Bubble universes and traversable wormholes

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Bubble universes and traversable wormholes in general relativity can be realized as two sides of the same concept. To exemplify it, we find, display, and study in a unified manner a Minkowski-Minkowski closed universe and a Minkowski-Minkowski traversable wormhole. By joining two 3-dimensional flat balls along a thin shell two-sphere of matter, i.e., a spherical domain wall, into a single spacetime one gets a Minkowski-Minkowski static closed universe, i.e., a bubble universe. By joining two 3-dimensional complements of flat balls along a thin shell two-sphere of matter, i.e., a spherical throat, into a single spacetime one gets a Minkowski-Minkowski static open universe which is a traversable wormhole. Thus, Minkowski-Minkowski bubble universes and wormholes can be seen as complementary to each other. It is also striking that these two spacetimes, the Minkowski-Minkowski bubble universe and the Minkowski-Minkowski traversable wormhole, have resemblances with two well-known static universes of general relativity. The Minkowski-Minkowski static closed universe, i.e., the Minkowski-Minkowski bubble universe, resembles in many aspects the Einstein universe, i.e., a static closed spherical universe homogeneously filled with dust matter and with a cosmological constant. The Minkowski-Minkowski static open universe, i.e., the Minkowski-Minkowski traversable wormhole, resembles the Friedmann static universe, i.e., a static open hyperbolic universe homogeneously filled with negative energy density dust and with a negative cosmological, which is a universe with two disjoint branes, or branches, and can be considered a failed wormhole. In this light, the Einstein static closed universe and the Friedmann static open universe should also be seen as the two sides of the same concept, i.e., they are complementary to each other. The scheme is completed by performing a linear stability analysis for the Minkowski-Minkowski bubble universe and the Minkowski-Minkowski traversable wormhole and also by comparing it to the stability of the Einstein static universe and the Friedmann static universe, respectively. This complementarity between bubble universes and traversable wormholes, that exists for these instances of static spacetimes, can be carried out for dynamical spacetimes, indicating that such a complementarity is quite general. The overall study suggests that bubble universes and traversable wormholes can be seen as coming out of the same concept, and thus, if one type of solution exists the other should also exist.

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I. INTRODUCTION

A. Minkowski-Minkowski bubble universe and Minkowski-Minkowski traversable wormhole

General relativity is an excellent theory to study universe solutions and wormhole solutions from which bubble universes and traversable wormholes can arise as complementary to each other. To see this, one can attempt to find within the theory Minkowski-Minkowski bubble universes and Minkowski-Minkowski traversable wormholes and study their properties. One picks up a Minkowski spacetime and at constant time cuts a ball in it, to obtain two spaces, namely, a 3-dimensional ball with a flat inside, and an infinite extended 3-dimensional flat space with a hole, which is the complement of the ball. Then one picks up another Minkowski spacetime and do the same, to get a second ball and a second infinite extended flat space with a hole. If one joins the two 3-dimensional balls along

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a 2-sphere, a shell containing matter, one obtains a single 3-space that including time makes altogether a static closed universe. If one joins the two complements, i.e., the two infinite extended 3-dimensional flat spaces with a hole in each, along a 2-sphere, a shell containing matter, one obtains a different single 3-space that including time makes altogether another universe, which is a traversable wormhole. Thus, one has a closed universe, which can be viewed as a bubble universe, and its complement, an open universe, which is a traversable wormhole. To implement the idea of a Minkowski-Minkowski closed universe, i.e., a bubble universe, and a Minkowski-Minkowski open universe, i.e., a traversable wormhole, one uses the equations of general relativity together with the appropriate thin shell formalism [1]. When one has a thin shell in an ambient spacetime, the normal vector field to the shell is an important quantity that provides a way to determine how the thin shell curves in that space, i.e., allows to determine the extrinsic curvature of the shell, which in addition to the induced metric, is the other quantity that completely characterize the embedded surface layer. These quantities play a pivotal part in the junction formalism. Indeed, to find all possible shell solutions in an ambient spacetime one has to understand the fact that the normal to a shell can have two relative directions, such that, for static spherically symmetric spacetimes, the normal to the shell may point toward or away from the center of the coordinates. For instance, in an ambient Minkowski-Schwarzschild spacetime, more precisely, for a shell with a Minkowski interior with a center and a Schwarzschild exterior, usually called a fundamental shell, if the normal points to increasing coordinate radius r in the exterior, one has a star shell, i.e., a shell that represents a star. In the same ambient Minkowski-Schwarzschild spacetime, if the normal points to decreasing coordinate radius r one has a tension shell black hole, i.e., a shell supported by tension that is in the other side of the Kruskal-Szekeres diagram as was noted by Katz and Lynden-Bell [2]. This can also be performed in an ambient Minkowski-Reissner-Nordström spacetime, yielding, instead of two fundamental electrically charged shell spacetimes, a bewildering variety of fourteen fundamental electrically charged shell spacetimes with different global spacetime structures [3]. Here, in place of using an ambient Minkowski-Schwarzschild or an ambient Minkowski-Reissner-Nordström we use an ambient Minkowski-Minkowski spacetime.

One possibility for a Minkowski-Minkowski spacetime is for a shell with a Minkowski interior with a center, i.e., a fundamental shell, such that the normal to the shell points toward decreasing radius r in the Minkowski exterior. One then finds the Minkowski-Minkowski closed universe, made of two 3-dimensional flat balls, or sheets, that are joined at some domain wall, i.e., a 2-sphere shell with matter, to make a Minkowski-Minkowski bubble universe. Note that for a shell with a Minkowski interior with a center such that the normal to the shell points toward increasing radius r in the Minkowski exterior yields the trivial global Minkowski spacetime with a zero shell.

There is yet another possibility for a Minkowski-Minkowski spacetime, different from the fundamental shell. It comes from an exotic shell, i.e., a shell attached to a Minkowski open interior, noting that interior is just a name since it could as well be called exterior. For a shell with a Minkowski open interior, when the normal to the shell points toward increasing r in the Minkowski exterior, one finds the Minkowski-Minkowski open universe, made of two 3-dimensional flat open infinite sheets that are joined at some 2-sphere with matter, to make a Minkowski-Minkowski traversable wormhole. Note that for a shell with a Minkowski open interior such that the normal to the shell points toward decreasing radius r in the Minkowski exterior yields the trivial global Minkowski spacetime with a zero shell.

Universes and wormholes are usually envisaged as distinct objects. The two Minkowski-Minkowski spacetimes demonstrate that they can be seen as complementary to each other, i.e., they are two sides of the same concept. The concept, i.e., a collection of two Minkowski spacetimes together, yields on one side a closed universe, i.e., a bubble universe, and on the other side an open universe which is a traversable wormhole. Surely, the collection of two Minkowski spacetimes can also lead to two separate Minkowski spacetimes, but this is the trivial case and needs not be considered.

B. Einstein static closed spherical universe and Friedmann static open hyperbolic universe

There are two paradigmatic static homogeneous universes in general relativity: the static closed spherical universe and the static open hyperbolic universe, with two separated branes or branches. To implement the idea of static uniform universes, one uses general relativity itself, i.e., Einstein equation modified to include a cosmological constant. From the staticity condition one imposes that neither the geometry nor the matter depend on time and from the homogeneous condition one imposes that the energy-density is a constant in space.

This implementation, that the Universe, in particular a static universe, could be described within general relativity, was put forward by Einstein. In devising a way to realize Mach's principle, a new interaction, namely, a cosmological constant with repulsion features, was postulated. General relativity with this new cosmological interaction is indeed the first modified theory of gravitation. This repulsive cosmological term, that counterbalances the self gravitational force due to the energy density of the matter supposed pressureless, was then used to find a unique static solution for the Universe which was also assumed to be closed, finite, and spheric [4]. In the limiting case that the universe would be spatially flat, the Einstein universe

disappeared in a Minkowski empty universe. The enforcing of Mach's principle in this way proved to be a dead end as exemplified by the de Sitter universe with no matter and only a cosmological constant [5], but the static closed universe of Einstein was of great impact as it indeed started the concept of universes. For instance, dynamic closed universes within general relativity, like the Friedmann [6] and Lemaître [7] expanding universes, came out of Einstein's static one, which in turn, due to its instability and propensity to grow, continued to be studied as a progenitor of expanding closed universes [8–13].

Remarkably, Friedmann in his second paper on universes and cosmology proposed to start with an open static hyperbolic universe, as to exhaust the possible static pressureless universes [14]. In the process, a cosmological constant, now with attraction features, was again introduced to counterbalance the self gravitational repulsive force due to a matter energy density necessarily negative. In the limiting case the universe would be spatially flat, the Friedmann universe disappeared in a Minkowski empty universe. The Friedmann static universe can be seen as an anti-Einstein universe and it inaugurated the concept of open universes. Indeed, it was used by Friedmann in [14] to continue the analysis into dynamic open hyperbolic universes and it was developed by Harrison [15] who, among many other universes, also studied its stability. Now, the Friedmann static universe, being hyperbolic, has two branches, or branes, which fail to communicate to each other by an infinitesimal separation. This means that it can be considered a wormhole, actually, a failed wormhole. It is not traversable but almost and it can be thought as an embryo of a wormhole.

The Einstein static closed universe and the Friedmann static open universe can be seen as complementary, i.e., they are two sides of the same concept. The concept here is the constant spatial curvature of the spacetime, one side gives positive spatial curvature, i.e., the Einstein universe, the other side gives negative spatial curvature, i.e., the Friedmann static universe. The trivial case here would also be two zero curvature spacetimes, i.e., two separate Minkowski spacetimes, and needs not be considered.

C. Bubble universes and traversable wormholes

Bubble universes and traversable wormholes have been proposed as structures that might arise if appropriate physical conditions are available. Indeed, the Universe in its early phases, of which the inflationary period is an example, filled with scalar and gauge fields, may have produced domain walls, cosmic strings, and monopoles, which can still exist as frozen topological remains of the symmetry breaking phase transition of that early era. In this connection, a setting allowed by the prevailing physical conditions of that early inflationary era, or even of an epoch before it, is that bubble universes might have unfolded within the Universe and also, conceivably, systems such as traversable wormholes might have materialized to connect distant parts of the Universe or distinct universes. In addition, a possibility also permitted by the laws of physics, is that bubble universes and traversable wormholes might be constructed if sufficient technology is available. General relativity is an excellent theory to study universe solutions and wormhole solutions from which bubble universes and traversable wormholes can emerge as complementary to each other, and so they can be seen as duals of each other, leading to a better understanding of both.

A bubble universe, a universe within the Universe, is a complete solution of the Einstein's equations. Bubble universes, together with baby universes, are universes in themselves, somehow attached to our one. They made their appearance in the physics of false vacuum decay within dynamic bubbles [16]. Its interest and uses within the inflation theory was seen in [17]. The idea of bubble universes taking off out from our Universe was developed in [18], general relativistic dynamic bubble universe solutions with matter were proposed in [19], several possible universe decays and corresponding expanding or contracting domain walls were thoroughly analyzed in [20], interesting scenarios with bubbles with different gravitational constants were proposed in [21], their intrinsic stability has not been analyzed, see however [22], and bubble universe astrophysical connections to black holes and their formation were studied in [23,24].

