model, elementary spatial excitations are scalar s-bosons with spin and parity $l^\pi = 0^+$ and vector l-bosons with spin and parity $l^\pi = 1^-$. In the finite-dimensional $SU(1,1)$ algebra, we have the generators, which satisfies the following commutation relations

$$[Q^0(l), Q^\pm(l)] = \pm Q^\pm(l), \tag{2a}$$

$$[Q^+(l), Q^-(l)] = -2Q^0(l). \tag{2b}$$

Now, we apply the affine $\widehat{SU(1,1)}$ algebra for the $U_s(1) \otimes U_{Q_1Q_2}(3) \otimes U_{\bar{Q}_3\bar{Q}_4}(3) \otimes U_J(3) - SO(10)$ transitional Hamiltonian. It is important to note that $Q_1Q_2\bar{Q}_3\bar{Q}_4$ responds, respectively, to $T_{4c} = [cc][\bar{c}\,\bar{c}]$, $T_{4b} = [bb][\bar{b}\,\bar{b}]$ and $T_{2bc} = [bc][\bar{b}\,\bar{c}]$ systems; and also that the quasispin algebras have been explained in detail in Refs. [13,16–18].

By using Eqs. (2a) and (2b) as generators of the $SU^l(1,1)$-algebra for tetraquarks, we have

$$Q^+(l) = \frac{1}{2} l^\dagger \cdot l^\dagger, \tag{3a}$$

$$Q^-(l) = \frac{1}{2} \tilde{l} \cdot \tilde{l}, \tag{3b}$$

$$Q^0(l) = \frac{1}{2}\left(l^\dagger \cdot \tilde{l} + \frac{2l+1}{2}\right), \tag{3c}$$

where $l^\dagger$ is the creation operator of an l-boson constituting the tetraquark, and $\tilde{l}_\nu = (-1)^\nu l_{-\nu}$.

It is accepted that the basis vectors of $U(2l+1) \supset SO(2l+1)$ and $O(2l+2) \supset O(2l+1)$ are simultaneously the basis vectors of $SU(1,1)^l \supset U(1)^l_s$ and $SU(1,1)^{sl} \supset U(1)^{sl}_s$, respectively. Their complementary relation for tetraquark states can be expressed as

$$\begin{aligned} &|Q_1, Q_2, \bar{Q}_3, \bar{Q}_4, N; n_l\nu_l, n_\Delta JM\rangle \\ &\quad = |Q_1, Q_2, \bar{Q}_3, \bar{Q}_4, N; \kappa_l\mu_l, n_\Delta JM\rangle, \end{aligned} \tag{4}$$

with $\kappa_l = \frac{1}{2}\nu_l + \frac{1}{4}(2l+1)$ and $\mu_l = \frac{1}{2}n_l + \frac{1}{4}(2l+1)$, where N, n_l, ν_l, J, and M are quantum numbers of $U(N)$, $U(2l+1)$, $SO(2l+1)$, $SO(3)$ and $SO(2)$, respectively. The quantum number n_Δ is an additional one needed to distinguish different states with the same J. However, the pairing models of multilevel are also characterized by an overlaid $U(n_1 + n_2 + \ldots)$ algebraic structure which has been described in detail in, for instance, Refs. [13,14,17,18]. This is to say either

$$\underset{N}{U(n)} \supset \underset{n}{U(n-1)}, \tag{5}$$

or

$$\begin{aligned} U\left(\sum_{i=1}^m n_i\right) &\supset \left\{SO\left(\sum_{i=1}^m n_i\right)\right\} \\ &\supset SO(n_i) \otimes \ldots \otimes SO(n_m) \\ &\supset SO_{i\ldots m}(3). \end{aligned} \tag{6}$$

Affine Lie algebras are famous among the infinite-dimensional Lie algebras and have widespread applications because of their representation theory. We know that the affine Lie algebra is far richer than that of finite-dimensional simple Lie algebras. Hence, in contrast to Eqs. (2a) and (2b), the operators under the corresponding $SU(1,1)$ irreducible representations satisfy the following commutation relations:

$$[Q^0_m(l), Q^\pm_n(l)] = \pm Q^\pm_{m+n}(l), \tag{7a}$$

$$[Q_m^+(l), Q_n^-(l)] = -2Q_{m+n+1}^0(l). \tag{7b}$$

According to the definitions, Q_m^Ω, with $\Omega = 0, \pm$ and $m = 0, \pm 1, \pm 2, \ldots$ generate the affine Lie algebra without central extension. The infinite dimensional $\widehat{SU(1,1)}$ Lie algebra is defined by

$$Q_n^\pm = c_{Q_1}^{2n+1} Q^\pm(Q_1) + c_{Q_2}^{2n+1} Q^\pm(Q_2) + c_{\bar{Q}_3}^{2n+1} Q^\pm(\bar{Q}_3) + c_{\bar{Q}_4}^{2n+1} Q^\pm(\bar{Q}_4), \tag{8a}$$

$$Q_n^0 = c_{Q_1}^{2n} Q^0(Q_1) + c_{Q_2}^{2n} Q^0(Q_2) + c_{\bar{Q}_3}^{2n} Q^0(\bar{Q}_3) + c_{\bar{Q}_4}^{2n} Q^0(\bar{Q}_4), \tag{8b}$$

where c_Q's and $c_{\bar{Q}}$'s are real-valued control parameters for tetraquarks, and n can be taken to be $1, 2, 3, \ldots$

