Hawking effect in an extremal Kerr black hole spacetime

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It is well known that extremal black holes do not Hawking radiate, which is usually realized by taking an extremal limit from the nonextremal case. However, one cannot perceive the same phenomenon using the Bogoliubov transformation method starting from an extremal black hole itself, i.e., without the limiting case consideration. In that case, the Bogoliubov coefficients do not satisfy the required normalization condition. In canonical formulation, which closely mimics the Bogoliubov transformation method, one can consistently reproduce the vanishing number density of Hawking quanta for an extremal Kerr black hole. In this method, the relation between the spatial near-null coordinates, imperative in understanding the Hawking effect, was approximated into a sum of linear and inverse terms only. In the present work, we first show that one can reach the same conclusion in canonical formulation even with the complete relationship between the near-null coordinates, which contains an additional logarithmic term. It is worth mentioning that in the nonextremal case, a similar logarithmic term alone leads to the thermal Hawking radiation. Secondly, we study the case with only the inverse term in the relation (i.e., when the spatial near-null coordinates associated to the past and future observers are inversely related to each other) to understand whether it is the main contributing term in vanishing number density. Third, for a qualitative realization, we consider a simple thought experiment to understand the corresponding Hawking temperature and conclude that the inverse term indeed plays a crucial role in the vanishing number density.

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I. INTRODUCTION

The Hawking effect remains to be one of the most pioneering results perceived through the use of quantum field theory in a black hole spacetime. In [1] Hawking showed that an asymptotic future observer in a black hole spacetime observes a thermal distribution of particles which mathematically was realized through the Bogoliubov transformation between the ingoing and outgoing field modes. These field modes are again described in terms of the null coordinates that must satisfy a logarithmic relation among themselves to discern the Planckian distribution of the Hawking quanta [1]. While this Bogoliubov transformation method is one of the most straightforward methods to realizing the Hawking effect, it is unable to provide satisfactory results in a few areas. One encounters one of such case while dealing with an extremal black hole. In this regard, it is known that the extremal black holes do not Hawking radiate [2–4], which is noticed by taking the extremal limit from the nonextremal case and also using procedures like the tunneling formulation, Euclidean path integral formalism [3–11]. However, in the Bogoliubov transformation method, starting from an extremal black hole, the coefficients do not satisfy the necessary consistency condition emerging from the commutator of the ladder operators of the field modes [12]. It debars one to obtain the number density of the Hawking quanta reliably.

The authors in [13] provided a Hamiltonian-based derivation of the Hawking effect in a static Schwarzschild black hole background. One of the primary difficulties implementing a Hamiltonian-based framework to realize the Hawking effect was the null coordinates that describe the field modes related to the Bogoliubov transformation. These null coordinates do not lead to a true matter field Hamiltonian that can provide the evolution of the field modes. In [13], a set of near-null coordinates with spacelike and timelike signatures was introduced to circumvent this issue and construct the field Hamiltonians. The canonical formulation [13–15] closely mimics the Bogoliubov transformation method and utilizing it one can consistently reproduce the vanishing number density of the Hawking quanta in an extremal Kerr black hole spacetime [15]. Using this Hamiltonian formulation, the Hawking effect in a Schwarzschild and nonextremal Kerr black hole spacetime was realized in [13–15]. In these works the relation between the spatial near-null coordinates of two asymptotic observers plays a crucial role in the realization of Hawking

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effect. In [15] the authors established an exact relation between the spatial near-null coordinates but they used an approximated part of the complete relation to arrive at the vanishing number density of the Hawking quanta in a consistent fashion. This approximated relation contains one linear and one inverse term. However, in the literature using Bogoliubov transformation method only a similar inverse term in the relation between the null coordinates were taken into account for the estimations [12,16]. This led to an inconsistency in satisfying the normalization condition between the Bogoliubov coefficients. Therefore, it inspires us to perform a study with the complete relation between the spatial near-null coordinates in the context of canonical formulation. It is important to reconfirm and solidify the claims of previous results, also to find out which part of the relation is truly responsible for the vanishing number density of the Hawking quanta.

In this work, we consider the exact and complete relation between the near-null coordinates in an extremal Kerr black hole spacetime. A detailed study with the complete relation is important in its own right. This relation is a sum of linear, logarithmic, and inverse functions. The logarithmic term is important since such a similar term leads to the thermal spectrum of Hawking radiation in case of nonextremal Kerr black holes. We use the canonical formulation to obtain the number density of the Hawking quanta. We arrive at the conclusion that this entire general relation is also capable of providing the same conclusion of vanishing number density consistently. Furthermore, we also consider only the inverse relation between the spatial near-null coordinates, as this brings a close relevance to most of the works in the literature, and observe that even in this case mathematically, one can consistently get the vanishing number density. We also point out the subtleties and conceptual barriers to describe Hawking effect with this inverse relation approximation. Finally, we present a proper understanding of the primary contributing term in the vanishing number density by a simple thought experiment. Here outgoing particles nearly escaping being trapped by the event horizon in a nonextremal Kerr black hole spacetime reach an asymptotic future observer with a wavelength inversely proportional to the temperature of the Hawking effect, which is the *Wien's displacement law* for thermal distribution. On the other hand, for an extremal Kerr black hole, this wavelength tends to infinity, thus suggesting that the temperature corresponding to Hawking effect is zero. Moreover, we notice that the inverse term in the relation between the near-null coordinates is the main contributing term in this vanishing temperature of the Hawking effect in an extremal Kerr black hole spacetime.

In Sec. II, we provide a brief introduction about the Kerr black hole spacetime. In particular, in this section we talk about the horizon structure and the condition for extremality and set up the background for studying a massless, minimally coupled, free scalar field in this spacetime. In the succeeding Sec. III we present an overview of the canonical formulation with the near-null coordinates. In Sec. IV we specifically consider the extremal situation in the Kerr black hole spacetime and, using the canonical formulation, estimate the consistency condition and the number density of the Hawking quanta. We mention that in different subsections of this particular section, we shall be estimating the consistency and the number density using disparate relevant relations among the near-null coordinates. Subsequently, in Sec. V we prepare a physically understandable setup to shed further light on the dominating term in the relation between the near-null coordinates, which contributes to the vanishing number density. We conclude with a discussion of our findings in Sec. VI.

In this work we consider the natural units, i.e., the speed of light in vacuum c and Planck constant $\hbar$ will have unit values.

II. HORIZONS, EXTREMALITY, AND SCALAR FIELD IN A KERR BLACK HOLE SPACETIME

In this section, first, we are going to give a brief overview of the Kerr black hole spacetime. In particular, we will represent its metric in terms of the Boyer-Lindquist coordinates, elucidate the position of the two horizons, and the condition of extremality when the two horizons merge to a single one. Second, we will discuss the characteristics of a massless minimally coupled scalar field in this spacetime. Specifically, we shall talk about the action of this scalar field in regions near the horizon and radial infinity. The purpose is to show that one can utilize the understandings of quantum field theory from the flat spacetime in these regions and realize the Hawking effect.

A. The Kerr black hole spacetime

The Kerr black hole spacetime represents an exact solution of the vacuum Einstein field equations outside of a rotating mass. Unlike the hypothetical charged (Reissner-Nordström) and charged rotating (Kerr-Newman) black holes, the Kerr black holes have gained immense astrophysical significance, especially after the detection of the gravitational waves [17–20] from the perceived merger of these rotating black holes. In particular, the mass M and angular momentum per unit mass a are the sole delineating parameters which describe a Kerr black hole. One can express the line element in this spacetime using the Boyer-Lindquist coordinates [21] as

$$ds^2 = -\frac{1}{\rho^2}(\Delta - a^2\sin^2\theta)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 - \frac{2a}{\rho^2}(r^2 + a^2 - \Delta)\sin^2\theta dt d\phi, \quad (1)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$, $\Delta = r^2 + a^2 - r_s r$, and $r_s = 2GM$ with G representing the Newton's gravitational constant [22–33]. It should be mentioned that in this spacetime one obtains the coordinate singularity for $\Delta = 0$ and the curvature singularity for $\rho^2 = 0$. In the latter case the *Kretschmann scalar* is singular and mere coordinate transformations cannot remove this type of singularity. On the other hand, from the condition of the coordinate singularity $\Delta = 0$ one can find out the positions of the apparent horizons $r = r_h$ and $r = r_c$ as

$$r_h = \frac{1}{2}\left(r_s + \sqrt{r_s^2 - 4a^2}\right), \quad r_c = \frac{1}{2}\left(r_s - \sqrt{r_s^2 - 4a^2}\right). \quad (2)$$

Here r_h and r_c represent the outer event horizon and the Cauchy horizon, with $\varkappa_h = \sqrt{r_s^2 - 4a^2}/(2r_s r_h)$ and $\varkappa_c = \sqrt{r_s^2 - 4a^2}/(2r_s r_c)$ being their respective surface gravities. An interesting phenomenon in a Kerr black hole spacetime is an inertial observer is not static due to the frame-dragging effect (for example see Chapter 5, page 188 of [24], and Chapter 11, page 310 of [26]) and experiences an angular velocity

$$\Omega \equiv \Omega(r,\theta) = \frac{g^{t\phi}}{g^{tt}} = \frac{arr_s}{\Sigma}. \quad (3)$$