A traversable wormhole, joining two otherwise distinct universes through two mouths and a throat, is also a complete solution of the Einstein's equations. A wormhole is a concept with a history of its own that in a sense was initiated by Einstein in the celebrated Einstein-Rosen bridge [25]. The concept had further developments related to the quantization of the spacetime geometry [26], and it was essential to understand the maximal extension of the Schwarzschild spacetime, now seen as a white hole being converted to a black hole through a nontraversable wormhole connecting two separated asymptotically flat spacetimes [27], which in turn gave rise to the notion of multiply connected spacetimes [28]. Wormholes, in particular traversable wormholes, abound in general relativity and in gravitational theories. Traversable wormhole solutions were found first in [29,30], were placed in a proper framework in [31], were studied as inflating solutions in [32], were proposed in an exercise as a junction of two Minkowski spacetimes in a remarkable encyclopedic and also didactic book exclusively dedicated to them [33], had their matter energy conditions and their connection to the flare-out condition for the geometry described for generic dynamic solutions in [34], were built from vacuum stress-energy tensors in [35], were studied with the aim of calculating the ground state energy of a scalar field on the throat of two asymptotically Minkowski spacetime regions [36], were embedded in a cosmological constant setting in [37], had the energy conditions at the wormhole's

throat generically reassessed in [38], were analyzed in relation to their shadows and quasinormal modes in [39], had collisions between their own mouths studied in [40], had a stability analysis performed in [41–43], see also [33], and had their possible connection to astrophysics being proposed in [44–46].

The bubble universe and traversable wormhole solutions just mentioned are only a few of the vast number of all the many interesting solutions found in the literature, and it is not our intention to proceed with an analysis of so vast a camp. Usually, bubble universes appear in dynamic contexts, whereas traversable wormholes are generically realized in static backgrounds, but of course they can both be static or dynamic. Bubble universes and traversable wormholes can appear as duals of each other, a possibility of when quantum gravity dominated or was still nonnegligible, bubble universes and traversable wormholes would form alike out of the spacetime foam, and would stay stable or metastable structures well into the classical regime. We stick to general relativity and to the two static Minkowski-Minkowski spacetimes and the two static homogeneous universes of Einstein and Friedmann. With these four spacetimes it is possible to have two levels of comparison. On a first level of comparison, on the one hand, one can compare the two Minkowski-Minkowski spacetimes between themselves by investigating their similarities, and on the other hand, one can also attempt to compare the two static, homogenewous, pressureless spacetimes of general relativity with cosmological constant between themselves. On a second level of comparison, the two Minkowski-Minkowski spacetimes are put face to face with the two static spacetimes of Einstein and Friedmann. Let us be specific. When performing the first level comparison between the two Minkowski-Minkowski spacetimes, i.e., the Minkowski-Minkowski bubble universe and the Minkowski-Minkowski traversable wormhole, one should take some steps, namely, one has to reveal in a unified manner the two possible nontrivial cases in a Minkowski-Minkowski spacetime, or more concretely, one has to find the fundamental shell spacetime, which is a closed bubble universe, and find also the exotic shell spacetime, which is an open traversable wormhole universe. In doing so, one classifies and analyzes the possible junctions of Minkowski spacetimes through a static, timelike, thin matter shell, which are the two nontrivial cases just mentioned, the trivial one being the no shell pure Minkowski spacetime. A linear stability study of these spacetimes completes the comparison. Through this example, bubble universes and traversable wormholes can now be understood in a unified light, in the sense that the Minkowski-Minkowski bubble universe and the Minkowski-Minkowski traversable wormhole are two sides of the same concept, in which instance, if one exists it makes a case to the existence of the other. When performing the first level comparison between the two static, homogeneous, pressureless spacetimes with cosmological constant, i.e., the Einstein closed universe and the Friedmann open universe, one should take some steps, namely, one has to reveal in a unified manner these two possible nontrivial cases, and display them in a new light. A linear stability study of these spacetimes completes the analysis. In this new light, the Einstein and Friedmann static universes can be compared, they are seen anew as a bubble universe and a failed wormhole, respectively. On the second level of comparison, the two Minkowski-Minkowski spacetimes are put face to face with the two static homogeneous spacetimes, to find that the Minkowski-Minkowski closed universe, a bubble universe, goes along with the Einstein closed universe, which can then be seen then as a bubble universe, and the Minkowski-Minkowski open universe, a traversable wormhole, goes along with the Friedmann open universe, which is a failed wormhole. This comparison shows some striking resemblances between those spacetimes on several grounds. In this sense, the Einstein and Friedmann static universes are really seen anew as a bubble universe and a failed wormhole, respectively, and also they are an example that bubble universes and traversable wormholes can be perceived in a unified light. The complementarity, or duality, between general relativistic bubble universes and traversable wormholes exists for these examples of static spacetimes. One can carry out this idea for dynamical spacetimes and show that the complementarity, or duality, considered here is quite generic. Moreover, following this rationale, if one finds inflating bubble universe solutions, one should be able to find the corresponding inflating wormhole solutions, and vice versa, so that, for instance, a given solution already found in one of the sides could help in looking for the complementary solution in the other side.

The paper is organized as follows. In Sec. II, we formalize in a unified way the two possible junctions of two identical Minkowski spacetime regions and we perform a linearized stability analysis of the Minkowski-Minkowski universes. We then build in detail the Minkowski-Minkowski static closed universe, i.e., a bubble universe, and the Minkowski-Minkowski static open universe, i.e., the Minkowski-Minkowski static traversable wormhole. In Sec. III, we formalize in a unified way the two possible static homogeneous universes and we display a linearized stability analysis of them. We then display in detail the Einstein static spherical closed universe and we compare explicitly the Minkowski-Minkowski static closed universe, i.e., the Minkowski-Minkowski bubble universe, with the Einstein static spherical closed universe, and display the Friedmann static hyperbolic open universe, i.e., the failed wormhole, and compare explicitly the Minkowski-Minkowski static open universe, i.e., the Minkowski-Minkowski static traversable wormhole, with the Friedmann failed wormhole. In Sec. IV we conclude. Throughout the paper we work in geometrized units system where the constant of gravitation G and the speed of light c are set to one, $G = 1$ and $c = 1$, and use the metric signature $(-+++)$.

II. MINKOWSKI-MINKOWSKI CLOSED UNIVERSE AND MINKOWSKI-MINKOWSKI OPEN UNIVERSE

A. Minkowski-Minkowski universes: Formal solutions and stability

1. Solutions

The Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\mathcal{R}$ is the Einstein tensor, $R_{\alpha\beta}$ and $\mathcal{R}$ are the Ricci tensor and Ricci scalar, respectively, $g_{\alpha\beta}$ is the metric tensor, $T_{\alpha\beta}$ is the stress-energy tensor, and Greek indices run from 0 to 3 with 0 representing a time component and 1, 2, and 3 representing space components. One wants to join consistently two solutions of Einstein field equations and the Israel formalism provides the method needed to make the junction between two different general relativistic spacetime regions [1]. Consider then two spacetime manifolds with boundary, one is $\mathcal{M}_\mathrm{i}$ with metric g_i and the other is $\mathcal{M}_\mathrm{e}$ with metric g_e. The spacetimes $(\mathcal{M}_\mathrm{i}, g_\mathrm{i})$ and $(\mathcal{M}_\mathrm{e}, g_\mathrm{e})$ are solutions of the theory of general relativity and are to be glued together at a common boundary, forming a new spacetime $\mathcal{M}$. In brief, $\mathcal{M}$ is partitioned by an hypersurface $\mathcal{S}$ into two regions, the regions $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$. The formalism applies directly to a hypersurface $\mathcal{S}$ that can be either timelike or spacelike, the extension to the case of a null boundary hypersurface can also be done with care.

We assume that it is possible to formally define a common coordinate system $\{x^\alpha\}$ on both sides of the hypersurface $\mathcal{S}$. We also admit the existence of a normal vector field n, well defined on both sides of $\mathcal{S}$, which is orthogonal to the matching hypersurface at each point. We choose n to point from $\mathcal{M}_\mathrm{i}$ to $\mathcal{M}_\mathrm{e}$ and, without loss of generality, $n^\alpha n_\alpha = \varepsilon$, where n^α are the components of n in the coordinate system $\{x^\alpha\}$ and ε is $+1$ or -1 depending on n being spacelike or timelike, respectively. The null case has $\varepsilon = 0$ and it would have to be treated separately which we do not do here. For a timelike normal vector field n one has that the corresponding hypersurface $\mathcal{S}$ is spacelike, and for a spacelike n one has that the corresponding hypersurface $\mathcal{S}$ is timelike. Then, assuming $\{y^a\}$ to represent a local coordinate system on $\mathcal{S}$, the normal vector field n must be orthogonal at each point to the tangent vectors to the hypersurface $\mathcal{S}$, $e_a \equiv \frac{\partial}{\partial y^a}$, such that $e^\alpha_a n_\alpha = 0$, with $e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a}$. The induced metric on $\mathcal{S}$ as seen from each region $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$, is $h_{\mathrm{i}\,ab} = g_{\mathrm{i}\,\alpha\beta} e^\alpha_a e^\beta_b$, $h_{\mathrm{e}\,ab} = g_{\mathrm{e}\,\alpha\beta} e^\alpha_a e^\beta_b$, respectively, where $g_{\mathrm{i}\,\alpha\beta}$ and $g_{\mathrm{e}\,\alpha\beta}$ are the components of the metrics g_i and g_e in the coordinate system $\{x^\alpha\}$. Notice that, in general, the induced metric on $\mathcal{S}$ by each metric g_i and g_e may not coincide, hence we use the notation $h_{\mathrm{i}\,ab}$ and $h_{\mathrm{e}\,ab}$ to refer to the metric induced by the spacetime structure of $\mathcal{M}_\mathrm{i}$ or $\mathcal{M}_\mathrm{e}$, respectively. The extrinsic curvature $K_{\mathrm{i}\,ab}$ or $K_{\mathrm{e}\,ab}$ of the hypersurface $\mathcal{S}$, as an embedded manifold in $\mathcal{M}_\mathrm{i}$ or $\mathcal{M}_\mathrm{e}$, respectively, is defined as $K_{\mathrm{i}\,ab} = e^\alpha_a e^\beta_b \nabla_{\mathrm{i}\,\alpha} n_\beta$, $K_{\mathrm{e}\,ab} = e^\alpha_a e^\beta_b \nabla_{\mathrm{e}\,\alpha} n_\beta$, where ∇_i and ∇_e represent the covariant derivatives with respect to g_i or g_e. Their traces are $K_\mathrm{i} = h^{ab}_\mathrm{i} K_{\mathrm{i}\,ab}$, and $K_\mathrm{e} = h^{ab}_\mathrm{e} K_{\mathrm{i}\,ab}$, respectively.