The lowest weight state of fully heavy tetraquarks should satisfy $Q^-(l)|lw\rangle = 0$. Then, we define $|lw\rangle$ by the following expression:

$$|lw\rangle = |Q_1, Q_2, \bar{Q}_3, \bar{Q}_4, N; \kappa_l \mu_l, n_\Delta JM\rangle, \tag{9}$$

where $N = 2k + \nu_{Q_1} + \nu_{Q_2} + \nu_{\bar{Q}_3} + \nu_{\bar{Q}_4}$.[1] Hence, we have

$$Q_n^0|lw\rangle = \Lambda_n^l|lw\rangle, \quad \Lambda_n^l = \sum_l c_l^{2n} \frac{1}{2}\left(n_l + \frac{2l+1}{2}\right). \tag{10}$$

It is apparent that the system is in the vibrational $U(9)$ and rotational $SO(10)$ transition region as the pairing strengths, c_l, vary continuously within the closed interval [0, 1]. The quantum phase transition occurs in the all-heavy tetraquark pairing model. The $U(9)$ limit is fulfilled by $c_{Q_1} = c_{Q_2} = c_{\bar{Q}_3} = c_{\bar{Q}_4} = 0$ the $SO(10)$ limit occurs when $c_{Q_1} = c_{Q_2} = c_{\bar{Q}_3} = c_{\bar{Q}_4} = 1$. In our calculation, we have extracted different values for the control parameters between $U(9)$ and $SO(10)$ limits, *viz.* c_{Q_i} and $c_{\bar{Q}_i}$ belong to the [0, 1] interval with $i = 1, \ldots, 4$.

The total Hamiltonian is represented in terms of the Casimir operators $\hat{C}_2$ by branching chains. The two first terms of the Hamiltonian, $Q_0^+ Q_0^-$ and Q_1^0, are related to the $SU(1,1)$ algebra and the remaining ones are constant according to the Casimir operators. In the duality relation for the tetraquarks, the irreducible representations reduce the quasi-spin algebra chains (8a) and (8b) as well, and the labels for the chains are connected through the duality relations. By employing the generators of algebra $SU(1,1)$, the proposed Hamiltonian for heavy tetraquark pairing model is

$$\hat{H} = gQ_0^+ Q_0^- + \alpha Q_1^0 + \beta \hat{C}_2(SO(9)) + \gamma_1 \hat{C}_2(SO(3)_R) + \gamma_2 \hat{C}_2(SO(3)_{Q_1 Q_2}) + \gamma_3 \hat{C}_2(SO(3)_{\bar{Q}_3 \bar{Q}_4}) + \gamma \hat{C}_2(SO(3)_J), \tag{11}$$

where g, α, β, γ_1, γ_2, γ_3, and γ are real-valued parameters. Note that R stand the rotation for second order Casimir operator in Hamiltonian.

To find the non-zero energy eigenstates with k-pairs, we exploit a Fourier Laurent expansion of the eigenstates of Hamiltonians which contain dependences on several quantities in terms of unknown c-number parameters x_i, and thus eigenvectors of the Hamiltonian for excitations can be written as

$$\begin{aligned} &|k; \nu_{Q_1} \nu_{Q_2} \nu_{\bar{Q}_3} \nu_{\bar{Q}_4} n_\Delta JM\rangle \\ &= \sum_{n_i \in Z} a_{n_1 n_2 \ldots n_k} \\ &= x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_k^{n_k} Q_{n_1}^+ Q_{n_2}^+ Q_{n_3}^+ \ldots Q_{n_k}^+ |lw\rangle, \end{aligned} \tag{12}$$

and

$$Q_{n_i}^+ = \frac{c_{Q_1}}{1 - c_{Q_1}^2 x_i} Q^+(Q_1) + \frac{c_{Q_2}}{1 - c_{Q_2}^2 x_i} Q^+(Q_2) + \frac{c_{\bar{Q}_3}}{1 - c_{\bar{Q}_3}^2 x_i} Q^+(\bar{Q}_3) + \frac{c_{\bar{Q}_4}}{1 - c_{\bar{Q}_4}^2 x_i} Q^+(\bar{Q}_4). \tag{13}$$

The coefficients x_i are determined through the following set of equations

$$\frac{\alpha}{x_i} = \sum_l \frac{c_l^2 (\nu_l + \frac{2l+1}{2})}{1 - c_l^2 x_i} - \sum_{j \neq i} \frac{2}{x_i - x_j}. \tag{14}$$

A similar structure to (13) was first used by Gaudin as an ansatz in finding exact solutions of a spin-spin interaction system [55], which is now confirmed to be a consistent operator form in building the Bethe ansatz wave function (14) for the current tetraquark system. The Bethe ansatz equation (BAE) is a non-linear equation for a k-pair excitation to get the energy spectra. The quantum number k-pair excitation corresponds to the total number of bosons N. This connection is made possible by seniority numbers, i.e., the quantum number ν_l of $SO(2l+1)$. It is well known that in Eq. (4), the allowed seniority numbers are $\nu_l = n_l$, $n_l - 2$, $n_l - 4$, … Further details on the determination of the model parameters and the way of solving the eigenvalue problem can be found in Appendix.