The effect of this frame dragging results in a nonzero $\dot{\phi} = a/\Delta$ in the null geodesics' governing equations [24], which is unlike the case in the Schwarzschild spacetime. The other governing equations for null geodesics here are $\dot{t} = (r^2 + a^2)/\Delta$, $\dot{r} = \pm 1$, $\dot{\theta} = 0$, where the overhead dot denotes derivatives with respect to some affine parameter. Using these equations one can perceive that along the ingoing null trajectories, the coordinates $v = t + r_\star$ and $\psi = \phi + r_\sharp$ are constants, while along the outgoing null trajectories the coordinates $u = t - r_\star$ and $\chi = \phi - r_\sharp$ are constants. The expressions of $r_\star$ (the *tortoise* coordinate) and $r_\sharp$ are obtained from

$$dr_\star = \frac{r^2 + a^2}{\Delta} dr, \qquad dr_\sharp = \frac{a}{\Delta} dr. \quad (4)$$

As we will study the Hawking effect in an extremal Kerr black hole spacetime, it is imperative to talk about the $r_\star$ in this case. In the extremal case, both the horizons from Eq. (2) merge together which corresponds to $a \to r_s/2$. In this case, from the definition (4) one can find out the expressions of the coordinates $r_\star$ and $r_\sharp$ for an extremal Kerr black hole as

$$r_\star = r + r_s \ln\left(\frac{2r - r_s}{r_s}\right) - \frac{r_s^2}{2r - r_s}; \quad r_\sharp = -\frac{2a}{2r - r_s}. \quad (5)$$

These expressions of $r_\star$ and $r_\sharp$ in terms of radial coordinate r differ depending on the extremal or nonextremal case. One also finds the tortoise coordinate to be singular [3,12,16,34] if one takes the limit $a \to r_s/2$ from the nonextremal case to get this expression.

B. Scalar field action in a Kerr black hole spacetime

To understand the consequences of semiclassical gravity in an extremal Kerr black hole spacetime we first consider a massless, minimally coupled free scalar field $\Phi(x)$ in a general spacetime background. The action of scalar field $\Phi(x)$ is given by

$$S_\Phi = \int d^4x \left[-\frac{1}{2}\sqrt{-g} g^{\bar{\mu}\nu} \nabla_{\bar{\mu}} \Phi(x) \nabla_\nu \Phi(x) \right]. \quad (6)$$

From [15] it is seen that this $(1+3)$ dimensional general action (6) can be transformed into a simple $(1+1)$ dimensional form in Kerr black hole spacetime in the regions near the event horizon and near scriplus and scriminus. For this purpose one can consider that due to the axial symmetry of the Kerr spacetime the scalar field can be decomposed as

$$\Phi(t,r,\theta,\phi) = \sum_{lm} e^{im\phi} \Phi_{lm}(t,r,\theta). \quad (7)$$

Putting this decomposition back into action (6) and integrating out the action over the azimuth angle ϕ one can further consider a redefinition of the field as

$$\Phi_{lm}(t,r,\theta) \equiv e^{-im\Omega t} \tilde{\Phi}_{lm}(t,r,\theta). \quad (8)$$

This will remove all the quantities with a single term of derivative with respect to time, i.e., terms with single $\partial_t \Phi_{lm}$. Then considering $\tilde{\Phi}_{lm}(t,r,\theta) = \mathscr{S}_{lm}(\theta)\varphi_{lm}(r_\star,t)/\sqrt{r^2+a^2}$, and using the spheroidal harmonics orthogonality condition $\int d(\cos\theta)\mathscr{S}_{lm}(\theta)\mathscr{S}^*_{l'm}(\theta) = \delta_{l,l'}$ one can reduce action S_Φ near horizon as well as near past and future null infinities as $S_\Phi = \sum_{lm} S_{lm}$. Here S_{lm} is given by

$$S_{lm} \simeq \int dt dr_\star \left[\frac{1}{2}\partial_t \varphi^*_{lm} \partial_t \varphi_{lm} - \frac{1}{2}\partial_{r_\star} \varphi^*_{lm} \partial_{r_\star} \varphi_{lm} \right], \quad (9)$$

which represents a scalar field action in $(1+1)$ dimensional flat Minkowski spacetime. We mention that with partial field decomposition for the angular coordinates, the scalar field action in a Kerr black hole spacetime is realized as an infinite collection of $(1+1)$ dimensional fields. This fact is well understood in the literature (for example, see the derivation of Eq. (6) in [35] and the discussions therein). As we have done in our work, one can further simplify the action by redefining the field, with relation like Eq. (8) and introducing a factor of $1/\sqrt{r^2+a^2}$; see also [15]. Then in asymptotically large r and near the event horizon, the field imitates the $(1+1)$ dimensional flat spacetime like shown in Eq. (9) with coordinates t and $r_\star$. It should be mentioned that for the Hawking effect the modes essential are constructed near the horizon. In this region if one assigns a frequency $\tilde{\omega}$ to the redefined field $\tilde{\Phi}_{lm}(t,r,\theta)$, i.e., $\tilde{\Phi}_{lm}(t,r,\theta) \sim e^{-i\tilde{\omega}t}$, then the physical field Φ_{lm} will have the frequency $\omega = \tilde{\omega} + m\Omega_h$. This fact can be realized from Eq. (8), with the identification of $\Omega = \Omega_h$ at the horizon, see [35–41]. Then for our subsequent discussions regarding the Hawking effect in an extremal Kerr black hole spacetime we shall consider the particular action (9) with the perceived transformation to the frequency ω of the physical field as

$$\tilde{\omega} = \omega - m\Omega_h. \tag{10}$$

To briefly outline the issues, we mention that the number density of the Hawking quanta in a nonextremal Kerr black hole spacetime and as seen by an asymptotic future observer (for a detailed analysis, see [15]) is given by

$$N_{\tilde{\omega}} = \frac{1}{e^{2\pi\tilde{\omega}/\varkappa_h} - 1} = \frac{1}{e^{2\pi(\omega - m\Omega_h)/\varkappa_h} - 1}, \tag{11}$$

where, as mentioned earlier, $\varkappa_h$ denotes the surface gravity at the event horizon. From this spectrum of particles, one may realize the characteristic temperature of the Hawking effect to be

$$T_H = \varkappa_h/(2\pi k_B) = \sqrt{r_s^2 - 4a^2}/(4\pi k_B r_s r_h), \tag{12}$$

where k_B denotes the Boltzmann constant. Now, one can observe that in the extremal limit $a \to r_s/2$, the surface gravity and thus the temperature vanishes. Then in this limit, the expression (11) confirms a vanishing number density. Thus taking the extremal limit from the nonextremal case, one can see that the Hawking effect ceases to exist. However, there remains a persisting debate whether one can compare the extremal black holes to the extremal limit of nonextremal ones (see [16] and the references therein and the discussions in [42,43]). Moreover, starting with an extremal black hole, it is observed in the literature [12,16] that the Bogoliubov transformation coefficients do not satisfy the consistency condition arising from the commutator brackets between the ladder operators related to the ingoing and outgoing field modes. Thus, the standard Bogoliubov transformation method remains inconclusive to providing any decisive outcome in this matter. On the other hand, using the canonical formulation [15], which closely mimics the Bogoliubov transformation method, one is able to perceive a vanishing number density of Hawking quanta in an extremal Kerr black hole spacetime, considering some simplifications in the analysis. Current work aims to understand this vanishing number density without these underlying simplifications and precisely point out the deciding factors behind this phenomenon.

III. CANONICAL FRAMEWORK

In the original work, [1] Hawking considered the Bogoliubov transformation between ingoing and outgoing field modes, which are delineated in terms of the respective null coordinates, to perceive the particle creation in a black hole spacetime. However, these null coordinates cannot be used as dynamical variables to describe a true field Hamiltonian, debarring one to get a canonical description of the phenomena. In this regard, in [13,15] the authors considered a set of near-null coordinates, obtained by slightly deforming the null coordinates, and constructed the necessary field Hamiltonians to realize the Hawking effect in the black hole spacetime. This procedure closely mimics the original formulation provided by Hawking. In particular, in [15] it was shown that using this canonical formulation, one can consistently reproduce the vanishing number density of Hawking quanta in an extremal Kerr black hole spacetime. In this work, we will consider this Hamiltonian formulation and first talk about the near-null coordinates necessary for its understanding.

A. The near-null coordinates

For an observer near the past null infinity $\mathscr{I}^-$, say observer $\mathbb{O}^-$, one defines the near-null coordinates as

$$\tau_- = t - (1-\epsilon)r_\star; \quad \xi_- = -t - (1+\epsilon)r_\star, \tag{13}$$

where ϵ is a very small valued dimensionless real positive parameter such that $\epsilon \gg \epsilon^2$. The near-null coordinates for an observer near the future null infinity $\mathscr{I}^+$, for observer $\mathbb{O}^+$, are defined as

$$\tau_+ = t + (1-\epsilon)r_\star; \quad \xi_+ = -t + (1+\epsilon)r_\star. \tag{14}$$

We mention that for the past observer $\mathbb{O}^-$ one has $dr_\star = dr$, as in that case, the black hole is not yet formed.