Now, we need to give the conditions under which the matching of the two spacetimes $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$ form a valid solution of the Einstein field equations, Eq. (1). Following the Israel formalism, to join the two spacetimes $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$ at $\mathcal{S}$, such that the union of g_i and g_e forms a valid solution to the Einstein field equations (1), two junction conditions must be verified at the matching surface $\mathcal{S}$: (i) The induced metric h_{ab} as seen from each region $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$, must be the same, i.e.,

$$[h_{ab}] = 0. \tag{2}$$

(ii) If the extrinsic curvature K_{ab} is not the same on both sides of the boundary $\mathcal{S}$, then a thin shell is present at $\mathcal{S}$ with stress-energy tensor S_{ab} given by

$$-\frac{\varepsilon}{8\pi}([K_{ab}] - h_{ab}[K]) = S_{ab}, \tag{3}$$

where $[K_{ab}]$ represents the difference of K_{ab} as seen from each submanifold at $\mathcal{S}$, i.e., $[K_{ab}] \equiv K_{\mathrm{e}\,ab}|_\mathcal{S} - K_{\mathrm{i}\,ab}|_\mathcal{S}$, and similarly for $[K]$, and we use the notation $K_{\mathrm{i}\,ab} \equiv K_{ab}(\mathcal{M}_\mathrm{i})$ and $K_{\mathrm{e}\,ab} \equiv K_{ab}(\mathcal{M}_\mathrm{e})$ to refer to K_{ab} defined in $\mathcal{M}_\mathrm{i}$ or $\mathcal{M}_\mathrm{e}$, respectively, and similarly for K.

The Minkowski spacetime with line element $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, where t and r are the time and radial coordinates, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the angular coordinates, is a solution of Einstein equations, see Eq. (1), in fact the simplest solution. We assume now that the interior and exterior spacetime have Minkowski line elements, and find and analyze all possible junctions of two Minkowski spacetimes through a static, thin matter shell. Using the Israel formalism, we consider two spacetimes, $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$, each endowed with the Minkowski metric tensor field glued together at a common hypersurface, $\mathcal{S}$. To apply the formalism, we will have to find the induced metric and extrinsic curvature induced on an embedded hypersurface of each spacetime, $\mathcal{M}_\mathrm{i}$ or $\mathcal{M}_\mathrm{e}$. We will start by making the analysis in the interior spacetime, $\mathcal{M}_\mathrm{i}$, and extend the results to the exterior spacetime, $\mathcal{M}_\mathrm{e}$.

The interior Minkowski spacetime, $\mathcal{M}_\mathrm{i}$, is characterized by the following line element, in spacetime spherical coordinates,

$$ds^2_\mathrm{i} = -dt^2_\mathrm{i} + dr^2_\mathrm{i} + r^2_\mathrm{i} d\Omega^2, \tag{4}$$

where t_i and r_i are the time and radial coordinates, respectively, measured by a free-falling observer in $\mathcal{M}_\mathrm{i}$,

and again $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the angular coordinates. On the one hand, for the solution itself we are interested in studying the case where the hypersurface $\mathcal{S}$ is timelike and static, i.e., static as seen from an observer free falling in the interior Minkowski spacetime. On the other hand for the stability analysis that we will take we have to allow for the hypersurface to be dynamical, so we compute the induced metric and extrinsic curvature of $\mathcal{S}$ allowing for a dynamical shell and when needed we take the static solution. In the study of the properties of the matter shell at $\mathcal{S}$, we will then restrict the setup to the static case. Since we assume the hypersurface to be timelike, it is convenient to choose the coordinates on $\mathcal{S}$ to be $\{y^a\} = (\tau, \theta, \varphi)$, where τ is the proper time measured by an observer comoving with $\mathcal{S}$. In this coordinate system, it follows that $e_\tau \equiv u$, where u is the 4-velocity of an observer comoving with the shell. The hypersurface $\mathcal{S}$, as seen from the interior $\mathcal{M}_\mathrm{i}$ spacetime, is parametrized by τ, such that the surface's radial coordinate is described by a function $R_\mathrm{i} = R_\mathrm{i}(\tau)$. Then, the 4-velocity u_i, where the subscript i is not an index and as before denotes interior, as seen from the interior spacetime is given by $u_\mathrm{i} = (\frac{dt_\mathrm{i}}{d\tau}, \dot{R}_\mathrm{i}, 0, 0)$, where overdot represents the derivative with respect to the proper time, i.e., $\dot{R}_\mathrm{i} \equiv \frac{dR_\mathrm{i}}{d\tau}$. Since $\mathcal{S}$ is a timelike hypersurface, it must verify $u_{\mathrm{i}\,\alpha} u_\mathrm{i}^\alpha = -1$, therefore we find $u_\mathrm{i} = (\sqrt{1+\dot{R}_\mathrm{i}^2}, \dot{R}_\mathrm{i}, 0, 0)$, where we chose $\frac{dt_\mathrm{i}}{d\tau} > 0$ as we assume u_i to point to the future. The expression for the 4-velocity of an observer comoving with $\mathcal{S}$ and the condition $e_a^\alpha n_\alpha = 0$ allow us to find the components of the normal vector field to $\mathcal{S}$, n, as seen from the interior spacetime $\mathcal{M}_\mathrm{i}$, $n_\mathrm{i} = \xi_\mathrm{i}(\dot{R}_\mathrm{i}, \sqrt{1+\dot{R}_\mathrm{i}^2}, 0, 0)$, where ξ_i is a normalization factor. Using $n^\alpha n_\alpha = \varepsilon$, Eq. (4), and the condition that the normal vector field n_i is spacelike, yields $\xi_\mathrm{i} = \pm 1$. Now, defining $\nabla_\mathrm{i} r_\mathrm{i}$ as the gradient of r_i, the choice $\xi_\mathrm{i} = +1$ or $\xi_\mathrm{i} = -1$ represents whether the inner product $g_\mathrm{i}(n_\mathrm{i}, \nabla_\mathrm{i} r_\mathrm{i}) > 0$ or $g_\mathrm{i}(n_\mathrm{i}, \nabla_\mathrm{i} r_\mathrm{i}) < 0$, respectively. Under the Israel formalism both values for ξ_i are possible and we shall consider both cases. Using the induced metric equation, $h_{\mathrm{i}\,ab} = g_{\mathrm{i}\,\alpha\beta} e_a^\alpha e_b^\beta$, and $u_\mathrm{i} = (\sqrt{1+\dot{R}_\mathrm{i}^2}, \dot{R}_\mathrm{i}, 0, 0)$, we find that the induced metric on $\mathcal{S}$ by the spacetime $\mathcal{M}_\mathrm{i}$, is such that the line element can be written as $ds^2|_{\mathcal{S}_\mathrm{i}} = -d\tau^2 + R_\mathrm{i}^2 d\Omega^2$. Gathering these results, we can compute the components of the extrinsic curvature of $\mathcal{S}$ as seen from $\mathcal{M}_\mathrm{i}$, K^i_{ab}. In the case where the matching surface $\mathcal{S}$ is timelike and spherically symmetric, the non-null components of the extrinsic curvature are given by, dropping here the superscript i to not overcrowd the notation, $K_{\tau\tau} = -a^\alpha n_\alpha$, $K_{\theta\theta} = \nabla_\theta n_\theta$, $K_{\varphi\varphi} = \nabla_\varphi n_\varphi$, where $a^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ represents the components of the 4-acceleration of an observer comoving with $\mathcal{S}$. Taking into account Eq. (4) and $u_\mathrm{i} = (\sqrt{1+\dot{R}_\mathrm{i}^2}, \dot{R}_\mathrm{i}, 0, 0)$, we find that the nontrivial components of the exterior curvature as seen from the interior Minkowski spacetime are given by $K_\mathrm{i}{}^\tau{}_\tau = \xi_\mathrm{i} \frac{\ddot{R}_\mathrm{i}}{\sqrt{1+\dot{R}_\mathrm{i}^2}}$, and $K_\mathrm{i}{}^\theta{}_\theta = K_\mathrm{i}{}^\varphi{}_\varphi = \xi_\mathrm{i} \frac{\sqrt{1+\dot{R}_\mathrm{i}^2}}{R_\mathrm{i}}$, where the induced metric h_i^{ab} associated with the hypersurface line element was used to raise the indices. Since we assume the shell to be static, one has $\dot{R}_\mathrm{i} = 0$: So in this static case, in brief, one has, that the 4-velocity u_i and the normal n_i are $u_\mathrm{i} = (1, 0, 0, 0)$ and $n_\mathrm{i} = \xi_\mathrm{i}(0, 1, 0, 0)$, the line element on the shell is

$$ds^2|_{\mathcal{S}_\mathrm{i}} = -d\tau^2 + R_\mathrm{i}^2 d\Omega^2, \tag{5}$$

and the extrinsic curvature is given by

$$K_\mathrm{i}{}^\tau{}_\tau = 0, \qquad K_\mathrm{i}{}^\theta{}_\theta = K_\mathrm{i}{}^\varphi{}_\varphi = \frac{\xi_\mathrm{i}}{R_\mathrm{i}}. \tag{6}$$

Under the Israel formalism both values for ξ_i are possible and we shall consider both cases.

The exterior Minkowski spacetime, $\mathcal{M}_\mathrm{e}$, is characterized by the following line element, in spacetime spherical coordinates,

$$ds_\mathrm{e}^2 = -dt_\mathrm{e}^2 + dr_\mathrm{e}^2 + r_\mathrm{e}^2 d\Omega^2, \tag{7}$$

where t_e and r_e are the time and radial coordinates, respectively, measured by a free-falling observer in $\mathcal{M}_\mathrm{e}$, and again $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the angular coordinates. Since the setup is the same as the one for the interior we will sketch the calculations briefly in order to be complete. For a timelike hypersurface it is convenient to choose the coordinates on $\mathcal{S}$ to be $\{y^a\} = (\tau, \theta, \varphi)$, where τ is the proper time measured by an observer comoving with $\mathcal{S}$. In this coordinate system, it follows that the 4-velocity u of an observer comoving with the shell is given by $e_\tau \equiv u$. The hypersurface $\mathcal{S}$, as seen from the exterior $\mathcal{M}_\mathrm{e}$ spacetime, is parametrized by τ, such that the surface's radial coordinate is described by a function $R_\mathrm{e} = R_\mathrm{e}(\tau)$. Then, $u_\mathrm{e} = (\frac{dt_\mathrm{e}}{d\tau}, \dot{R}_\mathrm{e}, 0, 0)$, where the subscript e is not an index and as before denotes exterior, and $\dot{R}_\mathrm{e} \equiv \frac{dR_\mathrm{e}}{d\tau}$. Since $\mathcal{S}$ is timelike, one has $u_{\mathrm{e}\,\alpha} u_\mathrm{e}^\alpha = -1$, so $u_\mathrm{e} = (\sqrt{1+\dot{R}_\mathrm{e}^2}, \dot{R}_\mathrm{e}, 0, 0)$, where we chose $\frac{dt_\mathrm{e}}{d\tau} > 0$ as we assume u_e to point to the future. From $e_a^\alpha n_\alpha = 0$, one finds the components of the normal vector field to $\mathcal{S}$, n, as seen from the exterior spacetime, namely $n_\mathrm{e} = \xi_\mathrm{e}(\dot{R}_\mathrm{e}, \sqrt{1+\dot{R}_\mathrm{e}^2}, 0, 0)$, where ξ_e is a normalization factor. Using $n^\alpha n_\alpha = 1$ and Eq. (7), yields $\xi_\mathrm{e} = \pm 1$. Defining $\nabla_\mathrm{e} r_\mathrm{e}$ as the gradient of r_e, the choice $\xi_\mathrm{e} = +1$ or $\xi_\mathrm{e} = -1$ represents whether the inner product $g_\mathrm{e}(n_\mathrm{e}, \nabla_\mathrm{e} r_\mathrm{e}) > 0$ or $g_\mathrm{e}(n_\mathrm{e}, \nabla_\mathrm{e} r_\mathrm{e}) < 0$, respectively. Using the induced metric equation, $h_{\mathrm{e}\,ab} = g_{\mathrm{e}\,\alpha\beta} e_a^\alpha e_b^\beta$, and $u_\mathrm{e} = (\sqrt{1+\dot{R}_\mathrm{e}^2}, \dot{R}_\mathrm{e}, 0, 0)$, we find that the induced metric on $\mathcal{S}$ by the spacetime $\mathcal{M}_\mathrm{e}$,