Our formalism and methods for masses of heavy tetraquarks are the same as the procedure in Refs. [14,17,18]. The representation (11) is totally symmetric, corresponding to the fact that the excitations (vibrations and rotations) are bosonic in nature. So, we have to define the number of bosons in our system. Here the boson number value depicts the total number of vibrational states in the representation $[N]$.

[1]We take the many-body system with a k-pair excitation; see the Appendix for further details.

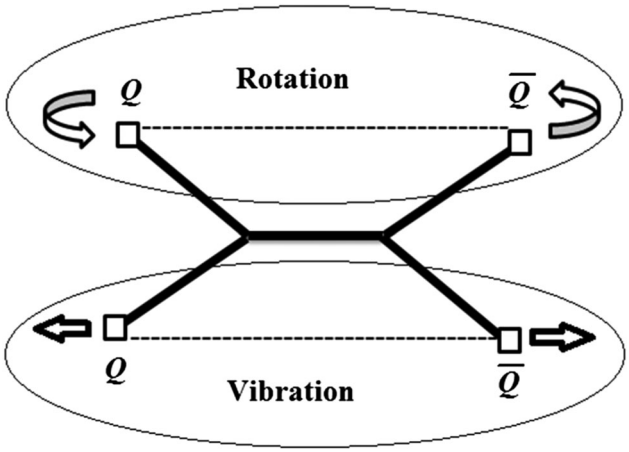

FIG. 1. Schematic representation of the rotational and vibrational degrees of freedom in the studied tetraquark systems.

The quantum phase transition occurs between vibrational and rotational limits in the fully heavy tetraquark pairing model. The quark (antiquark) configuration can perform vibrations and rotations (Fig. 1) defined by the quantum numbers ν_{Q_i}, $\nu_{\bar{Q}_i}$, and J. We do not study here bending and twisting of tetraquarks since these are required to lie at higher masses. The pure configuration problem of Fig. 1 is slightly complicated by the fact that quarks and antiquarks have *internal* degrees of freedom. The method of pairing strengths is applied herein, following Refs. [7,18]. The stringlike configuration of the tetraquark is shown in Fig. 1. Within the two-quarks configuration, the operator Q must be followed by $\bar{Q}$. Since we have pairing, for tetraquarks, one could also have the combinations QQ and $\bar{Q}\,\bar{Q}$. Finally, the quarks should be combined in such a way that an appropriate rotation-vibration pattern is provided satisfying the so-called number of pairing $N = 2k + \nu_{Q_1} + \nu_{Q_2} + \nu_{\bar{Q}_3} + \nu_{\bar{Q}_4}$.

III. RESULTS

In the diquark–antidiquark pairing model, the tetraquark mass can be determined by solving the eigenvalue problem of Eq. (11). Moreover, the quantum numbers that define a tetraquark state are the spins of diquark and antidiquark clusters, and the total spin, spatial inversion symmetry and charge conjugation of the system, i.e., the J^{PC} quantum numbers. Following Ref. [56], for a $Q_1Q_2\bar{Q}_3\bar{Q}_4$ system, the quantum labels are $J^{PC} = 0^{++}$, 1^{+-}, and 2^{++}, and thus we have:

(1) Two states for the scalar system:

$$|0^{++}\rangle = |0_{Q_1Q_2}, 0_{\bar{Q}_3\bar{Q}_4}; J = 0\rangle, \tag{15a}$$

$$|0^{++\prime}\rangle = |1_{Q_1Q_2}, 1_{\bar{Q}_3\bar{Q}_4}; J = 0\rangle. \tag{15b}$$

(2) Three states for the vector system:

$$|A\rangle = |0_{Q_1Q_2}, 1_{\bar{Q}_3\bar{Q}_4}; J = 1\rangle, \tag{16a}$$

$$|B\rangle = |1_{Q_1Q_2}, 0_{\bar{Q}_3\bar{Q}_4}; J = 1\rangle, \tag{16b}$$

$$|C\rangle = |1_{Q_1Q_2}, 1_{\bar{Q}_3\bar{Q}_4}; J = 1\rangle. \tag{16c}$$

Under charge conjugation, we have different configurations in which $|A\rangle$ and $|B\rangle$ interchange while $|C\rangle$ is odd. Thus, the $J^P = 1^+$ involves one C-even and two C-odd states:

$$|1^{++}\rangle = \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle), \tag{17a}$$

$$|1^{+-}\rangle = \frac{1}{\sqrt{2}}(|A\rangle - |B\rangle), \tag{17b}$$

$$|1^{+-\prime}\rangle = |C\rangle. \tag{17c}$$

Note here that we must select the appropriate values for the spin of $Q_1\bar{Q}_3$ and $Q_2\bar{Q}_4$. This means that the only state with $C = +$ is the one where $Q_1\bar{Q}_3$ has spin $S_{Q_1\bar{Q}_3} = 1$.