B. Field Hamiltonian and Fourier modes

From Eq. (9), one observes that in a Kerr black hole spacetime, the reduced scalar field action near the event horizon and the spatial infinities imitate the one from $(1+1)$ dimensional flat spacetime, described by the Minkowski metric $ds^2 = -dt^2 + dr_\star^2$. Then in terms of the near-null coordinates from Eqs. (13) and (14) one can obtain the line elements corresponding to $\mathbb{O}^-$ and $\mathbb{O}^+$, related to the conformally transformed flat metric $g^0_{\mu\nu}$, as

$$ds_\pm^2 = \frac{\epsilon}{2}\left[-d\tau_\pm^2 + d\xi_\pm^2 + \frac{2}{\epsilon}d\tau_\pm d\xi_\pm\right] \equiv \frac{\epsilon}{2} g^0_{\mu\nu} dx^\mu_\pm dx^\nu_\pm. \tag{15}$$

The subscript $+$ and $-$ denote the cases related to observer $\mathbb{O}^+$ and $\mathbb{O}^-$, respectively. The reduced scalar field action from (9) for both observers can now be expressed as

$$S_\varphi = \int d\tau_\pm d\xi_\pm \left[-\frac{1}{2}\sqrt{-g^0}g^{0\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \quad (16)$$

Here we have omitted the subscripts from the redefined field φ_{lm} for brevity of notation. From Eq. (15), one sees that corresponding lapse function, shift vector, and the determinant of the spatial metric are respectively given by $N = 1/\epsilon$, $N^1 = 1/\epsilon$, and $q = 1$. Then considering spatial slicing of the reduced spacetime with respect to $\tau_\pm$, the scalar field Hamiltonian for the two observers can be expressed as

$$H_\varphi^\pm = \int d\xi_\pm \frac{1}{\epsilon}\left[\left\{\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_{\xi_\pm}\varphi)^2\right\} + \Pi\partial_{\xi_\pm}\varphi\right], \quad (17)$$

with Π being the conjugate momentum to the field φ. The momentum Π can be obtained from the Hamilton's equation, given by

$$\Pi(\tau_\pm, \xi_\pm) = \epsilon\partial_{\tau_\pm}\varphi - \partial_{\xi_\pm}\varphi. \quad (18)$$

The field φ and the momentum Π satisfy the Poisson bracket

$$\{\varphi(\tau_\pm, \xi_\pm), \qquad \Pi(\tau_\pm, \xi'_\pm)\} = \delta(\xi_\pm - \xi'_\pm). \quad (19)$$

From Eq. (17) one can observe that the Hamiltonian becomes ill-defined at $\epsilon = 0$, representing the necessity of near-null coordinates for the realization of the Hawking effect using a Hamiltonian formulation. Now, as $\sqrt{q} = 1$ and $V_\pm = \int d\xi_\pm\sqrt{q}$ we consider finite fiducial box during the intermediate steps of computations to avoid dealing with diverging spatial volumes, with

$$V_\pm = \int_{\xi_\pm^L}^{\xi_\pm^R} d\xi_\pm\sqrt{q} = \xi_\pm^R - \xi_\pm^L, \quad (20)$$

and the Fourier transformations of the scalar field for the observers $\mathbb{O}^+$ and $\mathbb{O}^-$ are defined as

$$\varphi(\tau_\pm, \xi_\pm) = \frac{1}{\sqrt{V_\pm}}\sum_k \tilde{\phi}_k^\pm e^{ik\xi_\pm},$$
$$\Pi(\tau_\pm, \xi_\pm) = \frac{1}{\sqrt{V_\pm}}\sum_k \sqrt{q}\,\tilde{\pi}_k^\pm e^{ik\xi_\pm}, \quad (21)$$

where the complex-valued mode functions $\tilde{\phi}_k^\pm$, $\tilde{\pi}_k^\pm$ depend on $\tau_\pm$. Here the Kronecker and Dirac delta are defined as $\int d\xi_\pm\sqrt{q}e^{i(k-k')\xi_\pm} = V_\pm\delta_{k,k'}$ and $\sum_k e^{ik(\xi_\pm - \xi'_\pm)} = V_\pm\delta(\xi_\pm - \xi'_\pm)/\sqrt{q}$, and they imply $k \in \{k_s\}$ where $k_s = 2\pi s/V_\pm$ with s being nonzero integers. With the help of these definitions one can express the field Hamiltonians (17) in terms of the Fourier modes as $H_\varphi^\pm = \sum_k \frac{1}{\epsilon}(\mathcal{H}_k^\pm + \mathcal{D}_k^\pm)$ where

$$\mathcal{H}_k^\pm = \frac{1}{2}\tilde{\pi}_k^\pm\tilde{\pi}_{-k}^\pm + \frac{1}{2}k^2\tilde{\phi}_k^\pm\tilde{\phi}_{-k}^\pm, \quad (22)$$

and

$$\mathcal{D}_k^\pm = -\frac{ik}{2}(\tilde{\pi}_k^\pm\tilde{\phi}_{-k}^\pm - \tilde{\pi}_{-k}^\pm\tilde{\phi}_k^\pm), \quad (23)$$

respectively, denote the Hamiltonian densities and diffeomorphism generators and the corresponding Poisson brackets are now discretized as

$$\{\tilde{\phi}_k^\pm, \tilde{\pi}_{-k'}^\pm\} = \delta_{k,k'}. \quad (24)$$

C. Relation between Fourier modes

Since we are dealing with scalar field, the fact that $\varphi(\tau_-, \xi_-) = \varphi(\tau_+, \xi_+)$ makes it possible to express a particular field mode near $\mathscr{I}^+$ in terms of all the modes near $\mathscr{I}^-$. Similarly one can show that the field momentum at those regions obeys $\Pi(\tau_+, \xi_+) = (\partial\xi_-/\partial\xi_+)\Pi(\tau_-, \xi_-)$ [13]. This follows from Eq. (18) along with the fact that ingoing and outgoing modes travel keeping the null coordinates v and u constant, respectively. Using these the relations between the Fourier modes and their conjugate momenta follows the relations

$$\tilde{\phi}_\kappa^+ = \sum_k \tilde{\phi}_k^- F_0(k, -\kappa); \quad \tilde{\pi}_\kappa^+ = \sum_k \tilde{\pi}_k^- F_1(k, -\kappa), \quad (25)$$

where the coefficient functions $F_n(k, \kappa)$ are given by

$$F_n(k, \kappa) = \frac{1}{\sqrt{V_- V_+}}\int d\xi_+ \left(\frac{\partial\xi_-}{\partial\xi_+}\right)^n e^{ik\xi_- + i\kappa\xi_+}, \quad (26)$$

where $n = 0, 1$ and these coefficient functions mimic the Bogoliubov coefficients [1]. Using the general definition of the Dirac delta distribution $\delta(\mu) = \frac{1}{2\pi}\int dx e^{i\mu x}$ and by choosing $\mu = 1$, $x = (\pm k\xi_- + \kappa\xi_+)$ one gets

$$F_1(\pm k, \kappa) = \mp\left(\frac{\kappa}{k}\right)F_0(\pm k, \kappa). \quad (27)$$

So the evaluation of only one type of coefficient function, corresponding to $n = 0$ or $n = 1$, will be sufficient for our purpose. Also it is evident that the coefficient functions $F_n(\pm k, \pm\kappa)$ are complex conjugate to $F_n(\mp k, \mp\kappa)$. Later on we will use this fact to avoid mathematical complications.

D. Consistency condition and number density of Hawking quanta

Using Eq. (27) and demanding that the two different Poisson brackets $\{\tilde{\phi}_k^-, \tilde{\pi}_{-k'}^-\} = \delta_{k,k'}$ and $\{\tilde{\phi}_\kappa^+, \tilde{\pi}_{-\kappa'}^+\} = \delta_{\kappa,\kappa'}$ be simultaneously satisfied, we may express a consistency requirement among the coefficient functions as

$$\mathbb{S}_{-}(\kappa) - \mathbb{S}_{+}(\kappa) = 1, \tag{28}$$

where $\mathbb{S}_{\pm}(\kappa) = \sum_{k>0}(\kappa/k)|F_0(\pm k,\kappa)|^2$. One may find a similarity between this condition and the one from the Bogoliubov transformation method [12] with the latter one arising from the commutation relation of the ladder operators of the field modes. Using relations (25) and (27) one can express $\mathcal{H}^+_\kappa$ and $\mathcal{D}^+_\kappa$ as

$$\mathcal{H}^+_\kappa = h^1_\kappa + \sum_{k>0}\left(\frac{\kappa}{k}\right)^2[|F_0(-k,\kappa)|^2 + |F_0(k,\kappa)|^2]\mathcal{H}^-_k, \tag{29}$$

$$\mathcal{D}^+_\kappa = d^1_\kappa + \sum_{k>0}\left(\frac{\kappa}{k}\right)^2[|F_0(-k,\kappa)|^2 + |F_0(k,\kappa)|^2]\mathcal{D}^-_k, \tag{30}$$

where h^1_κ and d^1_κ are linear in ϕ^-_k and its conjugate momentum, i.e., the vacuum expectation values of their quantum counterpart vanish.