is such that the line element can be written as $ds^2|_{\mathcal{S}_e} = -d\tau^2 + R_e^2 d\Omega^2$. The non-null components of the extrinsic curvature are here given by, dropping here the superscript e to not overcrowd the notation, $K_{\tau\tau} = -a^\alpha n_\alpha$, $K_{\theta\theta} = \nabla_\theta n_\theta$, $K_{\varphi\varphi} = \nabla_\varphi n_\varphi$, where $a^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ represents the components of the 4-acceleration of an observer comoving with $\mathcal{S}$. Taking into account Eq. (7) and $u_e = (\sqrt{1+\dot{R}_e^2}, \dot{R}_e, 0, 0)$ the nontrivial components of the exterior curvature as seen from the exterior Minkowski spacetime are given by $K_e{}^\tau{}_\tau = \xi_e \frac{\ddot{R}_e}{\sqrt{1+\dot{R}_e^2}}$, and $K_e{}^\theta{}_\theta = K_e{}^\varphi{}_\varphi = \xi_e \frac{\sqrt{1+\dot{R}_e^2}}{R_e}$, where the induced metric h_e^{ab} associated with the hypersurface line element was used to raise the indices. Since we assume the shell to be static, one has $\dot{R}_e = 0$. So in this case one has that the 4-velocity u_e and the normal n_e are $u_e = (1,0,0,0)$ and $n_e = \xi_e(0,1,0,0)$, the line element on the shell is

$$ds^2|_{\mathcal{S}_e} = -d\tau^2 + R_e^2 d\Omega^2, \tag{8}$$

and the extrinsic curvature is given by

$$K_e{}^\tau{}_\tau = 0, \qquad K_e{}^\theta{}_\theta = K_e{}^\varphi{}_\varphi = \frac{\xi_e}{R_e}. \tag{9}$$

Both values for ξ_e are possible and we shall consider both cases.

To complete the solution we have find the properties of the matter at the thin shell. Indeed, having found previously the necessary expressions at the hypersurface $\mathcal{S}$ as seen from the $\mathcal{M}_i$ and $\mathcal{M}_e$ spacetimes, we can now use the Israel formalism to glue together the two spacetimes. Imposing the first junction condition, Eq. (2), and using Eqs. (5) and (8) for the induced metrics, we find that the radial coordinate of $\mathcal{S}$ as seen from the interior and exterior spacetimes, R_i and R_e, respectively, must be the same, $R_i = R_e$. We then denote by R the value of the radial coordinate of $\mathcal{S}$ as seen from both spacetimes,

$$R \equiv R_i = R_e. \tag{10}$$

We also assume that the stress-energy tensor S_{ab} of the thin shell on $\mathcal{S}$, can be cast in a perfect fluid form $S_{ab} = \sigma u_a u_b + p(h_{ab} + u_a u_b)$, where σ is the energy per unit area, p is the tangential pressure of the fluid, h_{ab} is the induced metric on $\mathcal{S}$, and u_a is the fluid's 3-velocity on $\mathcal{S}$. Using the appropriate equations we find $S^\tau{}_\tau = -\sigma$, and $S^\theta{}_\theta = S^\varphi{}_\varphi = p$. Having found previously the expressions for the extrinsic curvature of the hypersurface $\mathcal{S}$ as seen from the $\mathcal{M}_i$ and $\mathcal{M}_e$ spacetimes, and knowing that the stress-energy tensor of the shell is that of a perfect fluid, we can now impose the second junction condition, Eq. (3). Applying it to our spherically symmetric problem gives that the only nontrivial components of the stress-energy tensor $S^a{}_b$ are given by $S^\tau{}_\tau = \frac{1}{4\pi}[K^\theta{}_\theta]$, and $S^\theta{}_\theta = S^\varphi{}_\varphi = \frac{1}{8\pi}[K^\tau{}_\tau] + \frac{1}{4\pi}[K^\theta{}_\theta]$. Using then for the interior, $K_i{}^\tau{}_\tau = \xi_i \frac{\ddot{R}}{\sqrt{1+\dot{R}^2}}$, $K_i{}^\theta{}_\theta = K_i{}^\varphi{}_\varphi = \xi_i \frac{\sqrt{1+\dot{R}^2}}{R}$, and for the exterior, $K_e{}^\tau{}_\tau = \xi_e \frac{\ddot{R}}{\sqrt{1+\dot{R}^2}}$, and $K_e{}^\theta{}_\theta = K_e{}^\varphi{}_\varphi = \xi_e \frac{\sqrt{1+\dot{R}^2}}{R}$, we find $\sigma = (\xi_i - \xi_e)\frac{\sqrt{1+\dot{R}^2}}{4\pi R}$ and $p = -(\xi_i - \xi_e)\frac{R\ddot{R}+\dot{R}^2+1}{8\pi R\sqrt{1+\dot{R}^2}}$. From these two equations we derive the following conservation law for the shell $\dot{\sigma} + \frac{2\dot{R}}{R}(\sigma + p) = 0$. For a static shell, the time derivatives are zero and so, using directly if one wishes Eqs. (6) and (9),

$$\sigma = (\xi_i - \xi_e)\frac{1}{4\pi R}, \tag{11}$$

$$p = -(\xi_i - \xi_e)\frac{1}{8\pi R}. \tag{12}$$

From Eqs. (11) and (12) we derive

$$\sigma + 2p = 0. \tag{13}$$

The matter of the thin shell obeys necessarily this equation of state, namely, $p = -\frac{1}{2}\sigma$, for a Minkowski-Minkowski static spacetime. From Eqs. (11) and (12), we see that for $\xi_i = \xi_e$, and so $\frac{1}{2}(\xi_i - \xi_e) = 0$ we get the trivial case, $\sigma = 0$ and $p = 0$. For $\xi_i = 1$ and $\xi_e = -1$, and so $\frac{1}{2}(\xi_i - \xi_e) = 1$, we get $\sigma = \frac{1}{2\pi R}$ and $p = -\frac{1}{4\pi R}$. For $\xi_i = -1$ and $\xi_e = 1$, and so $\frac{1}{2}(\xi_i - \xi_e) = -1$, we get $\sigma = -\frac{1}{2\pi R}$ and $p = \frac{1}{4\pi R}$. All cases obey Eq. (13), i.e., the relation between the surface energy density σ and the surface pressure p is independent of ξ_i or ξ_e. Besides the trivial case, i.e., the Minkowski universe which has $\frac{1}{2}(\xi_i - \xi_e) = 0$, there are two possible universes at this juncture, the Minkowski-Minkowski static closed universe which has $\frac{1}{2}(\xi_i - \xi_e) = 1$, and the Minkowski-Minkowski static open universe which has $\frac{1}{2}(\xi_i - \xi_e) = -1$.

2. Linearized stability analysis for Minkowski-Minkowski universes

An important question regarding the Minkowski-Minkowski static universe solutions, i.e., the Minkowski-Minkowski closed and open universes, is if these are stable under perturbations. Here we will discuss the linear stability of the Minkowski-Minkowski solutions that we have found by analyzing the equation of motion of the shell near the static configuration.

To study the linear stability of the Minkowski-Minkowski static universe solution we have to find the evolution equation for the shell radius R and analyze the behavior of these solutions as we perturb the spacetime. The analysis can be done in a unified way by making use of

the parameter $\frac{1}{2}(\xi_i - \xi_e)$. The equation of motion of the thin shell previously found, namely, $\sigma = (\xi_i - \xi_e)\frac{\sqrt{1+\dot{R}^2}}{4\pi R}$, can be inverted and put in the form

$$\dot{R}^2 + V(R) = 0, \tag{14}$$

where a dot means derivative with respect to time t, and the potential $V(R)$ is given by

$$V(R) = 1 - \left(\frac{4\pi R\sigma}{\xi_i - \xi_e}\right)^2. \tag{15}$$

A thin matter shell is stable if and only if the potential $V(R)$ at the shell's position is at a local minimum, i.e., if $V'(R) = 0$ and $V''(R) \geq 0$, with the equality providing the marginal neutral case and where a prime denotes the derivative with respect to R. Thus, we have to calculate the matter properties and its derivatives. All these properties are functions of the shell radius R, namely, $\sigma = \sigma(R)$, $p = p(R)$, $\sigma' = \sigma'(R)$, and $p' = p'(R)$. To find an expression for $\sigma'(R)$, we consider the conservation law for the shell already found, namely, $\dot{\sigma} + \frac{2\dot{R}}{R}(\sigma + p) = 0$, i.e., $\dot{\sigma} = -\frac{2\dot{R}}{R}(\sigma + p)$. Using the inverse function theorem, we have $\sigma' = \frac{\dot{\sigma}}{\dot{R}}$, and so

$$\sigma' = -\frac{2(\sigma + p)}{R}. \tag{16}$$

We have also to analyze the derivatives of the potential at the static configuration. From Eq. (15) we get $V'(R) = -\frac{32\pi^2 R\sigma}{(\xi_i - \xi_e)^2}(\sigma + R\sigma')$. Taking another derivative we get $V''(R) = \frac{32\pi^2}{(\xi_i - \xi_e)^2}[(\sigma + R\sigma')(\sigma + 2p) + R\sigma(\sigma' + 2p')]$. Now, if we introduce Eq. (16) into $V'(R)$, we have $V'(R) = \frac{32\pi^2 R\sigma}{(\xi_i - \xi_e)^2}(\sigma + p)$. To analyze $V''(R)$ we have to find an expression for $p'(R)$. We assume that the thin matter shell is composed of cold matter such that it verifies a generic equation of state of the form $p = p(\sigma)$. Then, we can define the parameter $\eta(\sigma) = \frac{\partial p}{\partial \sigma}$ such that, $p' = \eta\sigma'$. Hence, using $\sigma' + 2p' = \sigma'(1 + 2\eta)$ and Eq. (16) we can write $V''(R)$ as $V''(R) = -\frac{32\pi^2}{(\xi_i - \xi_e)^2}[2\sigma(\sigma + p)(1 + 2\eta) + (\sigma + 2p)^2]$. In brief, the derivatives of the potential $V(R)$ are

$$V'(R) = \frac{32\pi^2 R\sigma}{(\xi_i - \xi_e)^2}(\sigma + p), \tag{17}$$

and

$$V''(R) = -\frac{32\pi^2}{(\xi_i - \xi_e)^2}[2\sigma(\sigma + p)(1 + 2\eta) + (\sigma + 2p)^2]. \tag{18}$$

where $\eta(\sigma) = \frac{\partial p}{\partial \sigma}$.

