

Holographic tetraquarks and the newly observed T_{cc}^+ at LHCb

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We describe a heavy and exotic tetraquark state as a holographic molecule, by binding the lightest heavy-light meson ($0^-, 1^-$) multiplet to a flavored sphaleron, in the bulk of the Witten-Sakai-Sugimoto model. Bound tetraquark states emerge as Efimov states in the heavy quark limit, with a binding energy for a charm tetraquark comparable to the T_{cc}^+ recently reported by the LHCb collaboration, but with a substantially smaller width, for a large but finite 't Hooft coupling. Fixing the parameters of the model at the empirical mass of T_{cc}^+ , allows for a prediction of the bindings of the undiscovered bottom-charm and bottom tetraquarks.

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I. INTRODUCTION

Hadrons composed of heavy (Q) and light (q) quarks have received considerable interest lately, due to the flurry of results stemming from electron and hadron colliders [1–6]. These hadrons embody in a remarkable way some key aspects of QCD: the spontaneous breaking of chiral symmetry for the light quarks, and a heavy quark spin flip symmetry [7,8]. In the heavy quark mass limit, a heavy hadron with spin up is degenerate with its counterpart with spin down, and the resulting doublets with even and odd parity are chiral partners of each other [9,10]. The result is the chiral doubling phenomenon first observed by the BABAR collaboration [11], and then confirmed by the CLEO collaboration [12]. Chiral doubling of heavy-light hadrons is likely more remarkable with b-quarks at LHCb and BESIII.

The many discoveries by several collaborations have led to renewed interest in exotic heavy-light hadrons. Spectacular discoveries are the $\chi_{c1}(3872)$, directly at the

D^{*0} and \bar{D}^0 threshold (minimal quark content $c\bar{c}q\bar{q}$), the family of Z states $Z_c(3900, 4020, 4050, 4200, 4430)$ (minimal quark content $c\bar{c}u\bar{d}$), some of their strange cousins $Z_{cs}(3985, 4000, 4220)$, and bottom analogues $Z_b(10610, 10650)$. These exotica (tetraquarks) are thought to be bound deuteronlike molecules of the type $(Q\bar{q})(\bar{Q}q)$ [13–20], although alternative explanations have been presented [8,21–25]. In the molecular scenario, the chemical-like bonding is thought to be mediated mostly by pion-exchange, and perhaps at the origin of the newly discovered and exotic charmed baryon-meson molecules of the type $(\bar{Q}q)(Qqq)$ such as the triplet $P_c((4312), (4440), (4457))$ and the newly noticed $P_c(4380)$, (although at the level of $3-4\sigma$ only), all with a minimal, pentaquark content $c\bar{c}uud$. Their strange cousin $P_{cs}(4459)$ has been also discovered. Note that all these states have a hidden heavy flavor.

The recent discovery of doubly heavy baryons of the ccq type suggests that an approximate Savage-Wise symmetry [26] may be at work, where a heavy and compact diquark QQ would be equivalent to a heavy anti-quark \bar{Q} . This heavy diquark-anti-quark supersymmetry allows for mass relations not only between heavy baryons and mesons such as $\bar{Q}q$ and QQq , but also between heavy baryons and tetraquarks with hidden heavy flavor such as Qqq and $\bar{Q}\bar{Q}qq$. The recent quark model estimates for this last state are remarkable [27,28]. It is suggested that for a compact $\bar{b}\bar{b}qq$ tetraquark, the binding energy is significant and about 200 MeV. If confirmed, this would be the first,

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nonmolecular and truly exotic tetraquark state outside the standard quark model classification.

Exploratory lattice QCD simulations appear to support the quark model prediction of a strongly bound b-tetraquark [29]. Given the difficulty to analyze QCD in the confining regime, it is not easy to identify the mechanism at work in the formation of these exotics. In recent years, holography has proven to be a useful framework for discussing QCD for a large number of colors N_c and strong coupling [32–30] λ . For hadrons, the formulation confines and breaks spontaneously chiral symmetry through geometry [33–38]. Its extension to a heavy quark exhibits explicit heavy-quark symmetry [39] (for earlier approaches see [40,41]). Light holographic baryons are instantons in bulk, while heavy holographic baryons are bound states of a heavy meson multiplet to the instanton. This mechanism is the dual of the Callan-Klebanov mechanism [42] in the context of the Skyrme model [43] which we will review for clarity below. For completeness, we note that tetraquarks in the context of a holographic string construction have been discussed in [44,45], and using light cone holographic QCD in [46].

In this work we revisit our recent holographic analysis of the charm and bottom tetraquark states [47], by refining their mass analysis and assessing their strong decay widths. We take advantage of the fact, that the first open charm exotic state has just been discovered. Our results are shown to be compatible, with the newly reported charmed tetraquark T_{cc}^+ by LHCb as shown in Fig. 1, for a large value of the 't Hooft coupling. Using this experimental value, more precise estimate of the masses are given for the mixed charm-bottom and bottom tetraquark states, yet to be observed. We also provide generic arguments, why the width of the T_{cc}^+ is so narrow.

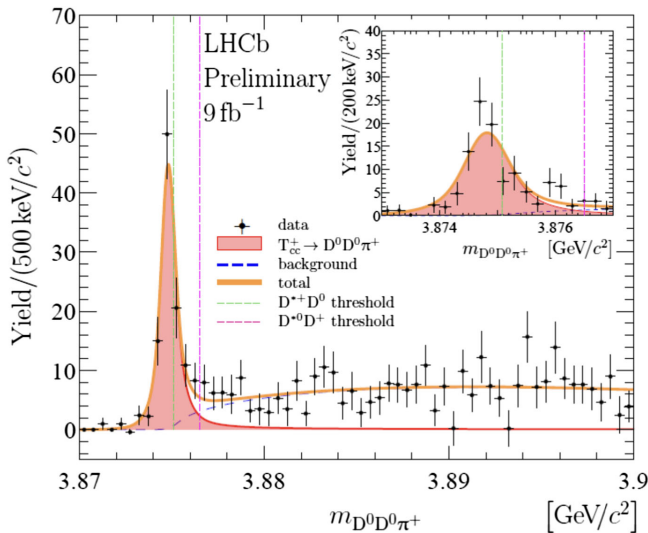


FIG. 1. Newly measured isoscalar charmed T_{cc}^+ tetraquark by LHCb [48].

