Almost relevant corrections for direct measurements of electron's g factor

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We revisit the observable used for the direct measurements of the electron's g factor. This is done by considering the subleading effects of the large magnetic background field and virtual Standard Model processes. We find substantial corrections to the Landau levels of the electron. Implications for the observed magnetic moment and the tension between direct and indirect measurement are discussed.

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I. INTRODUCTION

A. Electron $g-2$ in a nutshell

Quantum electrodynamics (QED) is one of the cornerstones of our understanding of the electron, its properties, and interactions [1,2]. This theory allows performance of the interplay between theoretical prediction and experimental confirmation to unprecedented precision. The anomalous magnetic moment of the electron plays a pivotal role in this ongoing success.

Schwinger [3] and Luttinger [4] made the first theoretical calculations suggesting that the magnetic moment of the electron should differ from two. Over the decades, these calculations became more and more precise [5–7], such that they currently include even tenth order in perturbation theory (for a review see [8]). For recent developments concerning the anomalous magnetic moment of the muon, see [9].

Such an amazing effort on the theoretical side only makes sense, if the experimental observations have a precision that is comparable to the theoretical uncertainty. Since the first experimental observation of $g \neq 2$ [10,11], the measurements have been continuously improved. Currently, the most precise direct measurement of the electron's g factor goes down to 13 significant digits [12–14]. These experiments, which will be discussed in more detail below, use Penning traps. A comparison between theory and experiment at such precision does not only allow for better testing of QED, but is an excellent tool to test the Standard Model of particle physics (SM) and to constrain models that go beyond the Standard Model of particle physics (BSM) [15–20].

These direct measurements can also be complemented by measurements of the fine structure constant α, which, by virtue of the theoretical predictions, allow inferring the value of g [21,22]. Three years ago, such a comparison showed a significant tension of 2.4 standard deviations between the directly measured and the inferred magnetic moment [23]. This triggered new hope for the search of physics beyond the Standard Model. Currently, the most prominent examples of these models, which are capable of resolving the tension, are given in [24–33]. However, last year a new experiment gave a discrepancy pointing in the opposite direction, but with a smaller absolute value, such that one cannot speak of a significant tension anymore [34].

In this paper, we revisit the observable that is used to determine the electron's anomalous magnetic moment in the most precise direct measurements [12–14]. The paper is organized as follows. In Sec. I B, we give an idealized and thus simplified summary of the key observable. We then motivate the study by discussing and comparing the orders of magnitude involved in this problem in Sec. I C. Section II uses an effective Lagrangian approach. It contains four parts in which the path from Lagrangian corrections to changes in the observable is described. In Sec. III, these findings are then applied and discussed in the context of a comparison between direct and indirect measurements of $(g-2)$. A summary and conclusion is

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given in Sec. IV. Recently, we noticed the preprint [35] also considers the effect of a finite magnetic field. It, however, mainly discusses the implications for the anomalous magnetic moment of the muon.

B. Idealized experimental setting

The motion of an electron in a real Penning trap has four eigenfrequencies, which are the spin, cyclotron, axial, and magnetron frequency. For the most part of this paper, it is, however, sufficient to consider the idealized case, where electric field and magnetron effects are absent.

The magnetic field strength, used in this experiment, can be inferred from the measured cyclotron frequency of about $\nu_c = \omega/(2\pi) \approx 149$ GHz [12],

$$|B| = \frac{2\pi\nu_c m}{e_0} \approx 5.3\ T. \tag{1}$$

Here, e_0 is the absolute value of the electron charge. The energy eigenvalues of this system are [36]

$$E_n^{\pm} = \frac{1}{2}(2n+1\pm 1)h\nu_c \pm \frac{a}{2}h\nu_c - \frac{1}{8}\frac{h^2\nu_c^2}{mc^2}(2n+1\pm 1)^2, \tag{2}$$

where the last term is the leading relativistic correction. The parameter of interest in (2) is the anomalous magnetic moment

$$a \equiv \frac{(g-2)}{2}. \tag{3}$$

Measurements of transitions between these energies are used to determine the value of a. It turns out that a particular ratio of such transition energies cancels several effects of imperfections and is thus particularly suited for this purpose [12–14],

$$a_d = \frac{E_0^+ - E_1^-}{E_1^+ - E_0^+ + 3h\delta/2}. \tag{4}$$

The subscript of a_d refers to “direct” measurement.

C. Corrections to expect

Quantum field theoretical descriptions are effective many-particle theories, which consider excitations of all fields $(\psi, A_\mu, \ldots)$. This leads to corrections to the quantum mechanical Dirac equation

$$i\hbar\psi = H_D\psi. \tag{5}$$

When discussing these corrections for the above experimental setting, we have three small dimensionless numbers at our disposal, which can be used to obtain good estimates. In order to compare these quantities, we introduce a common small expansion parameter ϵ.

(i) First, for the nonvanishing magnetic field in (1), there is

$$\delta_c \equiv \frac{h\nu_c}{mc^2} \sim 10^{-9} \sim \epsilon^4. \tag{6}$$

The Foldy-Wouthuysen transformation [37] is essentially based on an expansion in (6) since relativistic energy corrections to the rest mass are given as series of the form $1 + \delta_c + \delta_c^2 + \delta_c^3 \cdots$.

(ii) Second, there is the fine-structure constant

$$\alpha \sim 2\pi a \sim 0.007 \sim \epsilon. \tag{7}$$

Perturbation theory in QED is an expansion in (7). The famous anomalous magnetic moment contributes to the energies with a term $\alpha\delta_c$.

(iii) The third dimensionless parameter is the ratio between the electron mass m and the mass of the W boson

$$\delta_W = \frac{m}{m_W} \sim 10^{-5} \sim \epsilon^{2.5}. \tag{8}$$

The leading contribution to the Hamiltonian of these corrections appears at quadratic order in δ_W and is induced by loop corrections. It is suppressed by $\alpha\delta_c\delta_W^2$ or by $\alpha\delta_W^2$, as shown in (C4).

The above expansions are present in energy formulas like (2). To see the relevance of the different terms in this expansion it is instructive to compare them to the experimental precision of [13]. In this experiment “a” was determined by measuring a dimensionless ratio $\sim(\Delta E/(mc^2\delta_c))$ between different transition energies $(\Delta E, \delta_c)$ of the quantum mechanical system to a precision of 2.8×10^{-13}. This comparison of magnitudes is shown in Fig. 1. Remembering that the interpretation of the experimental measurement considers dimensionless corrections up to order $\sim\delta_c$, Fig. 1 gives some expected and some surprising insight. One sees that sub-sub-leading-order relativistic corrections ($\sim\delta_c^2$), or loop corrections with three external photon lines ($\alpha\delta_c^2$), are well beyond experimental reach. On the other hand, Fig. 1 shows that loop corrections with two external photon lines $\sim\alpha\delta_c$ and loop corrections involving a weak interaction and a single external photon line $\sim\alpha\delta_W$ can be expected to give relevant corrections to the interpretation of experimental results. From the point of view of a combined expansion in (6)–(8), there is no reason why such terms should be absent.

We will discuss possible corrections to the energies and to the observable [(2) and (4)] up to order $\alpha\delta_c^2$ in a systematic way. We address the issue of the seemingly relevant corrections $\sim\alpha\delta_c$ and $\alpha\delta_W^2$.