We now linearize the equation of motion for the shell given by Eq. (14) around a static solution. Defining R_0 as the circumferential radius of the static thin shell and assuming the potential V to be a differentiable function at R_0, we can expand the potential given in Eq. (15) around R_0 as

$$V(R) = V(R_0) + V'(R_0)(R - R_0) + \frac{1}{2}V''(R_0)(R - R_0)^2, \tag{19}$$

plus higher order terms of $\mathcal{O}[(R - R_0)^3]$. A thin matter shell with radius R_0 is stable or neutrally stable if and only if the potential $V(R)$ satisfies $V'(R_0) = 0$ and $V''(R_0) \geq 0$. For a shell with radius R_0 the static solutions found in the previous section are characterized generically by the following expressions, $\sigma = \frac{\xi_i - \xi_e}{4\pi R_0}$ and $p = -\frac{\xi_i - \xi_e}{8\pi R_0}$, see Eqs. (11) and (12), where $\xi_i \neq \xi_e$ for the nontrivial solutions. Substituting Eq. (11) into Eq. (15) we find $V(R_0) = 0$, as expected. Substituting Eqs. (11) and (12) into Eq. (17) and evaluating it at the static solution we find $V'(R_0) = 0$. Evaluating $V''(R)$ at the static solution, $R = R_0$, and using again Eqs. (11) and (12), we find $V''(R_0) = 0$. In brief,

$$V'(R_0) = 0, \tag{20}$$

and

$$V''(R_0) = 0. \tag{21}$$

Moreover, all higher order derivatives of the potential go to zero at R_0 for the static solutions.

Gathering these calculations, we conclude that, besides the trivial case, i.e., the Minkowski universe which has $\frac{1}{2}(\xi_i - \xi_e) = 0$ and is trivially neutrally stable, there is the Minkowski-Minkowski static closed universe which has $\frac{1}{2}(\xi_i - \xi_e) = 1$ and is nontrivially neutrally stable, and the Minkowski-Minkowski static open universe which has $\frac{1}{2}(\xi_i - \xi_e) = -1$ and is also nontrivially neutrally stable. This neutral stability means that if we slightly displace the thin shell, it will simply stay at the new radius. This confirms our expectation, as the interior and exterior spacetimes are both described by the Minkowski, i.e., flat, solution.

B. Minkowski-Minkowski universes: Geometry and physics

1. Minkowski-Minkowski static closed universe: A bubble universe

Here we display a Minkowski-Minkowski static closed universe as a solution of general relativity. We rely on the results presented above. We assume that the Minkowski

line element is valid for a region, which we call interior $\mathcal{M}_{\rm i}$, up to a radius R, i.e., $0 \leq r_{\rm i} \leq R$, where $r_{\rm i}$ denotes the interior radial coordinate. We join this region to another region, which we call exterior $\mathcal{M}_{\rm e}$, where the exterior radial coordinate is denoted by $r_{\rm e}$. The junction is done at a common hypersurface $\mathcal{S}$ with circumferential radius $r_{\rm i} = r_{\rm e} = R$. Thus, the whole spacetime is composed by the two regions plus the common hypersurface, which is a domain wall, i.e., a thin shell. The common hypersurface $\mathcal{S}$ is assumed to be static. Assuming the existence of a vector field n, normal, at each point, to the common hypersurface $\mathcal{S}$, we have found that the solution depends on the orientation of this normal field n. For each region, $\mathcal{M}_{\rm i}$ and $\mathcal{M}_{\rm e}$, the orientation of the normal is encoded in a single parameter, namely, one for the interior, $\xi_{\rm i}$, and one for the exterior, $\xi_{\rm e}$. In both cases, $\xi_{\rm i}$ and $\xi_{\rm e}$ can have values $+1$ or -1. The value $+1$ indicates that the normal points in the direction of increasing radial coordinate and the value -1 indicates that the normal points in the direction of decreasing radial coordinate. The solutions with $\xi_{\rm i} = \xi_{\rm e}$ are trivial as the resulting spacetime is simply the full Minkowski flat universe. Here we consider the first nontrivial solution, i.e., $\xi_{\rm i} = +1$ and $\xi_{\rm e} = -1$. This static solution represents the case where the normal vector field n points in the direction of increasing radial coordinate as seen from the interior spacetime $\mathcal{M}_{\rm i}$ and points in the direction of decreasing radial coordinate as seen from the exterior spacetime $\mathcal{M}_{\rm e}$. Since n is assumed to point from $\mathcal{M}_{\rm i}$ to $\mathcal{M}_{\rm e}$ this implies that in the exterior region one also has $0 \leq r_{\rm e} \leq R$. Thus, this solution is composed by two spatially compact Minkowski spacetime regions glued together at the common boundary $\mathcal{S}$. This solution then represents a Riemann flat spacetime everywhere except at $\mathcal{S}$. Overall it is a static closed Minkowski-Minkowski universe, for which the line element can be written as, see also Eqs. (4) and (7),

$$\begin{aligned} ds^2 &= -dt^2 + dr_{\rm i}^2 + r_{\rm i}^2 d\Omega^2, \qquad 0 \leq r_{\rm i} \leq R, \\ ds^2 &= -dt^2 + dr_{\rm e}^2 + r_{\rm e}^2 d\Omega^2, \qquad 0 \leq r_{\rm e} \leq R. \end{aligned} \tag{22}$$

Now we turn to the properties of the matter at the domain wall, or shell, at $\mathcal{S}$. In this case, putting $\xi_{\rm i} = +1$ and $\xi_{\rm e} = -1$ into Eqs. (11) and (12), the energy density σ and the tangential pressure p at the domain wall are given by

$$\sigma = \frac{1}{2\pi R}, \tag{23}$$

$$p = -\frac{1}{4\pi R}. \tag{24}$$

This solution is then characterized by the presence of a surface layer in the form of a domain wall at radius R, separating two Minkowski halves. The thin domain wall is composed of a perfect fluid with positive energy density and is supported by tension such that it obeys the equation of state $\sigma + 2p = 0$. Moreover, from Eqs. (23) and (24), we find that the following inequalities are verified: $\sigma \geq 0$, $\sigma + p \geq 0$, $\sigma + 2p \geq 0$ and $\sigma \geq |p|$, therefore, the matter at the domain wall verifies the null, weak, strong, and dominant pointwise energy conditions. Since the effective mass m can be defined by the quantity $\sigma + 2p$ and this latter is zero, one has $m = 0$. So the domain wall yields no total mass m as it should to have Minkowski spacetime on both sides of the domain wall. The volume of this universe is $V = \frac{8\pi}{3} R^3$.

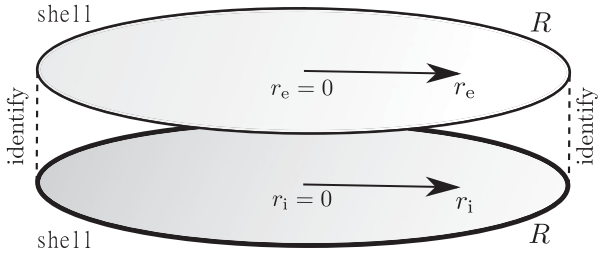

FIG. 1. Embedding diagram of a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Minkowski-Minkowski static closed universe in 3-dimensional Euclidean space. The interior coordinate is in the range $0 \leq r_{\rm i} \leq R$, the exterior coordinate is in the range $0 \leq r_{\rm i} \leq R$, the radius of the domain wall, or shell, is R, and the borders of the circumferences should be identified.

The spatial structure and the causal structure are also important to analyze. A time slice $t =$ constant of the spacetime gives that the 3-space is a highly squashed 3-sphere, i.e., it is made of two copies of two plane 3-balls joined at a 2-sphere. To see this one makes an embedding. The embedding of this 3-space can be easily done in 4-dimensional Euclidean space $\mathbb{R}^4$. In Fig. 1 we show an embedding diagram of a $\theta = \frac{\pi}{2}$ slice of the static Minkowski-Minkowski closed universe in a 3-dimensional Euclidean space, displaying clearly the squashed character of the 3-sphere. One can also make appropriate identifications between points in the interior and exterior spherical pieces to turn the space into a projective space. The causal structure of the resulting spacetime can be shown in a Carter-Penrose diagram, as in Fig. 2. We use the hash symbol # to represent the connected sum of the spacetime manifold in order to conserve the conformal structure in the Carter-Penrose diagram of the total spacetime. We see that it represents a universe in which the spatial sections are highly squashed 3-spheres, i.e., two copies of two plane 3-balls joined at a 2-sphere, such that if we include time the total spacetime is a squashed 3-cylinder, the time line times the squashed 3-sphere. A timelike geodesic, or a free-falling particle, initially moving along the radial coordinate toward increasing values of it in one half of the spacetime, would reach the domain wall at $r = R$ at some point, and then continue until it reaches the center of coordinates at the other half where it would continue its trajectory into the antipode point of the wall, and so on.

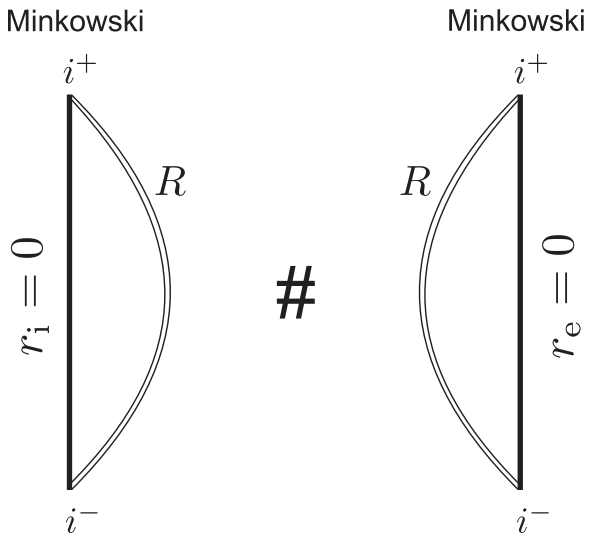

FIG. 2. Carter-Penrose diagram of the Minkowski-Minkowski static closed universe. The hash symbol represents the connected sum of the spacetime manifold. The symbols i^- and i^+ represent past and future causal infinity, respectively.

The Minkowski-Minkowski static closed universe is marginally stable. Indeed, one has from Eq. (21) that $V''(R_0) = 0$. The solution is in neutral equilibrium, meaning that for a slight displacement the thin shell stays at the new radius. This result confirms the expectations. as the interior and exterior spacetimes are both described by a Minkowski, i.e., flat, solution.

This Minkowski-Minkowski static closed universe is a bubble universe which is summarized in Eqs. (22)–(24) and in Figs. 1 and 2. Moreover, a fundamental shell is defined as a shell with Minkowski interior with a center and one of the three basic exterior spacetimes, Minkowski, Schwarzschild, and Reissner-Nordström. Thus the Minkowski-Minkowski static closed universe we just found completes the search of all the fundamental shells in the three basic ambient spacetimes, namely, Minkowski-Minkowski, Minkowski-Schwarzschild, and Minkowski-Reissner-Nordström, the latter two having been found previously.

The Minkowski-Minkowski static closed universe is a representative of the set of closed universes. It can be compared with other such closed universes. This will be done later.