Now for real-valued scalar field the Fourier modes satisfy $\tilde{\phi}^*_k = \tilde{\phi}_{-k}$, implying the real and imaginary parts of these Fourier modes not being independent. One may suitably select one of these real or imaginary parts in different domains of the Fourier modes [13,44] so that the previous reality condition is implemented. It is seen [13,44] that this makes $\mathcal{D}^-_k = 0$ and the Hamiltonian density to represent a simple harmonic oscillator

$$\mathcal{H}^\pm_k = \frac{1}{2}\pi^2_k + \frac{1}{2}k^2\phi^2_k, \qquad \{\phi^2_k, \pi^2_{k'}\} = \delta_{k,k'}, \tag{31}$$

where ϕ_k and π_k are the redefined Fourier field modes which are real-valued. Now a mode with wave vector k has frequency $|k|$ and the energy spectrum for each of these oscillator modes is given by $\hat{\mathcal{H}}^-_k|n_k\rangle = \left(\hat{N}^-_k + \frac{1}{2}\right)|k||n_k\rangle = \left(n_k + \frac{1}{2}\right)k|n_k\rangle$ where $\hat{N}^-_k$ is the corresponding number operator, $|n_k\rangle$ are its eigenstates with integer eigenvalues $n_k \geq 0$. It is also understood that for realizing the Hawking effect one has to evaluate the expectation value of the Hamiltonian density operator $\hat{\mathcal{H}}^+_\kappa \equiv (\hat{N}^+_\kappa + \frac{1}{2})$ for observer $\mathbb{O}^+$ in the vacuum state $|0_-\rangle = \Pi_k|0_k\rangle$ of the observer $\mathbb{O}^-$. Then using Eqs. (28) with (29) the expectation value of the number density operator corresponding to the Hawking quanta of frequency $\tilde{\omega} = \kappa$ is obtained as [13,15]

$$N_{\tilde{\omega}} = N_\kappa \equiv \langle 0_-|\hat{N}^+_\kappa|0_-\rangle = \mathbb{S}_+(\kappa). \tag{32}$$

IV. EXTREMAL KERR BLACK HOLE

To understand the particle creation in an extremal Kerr black hole spacetime, we have to first determine the relation between the spatial near-null coordinates ξ_+ and ξ_-, which requires the expression of the tortoise coordinate from Eq. (5). To derive this relation, we consider a spatial $\tau_- =$ constant hypersurface, with a pivotal point ξ^0_- on it, corresponding to observer $\mathbb{O}^-$. Then any spacelike interval can be expressed as

$$(\xi_- - \xi^0_-)_{|\tau_-} = 2(r^0_\star - r_\star)_{|\tau_-} = 2(r^0 - r)_{|\tau_-} \equiv \Delta, \tag{33}$$

where r^0 relates to ξ^0_-. On the other hand, corresponding to observer $\mathbb{O}^+$ a spacelike interval on a $\tau_+ =$ constant hypersurface can be expressed using Eq. (5), as

$$(\xi_+ - \xi^0_+)_{|\tau_+} = \Delta + 2r_s\ln\left(1 + \frac{\Delta}{\Delta_0}\right) - \frac{2r_s^2}{\Delta + \Delta_0} + \frac{2r_s^2}{\Delta_0}, \tag{34}$$

where using geometric optics approximation one can identify the interval $2(r - r^0)_{|\tau_+}$ as Δ and we have $\Delta_0 \equiv 2(r^0 - r_s/2)_{|\tau_+}$. Furthermore, choosing the pivotal value $\xi^0_- = \Delta_0$ and $\xi^0_+ = \xi^0_- + 2r_s\ln(\xi^0_-/\sqrt{2}r_s) - 2r_s^2/\xi^0_-$, one can express the above relation as

$$\xi_+ = \xi_- + 2r_s\ln\left(\frac{\xi_-}{\sqrt{2}r_s}\right) - \frac{2r_s^2}{\xi_-}. \tag{35}$$

The presence of the inverse term makes the relation qualitatively different from nonextremal case [15]. Furthermore, to keep track of contribution of the individual term in the right-hand side of Eq. (35) we introduce the parameters α_j, with $j = 1, 2, 3$, and express the relation as

$$\xi_+ = \alpha_1\xi_- + \alpha_2 2r_s\ln\left(\frac{\xi_-}{\sqrt{2}r_s}\right) - \alpha_3\frac{2r_s^2}{\xi_-}, \tag{36}$$

where α_j can only take values 0 or 1. In our subsequent analysis we shall be using this relation to evaluate the consistency condition and the number density of Hawking quanta in an extremal Kerr black hole spacetime. Later we shall also consider only the inverse term in the relation (36) to understand the consequences.

A. Understanding the Hawking effect using the complete relation among the spatial near-null coordinates

Here we shall be evaluating the coefficient functions of Eq. (26) with the complete relation between the spatial near-null coordinates ξ_+ and ξ_- from Eq. (36). Then we shall give a detailed understanding on the consistency condition from Eq. (28) and subsequently the number density (32) corresponding to the Hawking effect. It should be noted that in the general relation (36) by keeping only a certain α_j nonvanishing one can ascertain the role of individual terms contributing in the satisfaction of the consistency condition and also in the vanishing number density of the Hawking quanta. In our following study we shall also discuss few approximations on the relation (36) and their outcomes.

1. Evaluation of the coefficient functions

As already discussed one can either estimate $F_0(\pm k,\kappa)$ or $F_1(\pm k,\kappa)$ and get the expression of the other coefficient function from (27). In particular, here for the lucidity of calculation we estimate only $F_1(\pm k,\kappa)$. $F_0(\pm k,\kappa)$ will follow from relation (27). Putting the expression of ξ_+ from Eq. (36) in Eq. (26) with $n=1$ we get

$$F_1(\pm k,\kappa)=\frac{1}{\sqrt{V_-V_+}}\int_{\xi_-^L}^{\xi_-^R}d\xi_-\left(\frac{\xi_-}{\sqrt{2}r_s}\right)^{i\alpha_2 2\kappa r_s}\times e^{i(\pm k+\alpha_1\kappa)\xi_- -\frac{i\alpha_3 2\kappa r_s^2}{\xi_-}}. \quad (37)$$

We mention that this integrand is oscillatory in nature and for the limit $\xi_-^L\to 0$, $\xi_-^R\to\infty$ it is essentially divergent. Therefore, in this case we shall be introducing regulators with parameter "δ" to properly evaluate the integrals

$$F_1^\delta(\pm k,\kappa)=\frac{b}{\sqrt{V_-V_+}}\int_{x^L}^{x^R}dx x^{\sqrt{2}\alpha_2 b_0}\exp\left(-b_\pm x-\frac{\alpha_3 b_0}{x}\right), \quad (38)$$

where $b=\sqrt{2}r_s$, $b_\pm=b[\delta|\pm k+\alpha_1\kappa|-i(\pm k+\alpha_1\kappa)]$ and $b_0=b[\delta|\kappa|+i\kappa]$. Here we have represented the original integral in terms of the dimensionless variable $x=\xi_-/b$. It should be noted that in the limit $\delta\to 0$ Eq. (38) goes back to Eq. (37).

One can notice from Eq. (37) that there is a possibility of $b_-=0$ when $\alpha_1\kappa=k$. In that scenario the characteristic form of the integral will be different. So we will explore this particular situation separately. Also, we remind that k, $\kappa>0$. So this particular situation will not arise if $\alpha_1=0$. It is worth mentioning that the fact $F_n(\pm k,\kappa)$ is complex conjugate to $F_n(\mp k,-\kappa)$ does not get violated due to the inclusion of the regularization parameter δ.

a. Evaluation of $F_1(-\alpha_1\kappa,\kappa)$:
In this particular situation $\alpha_1\kappa=k$ and $b_-=0$. Then the integral form Eq. (37) becomes

$$F_1^\delta(-\alpha_1\kappa,\kappa)=\frac{1}{\sqrt{V_-V_+}}\int_{\xi_-^L}^{\xi_-^R}d\xi_-\left(\frac{\xi_-}{\sqrt{2}r_s}\right)^{i\alpha_2 2\kappa r_s}e^{-\frac{i\alpha_3 2\kappa r_s^2}{\xi_-}}. \quad (39)$$

In terms of the dimensionless variable along with the δ regulator, i.e., the expression from Eq. (38) in this scenario becomes

$$F_1^\delta(-\alpha_1\kappa,\kappa)=\frac{b}{\sqrt{V_-V_+}}\int_{x^L}^{x^R}dx x^{\sqrt{2}\alpha_2 b_0}e^{-\frac{\alpha_3 b_0}{x}}. \quad (40)$$

The explicit form of the above integral can be given in terms of *incomplete gamma* functions as

$$F_1^\delta(-\alpha_1\kappa,\kappa)=\frac{b(\alpha_3 b_0)^{\sqrt{2}\alpha_2 b_0+1}}{\sqrt{V_-V_+}}\left[\Gamma\left(-1-\sqrt{2}\alpha_2 b_0,\frac{\alpha_3 b_0}{x^R}\right)-\Gamma\left(-1-\sqrt{2}\alpha_2 b_0,\frac{\alpha_3 b_0}{x^L}\right)\right]. \quad (41)$$

The second *gamma* function within the square bracket will vanish as $x^L\to 0$. Following the properties of *incomplete gamma functions* [45] in the limit $x^R\to\infty$, we get the above expression to become

$$|F_1^\delta(-\alpha_1\kappa,\kappa)|^2\approx\frac{\gamma}{1+2\alpha_2^2 b^2\kappa^2}, \quad (42)$$

where $\gamma\equiv V_-/V_+$. In deriving the above result we have used the fact that in the limit $x^L\to 0, x^R\to\infty$, $V_-\sim x^R$.

b. Evaluation of $F_1(\pm k,\kappa)$, for $k\neq\alpha_1\kappa$:
When $k\neq\alpha_1\kappa$ one has $b_-\neq 0$, and the expression of the integral (38) can be obtained in terms of *modified Bessel* functions of second kind $\mathcal{K}(\mu,z)$ as

$$F_1^\delta(\pm k,\kappa)=\frac{2b}{\sqrt{V_-V_+}}\left(\frac{\alpha_3 b_0}{b_\pm}\right)^{\frac{1+\sqrt{2}\alpha_2 b_0}{2}}\times\mathcal{K}(-1-\sqrt{2}\alpha_2 b_0, 2\sqrt{\alpha_3 b_\pm b_0}). \quad (43)$$

Here we have considered the limits of the integral (38) from 0 to ∞ instead of x^L to x^R, which can be done by adding two boundary terms which tends to zero as $x^L\to 0$ and $x^R\to\infty$ [15]. We mention that these modified Bessel functions have the following approximate expressions [45] for different limits of their arguments:

$$\mathcal{K}(\mu,z)\sim\frac{1}{2}\left(\frac{2}{z}\right)^\mu\Gamma(\mu);\quad \text{as } z\to 0,$$
$$\mathcal{K}(\mu,z)\sim\sqrt{\frac{\pi}{2z}}e^{-z};\quad \text{as } z\to\infty. \quad (44)$$

Here μ and z can be both real as well as complex valued. Another relation satisfied by these modified Bessel functions is $\mathcal{K}(\mu,z)=\mathcal{K}(-\mu,z)$, which will also be relevant in our study.