II. STRANGE SOLITONIC BARYONS

In the large number of colors limit, QCD truncates to an effective theory of weakly coupled mesons where baryons are solitons. The meson effective theory is chiefly chiral, consisting of the light mesons. Once the effective mesonic description is fixed, the baryonic description follows with no new parameters. The soliton is usually characterized by a moduli following from the set of zero modes associated to the classical solution. The quantum numbers of the baryon follows by quantizing the moduli using the so-called collective coordinate method. This construction works well for two light flavors up-down, but when extended to strangeness, the method fails phenomenologically.

Callan and Klebanov [42] argued that the strange mass is somehow large, and therefore a strange quark as a kaon cloud is more likely to bind to the soliton owing to its short Compton wavelength. Specifically, the fast vibrational modes (kaon) do not decouple from the slow rotational moduli (soliton), and generate an effective potential (non-Abelian Berry phase) in the Born-Oppenheimer approximation. As a result, the spin of the rotating soliton is shifted by the isospin of the kaon. This construction fares better phenomenologically.

This construction has been extended to charm and bottom heavy baryons [49,50]. The difference with strangeness though is that the partners of the kaon, i.e., $(0^-, 1^-) = (D(B), D^*(B))$, are degenerate leading to a degenerate baryon multiplet $(\frac{1}{2}^+, \frac{3}{2}^+)$, in the heavy quark limit. Parity doubling suggests a nearby baryon multiplet $(\frac{1}{2}^-, \frac{3}{2}^-)$ by binding to the chiral partners $(0^+, 1^+) = (\tilde{D}(B), \tilde{D}^*(B))$ [51].

III. HOLOGRAPHIC LIGHT BARYONS

In holographic QCD, confinement and chiral symmetry breaking can be addressed simultaneously for instance in the Witten-Sakai-Sugimoto model [33]. Confinement follows from a stack of colored D4-branes, in the double limit of a large number of colors and strong coupling, with the string tension fixed by the apparent horizon. The spontaneous breaking of chiral symmetry arises from the geometrical fusion of a pair of flavored D8 – $\overline{D8}$ -branes in the probe approximation. The model with only two parameters—the brane tension κ and the Kaluza-Klein compactification scale M_{KK} —is in remarkable agreement with phenomenology [52].

Holographic baryons are flavor valued instantons in the probe D8 – $\overline{D8}$ branes. Their topological charge is identified with baryon charge, and their quantization follows from the quantization of the instanton moduli in bulk. Most noteworthy is the fact that the instanton size or equivalently the baryon core is fixed by geometry or equivalently the BPS condition, making it independent of the nature of the mesons retained and/or their derivatives thereby solving a key problem in the Skyrme model.

The quantum moduli for the flavored instanton is the standard $R^4 \times R^4/Z_2$ (flat space) [33]. We focus on R^4/Z_2 which corresponds to the size and global flavor SU(2) orientations, and denote by $y_I = \rho a_I$ the coordinates on R^4/Z_2 , with the SU(2) orientations parametrized by a_I subject to the normalization $a_I^2 = 1$, and ρ the instanton size. Note that the instanton size ρ is a dynamical variable playing the role of the radial coordinate in moduli space. The collective Hamiltonian in polar coordinates on the R^4/Z_2 moduli for the light holographic baryon, is [34]

$$\mathbf{H}_k = -\frac{1}{2m_k} \left(\frac{1}{\rho^3} \partial_\rho^2 \rho^3 + \frac{1}{\rho^2} (\nabla_{S^3}^2 - 2m_k Q(k)) \right) + \frac{1}{2} m_k \omega_k^2 \rho^2 \quad (1)$$

All scales are in units of the KK scale M_{KK} which is set to 1. The $k=1$ labels the instanton path with topological charge 1. The inertial parameters are $m_{k=1} = 16\pi^2 a N_c$, $\omega_{k=1}^2 = \frac{1}{6}$. The charge $Q(k=1) = N_c/(40\pi^2 a)$ with $a = 1/(216\pi^3)$, characterizes the U(1) topological self-repulsion within the instanton. The first two contributions in (1) are the kinetic Laplacian in R^4 , and the last harmonic contribution is the gravitational attraction induced by the warped holographic direction. A detailed derivation of (1) can be found in [34] (see Eq. 5.9) and will not be repeated here.

The eigenstates of (1) are $T_l(a)R_{lm}$, with $T_l(a)$ as spherical harmonics on S^3 with $\nabla^2 T_l = -l(l+2)T_l$.

Under $SO(4) \sim SU(2) \times SU(2)$ they are in the symmetric $(\frac{l}{2}, \frac{l}{2})$ representations, with the two SU(2) identified by the isometry $a_I \rightarrow V_L a_I V_R$. The left factor is the isospin rotation and the right factor is the space rotation with quantum numbers $I = J = \frac{l}{2}$. For instance the proton with spin-up carries a wave function $R(a) \sim (a_1 + ia_2)$ with a rotational tower ($M_1 = 8\pi^2 \kappa$)

$$M = M_1 + \left(\frac{(l+1)^2}{6} + \frac{2}{15} N_c^2 \right)^{\frac{1}{2}} + \frac{2}{\sqrt{6}} \quad (2)$$

IV. HOLOGRAPHIC EXOTIC BARYONS

Recently, two of us extended the holographic approach to the description of heavy-light mesons and baryons with manifest chiral and heavy quark symmetry [39]. Heavy baryons emerge by binding a 5-dimensional $(0^-, 1^-)$ spin-1 multiplet to the flavored instanton in bulk. In the heavy mass limit, the spin-1 meson transmutes to a spin- $\frac{1}{2}$ zero mode, leading to a rich heavy baryon spectrum including exotics, thereby extending the Callan-Klebanov mechanism to holography.