2. Minkowski-Minkowski static open universe: A traversable wormhole

Here we display a Minkowski-Minkowski static open universe as a solution of general relativity. We rely on the previous results. We assume that the Minkowski line element is valid for a region, which we call interior $\mathcal{M}_\mathrm{i}$, that goes from spatial infinity to a radius R, i.e., $R \leq r_\mathrm{i} < \infty$, where r_i denotes the interior radial coordinate. We join this region to another region, which we call exterior $\mathcal{M}_\mathrm{e}$, where the exterior radial coordinate is denoted by r_e. The junction is done at a common hypersurface $\mathcal{S}$ with circumferential radius $r_\mathrm{i} = r_\mathrm{e} = R$. Thus, the whole spacetime is composed by the two regions plus the common hypersurface, which is a thin shell. The common hypersurface $\mathcal{S}$ is assumed to be static. Assuming the existence of a vector field n, normal, at each point, to the common hypersurface $\mathcal{S}$, we have found that the solution depends on the orientation of this normal field n. For each region, $\mathcal{M}_\mathrm{i}$ and $\mathcal{M}_\mathrm{e}$, the orientation of the normal is encoded in a single parameter, namely, one for the interior, ξ_i, and one for the exterior, ξ_e. In both cases, ξ_i and ξ_e can have values $+1$ or -1. The value $+1$ indicates that the normal points in the direction of increasing radial coordinate and the value -1 indicates that the normal points in the direction of decreasing radial coordinate. The solutions with $\xi_\mathrm{i} = \xi_\mathrm{e}$ are trivial as the resulting spacetime is simply the full Minkowski flat universe. Here, we consider the second nontrivial solution, i.e., $\xi_\mathrm{i} = -1$ and $\xi_\mathrm{e} = +1$. This static solution represents the case where the normal vector field n points in the direction of decreasing radial coordinate as seen from the interior spacetime $\mathcal{M}_\mathrm{i}$ and points in the direction of increasing radial coordinate as seen from the exterior spacetime $\mathcal{M}_\mathrm{e}$. Since n is assumed to point from $\mathcal{M}_\mathrm{i}$ to $\mathcal{M}_\mathrm{e}$ this implies that in the exterior region one also has $R \leq r_\mathrm{e} < \infty$. Thus, this solution is composed by two spatially open Minkowski spacetime regions glued together at the common boundary $\mathcal{S}$. This solution then represents a Riemann flat spacetime everywhere except at $\mathcal{S}$. Overall it is a static closed Minkowski-Minkowski universe, for which the line element can be written as, see also Eqs. (4) and (7),

$$\begin{aligned} ds^2 &= -dt^2 + dr_\mathrm{i}^2 + r_\mathrm{i}^2 d\Omega^2, \qquad R \leq r_\mathrm{i} < \infty, \\ ds^2 &= -dt^2 + dr_\mathrm{e}^2 + r_\mathrm{e}^2 d\Omega^2, \qquad R \leq r_\mathrm{e} < \infty. \end{aligned} \tag{25}$$

Now we turn to the properties of the matter shell at $\mathcal{S}$. In this case, putting $\xi_\mathrm{i} = -1$ and $\xi_\mathrm{e} = +1$ into Eqs. (11) and (12), the energy density σ and the tangential pressure p at the thin shell are given by

$$\sigma = -\frac{1}{2\pi R}, \tag{26}$$

$$p = \frac{1}{4\pi R}. \tag{27}$$

This solution is then characterized by the presence of a surface layer or thin shell at radius R, separating two Minkowski open halves. The thin matter shell is composed of a perfect fluid with negative energy density and is supported by pressure such that it obeys the equation of state $\sigma + 2p = 0$. Moreover, from Eqs. (26) and (27), we find that the following inequalities are verified: $\sigma \leq 0$, $\sigma + p \leq 0$, $\sigma + 2p \leq 0$ and $\sigma \leq |p|$, therefore, the matter

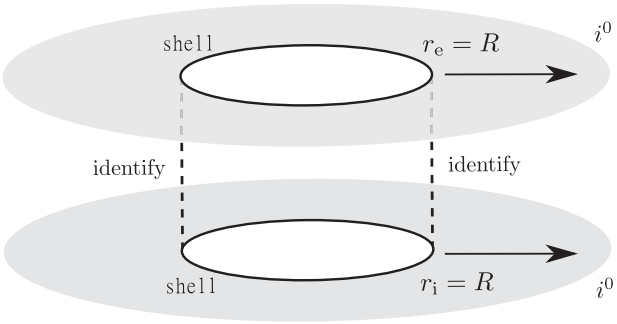

FIG. 3. Embedding diagram of a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Minkowski-Minkowski static open universe, or traversable wormhole, in 3-dimensional Euclidean space. The interior coordinate is $R \leq r_i < \infty$, the exterior coordinate is $R \leq r_e < \infty$, the radius of the shell is R, and the borders of the circumferences should be identified.

shell violates the null, weak, strong and dominant pointwise energy conditions, as it could be expected from imposing the flare-out condition for a traversable wormhole. Since the effective mass m can be defined by $\sigma + 2p$ and this latter is zero, one has $m = 0$. So the shell yields no total mass m as it should to have Minkowski spacetime on both sides of the shell. The volume of this spacetime, in its simplest personification, i.e., without making identifications for r, is infinite.

The spatial structure and the causal structure are also important to analyze. A time slice $t =$ constant of the spacetime gives that the 3-space is a universe in which the spatial sections are two copies of the complements of two plane 3-balls joined at a 2-sphere, the throat, yielding a nonsimply connected open universe, more precisely, a traversable wormhole. To see this one makes an embedding. The embedding of this 3-space can be easily done in 4-dimensional Euclidean space $\mathbb{R}^4$. In Fig. 3 we show an embedding diagram of a constant $\theta = \frac{\pi}{2}$ slice of the static Minkowski-Minkowski open universe, or traversable wormhole. One can also make appropriate identifications of the two open sheets, and turn the space into, e.g., a flat 3-torus, in which case the space is closed. The causal structure of the resulting spacetime can be shown in a Carter-Penrose diagram, as in Fig. 4. We use the hash symbol # to represent the connected sum of the spacetime manifolds in order to conserve the conformal structure in the Carter-Penrose diagram of the total spacetime. We see that it represents a universe in which the spatial sections are two copies of the complements of two plane 3-balls joined at a 2-sphere, the throat, yielding a traversable wormhole, such that if we include time, the total spacetime has the topology $\mathbb{R} \times \Sigma$, where Σ is a 3-manifold with nontrivial topology, whose boundary $\partial\Sigma \sim S^2$. A causal geodesic, or a free-falling particle, initially moving in the direction of decreasing radial coordinate in one half of the spacetime, would reach the shell at $r = R$ and then continue until it reaches infinity at the other sheet of the wormhole.

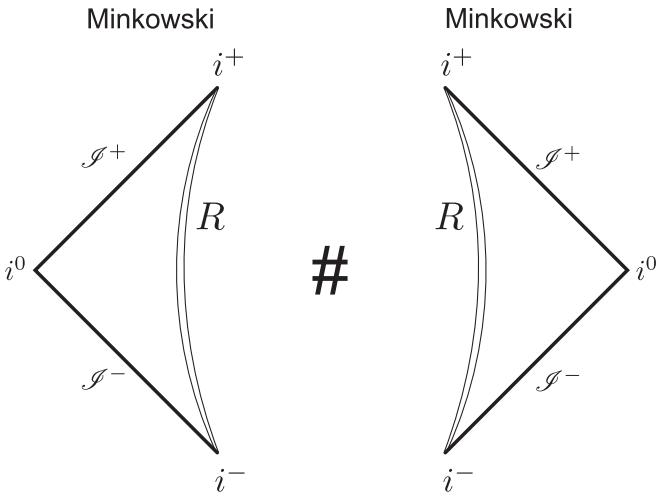

FIG. 4. Carter-Penrose diagram of the Minkowski-Minkowski static open universe, or traversable wormhole. The hash symbol represents the connected sum of the spacetime manifold. The symbols i^-, i^0, and i^+ represent past timelike infinity, spatial infinity, and future timelike infinity, respectively, and the symbols $\mathscr{I}^-$ and $\mathscr{I}^+$, represent past and future null infinity, respectively.

The Minkowski-Minkowski static open universe, i.e., the Minkowski-Minkowski traversable wormhole, is marginally stable. Indeed, one has from Eq. (21) that $V''(R_0) = 0$. The solution is in neutral equilibrium, meaning that for a slight displacement the thin shell stays at the new radius. This result confirms the expectations, as the interior and exterior spacetimes are both described by a Minkowski, i.e., flat, solution.

This Minkowski-Minkowski static open universe is a traversable wormhole which is summarized in Eqs. (25)–(27) and in Figs. 3 and 4. This open universe is an exotic shell spacetime rather than a fundamental shell spacetime, as its interior does not contain a center or origin, instead the interior opens up to infinity. This further possibility for a shell spacetime, i.e., that its interior opens up to infinity, implies that the spacetime is a traversable wormhole spacetime, and thus the matter properties of the shell must be exotic since they necessarily violate the energy conditions. The study of the Minkowski-Minkowski open spacetime, or traversable wormhole, introduces the prospect of analyzing all possible exotic shells, i.e., shells for which the Minkowski interior has no center, in the other two basic ambient spacetimes, namely, Schwarzschild and Reissner-Nordström spacetimes.

The Minkowski-Minkowski static open universe is a representative of the set of traversable wormholes. It can be compared with other such open universes and traversable wormholes. This will be done later.

C. Minkowski-Minkowski universes: One concept with two sides

The two, close and open, Minkowski-Minkowski spacetimes demonstrate the idea that they can be seen as

complementary to each other, i.e., they are two sides of the same concept. The concept, i.e., a collection of two Minkowski spacetimes together, that when cut into spherical regions yield on one side a closed universe, a bubble universe, and on the other side an open universe which is a traversable wormhole. The formalism presented in analyzing the two Minkowski-Minkowski universes is well suited to show this point. Indeed, from an algebraic point of view, one side is given by $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=1$, the other side is given by $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=-1$, where $\xi_{\rm i}$ and $\xi_{\rm e}$ are the characteristics of the interior and exterior normals to the shell, respectively. This algebraic side appears clearly in the evaluation of the matter properties as displayed in Eqs. (11) and (12). More formally, to implement the idea of a Minkowski-Minkowski closed universe, i.e., a bubble universe, and a Minkowski-Minkowski open universe, i.e., a traversable wormhole, one uses the equations of general relativity together with the appropriate thin shell formalism. For a shell with a Minkowski interior with a center, when the normal to the shell in the exterior region points toward decreasing r, i.e., $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=1$, one finds the Minkowski-Minkowski closed universe, made of two 3-dimensional flat balls, or sheets, that are joined at some domain wall, i.e., a 2-sphere shell with matter, to make a Minkowski-Minkowski bubble universe. For a shell with a Minkowski open interior, when the normal to the shell points toward increasing r in the Minkowski exterior, i.e., $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=-1$, one finds the Minkowski-Minkowski open universe, made of two 3-dimensional flat open infinite sheets that are joined at some throat, to make a Minkowski-Minkowski traversable wormhole. From a matter point of view the two universes show a form of complementarity, as for $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=1$ the matter obeys the energy conditions while for $\frac{1}{2}(\xi_{\rm i}-\xi_{\rm e})=-1$ the matter violates the energy conditions. From a geometrical point of view, the two sides of the concept appear when one picks up a Minkowski spacetime and at constant time cuts a ball in it, to obtain two spaces, namely, a 3-dimensional ball with a flat inside, and an infinite extended 3-dimensional flat space with a hole, which is the complement of the ball. Then one picks up another Minkowski spacetime and do the same, to get a second ball and a second infinite extended flat space with a hole. One side is given if one joins the two 3-dimensional balls along a 2-sphere, a shell containing matter, to obtain a single 3-space that including time makes altogether a static closed Minkowski-Minkowski universe, a bubble universe. The other side is given if one joins the two complements, i.e., the two infinite extended 3-dimensional flat spaces with a hole in each, along a 2-sphere, a shell containing matter, to obtain a different single 3-space that including time makes altogether another Minkowski-Minkowski universe, which is a traversable wormhole. Comparison of Fig. 1 with Fig. 3 for a spatial geometrical representation of the bubble universe and the traversable wormhole, respectively, displays the complementarity of the two spaces clearly, which can be further strengthened with the comparison of the spacetime drawings in the form of Carter-Penrose diagrams, given in Fig. 2 and in Fig. 4, respectively. From a stability point of view it is also interesting that both spacetimes are marginally stable, showing thus here some form of neutral complementarity.