Note that there are further dimensionless scales. In particular, if one would consider axial motion with the

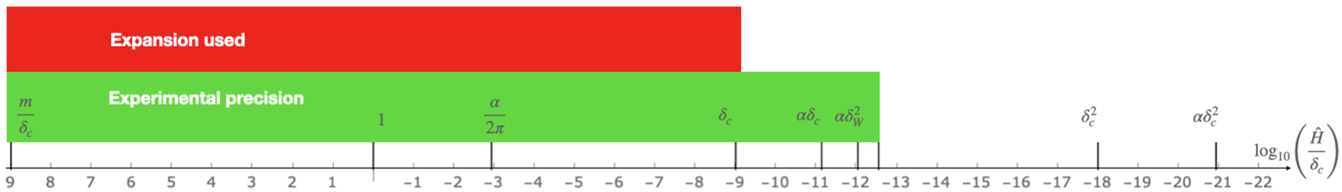

FIG. 1. Orders of magnitude of the expansion parameters [(6)–(8)]. The precision reached by current experiments is indicated by the green bar. The theoretical expansion order used to interpret the experimental results [(2) and (4)] is indicated by the red bar. The energies in this sketch are normalized to the dimensionless ratio $\sim(\Delta E/(mc^2\delta_c))$.

axial frequency ν_z, there would be a fourth small dimensionless parameter

$$\delta_z \equiv \frac{h\nu_z}{mc^2} \sim 10^{-13} \sim \epsilon^6. \tag{9}$$

This parameter is shortly mentioned in the discussion. There are, of course, further higher orders in fine-structure constant α, which are not shown in Fig. 1 since these corrections do not introduce new effective operators.

Note further that, in principle, this discussion of orders of magnitude also applies to the anomalous magnetic moment of the muon, but there are two crucial differences: First, the experimental observable is different from (4). Second, the muon is about 200 times heavier than the electron, which further suppresses δ_c by the same factor. Thus, most of the possible corrections discussed for the electron are irrelevant for the muon.

II. CORRECTIONS FROM EFFECTIVE LAGRANGIAN APPROACH

We address the problem of corrections to the anomalous magnetic moment of the electron with an effective Lagrangian approach. This approach allows one to obtain the energy spectrum as a result of the solutions of the effective equations of motion. Alternatively, the energies can be read off from the spectral structure of the exact electron propagator in the presence of a finite magnetic field, using Schwinger's proper-time representation [38]. We chose the former approach, because it allows us to include non-QED effects straightforwardly.

A. Hamiltonian from effective Lagrangian

In an effective field theory approach, the anomalous magnetic moment of the electron is given in terms of the effective Lagrangian [39]

$$\mathcal{L}_{\text{eff}} = \bar{\psi}\left[\gamma^\mu(i\hbar\partial_\mu - eA_\mu) - mc^2 + a\frac{e\hbar}{4m}\sigma^{\mu\nu}F_{\mu\nu} + \ldots\right]\psi. \tag{10}$$

The anomalous contribution to the g factor is encoded in the parameter (a). In a top-down approach, the corresponding last term in (10) arises from the sunset diagram with a single external photon line (see Fig. 2 with $y = 1$). In a bottom-up approach, one can also take (a) as a phenomenological parameter of the effective Lagrangian (10) and determine its value from an experiment. For this purpose, one uses (10) to derive the corresponding quantum mechanical Dirac equation

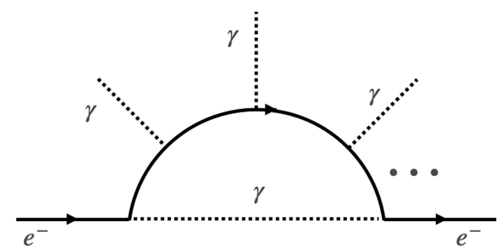

FIG. 2. Feynman diagram for coupling to $y = 3$ external photons.

$$\left[\gamma^\mu(i\hbar\partial_\mu - eA_\mu) - mc^2 + a\frac{e\hbar}{4m}\sigma^{\mu\nu}F_{\mu\nu} + \ldots\right]\psi = 0. \tag{11}$$

For a constant external magnetic field and $a = 0$, the energy levels and eigenfunctions of the system with a single electron are known exactly [40,41], with the eigenenergies given in (B1). With $a \neq 0$, the anomalous magnetic moment is described by an additional term,

$$\Delta_1 H = a\gamma^0\sigma^{\mu\nu}F_{\mu\nu}, \tag{12}$$

adding to the Dirac Hamiltonian. Since these are also eigenfunctions of the operator $\Delta_1 H$, the corresponding energy levels (B1) only get modified by an additive factor

$$E_n^\pm = mc^2\sqrt{1 + \frac{h\nu_c}{mc^2}(1 + 2n \pm 1)} \pm \frac{a}{2}h\nu_c. \tag{13}$$

Since our aim is to study corrections to (2), we expand the energies (13) up to powers of δ_c^3,

$$E_n^\pm = mc^2 + \frac{1}{2}(2n+1\pm 1)h\nu_c \pm \frac{a}{2}h\nu_c - \frac{1}{8}\frac{h^2\nu_c^2}{mc^2}(2n+1\pm 1)^2 + \frac{1}{16}\frac{h^3\nu_c^3}{m^2c^4}(2n+1\pm 1)^3. \tag{14}$$

One notes that this contains the constant mass term, the energy formula (2), and the additional last term $\sim\delta_c^3 \sim \epsilon^{12}$, which is the type of correction we are interested in. Schematically one can write

$$\frac{E_n^{\pm}}{mc^2\delta_c} = \mathcal{O}\left(\frac{1}{\delta_c} + 1 + \alpha + \delta_c + \delta_c^2 + \ldots\right). \qquad (15)$$

The contributions $\sim\delta_c$ lead to relative changes in the energy differences of $\sim 10^{-9}$ and are thus an essential part of the current measurements, which have a precision of about 10^{-13}. The corrections $\sim\delta_c^2$ lead to relative changes in the energy differences of $\sim 10^{-18}$; they will only be accessible to future experiments. One realizes that the solution (13) does not give rise to $\alpha\delta_c$ or $\alpha\delta_W^2$ terms. For these terms to appear, one needs to consider additional operators in the effective action (10).

In the following subsections, we will explore this possibility.

B. Between theory and experiment

There is a conceptual gap between the theoretical definition of the anomalous magnetic moment and the observable that is used to measure this quantity. From the theory side, one defines a in terms of the photon-electron vertex function $\Gamma_{e\bar{e}\rho}(p, -p, q)$, evaluated at zero momentum transfer [42,43]. From the experimental side, a is defined through Eq. (4) in terms energy eigenvalues of an electron in the background of a finite magnetic field.

These are different things, which only agree in certain limits and approximations. One way to discuss this issue is in terms of the effective Lagrangian (10) and the resulting field equations, as sketched in the previous subsection. The photon-electron vertex function $\Gamma_{e\bar{e}\rho}(p, -p, q)$ evaluated at zero momentum gives rise to the effective Hamiltonian contribution (12). This contribution is thus an unambiguous SM prediction, which is experimentally tested by measuring transitions between energy eigenvalues of (11) and plugging them into relation (4). This test is correct as long as additional contributions to the effective Lagrangian, symbolized by the dots in relations (10) and (11), are negligible or at least give negligible contributions to the energy eigenvalues in (4).

Higher-order corrections to the effective field equations (11), which are not included in the standard analysis because they do not arise from the photon-electron vertex function $\Gamma_{e\bar{e}\rho}(p, -p, q)$, are a consequence of three different aspects:

(i) The experiments are performed at finite magnetic field, which means that the assumption of zero momentum transfer is only approximately true. Since such corrections are typically quadratic in the momentum transfer, one can expect that they lead to $\sim\delta_c^2$ corrections in (15).

(ii) When allowing contributions to second or higher order in the finite external magnetic field, (11) will contain additional terms that have a field and Lorentz structure, which is not captured by (12). Such corrections can already appear at order $\sim\alpha\delta_c$ in (15).