More specifically, the instanton moduli described above is extended to include a spin- $\frac{1}{2}$ Grassmannian χ to account for the spin $1 \rightarrow \frac{1}{2}$ transmutation following the binding. The ensuing moduli for the exotic baryonic molecule follows also from (1) with the general charge

$$Q(k) = \frac{N_c}{40\pi^2 a} \left(q(k) + \frac{\lambda}{m_H} \frac{5\alpha_0(k)}{432\pi} \frac{N_Q}{N_c} + 30\alpha_1(k) \frac{N_Q}{N_c} + 5\alpha_2(k) \frac{N_Q^2}{N_c^2} \right) \quad (3)$$

with $N_Q = \chi^\dagger \chi$. For the instanton $q(1) = 1$ (topological charge) and $\alpha_0(1) = 0$ (self-dual). $\alpha_1(1) = -\frac{1}{8}$ characterizes the magnetic interaction of the heavy multiplet to the instanton, and $\alpha_2(1) = \frac{1}{3}$ captures the U(1) repulsion between the bound heavy mesons. $N_Q = 1, 2, \dots$ counts the number of bound mesons. More details regarding the charge (3) for $k=1$ can be found in [39] [last reference Eq. (40)].

The binding of any number of heavy mesons and anti-mesons follows from the substitution $N_Q \rightarrow N_Q - N_{\bar{Q}}$. In general, the isospin (**I**) and spin (**J**) now decouple, with the identification [39]

$$\mathbf{J} = -\mathbf{I} + \chi_Q^\dagger \mathbf{T} \chi_Q \quad (4)$$

The isospin-spin quantum numbers for the heavy exotic baryons are now shifted

$$IJ \equiv \left(\frac{l}{2}, \frac{l}{2} \right) \rightarrow \left(\frac{l}{2}, \frac{l}{2} \oplus_{i=1}^{N_Q} \frac{1}{2} \right) \quad (5)$$

V. HOLOGRAPHIC HEAVY TETRAQUARK

The predicted tetraquark in the context of the quark model is more challenging to describe using a topological molecular formulation since it is a boson and not a fermion. Here we propose to bind a heavy multiplet $(0^-, 1^-)$ to a sphaleron path as a topological tetraquark molecule, in total correspondence with the heavy holographic baryons described above. In the process quantum numbers get transmuted. This remarkable construction provides a topological realization for the Savage-Wise symmetry [26] whereby a fermion is continuously deformed to a boson along the sphaleron hill, in the holographic dual approach.

With this in mind, we observe that the instanton as an $O(4)$ gauge configuration belongs to a class of tunneling

paths with fixed Chern-Simons number, that cross the sphaleron hill, with the instanton at the bottom and the sphaleron at the top. These configurations are given by periodic elliptic functions that solve the same Yang-Mills equation with maximal $O(4)$ symmetry, with a tunneling period fixed by a parameter k [53–55]. For $k = 1$ the period is infinite and the solution is an instanton with Chern-Simons or topological charge 1, and for $k = 0$ the period is finite and the solution is a sphaleron with Chern-Simons $\frac{1}{2}$ [47].

The exact form of this family of solutions and their period will not be necessary for the rest of the paper as only the values of the parameters entering the charge (3) for $k = 0$ (sphaleron path) are needed, i.e., $\alpha_{0,1,2}(0) \approx (+6, -0.034, +0.165)$. The topological charge $q(0) = 0$, i.e., the sphaleron carries zero baryon number. It is a boson. The ratio of the sphaleron mass M_0 to the instanton mass M_1 is $M_0/M_1 = 3\pi/8\sqrt{2} = 0.83$. More details regarding this construction are presented in [47].

The explicit tetraquark states can now be obtained by seeking the eigenstates of (1) for $k = 0$. Specifically, the radial equation for the reduced wave function $R_{nl} = u_{nl}/\rho^{\frac{3}{2}}$ following from (1) after inserting (3), reads ($m_0/m_1 = M_0/M_1$)

$$-u_{nl}'' + \frac{g_l(0)}{\rho^2} u_{nl} + (m_0\omega_0\rho)^2 u_{nl} = e_{0,nl} u_{nl} \quad (6)$$

with the charge $g_l(0) = l(l+2) + 2m_0Q(0)$. The energies are $e_{0,nl} = 2m_0(E_{0,nl} - M_0 - N_Q m_H)$, with the binding energies as

$$\Delta_{nl}(0) = E_{0,nl} - N_Q m_H \quad (7)$$

The $1/\rho^2$ potential stems from the kinematical centrifugation plus the repulsion from the $U(1)$ charge at the sphaleron point, and is dominant at small distances.

The parameters λ, m_H, M_1 are all fixed in the holographic heavy baryon sector with $N_c = 3$ [39]. A numerical analysis shows that only for $l = 0$, the $N_Q \leq 3$ states are bound, i.e., open-flavor tetraquark $QQ\bar{q}\bar{q}$. The S-wave tetraquark states $QQ\bar{q}\bar{q}$ carry $IJ = 00, 01$ assignments and are degenerate. Heavier exotics are discussed more thoroughly in [47].