So, that the two Minkowski-Minkowski spacetimes demonstrate that they can be seen as complementary to each other, i.e., they are two sides of the same concept, is clear. It can be raised the point that the bubble universe has matter that obeys the energy conditions, whereas the traversable wormhole has matter that does not obey those conditions. This is obviously true, but there is no real problem with it. In an early era of the Universe, when quantum gravity dominates, there is really no obeyance to the classical energy conditions and the closed and open universes, created as bubble universes with domain walls and traversable wormholes with throats out of the spacetime foam must coexist together. Some kind of inflation would grow these objects to macroscopic dimensions turning them into new structures inhabiting the Universe itself, showing that bubble universes and traversable wormholes are distinct but connected objects, some obeying the energy conditions and others not.

III. EINSTEIN STATIC CLOSED UNIVERSE AND FRIEDMANN STATIC HYPERBOLIC OPEN UNIVERSE

A. Einstein and Friedmann static universes: Formal solutions and stability

1. Solutions

Two paradigmatic solutions of the theory of general relativity for static universes are the Einstein and the hyperbolic Friedmann spacetimes. These two solutions have various resemblances with the open and closed Minkowski-Minkowski universes studied in the previous section, and we will present them in a form suited for comparing their properties. Consider then the Einstein field equations with a nonvanishing cosmological constant Λ,

$$G_{\alpha\beta}+\Lambda g_{\alpha\beta}=8\pi T_{\alpha\beta}, \tag{28}$$

where $G_{\alpha\beta}=\mathcal{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\mathcal{R}$ is the Einstein tensor, $\mathcal{R}_{\alpha\beta}$ and $\mathcal{R}$ are the Ricci tensor and Ricci scalar, respectively, $g_{\alpha\beta}$ is the spacetime metric, and $T_{\alpha\beta}$ is the stress-energy tensor.

One assumes a static, homogeneous and isotropic spacetime and, in addition, one supposes that $T_{\alpha\beta}$ corresponds to that of a perfect fluid with energy density ρ and vanishing pressure p, i.e., a dustlike fluid. With these assumptions, the solution of the field equations (28) in spacetime spherical coordinates (t,r,θ,φ) is given by the line element

$$ds^2 = -dt^2 + dr^2 + R^2\left[\frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\frac{r}{R}\right)\right]^2 d\Omega^2, \quad (29)$$

where t is the time coordinate, r is the radial coordinate, $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the spherical angular coordinates, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, R is a positive constant scale factor representing a characteristic radius of the universe, and k is related with the Ricci curvature scalar by $\mathcal{R} = \frac{6k}{R^2}$ and may take the values $k = 1, 0, -1$. Furthermore, the field equations (28) also give expressions for the energy density of the fluid and for the cosmological constant, namely,

$$\rho = \frac{k}{4\pi R^2}, \quad (30)$$

$$\Lambda = \frac{k}{R^2}, \quad (31)$$

and $p = 0$. From Eqs. (30) and (31) one finds that

$$\rho - \frac{\Lambda}{4\pi} = 0, \quad (32)$$

and so Λ counteracts the gravitational pulling effects of ρ. Noticing that the term $\Lambda g_{\alpha\beta}$ can be thought as a perfect fluid contribution to the stress-energy tensor with energy density $\bar{\rho} = \frac{\Lambda}{8\pi}$ and pressure $\bar{p} = -\frac{\Lambda}{8\pi}$, this static homogeneous universe solution can then be seen as a solution of a two fluid system, one fluid with energy density $\rho = \frac{\Lambda}{4\pi}$ and pressure $p = 0$, and the other fluid, a vacuum fluid, with energy density $\bar{\rho} = \frac{\Lambda}{8\pi}$ and pressure $\bar{p} = -\frac{\Lambda}{8\pi}$, such that $\rho + \bar{\rho} + 3\bar{p} = 0$. Besides the trivial case, i.e., the Minkowski universe which has $k = 0$, there are two possible universes at this juncture, the Einstein static closed universe which has $k = 1$, and the Friedmann static open universe which has $k = -1$.

2. Linearized stability analysis for Einstein and Friedmann static universes

An important question regarding the static homogeneous universes, i.e., the Einstein and hyperbolic Friedmann static universes, is if these are stable under perturbations. Here we will discuss the linear stability of these cosmological solutions by analyzing the equation of motion of the universe near the static configuration. The result for the Einstein universe is well known, whereas the stability of the static hyperbolic Friedmann universe appears to be less known.

To study the linear stability of the static Einstein and Friedmann universes we have to find the evolution equation for the scalar factor R and analyze the behavior of these solutions as we perturb the spacetime. The analysis can be done in a unified way by making use of the parameter k. From the general relativity field equations we find the Friedmann equation, namely,

$$\dot{R}^2 + V(R) = 0, \quad (33)$$

where we have introduced the potential

$$V(R) = k - \frac{8\pi\rho + \Lambda}{3}R^2, \quad (34)$$

and the scale factor R is now a function of the time coordinate t, $R = R(t)$, and a dot represents derivative with respect to it, and again k represents the sectional curvature of constant time slices, such that $k = +1$ for the closed universe, $k = 0$ for the flat universe, and $k = -1$ for the open universe. Note anew that a universe is stable if, and only if, the potential $V(R)$ is at a local minimum, i.e., if $V'(R) = 0$ and $V''(R) \geq 0$, with the equality providing the marginal neutral case, where a prime denotes the derivative with respect to R. Thus, we have to calculate the matter properties and its derivatives. All these properties are functions of the shell radius R, namely, $\rho = \rho(R)$, $p = p(R)$, $\rho' = \rho'(R)$, $p' = p'(R)$, and the radius itself is a function of time $R(t)$. The conservation equation, which can be taken from the field equations, is $\dot{\rho} + \frac{3\dot{R}}{R}\rho = 0$, so that the equation for ρ', where prime denotes the derivative with respect to R, is

$$\rho' = -3\frac{\rho}{R}. \quad (35)$$

We have now to analyze the derivatives of the potential $V(R)$ given in Eq. (34) at the static configuration. From Eq. (34) we get $V'(R) = -\frac{1}{3}(8\pi\rho')R^2 - \frac{2}{3}(8\pi\rho + \Lambda)R$, where we assume that Λ is a constant. Taking the derivative of it we get $V''(R) = -\frac{1}{3}(8\pi\rho'')R^2 - \frac{4}{3}(8\pi\rho')R - \frac{2}{3}(8\pi\rho + \Lambda)$. Now, if we introduce Eq. (35) into $V'(R)$ we obtain $V'(R) = \frac{2}{3}(4\pi\rho - \Lambda)R$. Simplifying also $V''(R)$ we obtain $V''(R) = -\frac{2}{3}(8\pi\rho + \Lambda)$. In brief, the derivatives of the potential $V(R)$ are

$$V'(R) = \frac{2}{3}(4\pi\rho - \Lambda)R, \quad (36)$$

and

$$V''(R) = -\frac{2}{3}(8\pi\rho + \Lambda), \quad (37)$$

where we assume that $p = 0$ throughout, i.e., the cold generic equation $p = p(\rho)$ is the trivial one, so here $\eta(\rho) \equiv \frac{\partial p}{\partial \rho}$ is zero: $\eta(\rho) = 0$.

Following the usual reasoning, we linearize Eq. (33) around a static solution. Defining R_0 as the radius of the

static universe and assuming the potential V to be a differentiable function at R_0, we can expand the potential (34) around R_0 as

$$V(R) = V(R_0) + V'(R_0)(R - R_0) + \frac{1}{2}V''(R_0)(R - R_0)^2, \tag{38}$$

plus higher order terms of $\mathcal{O}[(R - R_0)^3]$. Now, a universe with radius R_0 is stable if, and only if, the potential, $V(R)$, is at a local minimum, i.e., if $V'(R_0) = 0$ and $V''(R_0) \geq 0$, with the equality providing the marginal neutral case. The static solutions found in the previous sections are characterized by the following expressions for the energy density and the cosmological constant of the fluid, $\rho = \frac{k}{4\pi R_0^2}$ and $\Lambda = \frac{k}{R_0^2}$, see Eqs. (30) and (31), with the fluid pressure p being zero, $p = 0$. Then putting Eq. (30) into Eq. (34) we find $V(R_0) = 0$, as expected. Substituting it into Eq. (36) and evaluating at the static solution we find $V'(R_0) = 0$. Evaluating $V''(R)$ at the static solution, $R = R_0$, we find $V''(R_0) = -\frac{2k}{R_0^2}$. In brief, defining R_0 as the value of the scale factor of the static Einstein and hyperbolic Friedmann universes, expanding the potential $V(R)$ around R_0, Eq. (38), we find that

$$V'(R_0) = 0, \tag{39}$$

and

$$V''(R_0) = -\frac{2k}{R_0^2}. \tag{40}$$

Gathering these calculations, we conclude that, besides the trivial case, i.e., the Minkowski universe which has $k = 0$ and is trivially neutrally stable, there is the Einstein static closed universe which has $k = 1$ and so is unstable, and the Friedmann static open universe which has $k = -1$ and so is stable. The instability of the $k = 1$ Einstein static closed universe means that if we slightly displace the scale radius R toward larger or smaller values, the universe will expand in the former displacement or collapse in the latter displacement, and the stability of the $k = -1$ Friedmann static closed universe means that if we slightly displace the scale radius R toward larger or smaller values, the universe will get back to the initial value R.

B. Einstein and Friedmann static universes: Geometry and physics

1. Einstein static closed universe

The Einstein universe is a solution of the general theory of relativity for a dust source with energy density ρ, pressure p equal to zero, a positive cosmological constant Λ, and positive curvature, $k = 1$. In spacetime spherical coordinates (t, r, θ, φ) it is characterized by the line element given in Eq. (29) with $k = 1$, i.e.,

$$ds^2 = -dt^2 + dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\Omega^2, \tag{41}$$

where t is the time coordinate, r is the radial coordinate with $0 \leq r \leq \pi R$, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the spherical angular coordinates, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. In addition, R is a positive scale factor which here is a constant, and which gives the characteristic radius of the universe. The Ricci scalar for the Einstein universe is given by $\mathcal{R} = \frac{6}{R^2}$. This solution then represents a static spacetime, a 3-dimensional sphere, and so is a closed universe. Now we turn to the properties of the matter in the Einstein universe. Assuming a perfect fluid made of dust, i.e., the matter has energy density ρ and pressure $p = 0$, the Einstein field equations with cosmological constant Λ for the line element given in Eq. (41) yield

$$\rho = \frac{1}{4\pi R^2}, \tag{42}$$

$$\Lambda = \frac{1}{R^2}, \tag{43}$$

see Eqs. (30) and (31) with $k = 1$. Note that $\rho - \frac{\Lambda}{4\pi} = 0$, see Eq. (32), and so Λ being positive is repulsive everywhere and thus assumes the function of a pressure that acts against the gravitational pull of the matter specified by ρ. This system can be seen as a two fluid system, one fluid with energy density ρ, the other fluid, a vacuum fluid, with energy density $\bar{\rho} = \frac{\Lambda}{8\pi}$ and pressure $\bar{p} = -\frac{\Lambda}{8\pi}$, such that $\rho + \bar{\rho} + 3\bar{p} = 0$. All the matter energy conditions are satisfied. The volume of this universe is $V = 2\pi^2 R^3$ and its mass is $m = 2\pi^2 R^3 \rho$.