(iii) Additional non-QED fields can also lead to operators with a different structure than (12) and which are thus not part of the standard analysis in the literature. If the additional field is a weak gauge boson, one expects $\sim\alpha\delta_W^2$ corrections to (15). If the additional field is some hypothetical beyond the Standard Model field, one expects a correction suppressed by the coupling and by the mass of this field.

In the following discussion, we will estimate these effects for the Standard Model of particle physics, but some of the findings can be useful for studies that go beyond this model.

C. Further gauge-invariant bilinear corrections

In a top-down approach, further corrections to the effective action (10) arise from irreducible Feynman diagrams with an arbitrary number of external legs. We are interested in the cases with two external fermion lines and y-external photon lines, as shown in Fig. 2. For $y = 1$, the only contribution is the one given in (10). Loop corrections induce numerous terms with $y > 1$. In an effective field theory approach, we study a class of possible terms in the extended Lagrangian that meet the following criteria:

(i) They are bilinear in the Dirac field $\bar{\psi}\mathcal{O}\psi$.

(ii) They form a Lorentz scalar.

(iii) The field content of the operator $\mathcal{O} = \mathcal{O}(A_\mu, F^{\mu\nu})$ only contains gauge-invariant contributions of the external electromagnetic field, which are either the field strength tensor $F^{\mu\nu}$ or the covariant derivative operator D_μ, thus $\mathcal{O} = \mathcal{O}(D_\mu, F^{\mu\nu})$.

(iv) The field content of $\mathcal{O}$ is only up to second power in the field and has at least one contribution from $F^{\mu\nu}$.

Since the operator $\mathcal{O}$ forms a 4×4 matrix in spinor space, it is useful to use $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5\}$ as complete five-dimensional basis for the classification of these operators. While the first three basis elements are naturally induced in QED, the last two elements require virtual non-QED contributions, such as weak interactions or BSM physics.

1. ~1 contributions

The term one can add to the effective Lagrangian (10), which is proportional to the identity matrix, is

$$\Delta_{1,\mathrm{FF}}\mathcal{L} = \alpha\frac{\xi_{1,\mathrm{FF}}}{m^3c^6}\bar{\psi}F_{\mu\nu}F^{\mu\nu}\psi, \qquad (16)$$

where ξ_i is the effective dimensionless coupling parameter, which is expected to be of order 1. The contributions of these operators to the energy spectrum can be estimated by using $|\phi_n^\pm\rangle$, the eigenfunctions of (11), which are given in (A1). These eigenfunctions are normalized as $\langle\phi_n^\pm|\phi_n^\pm\rangle = 1$. Since small terms like (16) only lead to small modifications of these wave functions, one can apply perturbation theory to estimate to which extent these modifications shift the energy levels $E_n^\pm$. These shifts are labeled $\Delta_{\ldots}E_n^\pm$. One finds

$$\Delta_{1,\mathrm{FF}}E_n^\pm = \frac{\xi_{1,\mathrm{FF}}\alpha}{m^3c^6}\langle\phi_n^\pm|\gamma^0 F_{\mu\nu}F^{\mu\nu}|\phi_n^\pm\rangle$$
$$= -\xi_{1,\mathrm{FF}}\alpha\frac{h^2\nu^2}{2mc^2\sqrt{1+\frac{h\nu}{mc^2}(2n+1\pm 1)}} \tag{17}$$

$$\approx \xi_{1,\mathrm{FF}}\alpha\left(-\frac{h^2\nu_c^2}{mc^2}+\frac{h^3\nu_c^3(2n+1\pm 1)}{m^2c^4}\right). \tag{18}$$

In terms of the series in Fig. 1, these contributions are of the order of

$$\Delta_{1,\mathrm{FF}}E_n^\pm/(mc^2\delta_c) \sim \xi_{1,\mathrm{FF}}\mathcal{O}(\alpha\delta_c + \alpha\delta_c^2). \tag{19}$$

Note that these corrections are also dependent on the other quantum number l, as long as $l \le n$. The above relation is for $l = 0$.

Note further that corrections of this type can also appear due to finite temperature effects, but these are so small that they do not interfere with our discussion.

2. $\sim\gamma^\mu$ contributions

When including one γ^μ, the first option is to consider one D_α and one $F^{\mu\nu}$. Since $D_\alpha\gamma^\alpha F_\mu^\mu = 0$, the only possibility, which is linear in the field strength, is

$$\Delta_{\gamma,\mathrm{DF}}\mathcal{L} = \frac{\xi_{\gamma,DF}}{m^2c^4}\bar\psi D_\alpha\gamma^\beta F_\beta^\alpha\psi \neq 0. \tag{20}$$

However, when integrating the corresponding contribution to the energy, one finds that it vanishes

$$\Delta_{\gamma,\mathrm{DF}}E_n^\pm = \frac{\xi_{1,DF}}{m^2c^4}\langle\phi_n^\pm|\gamma^0 D_\alpha\gamma^\beta F_\beta^\alpha|\phi_n^\pm\rangle = 0. \tag{21}$$

For second order in $F^{\mu\nu}$, there are two nonvanishing contributions

$$\Delta_{\gamma,\mathrm{DFF1}}\mathcal{L} = \alpha\frac{\xi_{\gamma,\mathrm{DFF1}}}{m^4c^8}\bar\psi D_\alpha\gamma^\alpha F_{\mu\nu}F^{\mu\nu}\psi, \tag{22}$$

with

$$\Delta_{\gamma,\mathrm{DFF1}}E_n^\pm = \xi_{\gamma,\mathrm{DFF1}}\alpha\frac{h^2\nu^2}{mc^2\sqrt{1+h\nu(1+2n\pm 1)}} \tag{23}$$

$$\approx\xi_{\gamma,\mathrm{DFF1}}\alpha\left(\frac{2h^2\nu^2}{mc^2}-\frac{h^3\nu^3(1+2n\pm 1)}{m^2c^4}\right) \tag{24}$$

and

$$\Delta_{\gamma,\mathrm{DFF2}}\mathcal{L} = \alpha\frac{\xi_{\gamma,\mathrm{DFF2}}}{m^4c^8}\bar\psi D_\alpha\gamma^\beta F_{\beta\nu}F^{\alpha\nu}\psi, \tag{25}$$

with

$$\Delta_{\gamma,\mathrm{DFF2}}E_n^\pm = -\Delta_{\gamma,\mathrm{DFF2}}\alpha\frac{h^3\nu^3(1+2n\pm 1)}{m^2c^4\sqrt{1+h\nu(1+2n\pm 1)}} \tag{26}$$

$$\approx -\xi_{\gamma,\mathrm{DFF2}}\alpha\frac{h^3\nu^3(1+2n\pm 1)}{m^2c^4}. \tag{27}$$

In terms of the series in Fig. 1, these contributions are of the order of

$$\Delta_{\gamma,\mathrm{DFF1}}E_n^\pm/(mc^2\delta_c) \sim \xi_{\gamma,\mathrm{DFF1}}\mathcal{O}(\alpha\delta_c + \alpha\delta_c^2), \tag{28}$$

$$\Delta_{\gamma,\mathrm{DFF2}}E_n^\pm/(mc^2\delta_c) \sim \xi_{\gamma,\mathrm{DFF2}}\mathcal{O}(\alpha\delta_c^2). \tag{29}$$