2. Consistency condition

So far we have determined the coefficient functions. Now we proceed to check for the consistency condition and the number density of the Hawking quanta. In this regard, first we need to estimate the left-hand side of Eq. (28) which reads

$$\mathbb{S}_{-}^{\delta}(\kappa) - \mathbb{S}_{+}^{\delta}(\kappa) = |F_1^{\delta}(-\alpha_1\kappa,\kappa)|^2 + \sum_{\substack{k>0\\ k\neq\alpha_1\kappa}} \left(\frac{k}{\kappa}\right)|F_1^{\delta}(-k,\kappa)|^2 - \sum_{k>0}\left(\frac{k}{\kappa}\right)|F_1^{\delta}(k,\kappa)|^2. \quad (45)$$

An explicit calculation using the functional form of the coefficient function F_1^{δ}, as derived in Eq. (43), leads to the following:

$$\begin{aligned} |F_1^{\delta}(k,\kappa)|^2 &\approx \frac{e^{-\pi\sqrt{2}b\kappa\alpha_2}}{V_-V_+}\frac{1}{(\alpha_1\kappa+k)^2}|z_+\mathcal{K}(\bar{\mu},z_+)|^2 \\ |F_1^{\delta}(-k,\kappa)|^2 &\approx \frac{e^{-\pi\sqrt{2}b\kappa\alpha_2}}{V_-V_+}\frac{1}{(\alpha_1\kappa-k)^2}|z_-\mathcal{K}(\bar{\mu},z_-)|^2; \quad \alpha_1\kappa>k \\ &= \frac{1}{V_-V_+}\frac{1}{(k-\alpha_1\kappa)^2}|z_-\mathcal{K}(\bar{\mu},z_-)|^2; \quad \alpha_1\kappa<k, \end{aligned} \quad (46)$$

where $\bar{\mu} = -1 - \sqrt{2}\alpha_2 b_0$. Also, z_- for $\kappa > k$ is not same as z_- for $\kappa < k$ and $z_\pm$ are given by

$$\begin{aligned} z_\pm &= 2\sqrt{\alpha_3 b_0 b_\pm} \\ &= 2\sqrt{\frac{\alpha_3|b_0|^2(i+\delta)[\delta|\alpha_1 m_\star \pm s| - i(\alpha_1 m_\star \pm s)]}{m_\star}}. \end{aligned} \quad (47)$$

Here we have used $k = 2\pi s/V_-$ and $m_\star = V_-\kappa/(2\pi)$. The first term in the right-hand side of Eq. (45) has already been calculated previously in Eq. (42). Now we shall concentrate in calculating the other two summations. Using Eqs. (46) and (47) the last two terms of the right-hand side of (45) become

$$\begin{aligned} \sum_{\substack{k>0\\ k\neq\alpha_1\kappa}}\left(\frac{k}{\kappa}\right)|F_1^{\delta}(-k,\kappa)|^2 - \sum_{k>0}\left(\frac{k}{\kappa}\right)|F_1^{\delta}(k,\kappa)|^2 = \frac{\gamma}{4\pi^2 m_\star}\Bigg[& e^{-\pi\sqrt{2}b\kappa\alpha_2}\sum_{\substack{s=1\\ s<\alpha_1 m_\star}}^{\alpha_1 m_\star -1}\frac{s}{(\alpha_1 m_\star - s)^2}|z_-\mathcal{K}(\bar{\mu},z_-)|^2 \\ & + \sum_{\substack{s=\alpha_1 m_\star+1\\ s>\alpha_1 m_\star}}^{\infty}\frac{s}{(s-\alpha_1 m_\star)^2}|z_-\mathcal{K}(\bar{\mu},z_-)|^2 - e^{-\pi\sqrt{2}b\kappa\alpha_2}\sum_{s=1}^{\infty}\frac{s}{(\alpha_1 m_\star + s)^2}|z_+\mathcal{K}(\bar{\mu},z_+)|^2\Bigg]. \end{aligned} \quad (48)$$

It is worth mentioning that z_- in the first and second summation is not exactly the same but it has to be calculated for $s < \alpha_1 m_\star$ and $s > \alpha_1 m_\star$, respectively. Performing a change in variables as $s - \alpha_1 m_\star = p, \alpha_1 m_\star - s = q$ and $s + \alpha_1 m_\star = r$, the above summation can be recast in the following way:

$$\begin{aligned} \frac{\gamma}{4\pi^2 m_\star}\Bigg[& e^{-\pi\sqrt{2}b\kappa\alpha_2}\sum_{q=1}^{\alpha_1 m_\star -1}\frac{\alpha_1 m_\star - q}{q^2}|\tilde{z}(q)\mathcal{K}(\bar{\mu},|\tilde{z}(q)|)|^2 + \sum_{p=1}^{\infty}\frac{\alpha_1 m_\star + p}{p^2}|\tilde{z}(p)\mathcal{K}(\bar{\mu},\tilde{z}(p))|^2 \\ & - e^{-\pi\sqrt{2}b\kappa\alpha_2}\sum_{r=\alpha_1 m_\star+1}^{\infty}\frac{r-\alpha_1 m_\star}{r^2}|\tilde{z}(r)\mathcal{K}(\bar{\mu},|\tilde{z}(r)|)|^2\Bigg]. \end{aligned} \quad (49)$$

Here $z_-(s)$ has been changed to $|\tilde{z}(q)| = \sqrt{4\alpha_3|b_0|^2 q/m_\star}$ for $\alpha_1 m_\star > s$. Similarly for $\alpha_1 m_\star < s$, $z_-(s)$ takes the form $\tilde{z}(p) = \sqrt{4\alpha_3|b_0|^2 p/m_\star}(i+\delta)$. Also, $z_+(s)$ goes to $|\tilde{z}(r)| = \sqrt{4\alpha_3|b_0|^2 r/m_\star}$. Rearranging the above summations leads to

$$\frac{\gamma}{4\pi^2}\sum_{s=1}^{\infty}\frac{\alpha_1}{s^2}[|\tilde{z}(s)\mathcal{K}(\bar{\mu},\tilde{z}(s))|^2 + e^{-\pi\sqrt{2}b\kappa\alpha_2}|\tilde{z}(s)\mathcal{K}(\bar{\mu},|\tilde{z}(s)|)|^2] + \frac{\gamma}{4\pi^2 m_\star}\sum_{s=1}^{\infty}\frac{1}{s}[|\tilde{z}(s)\mathcal{K}(\bar{\mu},\tilde{z}(s))|^2 - e^{-\pi\sqrt{2}b\kappa\alpha_2}|\tilde{z}(s)\mathcal{K}(\bar{\mu},|\tilde{z}(s)|)|^2], \quad (50)$$

where $\tilde{z}(s) = \sqrt{4\alpha_3|b_0|^2 s/m_\star}(i+\delta)$, and we also note that $\pi\sqrt{2}b\kappa\alpha_2 = 2\pi r_s\kappa\alpha_2$. So far we have not used any asymptotic expressions (44) of the Bessel function. From the above expression it is clear that in the infinite volume limit i.e., $V_- \to \infty$ which implies $m_\star \to \infty$, the second summation vanishes. Now using the approximations for modified Bessel functions, as mentioned in Eq. (44), when $\tilde{z} \to 0$, and $\mathcal{K}(-\bar{\mu}, z) = \mathcal{K}(\bar{\mu}, z)$ we get

$$|\tilde{z}(s)\mathcal{K}(\bar{\mu}, \tilde{z}(s))|^2 \approx \frac{2\pi r_s\kappa\alpha_2 e^{2\pi r_s\kappa\alpha_2}}{\sinh(2\pi r_s\kappa\alpha_2)}$$
$$|\tilde{z}(s)\mathcal{K}(\bar{\mu}, |\tilde{z}(s)|)|^2 \approx \frac{2\pi r_s\kappa\alpha_2}{\sinh(2\pi r_s\kappa\alpha_2)}. \tag{51}$$

Then from these expressions also one can observe that the second term in Eq. (50) vanishes when $\tilde{z} \to 0$ and one takes $m_\star \to \infty$. On the other hand, when $\tilde{z} \to \infty$ one can get the limiting expressions as

$$|\tilde{z}(s)\mathcal{K}(\bar{\mu}, \tilde{z}(s))|^2 \approx \sqrt{\frac{\alpha_3\pi^2|b_0|^2 s}{m_\star}}e^{-2\delta\sqrt{\frac{4\alpha_3|b_0|^2 s}{m_\star}}}$$
$$|\tilde{z}(s)\mathcal{K}(\bar{\mu}, |\tilde{z}(s)|)|^2 \approx \sqrt{\frac{\alpha_3\pi^2|b_0|^2 s}{m_\star}}e^{-2\sqrt{\frac{4\alpha_3|b_0|^2 s}{m_\star}}}. \tag{52}$$