(iii) One state for the tensor system:

$$|2^{++}\rangle = |1_{Q_1Q_2}, 1_{\bar{Q}_3\bar{Q}_4}; J = 2\rangle, \tag{18}$$

where this state has also $S_{Q_1\bar{Q}_3} = 1$.

A. The $[cc][\bar{c}\bar{c}]$ system

In the pairing tetraquark model, the rigid and nonrigid phases correspond, respectively, to the $SO(10)$ and $U(9)$ symmetry cases. Both are idealized situations and must coexist in the real world. Therefore, the $U(9) \leftrightarrow SO(10)$ transitional region is where the two phases coexist and vibrational-rotational modes appear.

The parameters in the transitional region are called the phase parameters since the case $c_{Q_i} = 1$, with $i = 1, \ldots, 4$, corresponds to the rotational mode, while the $c_{Q_i} = 0$ case corresponds to the vibrational mode. We first can calculate the mass spectrum of the pairing tetraquark model with fixed phase parameters. Then, the transitional spectra from one phase to the other can be obtained modifying the phase parameters within the closed interval [0, 1].

From a transitional theory point of view, the ideal way of extracting the values of the phase coefficients is looking at the meson-meson thresholds $\eta_c(1S)\eta_c(1S)$ and $J/\psi(1S)J/\psi(1S)$ for $J^{PC} = 0^{++}$, $\eta_c(1S)J/\psi(1S)$ for $J^{PC} = 1^{+-}$, and $J/\psi(1S)J/\psi(1S)$ for $J^{PC} = 2^{++}$. Our values are $c_{Q_1} = 0.92$, $c_{Q_2} = 1$, and $c_{\bar{Q}_3} = c_{\bar{Q}_4} = 0$, which results into the following masses

$$|0^{++\prime}\rangle = |1_{cc}, 1_{\bar{c}\,\bar{c}}; J=0\rangle : M = 5.978 \text{ GeV}, \quad (19)$$

$$|1^{+-\prime}\rangle = |1_{cc}, 1_{\bar{c}\,\bar{c}}; J=1\rangle : M = 6.155 \text{ GeV}, \quad (20)$$

$$|2^{++}\rangle = |1_{cc}, 1_{\bar{c}\,\bar{c}}; J=2\rangle : M = 6.263 \text{ GeV}, \quad (21)$$

for the T_{4c} tetraquark system.

B. The $[bb][\bar{b}\,\bar{b}]$ system

The situation here is very similar to the case above. This time, from a transitional theory point of view, the extraction phase coefficients must be performed attending to the meson-meson thresholds $\eta_b(1S)\eta_b(1S)$ and $\Upsilon(1S)\Upsilon(1S)$ for $J^{PC} = 0^{++}$, $\eta_b(1S)\Upsilon(1S)$ for $J^{PC} = 1^{+-}$, and $\Upsilon(1S)\Upsilon(1S)$ for $J^{PC} = 2^{++}$. Our numerical values are $c_{Q_1} = 0.97$, $c_{Q_2} = 1$, $c_{\bar{Q}_3} = 1$ and $c_{\bar{Q}_4} = 0$, which provide the following masses:

$$|0^{++\prime}\rangle = |1_{bb}, 1_{\bar{b}\,\bar{b}}; J=0\rangle : M = 18.752 \text{ GeV}, \quad (22)$$

$$|1^{+-\prime}\rangle = |1_{bb}, 1_{\bar{b}\,\bar{b}}; J=1\rangle : M = 18.805 \text{ GeV}, \quad (23)$$

$$|2^{++}\rangle = |1_{bb}, 1_{\bar{b}\,\bar{b}}; J=2\rangle : M = 18.920 \text{ GeV}, \quad (24)$$

for the T_{4b} tetraquark system.

C. The $[bc][\bar{b}\,\bar{c}]$ system

The final structure analyzed in this work is the $T_{2bc} = [bc][\bar{b}\,\bar{c}]$ tetraquark system. In this case, the $[bc]$ diquark spin can be either 0 or 1 and thus all states analyzed at the beginning of this section are possible. Again, from a transitional theory point of view, the best extraction procedure of the control parameters in the T_{2bc} tetraquarks are the corresponding meson-meson families which deliver the following values: $c_{Q_1} = c_{Q_2} = 1$, and $c_{\bar{Q}_3} = c_{\bar{Q}_4} = 0$. The masses computed in this case can be classified as follows:

(i) The $J^{PC} = 0^{++}$ contains two scalar states with masses

$$|0^{++}\rangle = |0_{bc}, 0_{\bar{b}\,\bar{c}}; J=0\rangle : M = 12.359 \text{ GeV}, \quad (25)$$

$$|0^{++\prime}\rangle = |1_{bc}, 1_{\bar{b}\,\bar{c}}; J=0\rangle : M = 12.503 \text{ GeV}. \quad (26)$$