3. $\sim\sigma^{\mu\nu}$ contributions

The contribution, which is first order in $F^{\mu\nu}$, is already considered by the coupling in (11). Second-order contributions such as $\sim F_\beta^\alpha\sigma^{\beta\mu}F_{\mu\alpha}$ are identically zero. A possible third-order contribution is

$$\Delta_{\sigma,\mathrm{FFF1}}\mathcal{L} = \alpha\frac{\xi_{\sigma,\mathrm{FFF1}}}{m^5c^{10}}\bar\psi\sigma^{\mu\nu}F_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\psi, \tag{30}$$

with

$$\Delta_{\sigma,\mathrm{FFF1}}E_n^\pm = \pm\xi_{\sigma,\mathrm{FFF1}}\alpha\frac{4h^3\nu^3}{m^2c^4}. \tag{31}$$

Similarly, there is

$$\Delta_{\sigma,\mathrm{FFF2}}\mathcal{L} = \alpha\frac{\xi_{\sigma,\mathrm{FFF2}}}{m^5c^{10}}\bar\psi\sigma^{\mu\nu}F_{\mu\alpha}F_{\nu\beta}F^{\alpha\beta}\psi, \tag{32}$$

with

$$\Delta_{\sigma,\mathrm{FFF2}}E_n^\pm = \pm\xi_{\sigma,\mathrm{FFF2}}\alpha\frac{2h^3\nu^3}{m^2c^4}. \tag{33}$$

Thus, one realizes that, in the spirit of the expansion in Fig. 1, all of these contributions are of order

$$\Delta_{\sigma\ldots}E_n^\pm/(mc^2\delta_c) \sim \xi_{\sigma\ldots}\mathcal{O}(\alpha\delta_c^2). \tag{34}$$

D. Parity-violating corrections

Once one leaves the realm of QED, there is also the possibility of forming Lorentz invariants with $\gamma^5, \gamma^\mu\gamma^5$, and through contractions with $\epsilon^{\mu\nu\alpha\beta}$.

1. $\sim\gamma^5$ contributions

One might consider mass contributions that have the form

$$\Delta_{\gamma^5}\mathcal{L} = mc^2\xi_{\gamma^5}\alpha\delta_W^2\bar{\psi}\gamma^5\psi. \tag{35}$$

They do not give corrections to the energy

$$\Delta_{\gamma^5}E_n^\pm = 0. \tag{36}$$

2. $\sim\gamma^5\gamma^\mu$ contributions

Parity-violating contributions to the minimal coupling can, for example, be introduced by virtual weak interactions

$$\Delta_{\gamma^5\gamma^\mu}\mathcal{L} = \xi_{\gamma^5\gamma^\mu}\alpha\delta_W^2\bar{\psi}\gamma^5\gamma^\mu D_\mu\psi. \tag{37}$$

Two examples are calculated in the Appendix C. Even though these terms come with the expected prefactor, they do not lead to corrections to the energy levels

$$\Delta_{\gamma^5\gamma^\mu}E_n^\pm = 0. \tag{38}$$

Similarly, there is the possibility of

$$\Delta_{\gamma^5\gamma^\mu\mathrm{FF}}\mathcal{L} = \alpha\frac{\xi_{\gamma^5\gamma^\mu\mathrm{FF}\delta_W^2}}{m^3c^6}\bar{\psi}\gamma^5\gamma^\alpha D_\alpha F^{\mu\nu}F_{\mu\nu}\psi, \tag{39}$$

also leading to vanishing energy corrections

$$\Delta_{\gamma^5\gamma^\mu\mathrm{FF}}E_n^\pm = 0. \tag{40}$$

Notably, all parity-violating contributions to the energy vanish at the considered order.

3. Considering the electric field

If one considers an electric field, e.g., the field confining the electrons in the z direction, the solutions of Eq. (11) will contain additional quantum numbers, like n_z referring to this degree of freedom,

$$\psi = \psi^\pm_{n,n_z,l,l_z}. \tag{41}$$

The strength of the electric field will depend on z, and it will be related to the dimensionless parameter (9); the energy contribution of the electric oscillations will be

$$E^\pm_{n,n_z} - E^\pm_{n,n_z=0} = n_z mc^2\delta_z + \mathcal{O}(\delta_z^2). \tag{42}$$

In order to avoid such complications, let us consider a constant electric field $\vec{E} = \hat{z}(mc^2)^2\delta_z$. Thus, as long as we consider only eigenstates with the same quantum number n_z (e.g., $n_z = 0$), we can reuse the eigenstates of the previous sections

$$\psi^\pm_{n,l} = \psi^\pm_{n,n_z=0,l,l_z=0}. \tag{43}$$

The discussion of the preceding sections remains valid in a perturbative sense, even though the electromagnetic stress tensor now reads

$$F_{\mu\nu} = (mc^2)^2\begin{pmatrix} 0 & 0 & 0 & \delta_z \\ 0 & 0 & \delta_c & 0 \\ 0 & -\delta_c & 0 & 0 \\ -\delta_z & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

The existence of the electric field $\sim(mc^2)^2\delta_z$ does not alter the differences between energy levels of the state (43). This invisibility becomes spoiled due to parity-violating contributions, analogous to the ones discussed above. One obtains, for example, an effective coupling

$$\Delta_{\sigma\tilde{F}}\mathcal{L} = \alpha\frac{\xi_{\sigma\tilde{F}}\delta_W^2}{mc^2}\sigma^{\mu\nu}\tilde{F}_{\mu\nu}, \tag{45}$$

where the dual electromagnetic tensor is given by $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. This coupling contributes to the energy levels with

$$\Delta_{\sigma\tilde{F}}E_n^\pm = \pm mc^2\alpha\delta_W^2\xi_{\sigma\tilde{F}}\delta_z. \tag{46}$$

E. Combined corrections

As a result of the previous two subsections, we have obtained eight corrections [(17), (23), (26), (31), (33), (36), (38), (40)] of the standard energy formula (2). Up to order δ_c^3 the expansion of the standard energy formula reads as (14). Within the same expansion, the combined correction arising from [(17), (23), (26), (31), (33), (36), (38), (40)] reads

$$\begin{aligned}\frac{\Delta E_n^\pm}{mc^2\delta_c} &= \alpha\delta_c^1(\xi_{\gamma^\mu\mathrm{FF1}} - \xi_{1,\mathrm{FF}}) \pm \alpha\frac{\delta_W^2\delta_z}{\delta_c}\xi_{\sigma\tilde{F}} \\ &\quad + \alpha\delta_c^2(\xi_{Id\mathrm{FF}}(2n \pm 1 + 1) \\ &\quad - 2n(\xi_{\gamma^\mu\mathrm{FF1}} + \xi_{\gamma^\mu\mathrm{FF2}}) - \xi_{\gamma^\mu\mathrm{FF1}} - \xi_{\gamma^\mu\mathrm{FF2}} \\ &\quad \mp (\xi_{\gamma^\mu\mathrm{FF1}} + \xi_{\gamma^\mu\mathrm{FF1}} - 2(\xi_{\sigma\mathrm{FFF1}} + \xi_{\sigma\mathrm{FFF2}}))) + \cdots.\end{aligned} \tag{47}$$

In Eq. (B8) a theoretical prediction for the proportionality factor of the leading coefficient of (47) is given $(\xi_{\gamma^\mu\mathrm{FF1}} - \xi_{1,\mathrm{FF}}) \approx 0.02$.