These expressions also confirm that the second term in Eq. (50) vanishes when $\tilde{z} \to \infty$ as one takes $m_\star \to \infty$. Here in the first equation one can easily see that in the case $\delta = 0$ the right-hand side term diverges out as $s \to \infty$. Finally, the left-hand side of the consistency relation Eq. (28) reads

$$\begin{aligned}\mathbb{S}_-^\delta(\kappa) - \mathbb{S}_+^\delta(\kappa) &= \frac{\gamma}{1+2\alpha_2^2 b^2\kappa^2} + \frac{\gamma}{4\pi^2}\sum_{s=1}^{\infty}\frac{\alpha_1}{s^2}[|\tilde{z}(s)\mathcal{K}(\bar{\mu}, \tilde{z}(s))|^2 + e^{-2\pi r_s\kappa\alpha_2}|\tilde{z}(s)\mathcal{K}(\bar{\mu}, |\tilde{z}(s)|)|^2] \\ &\approx \frac{\gamma}{1+2\alpha_2^2 b^2\kappa^2} + \frac{\alpha_1\gamma}{4\pi^2}y\coth(y)\int_1^{\lambda_1 m_\star}\frac{ds}{s^2} + \frac{\alpha_1\gamma}{4\pi^2}\pi a\int_{\lambda_2 m_\star}^{\infty}\frac{ds}{s^2}\sqrt{s}[e^{-4\delta a\sqrt{s}} + e^{-4a\sqrt{s}}],\end{aligned} \tag{53}$$

where $y = 4\pi r_s\kappa\alpha_2$ and $a = \sqrt{\frac{\alpha_3|b_0|^2}{m_\star}}$. In order to arrive at the above expression we have considered $\tilde{z} \ll 1$ when $s \in [1, \lambda_1 m_\star]$ and $\tilde{z} \gg 1$ when $s \in [\lambda_2 m_\star, \infty)$. After carrying out these integrals and then taking the infinite volume limit, $m_\star \to \infty$, the consistency relation becomes

$$\gamma\left[\frac{1}{1+4\alpha_2^2 r_s^2\kappa^2} + \frac{\alpha_1}{4\pi^2}y\coth(y)\right] = 1. \tag{54}$$

One can notice that this consistency condition does not contain the integral regulator δ, but it depends on the volume regulators V_+ and V_- through the expression of γ. Then this consistency condition basically says that these two volume regulators corresponding two different observers, namely observer $\mathbb{O}^+$ and $\mathbb{O}^-$, are not independent of each other and are related among themselves for a proper consistency of the estimations. In particular, it specifically ascertains that for a fixed frequency of the outgoing field mode κ, the volume V_- must be proportional to V_+, i.e., $V_- \propto V_+$. From Eq. (54) we also observe that if one takes the case $\alpha_1 = 1$ with the limit $\alpha_2 \to 0$, i.e., in the case $\lim_{y\to 0}[y\coth(y)] = 1$, the consistency condition exactly reduces to $\gamma = 1 + 1/12 = 13/12$ stating the same requirement for consistency. This latter case signifies making the contribution of the logarithmic term to vanish in the relation between the spatial near-null coordinates of Eq. (36), which was obtained in [15]. Therefore, from our present calculations one can get back the results of [15] in a straightforward manner.

3. Number density of Hawking quanta

As already observed in Eq. (32) the expectation value of the number density operator corresponding to the Hawking quanta is give by $N_{\tilde{\omega}} = \mathbb{S}_+(\kappa)$, which in the present scenario can be expressed in the form

$$\begin{aligned}N_{\tilde{\omega}} &= \sum_{k>0}\left(\frac{k}{\kappa}\right)|F_1^\delta(k,\kappa)|^2 \\ &= \frac{\gamma e^{-\pi\sqrt{2}b\kappa\alpha_2}}{4\pi^2 m_\star}\sum_{s=1}^{\infty}\frac{s}{(s+\alpha_1 m_\star)^2}|z_+(s)\mathcal{K}(-1-\sqrt{2}\alpha_2 b_0, z_+(s))|^2 \\ &= \frac{\gamma e^{-\pi\sqrt{2}b\kappa\alpha_2}}{4\pi^2 m_\star}\sum_{s=1+\alpha_1 m_\star}^{\infty}\frac{s-\alpha_1 m_\star}{s^2}|\tilde{z}(s)\mathcal{K}(-1-\sqrt{2}\alpha_2 b_0, |\tilde{z}(s)|)|^2,\end{aligned} \tag{55}$$

where $|\tilde{z}(s)| = \sqrt{4\alpha_3|b_0|^2 s/m_\star}$. This expression of number density from Eq. (55) can be expressed in terms of the sum of two quantities. First, a sum over a quantity with a factor of $1/s$ in it. Second, the sum over a quantity with a multiplicative factor of $1/s^2$ in it. It should be mentioned that there is a term $1/m_\star$ multiplied with the first term, which makes the term vanish in the limit of $m_\star \to \infty$, i.e., in the infinite volume limit. On the other hand, when $\alpha_1 = 0$ the second term vanishes. Let us evaluate the second sum when $\alpha_1 \neq 0$. One can express this sum as

$$\frac{\alpha_1\gamma e^{-\pi\sqrt{2}b\kappa\alpha_2}}{4\pi^2}\sum_{s=1+\alpha_1 m_\star}^{\infty}\frac{1}{s^2}|\tilde{z}\mathcal{K}(-1-\alpha_2 b_1,|\tilde{z}|)|^2 = \frac{\alpha_1\gamma e^{-\pi\sqrt{2}b\kappa\alpha_2}}{4\pi^2}\left[\sum_{s=1+\alpha_1 m_\star}^{\lambda m_\star -1}\frac{1}{s^2}|\tilde{z}\mathcal{K}(-1-\alpha_2 b_1,|\tilde{z}|)|^2\sum_{s=\lambda m_\star}^{\infty}\frac{1}{s^2}|\tilde{z}\mathcal{K}(-1-\alpha_2 b_1,|\tilde{z}|)|^2\right]$$

$$\approx \frac{\alpha_1\gamma e^{-\pi\sqrt{2}b\kappa\alpha_2}}{4\pi^2}\left[d_1\int_{1+\alpha_1 m_\star}^{\lambda m_\star -1}\frac{ds}{s^2} + \int_{\lambda m_\star}^{\infty}\frac{ds}{s^2}\left(\frac{\pi|\tilde{z}|}{2}\right)e^{-2|\tilde{z}|}\right], \tag{56}$$

which after carrying out the integration and in the limit of $m_\star \to \infty$ becomes 0. This implies that $N_{\tilde{\omega}} = \lim_{m_\star\to\infty}\mathbb{S}_+(\kappa) = 0$. Therefore, it can be shown that the number density of the Hawking quanta in an extremal Kerr black hole spacetime vanishes is the number density of the Hawking quanta in is zero.

We mention that in Eq. (36) if one puts $\alpha_2 = 1$, and $\alpha_1 = 0 = \alpha_3$ then the relation becomes the same as the ones considered in [13] for Schwarzschild and in [15] for nonextremal Kerr black holes. This relation is bound to give a Planckian distribution of particles. Therefore, only the logarithmic term in the relation (36) is incapable of providing vanishing number density of Hawking quanta in an extremal Kerr black hole spacetime. In the subsequent study we shall specifically choose the inverse term in the relation (36) to understand the consequences.

B. Inverse relation approximation $\xi_+ \approx -\frac{2r_s^2}{\xi_-}$

In the preceding studies we have performed the calculations with a complete relation (36) between ξ_+ and ξ_- and we have seen that the expectation value of the number density operator corresponding to the Hawking effect comes out to be zero in a consistent manner. However, this analysis along with the one in [15] do not explicitly present the situation when the relation between the spatial near-null coordinates contains only the inverse term. On the other hand, from our current calculation of previous subsections we have already observed that this inverse term in the full relation dominates over the other two terms in providing the vanishing number density. Moreover, in literature also it is widely believed that similar inverse terms are responsible for vanishing number density of Hawking quanta for extremal black holes. But there are certain ambiguities regarding the semiclassical approach [12]. This motivates us to perform a precise investigation keeping only the inverse term in the relation (36), i.e., by setting $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$. It is to be noted that using the entire relation of Eq. (36) with α_1, α_2, and α_3 all set to 1, as one takes the limits $\xi_- \to 0$ and $\xi_- \to \infty$ the spatial near-null coordinate ξ_+ respectively gives $-\infty$ and ∞. This remains true even if one sets $\alpha_2 = 0$. On the other hand, with only the inverse relation between the near null coordinates, i.e., when only $\alpha_3 = 1$, one gets the corresponding limits of ξ_+ as $-\infty$ and 0. It signifies the difference of this particular case. Here one may expect that the relation (27) will not hold true as ξ_+ does not have a full range $(-\infty,\infty)$ for the considered range of ξ_-. Therefore, we shall proceed with caution in this case and evaluate both $F_0(\pm k,\kappa)$ and $F_1(\pm k,\kappa)$ for our study.