(ii) The $J^{PC} = 1^{+-}$ contains two states with masses

$$|1^{+-}\rangle = \frac{1}{\sqrt{2}}(|0_{bc}, 1_{\bar{b}\,\bar{c}}; J=1\rangle - |1_{bc}, 0_{\bar{b}\,\bar{c}}; J=1\rangle) : M = 12.896 \text{ GeV}, \quad (27)$$

$$|1^{+-\prime}\rangle = |1_{bc}, 1_{\bar{b}\,\bar{c}}; J=1\rangle : M = 12.016 \text{ GeV}. \quad (28)$$

(iii) The $J^{PC} = 1^{++}$ contains one state with mass

$$|1^{++}\rangle = \frac{1}{\sqrt{2}}(|0_{bc}, 1_{\bar{b}\,\bar{c}}; J=1\rangle + |1_{bc}, 0_{\bar{b}\,\bar{c}}; J=1\rangle) : M = 12.155 \text{ GeV}. \quad (29)$$

(iv) The $J^{PC} = 2^{++}$ contains one state with mass

$$|2^{++}\rangle = |1_{bc}, 1_{\bar{b}\,\bar{c}}; J=2\rangle : M = 12.897 \text{ GeV}. \quad (30)$$

IV. DISCUSSION

In the calculation procedure, we fix the Hamiltonian parameters and allow the phase parameters to vary during the transition. In Ref. [8] we showed that the quantum number of the amount of bosons can be taken in the $N \to \infty$ limit. Moreover, it was adequate to take N large enough to cover all known and unknown states up to a maximum value of the quantum number of the angular momentum, and other quantum numbers related to the applications. In the present investigation, we take this to be the same as that used in Ref. [8] with $N = 100$.

The trend of the Hamiltonian is similar to that of the $O(4)$ limit condition proposed in mesons where the control parameter is taken to be 1. Most importantly, our investigation shows that the control parameters $c_{\bar{Q}_3}$ and $c_{\bar{Q}_4}$ cannot be taken to be 1 when heavy antiquarks are involved except for T_{4b} tetraquarks. This is because the masses of T_{4b} tetraquarks are 2–3 times heavier than T_{2bc} and T_{4c} ones. In this condition for heavy mass tetraquarks, the effect of pairing strength is strong, which can be seen in the fact that c_{Q_1} for T_{2bc} is larger than in the T_{4c} case.

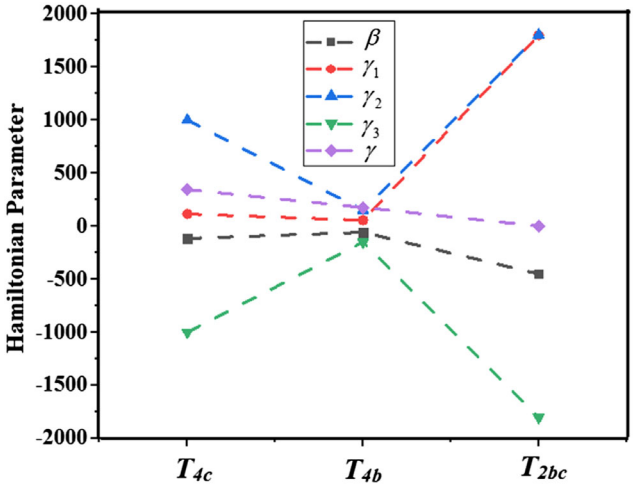

FIG. 2. The resulting parameters of the Hamiltonian when predicting the tetraquark masses based on the diquark-antidiquark pairing model. In the calculation, the effective g-factor is taken to be 1.

TABLE I. Masses of fully-heavy tetraquark systems as computed within the theoretical framework presented herein. The meson-meson threshold is $E_{\rm th}$, and $\Delta = M - E_{\rm th}$ represents the energy distance of the tetraquark with respected its lowest meson-pair threshold. The notation s and a indicates scalar and axial-vector diquarks.