F. Anomalous magnetic moment

Experimentally, the value of a_d is obtained from measuring transition energies and applying relation (4). This formula is valid for energies expanded up to δ_c^2. The value inferred from the direct Penning trap experiment by using this relation is [13]

$$a_d = 0.00115965218073(28). \tag{48}$$

If one wants to include the relativistic δ_c^3 energy corrections (14), one has to adopt formula (4) to

$$a_d = \frac{E_0^+ - E_1^-}{E_1^+ - E_0 + \frac{3}{2}mc^2\delta_c - \frac{7}{2}mc^2\delta_c^2}. \tag{49}$$

If one includes now, on top of the relativistic improvement (14), the combined corrections (47), the actual energies read

$$\tilde{E}_n^\pm = E_n^\pm + \Delta E_n^\pm. \tag{50}$$

Using these energies in the relativistically improved formula (49), one finds the inferred direct anomalous moment a_d [the lhs of (49)] differs from the actual magnetic moment a_a [the one appearing in (10)]. One finds

$$a_d = a_a + \frac{2\alpha\delta_W^2\delta_z}{\delta_c}\xi_{\sigma\tilde{F}} - 2\alpha\delta_c^2(a_a(\xi_{Id\mathrm{FF}} - \xi_{\gamma^\mu\mathrm{FF1}} - \xi_{\gamma^\mu\mathrm{FF2}}) - 2(2\xi_{\sigma\mathrm{FFF1}} + \xi_{\sigma\mathrm{FFF2}})) + \mathcal{O}(\epsilon^{11}). \tag{51}$$

The above relation used a combined expansion in the small parameter ϵ (6)–(8). This relation can be inverted to give the actual value of the anomalous magnetic moment a_a as a function of $a_a = a_a(a_d, \xi_i)$,

$$a_a = a_d - \frac{2\alpha\delta_W^2\delta_z}{\delta_c}\xi_{\sigma\tilde{F}} + 2\alpha\delta_c^2(a_d(\xi_{Id\mathrm{FF}} - \xi_{\gamma^\mu\mathrm{FF1}} - \xi_{\gamma^\mu\mathrm{FF2}}) - 2(2\xi_{\sigma\mathrm{FFF1}} + \xi_{\sigma\mathrm{FFF2}})). \tag{52}$$

It is now the task of theory to predict the precise values of ξ_i and the task of experiment to determine and/or constrain these values. One way of doing this is via a comparison with indirect $g-2$ measurements.

III. DISCUSSION

A. Indirect $g-2$ measurements

Theoretical calculations determine a relation between the value of $(g-2)$ and the fine-structure constant up to fifth order in α,

$$a(\alpha) = C_2\left(\frac{\alpha}{\pi}\right) + C_4\left(\frac{\alpha}{\pi}\right)^2 + C_6\left(\frac{\alpha}{\pi}\right)^3 + C_8\left(\frac{\alpha}{\pi}\right)^4 + C_{10}\left(\frac{\alpha}{\pi}\right)^5 + a_{\mu\tau} + a_{\mathrm{had}} + a_{\mathrm{weak}}, \tag{53}$$

where the constants C_i are given in [14] (consider the erratum). Thus, an independent measurement of α (and the lepton masses), also allows one to infer a value for $g-2$, and vice versa.

In recent years, this has been done by Parker *et al.* [23] and Morel *et al.* [34]. They obtained α-induced values

$$a_P(\alpha) = 0.001159652181610(230), \tag{54}$$

$$a_M(\alpha) = 0.001159652180252(95). \tag{55}$$

In contrast to this, the direct measurement gave (48). This situation is shown in Fig. 3, where we plot the value of $a(\alpha)$ relative to the value a_d, which was inferred from the direct measurement. Since the latter measurement of α has much smaller error bars, there seems to be no significant tension between a_d and $a_M(\alpha)$. The (non)tension between the direct measurement and the indirect measurements can be quantified by

$$\frac{a_{P/M}(\alpha) - a_d}{\sqrt{\delta a_d^2 + \delta a_{P/M}^2(\alpha)}} = \begin{cases} t_{i,P} = +2.4 \\ t_{i,M} = -1.6 \end{cases}. \tag{56}$$

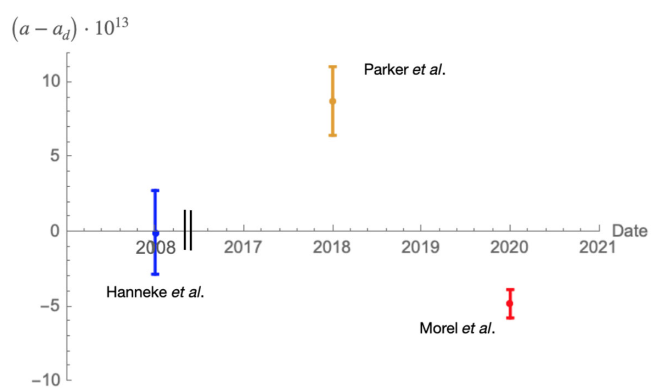

FIG. 3. Results for anomalous magnetic moment $(a_{P/M}(\alpha) - a_d) \cdot 10^{13}$ plotted as a function of time. The blue point is the direct measurement [13]; the red and orange points are the indirect measurements [23,34].

The values of $t_{i,P/M}$ are measured in standard deviations. Here the sign indicates whether $a(\alpha)$ is smaller or larger than a_d.

In the following subsections, we will explore a comparison between a_d and $a(\alpha)$ in light of the difference between a_d and a_a (52). In this comparison, we will identify $a(\alpha)$ with a_a. This identification assumes implicitly that there are no corrections to the measurements of the fine-structure constant α.

B. Benchmark without parity violation

Let us first consider a benchmark without parity violation by setting $\xi_{\sigma\tilde{F}} = 0$. The result in Sec. II suggests correcting the interpretation of the direct measurement by using the value (51): $a_d \to a_a = a_a(\xi_i)$ and the error induced by this correction.

As the first benchmark, let us treat all parity-conserving contributions with the same numerical factor ξ_i. For this benchmark, one can study how the comparison between direct measurement and indirect measurement depends on the parameter ξ_i. If one assumes these corrections come without an increased error bar, one can plot $a_a(\xi_i)$ relative to a_d and in comparison with the indirect results $a_{P/M} - a_d$. This is shown in Fig. 4.

From Fig. 4 one sees that the parity-conserving corrections only have a relevant effect on the comparison between direct and indirect observation, if the dimensionless parameters ξ_i are of the order of 10^6. However, the prefactors of the effective operators are chosen such that within the SM the parameters $\xi_{i,\ldots}$ are expected to be of order 1. Thus, Fig. 4 means that there are 6 orders of magnitude missing in the sensibility before one can appreciate the effect of these SM contributions.

This finding can be compared with the QED corrections for finite momentum transfer. The one-loop QED form factor for finite photon momentum squared q^2 is given by

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2 xy}$$
$$= \frac{\alpha}{2\pi}\left(1 - \frac{q^2}{6m^2}\right). \quad (57)$$

If one evaluates the correction due to the second term with the momentum scale given by the magnetic background field $q^2 \approx (h\nu_c)^2$, one obtains

$$F_2 \approx \frac{\alpha}{2\pi}\left(1 - \frac{\delta_c^2}{6}\right). \quad (58)$$

Thus, one also finds that finite momentum effects are suppressed by $\alpha\delta_c^2 \approx 10 \times 10^{-20}$, which is also 6 orders of magnitude below the current experimental precision.

From a phenomenological perspective (e.g., for BSM models), one has to demand that $|a_a - a_d| \leq 10 \times 10^{-13}$, which constrains the ξ_i parameter of this benchmark to

$$\xi_i \leq 1 \times 10^6. \quad (59)$$

The parameters ξ_i are thus useful mediators between BSM models and the precision measurement of a_d.

C. Benchmark with parity violation

In a second benchmark, we set all parity-conserving corrections to zero and only retain $\xi_{\sigma\tilde{F}}$. Figure 5 shows how the comparison between direct and indirect measurements is affected by the remaining parameter. One notes that a natural $\xi_{\sigma\tilde{F}} = \mathcal{O}(1)$ is 3 orders of magnitude too small for observation under the current simplifying assumptions.