Using the general expression from Eq. (26) the explicit forms of the coefficient functions using this inverse relation reads

$$F_n(\pm k,\kappa) = \frac{1}{\sqrt{V_- V_+}}\int_{\xi_-^L}^{\xi_-^R} d\xi_-\left(\frac{2r_s^2}{\xi_-^2}\right)^{1-n} e^{\pm ik\xi_- - \frac{i2\kappa r_s^2}{\xi_-}}. \tag{57}$$

Here n can only take values 0 or 1. In terms of the dimensionless variable $x = \xi_-/b = \xi_-/\sqrt{2}r_s$, this integral can be represented as

$$F_n^\delta(\pm k,\kappa) = \frac{b}{\sqrt{V_- V_+}}\int_{x^L}^{x^R} dx\left(\frac{1}{x^2}\right)^{1-n} e^{-b_\pm x - \frac{b_0}{x}}. \tag{58}$$

The above integral of Eq. (57) is formally divergent for both of the n values. We remove that divergence by introducing a δ regulator in $b_\pm = b[\delta|\pm k| - i(\pm k)]$ and $b_0 = b[\delta|\kappa| + i\kappa]$. Performing the above integration one gets

$$F_0^\delta(\pm k,\kappa) = \frac{2b}{\sqrt{V_- V_+}}\sqrt{\frac{b_\pm}{b_0}}\mathcal{K}(1, 2\sqrt{b_\pm b_0})$$

$$F_1^\delta(\pm k,\kappa) = \frac{2b}{\sqrt{V_- V_+}}\sqrt{\frac{b_0}{b_\pm}}\mathcal{K}(1, 2\sqrt{b_\pm b_0}). \tag{59}$$

However, from the above expressions one can easily find a relation between the coefficient functions and this reads

$$F_0^\delta(\pm k,\kappa) = \frac{k}{\kappa}\left(\frac{\mp i + \delta}{i + \delta}\right)F_1^\delta(\pm k,\kappa). \tag{60}$$

In the limit $\delta \to 0$ the above relation takes the form of Eq. (27).

1. Consistency condition

Here using the expressions of these coefficient functions we shall look into the consistency condition of Eq. (28). In particular, the left-hand side of Eq. (28) can be expressed as

$$\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) = \sum_{k>0} \left(\frac{k}{\kappa}\right) [|F_1^\delta(-k,\kappa)|^2 - |F_1^\delta(k,\kappa)|^2]. \quad (61)$$

We mention that unlike the previous situation, here we shall not get any special term for $k = -\kappa$. Now using Eq. (59), one can easily show that

$$\left(\frac{k}{\kappa}\right) |F_1^\delta(\pm k, \kappa)|^2 = \frac{\gamma}{4\pi^2 m_\star} \frac{1}{s} |z_\pm \mathcal{K}(1, \bar{z}_\pm)|^2, \quad (62)$$

where $\bar{z}_- = 2\sqrt{|b_0|^2 s/m_\star}(\delta + i)$, $\bar{z}_+ = |\bar{z}_-| = 2\sqrt{|b_0|^2 s/m_\star}$, $k = 2\pi s/V_-$, $m_\star = \kappa V_-/(2\pi)$, and $\gamma = V_-/V_+$. Then Eq. (61) simplifies to the form

$$\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) = \frac{\gamma}{4\pi^2 m_\star} \sum_{s=1}^{\infty} \frac{1}{s} [|\bar{z}_-(s)\mathcal{K}(1, \bar{z}_-(s))|^2 - |\bar{z}_+\mathcal{K}(1, \bar{z}_+(s))|^2]. \quad (63)$$

It is important to notice that Eq. (63) is exactly similar to the second summation term in Eq. (50) (with $\alpha_2 = 0$ and $\alpha_3 = 1$), where we had neglected its contribution in the infinite volume limit, $m_\star \to \infty$. Here we shall analyze its behavior for small as well as large values of s a bit more carefully. For small values of s i.e., when $\bar{z}_\pm \to 0$, $|\bar{z}_\pm K_1(\bar{z}_\pm)|^2 \sim 1$ and the terms inside the square bracket cancels out exactly. Then only for large values of Bessel function's argument we get nonzero contributions and that is

$$\begin{aligned}\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) &= \frac{\gamma \bar{a}}{8\pi m_\star} \int_{s_\star}^{\infty} \frac{1}{\sqrt{s}} [e^{-2\delta\bar{a}\sqrt{s}} - e^{-2\bar{a}\sqrt{s}}] ds \\ &= \frac{\gamma}{8\pi m_\star} \left[\frac{1}{\delta} e^{-2\delta\bar{a}\sqrt{s_\star}} - e^{-2\bar{a}\sqrt{s_\star}}\right], \quad (64)\end{aligned}$$

where $\bar{a} = 2\sqrt{|b_0|^2/m_\star}$. Here we have considered that in the range $s \in [s_\star, \infty)$, $\bar{z}_\pm \gg 1$. Then from Eq. (64) the only dominating quantity, for small δ, contributing in the consistency condition is

$$\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) = \frac{1}{\delta} \frac{\gamma}{8\pi m_\star} = 1, \quad (65)$$

which was previously neglected in the $m_\star \to \infty$ limit considering a nonzero positive regularizing parameter δ. However, in this case we cannot neglect it, otherwise the left-hand side of (65) vanishes thus failing the consistency condition. One can further simplify this consistency condition to get, $4\kappa\delta V_+ = 1$, which says that for a fixed frequency κ of the outgoing wave mode the integral regulator δ and the volume regulator V_+ are not independent and one is inversely proportional to the other. Then as $V_+ \to \infty$ one should make the integral regulator $\delta \to 0$, which is in agreement with our initial assumptions on the regulators. We shall then keep this particular phenomena in mind in the estimation of the number density of the Hawking quanta in this case. Then we shall not make the quantities vanish when there is a $\gamma/(\delta m_\star)$ factor multiplied in any quantity.

2. Number density

Using the expression of Eq. (62) in Eq. (32) one can get the number density of Hawking quanta, now can be written as

$$\begin{aligned}N_{\tilde{\omega}} = \mathbb{S}_+(\kappa) &= \sum_{k>0} \left(\frac{k}{\kappa}\right) |F_1^\delta(k,\kappa)|^2 \\ &= \frac{\gamma}{4\pi^2 m_\star} \sum_{s=1}^{\infty} \frac{1}{s} |\bar{z}_+\mathcal{K}(1, \bar{z}_+(s))|^2 \\ &\approx \frac{\gamma}{4\pi^2 m_\star} \left[\sum_{s=1}^{s_L} \frac{1}{s} + \sum_{s_*}^{\infty} \frac{\pi\bar{a}}{2\sqrt{s}} e^{-2\bar{a}\sqrt{s}}\right], \quad (66)\end{aligned}$$

where we have considered $\bar{z}_+ \ll 1$ when $s \in [1, s_L]$ and $\bar{z}_+ \gg 1$ for $s \in [s_\star, \infty)$. Here we observe that there is no δ in the denominator in the factor outside and the function inside the sum also do not contain δ. Then we do not have any problem in taking the limit $m_\star \to \infty$ which makes this quantity to vanish. Another way to realize this is to use the consistency relation (65) into (66) which follows

$$N_{\tilde{\omega}} = \frac{2}{\pi} \delta \left[\sum_{s=1}^{s_L} \frac{1}{s} + \sum_{s_*}^{\infty} \frac{\pi\bar{a}}{2\sqrt{s}} e^{-2\bar{a}\sqrt{s}}\right]. \quad (67)$$

In the limit $\delta \to 0$ the number density vanishes. This implies that in this situation also one can get a vanishing number density of Hawking quanta, i.e., $N_{\tilde{\omega}} = \lim_{m_\star \to \infty} \mathbb{S}_+(\kappa) = 0$.

V. WIEN'S DISPLACEMENT LAW AND THE UNDERSTANDING FOR A VANISHING NUMBER DENSITY

In this part we intend to provide a description of a particle as it travels from $\mathscr{I}^-$ to $\mathscr{I}^+$ escaping getting trapped by the formation of the horizon. Through this description we expect to get an idea about the changes in a particle's characteristics due to the presence of the horizon as observed at the asymptotic future infinity. With this analysis we intend to provide a physical reasoning and identify the main contributing term behind the vanishing number density of the Hawking quanta in an extremal Kerr black hole spacetime. For the convenience of understanding we shall first consider the case of a nonextremal Kerr black hole and shall get into the extremal case subsequently. In a nonextremal Kerr black hole spacetime the relation among the spacelike near-null coordinates [15] is

$$\xi_+ = \xi_- + \frac{1}{\kappa_h}\ln[\kappa_h\xi_-] - \frac{1}{\kappa_c}\ln\left[1 + \frac{\kappa_h\xi_-}{\sigma}\right], \quad (68)$$

where, κ_h and κ_c are respectively the surface gravities at the event horizon and at the inner Cauchy horizon. Here, $\sigma = \kappa_h(\Delta_c - \Delta_h)$, and $\Delta_c \equiv 2(r^0 - r_c)$, $\Delta_h \equiv 2(r^0 - r_h)$ with r_c, r_h respectively denoting the radius of the inner Cauchy horizon and the event horizon, and r^0 is a radial pivotal value considering which the geometric ray tracing is done to obtain the above relation among the spatial near-null coordinates. It should be mentioned that to obtain this relation the expression of the tortoise coordinate $r_\star = r + (1/2\kappa_h)\ln[(r - r_h)\kappa_h] - (1/2\kappa_c)\ln[(r - r_c)\kappa_c]$, in a nonextremal Kerr black hole spacetime is utilized. On a constant time hypersurface using Eq. (68) one can obtain the expression

$$\Delta\xi_+ = \Delta\xi_-\left[1 + \frac{1}{\kappa_h\xi_-} - \frac{\frac{\kappa_h}{\kappa_c\sigma}}{1 + \frac{\kappa_h\xi_-}{\sigma}}\right], \quad (69)$$

where, $\Delta\xi_+$ can be identified to the de Broglie wavelength λ_o of the particle at future null infinity, and $\Delta\xi_-$ to the de Broglie wavelength λ_e of the particle at past null infinity. Then for a particle starting its journey from a spatial point ξ_-^e the relation relating the wavelengths is

$$\lambda_o = \lambda_e\left[1 + \frac{1}{\kappa_h\xi_-^e} - \frac{\frac{\kappa_h}{\kappa_c\sigma}}{1 + \kappa_h\xi_-^e/\sigma}\right]. \quad (70)$$