Structure	Configuration	J^{PC}	$M_{\rm tetra}$ in this work (GeV)	Threshold	$E_{\rm th}$ (GeV)	Δ (GeV)
$T_{4c} = [cc][\bar{c}\,\bar{c}]$	$a\bar{a}$	0^{++}	5.978	$\eta_c(1S)\eta_c(1S)$	5.968	0.01
				$J/\psi(1S)J/\psi(1S)$	6.194	−0.216
		1^{+-}	6.155	$\eta_c(1S)J/\psi(1S)$	6.081	0.074
		2^{++}	6.263	$J/\psi(1S)J/\psi(1S)$	6.194	0.069
$T_{4b} = [bb][\bar{b}\,\bar{b}]$	$a\bar{a}$	0^{++}	18.752	$\eta_b(1S)\eta_b(1S)$	18.797	−0.045
				$\Upsilon(1S)\Upsilon(1S)$	18.920	−0.168
		1^{+-}	18.808	$\eta_b(1S)\Upsilon(1S)$	18.859	−0.051
		2^{++}	18.920	$\Upsilon(1S)\Upsilon(1S)$	18.920	0.0
$T_{2bc} = [bc][\bar{b}\,\bar{c}]$	$a\bar{a}$	0^{++}	12.503	$\eta_b(1S)\eta_c(1S)$	12.383	0.12
				$J/\psi(1S)\Upsilon(1S)$	12.557	−0.054
				$B_c^{\pm}B_c^{\mp}$	12.550	−0.047
				$B_c^{*\pm}B_c^{*\mp}$	12.666	−0.163
		1^{+-}	12.016	$\eta_c(1S)\Upsilon(1S)$	12.444	−0.428
				$J/\psi(1S)\eta_b(1S)$	12.496	−0.48
				$B_c^{\pm}B_c^{*\mp}$	12.608	−0.592
				$B_c^{*\pm}B_c^{*\mp}$	12.666	−0.65
		2^{++}	12.897	$J/\psi(1S)\Upsilon(1S)$	12.557	0.34
				$B_c^{*\pm}B_c^{*\mp}$	12.666	0.231
	$\frac{1}{\sqrt{2}}(a\bar{s} \pm s\bar{a})$	1^{++}	12.155	$J/\psi(1S)\Upsilon(1S)$	12.557	−0.402
				$B_c^{\pm}B_c^{*\mp}$	12.608	−0.453
				$B_c^{*\pm}B_c^{*\mp}$	12.666	−0.511
		1^{+-}	12.896	$\eta_c(1S)\Upsilon(1S)$	12.444	0.452
				$J/\psi(1S)\eta_b(1S)$	12.496	0.4
				$B_c^{\pm}B_c^{*\mp}$	12.608	0.288
				$B_c^{*\pm}B_c^{*\mp}$	12.666	0.23
	$s\bar{s}$	0^{++}	12.359	$\eta_c(1S)\eta_b(1S)$	12.383	−0.024
				$J/\psi(1S)\Upsilon(1S)$	12.557	−0.198
				$B_c^{\pm}B_c^{\mp}$	12.550	−0.191
				$B_c^{*\pm}B_c^{*\mp}$	12.666	−0.307

The values of the parameters in the Hamiltonian for the mentioned structures are given in Fig. 2. In the transition region, α is taken to be 1.5. Since the vibrational-rotational transition within the pairing model is a second-order quantum phase transition, the masses of the wave functions in the $U(9)$ model of studied tetraquarks behave smoothly with respect to changes in the parameters, which allows us to fix them in the transition region.

Table I shows the difference between the calculated tetraquark masses and the meson-pair threshold. We present the values of $\Delta = M_{\rm tetra} - E_{\rm th}$, where $M_{\rm tetra}$ and $E_{\rm th}$ are the tetraquark mass and its lowest meson-meson threshold, respectively. A negative Δ indicates that the tetraquark state lies below the threshold of the fall-apart decay into two mesons and consequently should be stable. Besides, a state with a small positive value for the Δ could also be observed as a resonance since the phase space would suppress its partial decay width. The remaining states, with large positive Δ values, are supposed to be broad and challenging to recognize in experimental analyses.

Our analysis confirms that the control parameter c_{Q_1} deviating a little from 1 turns out to be more appropriate in the extraction of the tetraquark masses, specially for comprehensive T_{2bc} families. One can also see that, in the $T_{4b} = [bb][\bar{b}\,\bar{b}]$ states, the higher contribution comes from the pairing of $c_{\bar{Q}_3}$ and $c_{\bar{Q}_4}$ quarks. This means that at high energy, around 18–19 GeV, phase parameters for $\bar{Q}_3$ and $\bar{Q}_4$ quarks begin to play an essential role in computing tetraquark masses; while in the low energy regime, there is a competition between the Q_1 and Q_2.

According to the above definition, it can be claimed that the energy spectra of the studied fully-heavy tetraquarks in which $c_{Q_i} \sim 0.9$–1.0 corresponds to a rotational phase. Note also that a change of $\pm 15\%$ in all coefficients produce a maximum variation of 30%, 23%, 17% in a particular channel's mass of T_{4c}, T_{4b}, and T_{2bc} tetraquark systems, respectively; having that all remaining masses experience lesser modifications.

Finally, the results obtain herein with the pairing model are compared with the prediction of previous theoretical calculations in Tables II and III. One can deduce that the theory fairly reproduces the other works, indicating that our solvable model could still play an essential role in the

TABLE II. Comparison of our results with theoretical predictions for the masses of $T_{4b} = [bb][\bar{b}\,\bar{b}]$, and $T_{4c} = [cc][\bar{c}\,\bar{c}]$ tetraquarks. All results are in GeV.