VI. EFIMOV STATES

For small distances and S-waves, (6) reduces to

$$-u_{n0}'' + \frac{g_0(0)}{\rho^2} u_{n0} \approx e_{0,n0} u_{n0} \quad (8)$$

For $g_0(0) + \frac{1}{4} < 0$, the potential in (8) is singular but attractive and leads *a priori* to infinitely many bound states, due to the conformal or scale invariance. The

quantization condition converts this continuous symmetry into a discrete one—the states accumulate at the rate

$$\frac{e_{0,(n+1)0}}{e_{0,n0}} = e^{-\frac{2\pi}{\nu_0}} \quad (9)$$

with $\nu_0 = \sqrt{-1/4 - g_0(0)}$. This is the essence of the Efimov phenomenon [56,57].

Historically, the Efimov effect originates from the Borromean effect, which allows binding of a three-body state even though the two-body state is unbound. The Efimov equation is usually written in configuration space, and the binding depends on the sign of the potential $V(R) = \frac{1}{R^2}(s_n^2 - 1/4)$, where R is a hyperspherical coordinate stemming from the Jacobi variables for the three-body problem [57] (see Eq. 2.32). It is remarkable that a similar equation appears in a holographic description of an exotic hadron, especially that the physical origin of the $1/\rho^2$ term is different—here it comes from the $U(1)$ Coulomb law in $1 + 4$ dimensions. For the details of the renormalization of the equation for the Efimov states we refer to [47], and here we only state the main results.

Numerically, the minimal value $\nu_0 \approx \frac{6}{5}$ occurs on the sphaleron path, for $N_c = 3, N_Q = 2$ and $m_H \rightarrow \infty$. The binding energies for $QQ\bar{q}\bar{q}$ depend on the strong 't Hooft coupling λ as listed in Table I for $M_{KK} = 1$ GeV [47], and in Table II for $M_{KK} = 0.475$ GeV [58,59]. The explicit dependence of the binding energy versus λ is given in Figs 2 and 3, for both values of considered values of M_{KK} , respectively. For both values of M_{KK} the binding energy of the charm tetraquark is few MeV for $\lambda \sim 30$. Since $e^{-2\pi/\nu_0} \approx 10^{-3}$, (9) shows that the radially excited states

TABLE I. Binding energies for tetraquarks versus the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ with $M_{KK} = 1$ GeV.

λ	$QQ\bar{q}\bar{q}$ (GeV)	$bb\bar{q}\bar{q}$ (GeV)	$bc\bar{q}\bar{q}$ (GeV)	$cc\bar{q}\bar{q}$ (GeV)
10	-0.097	-0.088	-0.080	-0.072
15	-0.107	-0.091	-0.077	-0.062
20	-0.108	-0.085	-0.064	-0.041
25	-0.103	-0.073	-0.045	-0.018
30	-0.093	-0.056	-0.024	-0.0016
32	-0.089	-0.048	-0.015	0.00073

TABLE II. Binding energies for tetraquarks versus the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ with $M_{KK} = 0.475$ GeV.

λ	$QQ\bar{q}\bar{q}$ (GeV)	$bb\bar{q}\bar{q}$ (GeV)	$bc\bar{q}\bar{q}$ (GeV)	$cc\bar{q}\bar{q}$ (GeV)
10	-0.046	-0.044	-0.042	-0.040
15	-0.051	-0.047	-0.044	-0.040
20	-0.051	-0.046	-0.040	-0.035
25	-0.049	-0.042	-0.035	-0.028
30	-0.045	-0.035	-0.027	-0.019
40	-0.031	-0.018	-0.0076	0.0011

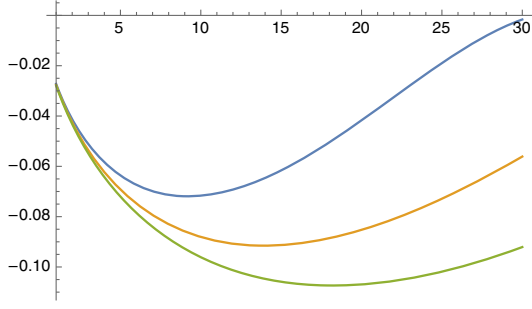


FIG. 2. Tetraquarks binding energies in GeV as a function of the 't Hooft coupling λ for $M_{KK} = 1$ GeV: The upper-blue curve is for T_{cc} , the middle-orange curve is for T_{bb} and the lower-green curve is for infinitely heavy quarks.

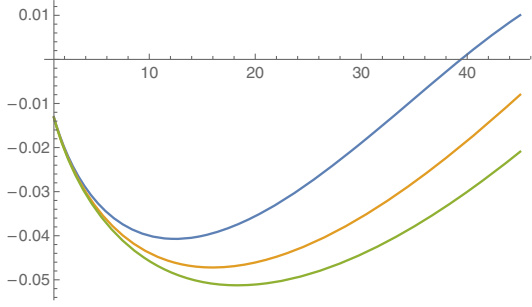


FIG. 3. Tetraquarks binding energies in GeV as a function of the 't Hooft coupling λ for $M_{KK} = 0.475$ GeV: The upper-blue curve is for T_{cc} , the middle-orange curve is for T_{bb} and the lower-green curve is for infinitely heavy quarks.

rapidly unbind. The leading λ/m_H heavy mass correction in (3) is repulsive, and penalizes the binding of $cc\bar{q}\bar{q}$ more than $bb\bar{q}\bar{q}$.

$$[T_{cc}(3880)(01^+)] \rightarrow [\text{Core}(M_0)[00^+]] + \left[D(1870)\frac{1}{2}0^- \right] + \left[D^*(2010)\frac{1}{2}1^- \right] \quad (11)$$

The unbound isoscalar-scalar flavor sphaleron core $[00^+]$ is unstable and decays subsequently to multipions, say minimally to two pions as illustrated in Fig. 4. Formally, the decay width is

$$\Gamma_T(S) = \frac{3g_H^2 V^2}{(2S+1)} (2\pi)^4 \int d\Phi_3 |\mathcal{A}(\vec{k}, \vec{p})|^2, \quad (12)$$

with $g_H \sim 0.67$, the $D^\dagger \partial_i \pi D_i$ coupling [60]. Here S is the total spin of the QQ system, and we have used the condition $l = 0$. We now estimate the 3-body phase space $d\Phi_3$, the decay amplitude \mathcal{A} , and the coupling V to the sphaleron core $[00^+] \rightarrow \pi\pi$.