It is to be noted from (68) that the modes responsible for the Planckian distribution of the Hawking effect are emitted from the region $\xi_- \ll 1/\kappa_h$ which results in the relation (69) to be $\xi_+ \approx (1/\kappa_h)\ln[\kappa_h\xi_-]$. This phenomena can also be realized from Eq. (70) where it is noticed that for $\xi_- \gg 1/\kappa_h$ one has $\lambda_o \approx \lambda_e$, i.e., there is not much change in the characteristics of a particle that have passed long before the horizon formation. Then the particles that narrowly escape the formation of the event horizon can only contribute to the Planckian distribution of the Hawking effect. On the other hand, the point of emission ξ_-^e cannot be made more accurate than the ingoing particle's de Broglie wavelength λ_e, i.e., $\xi_-^e \approx \lambda_e$. Here again λ_e can be expressed in terms of the ingoing mode's frequency $\lambda_e = hc/E_k^0 = 2hc/|k|$. Therefore, for the smallest possible wavelength of the particle one must consider the largest possible $|k|$, i.e., $|k| \to \infty$. Then in this limit it is seen that the wavelength of the particle observed at future null infinite is

$$\lambda_o \approx \frac{1}{\kappa_h} = \frac{2\pi}{T_H}, \quad (71)$$

which, signifies the Wien's displacement law for blackbody radiation, i.e., the characteristic temperature corresponding to a blackbody distribution is inversely proportional to the wavelength.

On the other hand, in an extremal Kerr black hole spacetime the relation among the spatial near-null coordinates is

$$\xi_+ = \xi_- + 2r_s\ln\left[\frac{\xi_-}{\sqrt{2}r_s}\right] - \frac{2r_s^2}{\xi_-}, \quad (72)$$

where the expression of the tortoise coordinate $r_\star = r + r_s\ln[\xi_-/(\sqrt{2}r_s)] - 2r_s^2/(2r - r_s)$ is used to obtain this relation. From this expression one can obtain the relation among the observed and emitted wavelength of a particle, similar to Eq. (70), as

$$\lambda_o = \lambda_e\left[1 + \frac{2r_s}{\xi_-^e} + \frac{2r_s^2}{\xi_-^{e^2}}\right]. \quad (73)$$

In this case if one adheres to the same concepts that the contribution significant to the Hawking effect comes from the high frequency modes nearly escaping the horizon formation, then it is convenient to choose $\xi_-^e \approx \lambda_e$ and then to take the limit $\lambda_e \to 0$. Unlike Eq. (71) in this case we observe that

$$\lambda_o \to \infty, \quad (74)$$

which corresponds to a characteristic temperature $T_H = 0$ of the Hawking effect. It should be noted that the result of (74) is due to the third quantity of the right-hand side of Eq. (73). Then it is evident that the inverse term in the relation between the spatial near-null coordinates in Eq. (72) is the dictating quantity. Presence of this inverse term in that relation results in a vanishing number density of the Hawking quanta for extremal Kerr black hole spacetime.

VI. DISCUSSION

In this work, we have provided a detailed derivation of the vanishing number density of Hawking quanta in an extremal Kerr black hole spacetime using the canonical formulation [13–15]. In [15] the authors used the canonical derivation to study the Hawking effect for both nonextremal and extremal Kerr black holes. However, the extremal case needed further studies to understand its origin of zero temperature better. In the derivation of the Hawking effect, the relation (35) between the near-null coordinates near past and future null infinities plays a crucial role. In [15] an approximation was made on that relation for mathematical simplification. In the present work, we started with the full relation and consistently arrived at the zero temperature conclusion, solidifying the results of [15]. We also discussed the effects of different approximations of the relation on the final result and the consistency condition. Furthermore, we presented an argument to visualize the zero temperature from a physically understandable point of view, pinpointing the particular term in the relation mentioned above responsible for the phenomenon.

It is to be noted that the concept of the spacelike and timelike near-null coordinates, obtained by slightly deforming the null coordinates, is of vital importance to describe the dynamics of a matter field Hamiltonian in canonical formulation. Here we aimed to identify the term which contributes to the vanishing number density of the Hawking quanta in the relation between the spatial near-null coordinates. In this regard, we have first considered the entire relationship between the spatial near-null coordinates (36) without any approximations and consistently obtained a vanishing number density of Hawking quanta. In the Bogoliubov transformation method [34,46,47] one can obtain a similar relation like the inverse term from Eq. (36) between the null coordinates. However, to the best of our knowledge, there is no detailed study with the complete relation. And this makes our study important and unique to a great extent. In this Hamiltonian formulation we first established the consistency condition which results from the simultaneous satisfaction of the Poisson brackets of the field modes and their conjugate momenta for the past and future observer. Using the relation (36) to understand the Hawking effect, we have introduced a few parameters α_j (with $j = 1, 2, 3$) which can have values 0 or 1 to keep track of the contributions of each term in the complete relation. However, we noticed from the final form of the consistency condition that one can not make $\alpha_1 = 0$ because we have already utilized a substitution containing α_1 to arrive at the result and making it zero afterward will render the mathematics incorrect. We also observed that when $\alpha_2 = 0$ in Eq. (36), i.e., the contribution of the logarithmic term is neglected, the consistency condition arrives to be the same as that was obtained in [15] and also the number density vanishes. Therefore from this first analysis, we conclude that the final result regarding the Hawking effect in an extremal Kerr black hole spacetime remains impervious to additional generalizations to the relation between spatial near-null coordinates as considered in [15].

Furthermore, we have also considered only the inverse term in the relation (36) for the study of the Hawking effect. In that case, we observed that for the domain $[0, \infty)$ of ξ_- (which corresponds to the situation after the formation of the black hole event horizon), the other spatial near null coordinate ξ_+ does not cover the entire $(-\infty, \infty)$ instead covers a reduced domain $(-\infty, 0]$. This observation has its physical shortcomings as it proclaims that not all future observers are eligible to comment about the Hawking effect (not even an observer at future timelike infinity). Nevertheless, mathematically one can pursue this case to obtain a vanishing number density of the Hawking quanta in a consistent manner. Our study solidifies the claims of a semiclassical approach in an extremal black hole spacetime.

This Hamiltonian-based formulation closely mimics the Bogoliubov transformation method of Hawking's original derivation [1]. It opens the avenue to study the effects of other quantization techniques, like Polymer quantization [44], into the picture. Furthermore, from our current work, one can observe that it has a robust mathematical structure that can consistently answer some longstanding questions related to particle creation in extremal black hole backgrounds.

Finally, to provide a firm physical reasoning to recognize the main contributing term in the number density, we presented a thought experiment to realize the Hawking effect in a Kerr black hole spacetime. First, in a nonextremal Kerr black hole spacetime, we observed that the particles nearly escaping the formation of the event horizon and arriving at the scri-plus to be detected as the Hawking quanta must possess the final wavelength inversely proportional to the temperature of the Hawking effect. It establishes the *Wien's displacement law* for thermal distribution of particles corresponding to the Hawking effect in a nonextremal Kerr black hole spacetime. Second, with the same setup in the extremal Kerr black hole spacetime, we found that the final wavelength of the particles approaches infinity. Compared to the nonextremal case, one can then associate the corresponding Hawking temperature to zero in the extremal scenario. We observed that this result is completely dictated by the inverse term in the relation (36), thus providing us with a definitive understanding of the vanishing number density of the Hawking quanta.

Our analysis has presented many intricate findings regarding the Hawking effect in an extremal Kerr black hole spacetime. We believe that these results will further improve the understanding of particle creation in extremal black hole spacetimes. It is to be noted that both Schwarzschild and the Kerr black hole spacetimes describe asymptotically flat geometry in regions far from the event horizon. A massless minimally coupled scalar field in these regions and near the horizon behaves like an infinite collection of fields from the flat spacetime, allowing one to construct the near-null coordinates and realize the Hawking effect using the Hamiltonian formulation [13,15]. In flat spacetime, one can realize the scalar field as an infinite sum of simple harmonic oscillators in the Fourier domain, which enables one to express the number density of Hawking quanta in terms of the Hamiltonian corresponding to each Fourier field mode. While one can successfully pursue this Hamiltonian-based formulation in flat spacetime [48], Schwarzschild, and Kerr black hole backgrounds, one cannot demand that the same will be true in other nontrivial black hole backgrounds. For example, in a de Sitter black hole background, the regions of radial infinity are not essentially flat, then one cannot readily apply the machinery of quantum field theory from flat spacetime in this background. But that remains an open direction to study further with the current approach.

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