	$bb\bar{b}\,\bar{b}$			$cc\bar{c}\,\bar{c}$		
Reference	0^{++}	1^{+-}	2^{++}	0^{++}	1^{+-}	2^{++}
This paper	18.752	18.808	18.920	5.978	6.155	6.263
[27]	18.460–18.490	18.320–18.540	18.320–18.530	6.460–6.470	6.370–6.510	6.370–6.510
[57]	18.690	…	…	…	…	…
[58]	18.748	18.828	18.900	5.883	6.120	6.246
[59]	18.750	…	…	<6.140	…	…
[60,61]	18.754	18.808	18.916	5.966	6.051	6.223
[48,62]	18.826	…	18.956	6.192	…	6.429
[63,64]	18.840	18.840	18.850	5.990	6.050	6.090
[65]	19.178	19.226	19.236	…	…	…
[66]	19.237	19.264	19.279	6.314	6.375	6.407
[67]	19.247	19.247	19.249	6.425	6.425	6.432
[68,69]	19.322	19.329	19.341	6.487	6.500	6.524
[49]	19.329	19.373	19.387	6.407	6.463	6.486
[70]	19.255	19.251	19.262	6.542	6.515	6.543
[26]	20.155	20.212	20.243	6.797	6.899	6.956
[71,72]	…	…	…	5.969	6.021	6.115
[73]	…	…	…	6.695	6.528	6.573
[74]	…	…	…	6.480	6.508	6.565
[75]	19.666	19.673	19.680	6.322	6.354	6.385
[76]	…	…	…	6.510	6.600	6.708
[77]	18.981	18.969	19.000	6.271	6.231	6.287
[78]	19.314	19.320	19.330	6.190	6.271	6.367
[50] set. I	18.723	18.738	20.243	5.960	6.009	6.100
[50] set. II	18.754	18.768	18.797	6.198	6.246	6.323
[79]	19.226	19.214	19.232	6.476	6.441	6.475

TABLE III. Comparison of our results with theoretical predictions for the masses of $T_{2bc} = [bc][\bar{b}\,\bar{c}]$ tetraquarks. All results are in GeV.

	$a\bar{a}$			$\frac{1}{\sqrt{2}}(a\bar{s} \pm s\bar{a})$		$s\bar{s}$
Reference	0^{++}	1^{+-}	2^{++}	1^{++}	1^{+-}	0^{++}
This paper	12.503	12.016	12.897	12.155	12.896	12.359
[60]	12359	12424	12566	12485	12488	12471
[58]	12374	12491	12576	12533	12533	12521
[59]	<12620	…	…	…	…	…
[80]	12746	12804	12809	…	12776	…
[49]	12829	12881	12925	…	…	…
[68]	13035	13047	13070	13056	13052	13050
[26]	13483	13520	13590	13510	13592	13553

prediction of fully-heavy tetraquark mesons. In order to do so, a possible improvement is to include the large-N limit of the pure pairing Hamiltonian to gain a better understanding of the multiquark dynamics.

V. SUMMARY

Inspired by the problem of solving the interacting sl-boson system in the transitional region, a solvable extended Hamiltonian that includes multipair interactions has been considered to provide the mass spectra of fully-heavy tetraquarks. Within an algebraic model in which the Bethe *ansatz* is adopted, numerical extractions of T_{4c}, T_{4b}, and T_{2bc} ground state masses were carried out to test the theory. The results reveal that the $U(9) \to SO(10)$ Hamiltonian could predict spectra in fair agreement with other theoretical approaches.

Finally, the solvable technique introduced in this manuscript may also be helpful in diagonalizing more general multiquark systems, which will be considered in future work.

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APPENDIX: ON THE DETERMINATION OF THE MODEL PARAMETERS

We have distinct bases states with different quantum numbers ν_l of $SO(2l+1)$ (seniority numbers) for tetraquarks in which the seniority numbers make an essential role in the Bethe ansatz equation (BAE), Eq. (14). Moreover, after fixing and solving the BAE, we have used the least square fit to get the masses of the tetraquarks. This is because the second order Casimir operator $\hat{C}_2(SO(2l+1)) = \nu_l(\nu_l + 2l - 1)$ is proportional to seniority numbers. In the process of extraction of the parameters in the Hamiltonian for different tetaquarks with different quantum labels of seniority numbers and angular momentum, it is expedited to get different values of coefficients. We have used LSF for each group of tetraquarks, distinctively, considering the fact that three different nuclear Hamiltonians are used. This means that the first two terms of the Hamiltonian correspond to the BAE which are different for each tetraquark. It is clear that due to different masses for tetraquarks, different values of the Hamiltonian parameters would correspond to the different tetraquarks.

A valuable and straightforward numerical algorithm for solving the BAE (14) and extracting the constants in comparison with masses of considered tetraquarks is based on the use of *Mathematica* software which will be outlined simultaneously. To get the roots of the BAE with defined values of seniority numbers, we solve Eq. (14) with fixed values of α for $i = 1$ and then use the function *FindRoot* in *Mathematica* to obtain all roots.

Let us explain the numerical algorithm for fixing the parameters of the Hamiltonian. Eigenenergies and the corresponding wave functions can easily be determined for limiting cases; in global problems, Eq. (14) is a set of nonlinear equations with k unknowns for a k-pair excitation. The method and numerical procedure are related to two steps: (i) solve the non-linear BAEs, and (ii) extract by using LSF. Accordingly, the infinite-dimensional algebraic expansion (12) along with the energy eigenequation defines the functional $S^+(x_i)$ and the possible values that the spectral parameters x_i can take on.