The first part of the decay process $T_{QQ} \rightarrow [00^+] + D + D^*$ is described by the bulk Chern-Simons term [59]

$$-\frac{i}{16\pi^2} \epsilon_{MNPQ} \Phi_M^\dagger \Phi_N \Phi_P^\dagger \partial_i \Phi_Q + \text{c.c.}, \quad (13)$$

Theoretical predictions for the charmed tetraquark $cc\bar{q}\bar{q}$ are not concise—they vary from binding at the level of 200–300 MeV to unbinding with 200–300 MeV surplus. Lattice and phenomenological estimates suggest that the double-bottom tetraquark state is deeply bound with $\Delta_{BB} = -(0.15-0.2)$ GeV [29] (lattice) and $\Delta_{BB} = -(0.17)$ GeV [28] (quark model). The same lattice analysis suggests that the mixed charm-bottom tetraquark state is bound $\Delta_{CB} = -(0.061-0.015)$ GeV, but the double-charm tetraquark state is not [29]. Our holographic results support doubly charmed state, provided we fix the value of 't Hooft coupling at $\lambda \sim 30$, and allows us to make predictions for binding for bottom and mixed bottom-charm states (see Tables I and II for precise numbers).

VII. DECAY WIDTH

Recently LHCb has reported a narrow tetraquark T_{cc}^+ with a quark content $cc\bar{u}\bar{d}$ and isospin-spin-parity assignment (01^+) [48]. This is consistent with the holographic prediction of bound and degenerate charm tetraquark states $(00^+, 01^+)$ in the heavy quark limit and a strong 't Hooft coupling. The empirical binding energy Δ_{CC} and width Γ_{CC} are relatively small and narrow as illustrated in Fig 1, with

$$\begin{aligned} \Delta_{CC}(T_{cc}^+) &= -360 \pm 40 \text{ keV} \\ \Gamma_{CC}(T_{cc}^+) &= +48 \pm 2 \text{ keV} \end{aligned} \quad (10)$$

The holographic tetraquarks is a bound heavy-light vector multiplet $[0^-, 1^-]$ to a flavor sphaleron core in bulk with (00^+) assignment. This is the holographic dual to a molecule composed of heavy-light $[D, D^*]$ mesons strongly bound by light meson exchanges on the boundary. The strong decay mode of this molecule is natural through

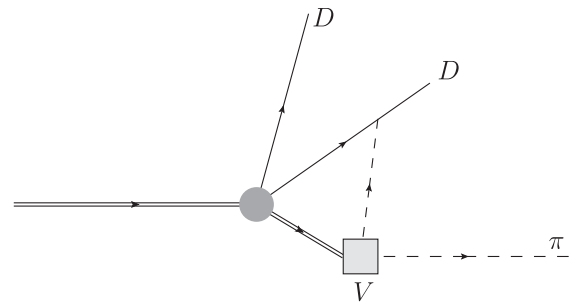


FIG. 4. The holographic tetraquark T_{QQ} decay process takes place in two steps: first into core + $D + D^*$, then into $D + D + \pi$ when the unstable flavor sphaleron core decays core $\rightarrow \pi + \pi$, following the recombination $D^* + \pi \rightarrow D$. The incoming-double line denotes the tetraquark T_{QQ} , the outgoing double line the core, the solid lines denote the out-going heavy-mesons and the dotted lines the pions.

in leading order in λ and m_H . It is of order $1/m_H$ since $\Phi_M \rightarrow \Phi_M/\sqrt{m_H}$, and $\partial_t \Phi_M \rightarrow -im_H \Phi_M$ in the heavy quark limit [39]. The remaining contributions are suppressed by $1/m_H^2$. In (13) two of the heavy-light

fields are valued in the moduli, while the remaining two fields give rise to the decaying heavy-light mesons. More specifically, using the results in [39], (13) gives

$$\frac{m_H}{8\pi^2} \left(\frac{1}{\sqrt{16m_H a N_c}} \right)^2 \phi_n^2(Z) f_{k=0}^2(X - \bar{X}, Z - \bar{Z}) u_{Q,s}^\dagger \vec{\sigma} \epsilon \cdot (u_{Q,s'}^\dagger \vec{\sigma} \times \vec{p} \tau^a \epsilon') a_{Q,s}^\dagger a_{Q,s'}^\dagger + \text{c.c.}, \quad (14)$$

with $a = 1/(216\pi^3)$. Here $\phi_n(Z)$ is the wave function for the heavy-light mesons [39]

$$\phi_n(Z) = \frac{1}{\sqrt{2\kappa}} \frac{1}{\sqrt{2^n n!}} \left(\frac{\sqrt{2}m_H}{2\pi} \right)^{\frac{1}{4}} e^{-\frac{\sqrt{2}Z^2}{4}} H_n(\tilde{Z}) \quad (15)$$

with $\tilde{Z} = \sqrt{m_H} Z$, and

$$f_{k=0}^2(X, Z) = \frac{\rho^3}{(X^2 + Z^2)^{\frac{3}{2}}}, \quad (16)$$

the profile function of the fermionic zero mode at the sphaleron point with $k = 0$. In (14), the collective sphaleron positions are (\bar{X}, \bar{Z}) , ϵ and ϵ' are the polarization-isospin vectors for the heavy-light doublet $(0^-, 1^-)$, and $a_{Q,s}^\dagger$ is the creation-operator for the heavy-quarks in the bound state.