D. What happened?

In Sec. I C, it has been shown that background field effects ($\sim\alpha\delta_c$) and effects of the weak interaction ($\sim\alpha\delta_W^2$) can be expected to give important contributions to the energy

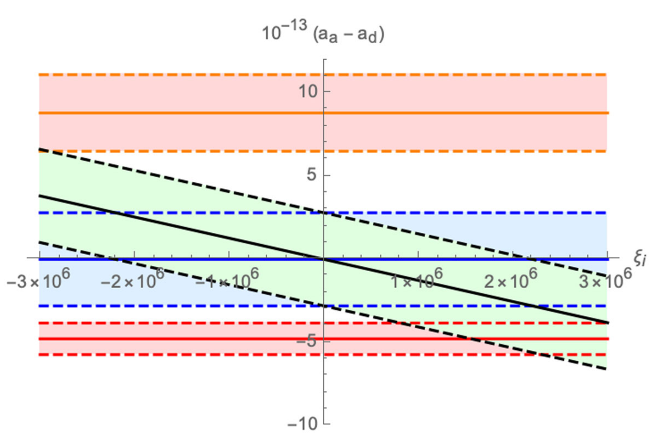

FIG. 4. The red line shows $(a_M - a_d)$, with the error bars of a_M [34]. The orange line shows $(a_P - a_d)$, with the error bars of a_P [23]. The blue region are the error bars of a_d [13]. The black line shows $(a_a(\xi_i) - a_d)$, with the error bars of a_d.

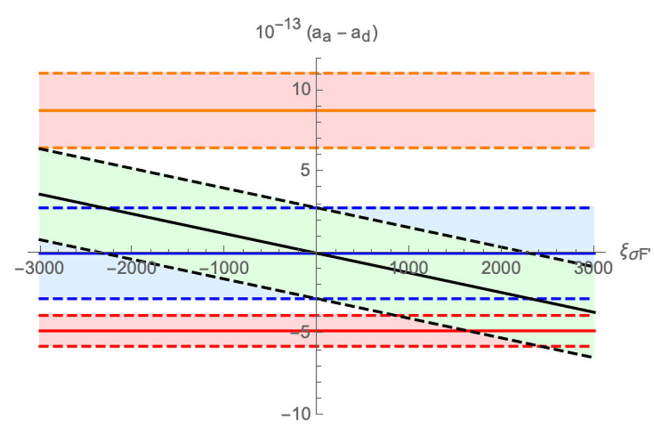

FIG. 5. The red line shows $(a_M - a_d)$, with the error bars of a_M [34]. The orange line shows $(a_P - a_d)$, with the error bars of a_P [23]. The blue region shows the error bars of a_d [13]. The black line shows $(a_a(\xi_i) - a_d)$, with the error bars of a_d.

levels $E_n^{\pm}/(mc^2\delta_c)$, as depicted in Fig. 1. These energy levels are used to determine a_d. However, when studying the actual comparison between direct and indirect measurements, the effect of these corrections was rather negligible, as shown in Fig. 4. The disappearance of these seemingly substantial contributions happens for different reasons:

(i) The parity-violating contributions, arising from virtual weak interactions, introduce new types of effective couplings [(35), (37), (39)]. These couplings do not, however, contribute to the leading-order corrections of the energy levels $E_n^{\pm}/(mc^2\delta_c)$. This is due to the particular off-diagonal nature of the $\gamma^0 \cdot \gamma^5$ matrix.

(ii) The parity-conserving corrections [(16), (20), (22), (25), (30), (32)] do all contribute to $E_n^{\pm}/(mc^2\delta_c)$ and they do so with corrections $\sim\alpha\delta_c^1$ and $\sim\alpha\delta_c^2$. The problem is that all the $\sim\alpha\delta_c^1$ corrections are independent of the quantum numbers $(\pm, n)$. Thus, when evaluating energy differences in (49), only the much smaller $\sim\alpha\delta_c^2$ corrections survive. It remains to be seen whether larger corrections $\sim\alpha\delta_c^1$ can be noted by a different type of experimental setup.

(iii) The largest corrections arise from the parity-violating contributions (45). Under the simplifying assumptions used above, they are still 3 orders of magnitude below the sensibility threshold. Concerning this result, one has to consider two additional aspects. On the one hand, one could imagine an experimental setup with an increased electric field, which would then reach the observability limit. On the other hand, one must consider that the assumption of a homogenous and constant electric field is not realistic. In a more realistic scenario, it is likely that contributions from positive and negative electric fields cancel each other.

IV. CONCLUSION

Penning trap experiments allow for a precise direct measurement of the electron's anomalous magnetic moment $a = (g-2)/2$. This is done by relating observed transition energies between Landau levels, as shown in Eq. (4).

In this paper, we studied the impact of parametrized corrections to the effective Lagrangian (10) on the electron's anomalous g factor, deduced from the observable (49). It is found that the presence of these parameters leads to a difference between the deduced anomalous magnetic moment a_d and the actual magnetic moment a_a given by (51). This generic result is then studied for two benchmarks.

It is shown that there are QED-loop-induced parity-conserving corrections to the energies $E_n^{\pm}$, which are of the order of $\alpha\delta_c^2$. Such corrections are, in principle, numerically very relevant since $\Delta a \sim \Delta E/\delta_c \sim \alpha\delta_c \sim 10^{-11}$. However, these leading corrections do not depend on the quantum numbers $(n, \pm)$ and thus they get canceled in the actual observable (49), where only energy differences are considered. The remaining subleading effects $\Delta a_d \sim \delta_c^2$ are, at the current stage, not observable. They are about 6 orders of magnitude smaller than the experimental precision.

Most parity-violating corrections to the energy vanish unless one considers effects of the electric field. An example for such a contribution is given in relation (45). Whether this scenario has the potential of studying parity violation with $g-2$ experiments has to be investigated with a detailed study considering a spatial variation of the electric field.

While this paper has its focus on SM contributions, the relations (47) are perfectly suited to test and constrain BSM effects.

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APPENDIX A: EXPLICIT SOLUTIONS

The exact solutions of (11) are well known [41]. Since we made massive use of the particle eigenstates $\psi^{\pm}_{n=(0,1),l=0}$, these functions in symmetric gauge are given explicitly

$$\psi^+_{0,0} = \left(\frac{\rho}{2\sqrt{2\pi}}\sqrt{1+\frac{m}{\sqrt{m^2+\rho^2}}}, 0, 0, \frac{i\rho^3(x+iy)}{4\sqrt{2\pi}\sqrt{m(\sqrt{m^2+\rho^2}+m)+\rho^2}}\right)e^{-\frac{1}{8}\rho^2(x^2+y^2)-it\sqrt{m^2+\rho^2}},$$

$$\psi^-_{0,0} = \left(0, \frac{\rho}{2\sqrt{\pi}}, 0, 0\right)e^{-\frac{1}{8}\rho^2(x^2+y^2)-itm},$$

$$\psi^+_{1,0} = \left(\frac{\rho^2(x+iy)}{4\sqrt{2\pi}}\sqrt{1+\frac{m}{\sqrt{m^2+\rho^2}}}, 0, 0, \frac{i\rho^4(x+iy)^2}{8\sqrt{2\pi}\sqrt{m(\sqrt{m^2+2\rho^2}+m)+2\rho^2}}\right)e^{-\frac{1}{8}\rho^2(x^2+y^2)-it\sqrt{m^2+2\rho^2}},$$

$$\psi^-_{1,0} = \left(\frac{\rho^2(\sqrt{m^2+\rho^2}+m)(x+iy)}{4\sqrt{2\pi}\sqrt{m(\sqrt{m^2+2\rho^2}+m)+2\rho^2}}, 0, 0\frac{-i\rho^2}{2\sqrt{2\pi}\sqrt{m(\sqrt{m^2+2\rho^2}+m)+2\rho^2}}\right)e^{-\frac{1}{8}\rho^2(x^2+y^2)-it\sqrt{m^2+\rho^2}}, \quad \text{(A1)}$$

where the abbreviation $\rho^2 = 2mh\nu$ and $c \equiv 1$ was used.