The challenge is that quantities of physical interest are expressed in terms of solutions of the BAE; and the solution of BAE is usually difficult. Many methods have been introduced for extracting solutions from these equations. A typical example for solving the BAE is the Hubbard model. On the other hand, an analysis similar to that used in [13,14] is helpful in understanding the behavior of the solutions. Here, a valuable numerical algorithm for solving the BAE using *Mathematica* will be outlined. Also, because of the S_k symmetry with respect to permutations among the $\{x_1, x_2, \ldots, x_k\}$, in the following we eliminate those solutions that can be obtained by such root permutations, keeping only one since the others correspond to the same eigenenergy and wave function. First, we want to identify various sets of solutions of BAE that exist, excluding those that can be obtained by permutations of the roots of S_k. Suppose that the total number of such solutions is p. The roots in BAE can then be arranged as $x_1^{(\zeta)}, x_2^{(\zeta)}, \ldots, x_k^{(\zeta)}$, with $\zeta = 1, 2, \ldots, p$. Now we apply the pairing theorem to tetraquarks. Each set of roots corresponds to a unique eigenvector. These eigenvectors span a subspace, which is called the diagonalized configuration subspace. For k-pair excitation, besides solutions achieved by root permutations of the S_k, BAE has $k+1$ different sets of solutions, namely $p = k+1$. According to the pairing theorem, a k-pair excitation BAE has $(k+1)!$ solutions, while only $k+1$ of them are solutions to the eigenvalue problem of the tetraquarks.

In what follows, we apply a solid case to prove how to numerically find all roots of BAE. We take the $cc\bar{c}\,\bar{c}$ system with a k-pair excitation as an example, in which the parameters $\nu_{Q_1} = 0$, $\nu_{Q_2} = 1$, $\nu_{\bar{Q}_3} = 1$, $\nu_{\bar{Q}_4=0}$, $\alpha = 1.5$ and $c_{Q_1} = 0.92$, $c_{Q_2} = 1$, $c_{\bar{Q}_3} = 0$, and $c_{\bar{Q}_4} = 0$. Thus, BAE (14) becomes

$$\frac{1.5}{x_i} = \frac{0.42}{1-0.84x_i} + \frac{2.5}{1-x_i} - \sum_{j\neq i}\frac{2}{x_i - x_j} \qquad i = 1, 2, 3, \ldots k. \tag{A1}$$

We employ the notation $[k]$ to express this equation. It can be confirmed that the solutions of $[k+1]$ can be achieved by applying *FindRoot* in *Mathematica* from those for $[k]$. Starting from $[k=1]$, which can be determined by using *Solve* in our code, one can get solutions of $[k=2]$. Consequently, an iterative method can be set up for obtaining solutions of $[m+1]$ from those of $[m]$. Excluding roots that can be obtained from S_k symmetry, all inequivalent roots up to $[k=3]$ are listed in Table IV. It can easily be seen from Table IV that the roots of $[k]$ have the following properties:

TABLE IV. All inequivalent roots up to $[k=3]$. Those roots that can be obtained from S_k symmetry are excluded.

k	N	x_i
1	5	$\begin{cases} x_1^{(1)}=1.8156 \\ x_2^{(1)}=0.2227 \end{cases}$
2	7	$\begin{cases} x_1^{(1)}=1.1745 \\ x_2^{(1)}=1.1699 \end{cases}$
		$\begin{cases} x_1^{(2)}=1.1587 \\ x_2^{(2)}=0.1924 \end{cases}$
		$\begin{cases} x_1^{(3)}=0.1369 \\ x_2^{(3)}=0.1123 \end{cases}$
3	9	$\begin{cases} x_1^{(1)}=1.1250 \\ x_2^{(1)}=1.0913 \\ x_3^{(1)}=1.0533 \end{cases}$
		$\begin{cases} x_1^{(2)}=0.7458 \\ x_2^{(2)}=0.6051 \\ x_3^{(2)}=0.4929 \end{cases}$
		$\begin{cases} x_1^{(3)}=0.4019 \\ x_2^{(3)}=0.2475 \\ x_3^{(3)}=0.1813 \end{cases}$
		$\begin{cases} x_1^{(4)}=0.2070 \\ x_2^{(4)}=0.1908 \\ x_3^{(4)}=0.1406 \end{cases}$

(1) The two roots of $[k=1]$ are very different.
(2) The two roots of $[k=1]$ are fundamental because all other roots of $[k]$ are in the vicinity of these two roots.
(3) Assume that the larger root of $[k=1]$ corresponds to the basis vector Q_1 and others from larger to smaller corresponds to the basis vector Q_2, Q_3, and Q_4, respectively.

Now a is the largest root and b, c and d are smaller roots of $[k]$ corresponding to the basis vector of tetraquarks $Q_1^a Q_2^b Q_3^c Q_4^d$ $(a+b+c+d=k)$ in the k-dimensional diagonalized subspace. These properties can be used to set up a procedure for finding all inequivalent roots of $[k]$ using a solvable code.

While we have seen the exact solution for a particular numerical example, the conclusions apply to the cases with $\alpha > 0$, and $0 < c_Q < 1$ as well. In all of these cases the roots of BAE are real. After getting these roots, we have inserted to the Hamiltonian with considering the seniority numbers and total angular momentum. Here is half of the method for optimizing the set of parameters. A detailed discussion of the numerical results in the Hamiltonian includes carrying out the least square fit in the excitation masses of selected states from tetraquarks. We have repeated these processes with different values of considered quantities to optimize the values.

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