The second part of the decay process stems from the produced sphaleron core which is unstable once the heavy quarks are released (recall that stability follows from heavy quark binding). As a result the isoscalar-scalar flavor core decays through $[00^+] \rightarrow \pi + \pi$, for a core mass $M_0/M_1 = 3\pi/8\sqrt{2} = 0.83$. This decay is captured by the

standard chiral Lagrangian at the boundary, with the sphaleron core with $k = 0$, described by the monodromy along the holographic direction [33]

$$U_{k=0}(x) = e^{i \int_0^\infty dz A_z(\vec{x}, z; k=0)} = e^{i\pi \vec{\tau} \cdot \hat{x}}. \quad (17)$$

The minimal $\pi\pi$ -coupling to the monodromy is through the chiral symmetry breaking term in the standard chiral Lagrangian, with $V_{\pi\pi} \sim m_\pi^2$, hence $V \sim \sqrt{2}M_0 V_{\pi\pi}$. This is an estimate, since the breaking of chiral symmetry in bulk is expected to modify the monodromy for the sphaleron at the boundary, away from the chiral limit (much like for the instanton at $k = 1$ [33]).

Finally, combining the two decay processes, and using the identity

$$\sum_M |u_{Q,s}^\dagger \vec{\sigma} \epsilon \cdot (u_{Q,s'}^\dagger \vec{\sigma} \times \vec{a} \tau^a \epsilon') C_{ss'}^{SM}|^2 = 6|\vec{a}|^2, \quad (18)$$

with the spin-isospin modular wave functions $T_l(a_4, \vec{a})$ defined in [47], the spin-isospin averaged squared decay amplitude for the total process shown in Fig. 4, reads

$$|\mathcal{M}(T_{Q0} \rightarrow D(k) + D(k') + \pi(p))|^2 = \frac{g_H^2 V^2}{2(2S+1)} \times 12M_T \times |\mathcal{A}(\vec{k}, \vec{p})|^2 \quad (19)$$

with

$$\begin{aligned} \mathcal{A}(\vec{k}, \vec{p}) &= \int \frac{d^4 p'}{(2\pi)^4} \frac{F(\vec{p}' + \vec{p})}{128\pi^2 a N_c} \frac{\vec{p}'}{(k-p')^2 - m_{D^*}^2 + i0} \frac{1}{(p')^2 - m_\pi^2 + i0} \equiv \frac{\lambda\sqrt{\pi}}{64\pi\kappa} \langle N_Q = 2 | \rho^3 | N_Q = 0 \rangle \\ &\times \int \frac{d^3 \vec{p}'}{4(2\pi)^3 E_{\vec{p}'} E_{\vec{k}-\vec{p}'}} e^{\frac{1}{8}\sqrt{\frac{3}{2}}(\vec{p}+\vec{p}')^2} K_0\left(\frac{1}{8}\sqrt{\frac{3}{2}}(\vec{p}+\vec{p}')^2\right) \left(\frac{\vec{p}'}{E_{\vec{k}} - E_{\vec{p}'} - E_{\vec{k}-\vec{p}'}} - \frac{\vec{p}'}{E_{\vec{k}} + E_{\vec{p}'} + E_{\vec{k}-\vec{p}'}} \right), \quad (20) \end{aligned}$$

and $\kappa = \lambda a N_c$ in units of M_{KK} . Here $K_0(x)$ is a modified Bessel function. The momentum dependence of the transition form factor is independent of the modular wave function in the ρ -direction. Also the formfactor decays as $1/|p|$ at large p , which is sufficient for the integral to converge, both at large \vec{p}' and $\vec{p}' = -\vec{p}$. The ensuing decay width Γ_T is given by (12), with the three-body phase space measure

$$d\Phi_3 = \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} \frac{\delta(M_T - E_{\vec{k}} - E_{\vec{p}} - E_{\vec{k}+\vec{p}})}{(2\pi)^3 2E_{\vec{k}+\vec{p}}}. \quad (21)$$

For an estimate of the amplitude, let us consider the special point in the phase space where the pion in the final state is at rest, or $\vec{p} = 0$. In this case the momentum of the D mesons are opposite in direction and equal in absolute

value: $|k| = |k'| = 8.20$ MeV, assuming that the tetraquark mass $M_T = 3880$ MeV and the pion mass is $m_\pi = 139$ MeV. Furthermore, since the mass of the D^* meson is much larger than the pion and the final state

momentum, and that the pion mass is also much larger than the final state momentum, the momentum \vec{p}' can be approximated by $\frac{m_{D^*}}{m_{D^*} + m_\pi} \vec{k} \approx \vec{k}$. Therefore,

$$\int \frac{d^3 \vec{p}' e^{\frac{1}{2} \sqrt{3} (\vec{p}')^2}}{4(2\pi)^3 E_{\vec{p}'} E_{\vec{k}-\vec{p}'}} K_0 \left(\frac{1}{8} \sqrt{\frac{3}{2}} (\vec{p}')^2 \right) \left(\frac{\vec{p}'}{E_{\vec{k}} - E_{\vec{p}'} - E_{\vec{k}-\vec{p}'}} - \frac{\vec{p}'}{E_{\vec{k}} + E_{\vec{p}'} + E_{\vec{k}-\vec{p}'}} \right) = -\frac{m_{D^*} \vec{k}}{m_{D^*} + m_\pi} \frac{6.2}{4(2\pi)^2} \approx -0.4 \vec{k}, \quad (22)$$

which is about 3.2 MeV in magnitude. \mathcal{A} is about constant, and the phase space volume is then generic

$$(2\pi)^4 \int d\Phi_3 = \frac{0.156}{32\pi^3} \text{ MeV}^2 \quad (23)$$

which amounts to the following contributions to the decay width

$$\Gamma_T(S) \approx \left[\frac{3g_H^2 V^2}{2S+1} \right] \times [\langle 2|\rho^3|0 \rangle]^2 \times \left[\frac{0.4|\vec{k}|\sqrt{\pi}}{64\pi\kappa/\lambda} \right]^2 \times \left[\frac{0.156}{32\pi^3} \text{ MeV}^2 \right]. \quad (24)$$