APPENDIX B: QED BACKGROUND CORRECTIONS FROM SUZUKI APPROACH

In the effective Lagrangian approach (10), ξ_i are phenomenological parameters that have to be determined by experiment. In a full quantum field theory (QFT) approach, with nonvanishing background fields, these parameters can be predicted. Below we will sketch how such a calculation relates to the above results.

Complementary to using the effective action (10), one can also calculate the loop corrections to the different energies $E_n^\pm$ directly in the underlying quantum field theory. Usually, this is done to leading order in the magnetic field. For a comparison with the results obtained in the previous sections, this has to be done up to second order in B. While the complete calculation of these QED corrections goes beyond the scope of this paper, one can use the known shift in the ground state energy to get an estimate of the uncertainty. The finite ground state corrections, which are proportional to B^2, are given by Suzuki [41]. Strong field effects in a broader sense are discussed in [44].

Suzuki uses the exact quantum mechanics (QM) solution for the energy levels of the electron [41]

$$E_n^\pm = mc^2\sqrt{1+\frac{h\nu_c}{mc^2}(1+2n\pm 1)}. \tag{B1}$$

Note that, when comparing the above formula with the energies given in [39,41], one needs to do the identification $n \to n+1$. The energies (B1) are identical to (13) for $a = 0$.

The actual (QFT corrected) energies are then labeled $\mathcal{E}_n^\pm$; they can be expressed as corrections to (B1). One finds

$$\Delta E_n^\pm \equiv \mathcal{E}_n^\pm - E_n^\pm = \int d^3x \int d^3x' \bar{\psi}_{E,n,\pm}(x)\mathcal{M}(E_n^\pm, \vec{x}, \vec{x}')\psi_{E,n,\pm}(x'), \tag{B2}$$

where $\mathcal{M}$ is the mass operator and $\psi_{E,n,\pm}$ are the eigenfunctions for the QM eigenvalues (B1). The calculation is tedious, but for a small magnetic field $h\nu_c \ll mc^2$, the result can be written as an expansion

$$\Delta E_n^\pm = \frac{a}{2}h\nu_c\left(1+\frac{h\nu_c}{mc^2}(b_0^\pm + b_1^\pm n)\right) + \mathcal{O}\left(\frac{h\nu_c}{mc^2}\right)^3. \tag{B3}$$

In the recent paper of Kim *et al.* [35], this type of expansion is given for the ground state energy and the coefficients are calculated explicitly. However, the above formula allows also for $n \neq 0$. These energy contributions do, in principle, depend on the quantum numbers $(n, \pm)$, since the eigenfunctions $\psi_{n,\pm}$ depend on these quantum numbers. This dependence on n does not have to be linear, but since (4) only considers $n = (0, 1)$, a distinction with different powers of n is not necessary. Here, $a, b_i^\pm$ are numerical coefficients that can be obtained in a loop expansion

$$a = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2), \tag{B4}$$

$$b_i^\pm = b_{i,0}^\pm + \alpha \cdot b_{i,1}^\pm + \mathcal{O}(\alpha^2). \tag{B5}$$

The coefficients of a are very well studied to high order, while the coefficients $b_i^\pm$ are hard to find in the literature. Their order of magnitude can be estimated from the $\sim B^2$ correction to the ground state energy given in [41]. Using the leading-order expansion in ϵ and minimal subtraction, this ground state correction is

$$\Delta E_0^-|_{B^2} = \alpha B^2 \frac{1259}{16800}\frac{e^2\hbar^2}{m^3c^2\pi} + \mathcal{O}(\epsilon^8). \tag{B6}$$

From this, one can read off the numerical value of the first coefficient of (B5),

$$b_0^- = \frac{1159}{4200} \approx 0.3 + \mathcal{O}(\alpha). \tag{B7}$$

From Eq. (47) and the preceding discussion, one can expect that the coefficients that are proportional to n vanish at order δ_c $b_{b_1}^\pm = 0$. Further, one can extract from a comparison between (B7) and (47) a theoretical prediction for a combination of the parameters $\xi_{\gamma^\mu \mathrm{FF1}}$ and $\xi_{id\mathrm{FF}}$,

$$\xi_{\gamma^\mu \mathrm{FF1}} - \xi_{1,\mathrm{FF}} = \frac{b_0-}{4\pi} \approx 0.02. \tag{B8}$$

APPENDIX C: PARITY-VIOLATING SM PROCESSES

There are SM processes that can contribute to parity violation in low energy QED processes (see, for example, [45–48]).

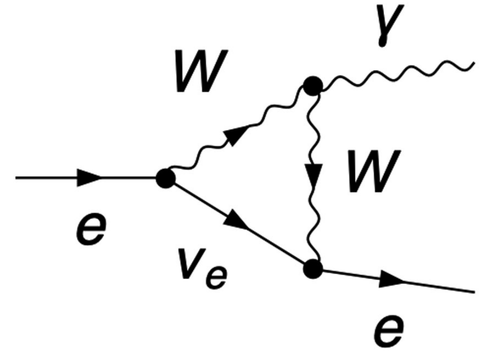

FIG. 6. Feynman diagram weak subprocess.

To quantify the magnitude of similar effects in the context of the observable (49), one needs to consider the weak process shown in Fig. 6. The prefactor of the Feynman amplitude of this process gives the magnitude one should expect for the parameters $\xi_{\gamma^5\ldots}$. As discussed in the previous section, the contribution, relevant for the $g-2$ observable, is $\xi_{\gamma^5\gamma^\mu}$. The amplitude shown in Fig. 6 can be calculated with the software packages based on [49–51]. It reads

$$\begin{aligned} A = \frac{2i\pi^3\alpha e}{\sin^2(\theta_W)}&(-2(-2B_0(m^2,0,m_W^2)+B_0(0,m_W^2,m_W^2)-2m_W^2C_0(0,m^2,m^2,m_W^2,m_W^2,0)\\ &+2m^2C_0(0,m^2,m^2,m_W^2,mW^2,0)-DC_{00}(m^2,0,m^2,0,m_W^2,m_W^2))+2C_{00}(m^2,0,m^2,0,m_W^2,m_W^2)\\ &+3m^2C_1(m^2,0,m^2,0,m_W^2,m_W^2)\cdot\bar{\psi}_2\gamma^\mu\gamma^7\psi_1,\\ &m\bar{\psi}_2\gamma^7\psi_1\cdot(D-2)p_2^\mu((D-8)C_1(m^2,0,m^2,0,m_W^2,m_W^2)2(D-2)C_{12}(m^2,0,m^2,0,m_W^2,m_W^2))),\\ &m\bar{\psi}_2\gamma^6\psi_1\cdot((D-2)p_1^\mu(C_1(m^2,0,m^2,0,m_W^2,m_W^2)+2C_{11}(m^2,0,m^2,0,m_W^2,m_W^2)),\\ &p_2^\mu((D-8)C_1(m^2,0,m^2,0,m_W^2,m_W^2)+2(D-2)C_{12}(m^2,0,m^20,m_W^2,m_W^2)))\\ &+6m^2C_1(m^2,0,m^2,0,m_W^2,m_W^2)\cdot\bar{\psi}_2\gamma^\mu\gamma^6\psi_1), \end{aligned} \tag{C1}$$

where $D=4-\tilde{\epsilon}$ is the spacetime dimension of dimensional regularization, $\gamma^6=(1+\gamma^5)/2$, $\gamma^7=(1-\gamma^5)/2$, and C_{ij} are the Passarino-Veltman loop integral functions, which are evaluated with [52,53]. To reduce the resulting expression to the parity-violating terms, which are relevant for our analysis, we use the following simplifications:

(i) Retain only terms proportional to $\bar{\psi}_2\gamma^\mu\gamma^5\psi_1$.
(ii) Subtract divergent terms $\sim 1/\tilde{\epsilon}$.
(iii) Perform an expansion in δ_W^2 in which $A\sim\delta_W^0+\delta_W^2+\delta_W^4$. Since the order δ_W^0 is controlled by the dimensional regularization parameters ($\tilde{\epsilon}$, μ) and since the order δ_W^4 is negligible, we keep the $\sim\delta_W^2$ terms.
(iv) Set $D\to 4$.