The first bracket is from the decay couplings, the second bracket originates from the modular transition vertex, the third bracket is from the loop integral, and the last bracket is from the integration over the final phase space. $\kappa = a\lambda N_c$ is fixed by the nucleon mass $M_1 = 8\pi^2 \kappa \equiv M_N$ in units of $M_{KK} = m_v/\sqrt{0.67} \sim 1$ GeV [33], where m_v is a mass of the vector meson ρ . An estimate for the ρ^3 -modular transition matrix element is subtle, since the $|0\rangle$ solution of the Efimov equation is singular and depends on the cutoff [47]. With this in mind, we can estimate the modular transition matrix element as

$$\langle 2|\rho^3|0 \rangle \sim (\xi \langle 2|\rho|2 \rangle)^3 \sim \xi^3 \left(\frac{1}{m_0 \omega_1} \right)^{\frac{3}{2}} \quad (25)$$

where the last equality is set by the range of the conformal potential in [47]. Here ξ sets the range of our estimate, with $(\lambda m_0/2)/M_1 = M_0/M_1 = 3\pi/8\sqrt{2} = 0.83$, and $\omega_1 = 1/\sqrt{6}$ also in units of M_{KK} . Hence

$$\left(\frac{0.4|\vec{k}|\sqrt{\pi}}{64\pi\kappa} \right)^2 \approx 4.9 \times 10^{-6} \langle 2|\rho^3|0 \rangle^2 = \xi^6 \lambda^3 \times 3.18 \text{ GeV}^{-6} \quad V^2 = M_0 \times 7.68 \times 10^{-4} \text{ GeV}^4 \quad (26)$$

which amounts to a relatively small width

$$\Gamma_T(S) = \frac{3\lambda^5 g_H^2}{2S+1} \times M_0 \times \xi^6 \times 1.88 \times 10^{-18} \sim \frac{1 \text{ keV}}{2S+1} \quad (27)$$

with $\xi \sim 5$, $M_0/M_N = 0.83$, $g_H \sim 0.67$ and $\lambda \sim 30$. Note that increasing the value of the parameter ξ by 50% increases the width by an order of magnitude. The main observation is that for any ‘‘natural’’ value of ξ , e.g., not exceeding 10, the smallness of the observed width stems mostly from the very small phase space, combined with the additional suppression of the core due to the chiral limit.

In sum, the final numerical width is very small with $\Gamma_T(S) \approx 1 \text{ keV}/(2S+1)$. Our results should be considered as an estimate, and not an absolute prediction, taking into account the large sensitivity of the decay width to the modular transition matrix element, and the estimate for the chiral coupling V . However, the qualitative smallness of the width is generic in our analysis, and results from the very

small available phase-space, and the suppression of the core decay constant V in the chiral limit. The holographic tetraquark spectrum to order λ/m_H in (3), does not discriminate between the intrinsic heavy quark spin $S = 0, 1$, making the tetraquark assignments $(00^+, 01^+)$ degenerate by heavy quark symmetry. The degeneracy is lifted by spin-orbit interactions as in [58] for pentaquarks.

However, we note that in our original work on heavy-baryons, the zero-mode moduli of the heavy-meson field was quantized as a fermion, mostly due to the fact that the leading order Lagrangian in $1/m_H$ is linear in time-derivative. On the other hand, at subleading order, there are quadratic terms in time derivatives, supporting an alternative quantization as a boson. In the Appendix, we

show how this is implemented for $k = 1$ (instanton point), as it carries verbatim for $k = 0$ (sphaleron point). Most notably, the tetraquark spectrum remains unchanged at quadratic order (the equation of motion remains the same linear equation in the two cases). The decay width assessment is also unchanged at this order. However, the quantization of the χ -moduli as a *boson*, eliminates the (00^+) tetraquark state (intrinsic spin $S = 0$ antisymmetric state), leaving only a single and non-degenerate (01^+) tetraquark state (intrinsic spin $S = 1$ symmetric state). This alternative quantization scheme within holography, appears to be favored by the current experimental reporting of a single and nondegenerate T_{cc}^+ state by LHCb.

VIII. DISCUSSIONS AND CONCLUSIONS

We have suggested that a heavy and strongly coupled tetraquark emerges in holography as an Efimov state by binding a heavy meson multiplet $(0^-, 1^-)$ to a sphaleron path in a $D8 - \overline{D8}$, with quantum numbers $(00^+, 01^+)$ (fermionic moduli) or (01^+) (bosonic moduli). For a charmed tetraquark the small binding appears to be consistent with the recently measured tetraquark mass for a strong 't Hooft coupling with $\lambda \sim 30$. However, its decay width is small, mostly due to the smallness of the available phase space, and the suppression of the remaining core decay constant V to pions in the chiral limit. The single and narrow T_{cc}^+ state recently reported by LHCb is compatible with the (01^+) holographic pentaquark (bosonic moduli).

In our construction, the tetraquark binding mechanism is the holographic dual of the Callan-Klebanov mechanism, albeit for heavier mesons around a topological configuration with fractional Chern-Simons number. We have also found the geometrical analogue of the Savage-Wise “supersymmetry” between a heavy antiquark, and a heavy diquark formulated in quark models.

The Efimov effect requires that the modulus of the scattering wave, is much larger than the range for asymptotically weak or power like decaying potentials. In real physical systems, the infinite Efimov series truncates to few terms. Actually, the experimental confirmation of the

longer hierarchy of states in the Efimov effect was possible only after the discovery of artificial quantum systems on optical lattices, where one can control the range and scattering length through external parameters [61]. The Efimov “window” in our case, is very narrow too. It is limited by the size of the heavy meson Compton wavelength in relation to the bound state width controlled by the binding energy. The exponential penalty factor suggests at most two bound states, and most probably one, with a typical binding of order few MeV for charm tetraquarks for $\lambda \sim 30$.

Our holographic tetraquarks are different from the molecules mediated by pion exchange (deuson with zero heavy flavor) or baryon-antibaryon states (baryonium) and if also discovered in the bottom sector, will provide the first evidence of a nonconventional, strongly bound cluster different either from a standard meson or a baryon. Our conclusion is in line with similar recent claims [28], but the present description is less restrictive (comparing to the quark models) when it comes to the spin and parity assignment. The reason is that in our case, the fused heavy quarks are still very strongly correlated with the light flavor degrees of freedom. Needless to say that the holographic construction is predictive and therefore falsifiable.

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APPENDIX: BOSONIC QUANTIZATION OF THE MODULI

In this appendix we consider an alternative quantization of the χ -moduli for $k = 1$ (instanton point). The same reasoning holds for $k = 0$ (sphaleron point). To order $1/m_H$, the full Lagrangian for the quadratic χ -moduli, can be read from Eqs. (A7) and (A21) in [58],

$$\mathcal{L} = \frac{1}{2m_H} \dot{\chi}^\dagger \dot{\chi} + i \left(1 + \frac{3}{2m_H m_y \rho^2} \right) \chi^\dagger \dot{\chi} + \frac{3}{2m_y \rho^2} \chi^\dagger \chi + \frac{49}{40m_H m_y^2 \rho^4} \chi^\dagger \chi + \frac{33i}{40m_H m_y \rho^2} \chi^\dagger \tau^a \chi \chi^a - \frac{37 + 12 \frac{Z^2}{\rho^2}}{192m_H} \chi^\dagger \chi. \quad (\text{A1})$$

with all notations defined therein. By the replacement $\chi \rightarrow e^{im_H t} \sqrt{m_H}$, (A1) simplifies

$$\mathcal{L} = \frac{1}{2} \dot{\chi}^\dagger \dot{\chi} + \frac{3i}{2m_y \rho^2} \chi^\dagger \dot{\chi} - \frac{m_H^2}{2} \chi^\dagger \chi + \left(\frac{49}{40m_y^2 \rho^4} - \frac{37 + 12 \frac{Z^2}{\rho^2}}{192} \right) \chi^\dagger \chi + \frac{29i}{40m_y \rho^2} \chi^\dagger \tau^a \chi \chi^a. \quad (\text{A2})$$

This can be interpreted as a system of two harmonic oscillators, in a ρ dependent background magnetic field, coupled by the spin-orbit term. To proceed, we define

$$\chi = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}, \quad (\text{A3})$$

and (A2) can be rearranged

$$\mathcal{L} = \frac{1}{2}(\dot{\vec{x}}_1^2 + \dot{\vec{x}}_2^2) + \frac{3}{2m_y \rho^2}(y_1 \dot{x}_1 - x_1 \dot{y}_1 + y_2 \dot{x}_2 - x_2 \dot{y}_2) - \frac{m_H^2 + \Omega^2(\rho)}{2}(\vec{x}_1^2 + \vec{x}_2^2) + \text{Spin-Orbit}. \quad (\text{A4})$$

where $\vec{x}_1 = (x_1, y_1)$, $\vec{x}_2 = (x_2, y_2)$ and

$$\Omega^2(\rho) = -\frac{49}{20m_y^2 \rho^4} + \frac{37 + 12 \frac{Z^2}{\rho^2}}{96}. \quad (\text{A5})$$

Here we quantize the theory in the Born-Oppenheimer approximation. We fix the modular coordinates y_l and Z , and first quantize the \vec{x}_1 and \vec{x}_2 coordinates. This can be justified in the heavy quark limit with m_H large, where χ is fast-moving at frequency m_H , while the other degrees of freedom are in slow motion with a typical frequency $\omega_y = \frac{1}{\sqrt{6}} M_{KK}$.

For the $l = 0$ state, the spin-orbit coupling vanishes. In this case \vec{x}_1 and \vec{x}_2 decouple, and we have two identical harmonic oscillators in the background field

$$\vec{A} = \omega_c(y, -x), \quad \omega_c = \frac{3}{2m_y \rho^2}. \quad (\text{A6})$$

The spectrum of this system is readily found

$$E = \left(n_+ + \frac{1}{2}\right)\Omega_+ + \left(n_- + \frac{1}{2}\right)\Omega_-, \quad (\text{A7})$$

with

$$\Omega_{\pm} = \sqrt{m_H^2 + \Omega^2 + \omega_c^2} \pm \omega_c. \quad (\text{A8})$$

At large m_H , one has

$$\Omega_{\pm} = m_H \pm \omega_c + \frac{\Omega^2(\rho) + \omega_c^2}{2m_H} + \mathcal{O}\left(\frac{1}{m_H^2}\right). \quad (\text{A9})$$

Clearly, the \pm solutions can be interpreted as *bosonic particle and antiparticles*. To leading order in $1/m_H$, the two frequencies are identical to the energies following from the fermionic quantization of χ . To be more explicit, one has

$$\Omega_{\pm} - m_H = \pm \frac{3}{2m_y \rho^2} - \frac{1}{10m_H m_y^2 \rho^4} + \frac{37 + \frac{Z^2}{\rho^2}}{192m_H} \quad (\text{A10})$$

and the leading order result $\frac{3}{2m_y \rho^2}$ is simply the coefficient of the quadratic term of the fermionic Lagrangian.

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