The result is then

$$A|_{\mathrm{WW}} = -i\frac{5e^3m^2}{96\pi^2m_W^2\sin^2(\theta_W)}\bar{\psi}_2\gamma^\mu\gamma^5\psi_1. \tag{C2}$$

The same procedure can be repeated with the triangle diagram with a single Z boson in the loop, giving

$$A|_Z = i\frac{e^3m^2(6\log(m_Z^2/m^2)-1)(4\sin^2(\theta_W)-1)}{192\pi^2m_Z^2\sin^2(\theta_W)\cos^2(\theta_W)}\bar{\psi}_2\gamma^\mu\gamma^5\psi_1. \tag{C3}$$

Thus, one can conclude that parity-violating corrections to the QED couplings are indeed suppressed by

$$\sim\alpha\frac{m^2}{m_{W/Z}^2}. \tag{C4}$$

[1] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[2] R. P. Feynman, Phys. Rev. **80**, 440 (1950).
[3] J. S. Schwinger, Phys. Rev. **73**, 416 (1948).
[4] J. M. Luttinger, Phys. Rev. **74**, 893 (1948).
[5] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Phys. Rev. D **77**, 053012 (2008).
[6] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Phys. Rev. Lett. **109**, 111807 (2012).
[7] T. Aoyama, T. Kinoshita, and M. Nio, Phys. Rev. D **97**, 036001 (2018).
[8] T. Aoyama, T. Kinoshita, and M. Nio, Atoms **7**, 28 (2019).
[9] T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C. M. Carloni Calame, M. Cè, G. Colangelo *et al.* Phys. Rep. **887**, 1 (2020).
[10] H. M. Foley and P. Kusch, Phys. Rev. **73**, 412 (1948).
[11] P. Kusch and H. M. Foley, Phys. Rev. **74**, 250 (1948).
[12] D. Hanneke, S. F. Hoogerheide, and G. Gabrielse, Phys. Rev. A **83**, 052122 (2011).
[13] D. Hanneke, S. Fogwell, and G. Gabrielse, Phys. Rev. Lett. **100**, 120801 (2008).
[14] G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, and B. C. Odom, Phys. Rev. Lett. **97**, 030802 (2006); **99**, 039902(E) (2007).

[15] A. Broggio, E. J. Chun, M. Passera, K. M. Patel, and S. K. Vempati, J. High Energy Phys. 11 (2014) 058.
[16] R. Essig, J. Mardon, M. Papucci, T. Volansky, and Y. M. Zhong, J. High Energy Phys. 11 (2013) 167.
[17] M. Endo, K. Hamaguchi, and G. Mishima, Phys. Rev. D **86**, 095029 (2012).
[18] G. F. Giudice, P. Paradisi, and M. Passera, J. High Energy Phys. 11 (2012) 113.
[19] Y. Kahn, G. Krnjaic, S. Mishra-Sharma, and T. M. P. Tait, J. High Energy Phys. 05 (2017) 002.
[20] M. Raggi and V. Kozhuharov, Riv. Nuovo Cimento **38**, 449 (2015).
[21] R. Bouchendira, P. Clade, S. Guellati-Khelifa, F. Nez, and F. Biraben, Phys. Rev. Lett. **106**, 080801 (2011).
[22] G. Amelino-Camelia, F. Archilli, D. Babusci, D. Badoni, G. Bencivenni, J. Bernabeu, R. A. Bertlmann, D. R. Boito, C. Bini, C. Bloise *et al.* Eur. Phys. J. C **68**, 619 (2010).
[23] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller, Science **360**, 191 (2018).
[24] G. Amelino-Camelia, Living Rev. Relativity **16**, 5 (2013).
[25] A. Crivellin, M. Hoferichter, and P. Schmidt-Wellenburg, Phys. Rev. D **98**, 113002 (2018).
[26] H. Davoudiasl and W. J. Marciano, Phys. Rev. D **98**, 075011 (2018).
[27] M. Badziak and K. Sakurai, J. High Energy Phys. 10 (2019) 024.
[28] M. Bauer, M. Neubert, S. Renner, M. Schnubel, and A. Thamm, Phys. Rev. Lett. **124**, 211803 (2020).
[29] M. Endo and W. Yin, J. High Energy Phys. 08 (2019) 122.
[30] J. Liu, C. E. M. Wagner, and X. P. Wang, J. High Energy Phys. 03 (2019) 008.
[31] G. Hiller, C. Hormigos-Feliu, D. F. Litim, and T. Steudtner, Phys. Rev. D **102**, 071901 (2020).
[32] X. F. Han, T. Li, L. Wang, and Y. Zhang, Phys. Rev. D **99**, 095034 (2019).
[33] B. Dutta and Y. Mimura, Phys. Lett. B **790**, 563 (2019).
[34] L. Morel, Z. Yao, P. Cladé, and S. Guellati-Khélifa, Nature (London) **588**, 61 (2020).
[35] G. Kim, T. Demircik, D. K. Hong, and M. Järvinen, arXiv:2110.01169.
[36] L. S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 233 (1986).
[37] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[38] A. Ayala, J. D. Castaño-Yepes, M. Loewe, and E. Muñoz, Phys. Rev. D **104**, 016006 (2021).
[39] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (Dover Publications, New York, 2006), ISBN 9780070320710, p. 67.
[40] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin, 1992), ISBN-13:978-3540548836.
[41] J. Suzuki, arXiv:hep-th/0512329.
[42] A. Czarnecki, B. Krause, and W. J. Marciano, Phys. Rev. D **52**, R2619 (1995).
[43] S. Heinemeyer, D. Stockinger, and G. Weiglein, Nucl. Phys. **B699**, 103 (2004).
[44] G. V. Dunne, Eur. Phys. J. D **55**, 327 (2009).
[45] S. Dittmaier, Phys. Lett. B **409**, 509 (1997).
[46] V. A. Zykunov, J. Suarez, B. A. Tweedie, and Y. G. Kolomensky, arXiv:hep-ph/0507287.
[47] G. M. Jones, A Precision Measurement of the Weak Mixing Angle in Møller Scattering at Low Q^2, 10.7907/S506-PZ85.
[48] Y. Du, A. Freitas, H. H. Patel, and M. J. Ramsey-Musolf, Phys. Rev. Lett. **126**, 131801 (2021).
[49] A. Denner, Fortschr. Phys. **41**, 307 (1993).
[50] T. Hahn, Comput. Phys. Commun. **140**, 418 (2001).
[51] T. Hahn, S. Paßehr, and C. Schappacher, *Proc. Sci.*, LL2016 (2016) 068 [arXiv:1604.04611].
[52] A. Denner and S. Dittmaier, Nucl. Phys. **B734**, 62 (2006).
[53] H. H. Patel, Comput. Phys. Commun. **197**, 276 (2015).