The spatial and causal structure of the spacetime can also be presented. Considering a slice of constant time of the spacetime, $t =$ constant, we find that the 3-space is diffeomorphic to a 3-sphere. To show this, we can embed the 3-space in 4-dimensional Euclidean space, $\mathbb{R}^4$. Defining the Euclidean spatial coordinates (w, x, y, z) as $w = R\cos\frac{r}{R}$, $x = R\sin\frac{r}{R}\sin\theta\cos\phi$, $y = R\sin\frac{r}{R}\sin\theta\sin\phi$, and $z = R\sin\frac{r}{R}\cos\theta$, the line element of the embedded surface is given by $ds^2 = dw^2 + dx^2 + dy^2 + dz^2$, and the surface verifies the equation $w^2 + x^2 + y^2 + z^2 = R^2$, showing that indeed it can be regarded as a 3-sphere in $\mathbb{R}^4$. To visualize the embedding one makes a $\theta = \frac{\pi}{2}$ slice, i.e., $z = 0$ in the Euclidean coordinates. In Fig. 5 we show such an embedding for the static spherical Einstein universe. By making appropriate identifications between points in the two hemispheres, the spherical space turns into a projective spherical space also called an elliptical space. In Fig. 6 we show the causal structure of the resulting spacetime in a Carter-Penrose diagram. The Einstein universe, a static spacetime, models

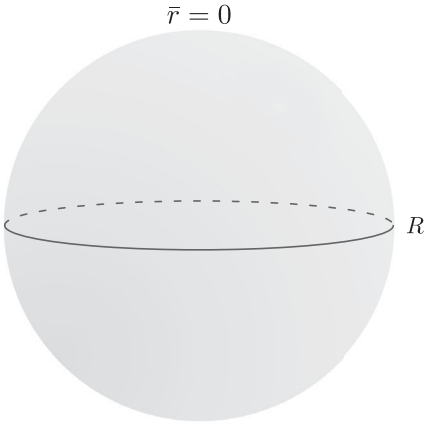

FIG. 5. Embedding diagram of a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Einstein static closed universe in 3-dimensional Euclidean space. The radial coordinate $\bar{r}$ related to the area defined by it, namely, $\bar{r} = R\sin(\frac{r}{R})$, is the radial coordinate used in the diagram. This coordinate runs from 0 at one pole, to R at the equator, and then back to 0 at the other pole, with R being the characteristic radius of the Einstein universe.

a universe with spherical spatial sections such that if we include time the total spacetime is a 3-cylinder, $\mathbb{R} \times S^3$. A timelike geodesic, or a free-falling particle, initially moving from $r = 0$ in the direction of increasing radial coordinate would reach the other pole at $r = \pi R$ and then continue until it reaches back the center of coordinates and so forth.

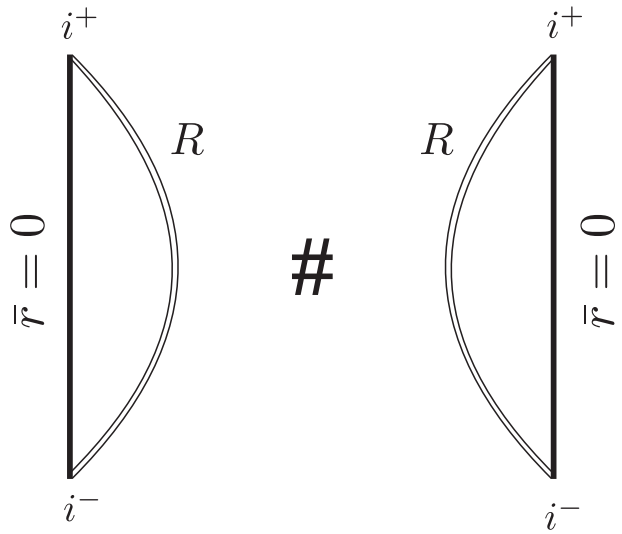

FIG. 6. Carter-Penrose diagram of the Einstein static closed universe. The vertical lines represent the two poles of the sphere. The radial coordinate $\bar{r}$ related to the area defined by it, namely, $\bar{r} = R\sin(\frac{r}{R})$, is the radial coordinate used in the diagram. One pole is situated at the origin with $\bar{r} = 0$. The other pole has also $\bar{r} = 0$. The two lines denoted by R constitute the equator. The hash symbol represents the connected sum of the spacetime manifold. The symbols i^- and i^+ represent past and future causal infinity, respectively.

The Einstein static closed universe is unstable. Indeed, from Eq. (40) one has that for $k = 1$, $V''(R_0) < 0$, recovering the well known result that the Einstein static closed universe is unstable under perturbations. This result confirms the expectations. For the static Einstein universe a small increase in the radius of the universe means less gravitational field due to matter and more cosmological repulsion field from Λ, so it is a runaway expanding unstable solution, and reversing the argument for a small decrease in the radius one finds a runaway contracting unstable solution. So, although it obeys the energy conditions and is *a priori* not problematic, it is unstable, giving rise to an expanding bubble universe.

This Einstein static closed universe is well known. It was extremely important in initiating the science of cosmology. The requirement that the boundary conditions on the gravitational field should be finite and consistent led to a closed universe, which in turn was also relevant to make the point that general relativity could be Machian, i.e., that geometry and inertia would arise solely from matter. The requirement that the universe should be static, as was thought at the time, yielded a new constant to physics, the cosmological constant. The corresponding cosmological term added to the original general theory of relativity provided in turn the first modified gravitational theory.

We can now make a comparison between the Minkowski-Minkowski static closed universe and the Einstein universe. Although the two universes are, of course, totally distinct solutions of the general relativistic field equations, there are differences and also striking similarities between them. In relation to the matter properties, the Minkowski-Minkowski closed universe is highly nonuniform, it is vacuum everywhere except at a thin shell with circumferential radius R, made of a perfect fluid with a positive energy density σ and a positive, repulsive, pressure p to hold it static against gravitational collapse or expansion. The Einstein universe, with characteristic radius R, is uniform, permeated by a fluid with a positive energy density ρ and a repulsive cosmological constant to hold it static against gravitational collapse or expansion. Thus, both universes obey the energy conditions, they have positive densities and have some form of pressure, negative tangential shell pressure in one case and positive cosmological constant pressure in the other case, to hold them static. In relation to the geometric and causal properties, one can compare the figures drawn, namely, a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Minkowski-Minkowski closed universe and the corresponding Carter-Penrose diagram shown in Figs. 1 and 2, respectively, and a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Einstein closed universe and the corresponding Carter-Penrose diagram shown in Figs. 5 and 6. The comparison leads to the conclusion that the two universes have an evident similar causal structure. The Minkowski-Minkowski closed universe models a universe with squashed spherical spatial sections such that the total

spacetime is a squashed 4-cylinder with $\mathbb{R} \times S^3$, with the possibility of further identifications in S^3. The Einstein closed universe models a universe with spherical spatial sections such that the total spacetime is a 4-cylinder $\mathbb{R} \times S^3$, with the possibility of further identifications in S^3. Both universes have thus spherical spatial topology and similar causal structures. The Minkowski-Minkowski closed universe is the result of compressing, in a sense, the evenly distributed matter of the Einstein universe into a thin shell leaving the rest of the spacetime empty. The Minkowski-Minkowski closed universe is stable, marginally, and the Einstein closed universe is unstable, so, since there are no topological obstructions, a possible endpoint of the Einstein closed universe, if perturbed at constant universe radius, could be the Minkowski-Minkowski closed universe.

2. Friedmann static hyperbolic open universe: A failed wormhole

The Friedmann static universe is a solution of the general theory of relativity for a dust source with negative energy density ρ, pressure p equal to zero, a negative cosmological constant Λ, and negative curvature $k = -1$. In spacetime hyperspherical coordinates (t, r, θ, φ) it is characterized by the line element given in Eq. (29) with $k = -1$, i.e.,

$$ds^2 = -dt^2 + dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\Omega^2, \tag{44}$$

where t is the time coordinate, r is the radial coordinate with $0 \leq r < \infty$, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, with θ and φ being the spherical angular coordinates, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. In addition, R is a positive scale factor which here is a constant, and which gives the characteristic radius of the universe. The Ricci scalar for the Friedmann universe is given by $\mathcal{R} = -\frac{6}{R^2}$, so the Friedmann universe is a negative constant curvature spacetime. Clearly it is a static hyperbolic spacetime, and so an open universe. Now we turn to the properties of the matter in the static Friedmann universe. Assuming a perfect fluid made of dust, i.e., the matter has energy density ρ and pressure $p = 0$, the Einstein field equations with negative cosmological constant Λ for the line element given in Eq. (44) yield

$$\rho = -\frac{1}{4\pi R^2}, \tag{45}$$

$$\Lambda = -\frac{1}{R^2}, \tag{46}$$

see Eqs. (30) and (31) with $k = -1$. Note that $\rho - \frac{\Lambda}{4\pi} = 0$, see Eq. (32), and so Λ being negative is attractive everywhere, and thus assumes the function of a tension that acts against the gravitational push of the matter specified by a negative energy density ρ. This system can be seen as a two fluid system, one fluid with negative energy density ρ, the other fluid, a vacuum fluid, with negative energy density $\bar{\rho} = \frac{\Lambda}{8\pi}$ and tension $\bar{p} = -\frac{\Lambda}{8\pi}$ such that $\rho + \bar{\rho} + 3\bar{p} = 0$. The matter energy conditions are violated. The volume of this hyperbolic universe, in its open form, is $V = \infty$ and its mass is also infinite.

The spatial and causal structure of the spacetime can also be presented. Considering a time slice $t =$ constant of the spacetime gives two copies of the hyperbolic 3-space. To see this, one makes an embedding. The hyperbolic 3-space cannot be embedded in the 4-dimensional Euclidean space, but it can be embedded to an open region of the 4 dimensional Minkowski spacetime. Defining the Minkowski coordinates (w, x, y, z), where w is a time coordinate and (x, y, z) are spatial coordinates, as $w = R\cosh\frac{r}{R}$, $x = R\sinh\frac{r}{R}\sin\theta\cos\phi$, $y = R\sinh\frac{r}{R}\sin\theta\sin\phi$, and $z = R\sinh\frac{r}{R}\cos\theta$, the line element of the embedded surface is given by $ds^2 = -dw^2 + dx^2 + dy^2 + dz^2$, and the surface verifies the equation $w^2 - x^2 - y^2 - z^2 = R^2$. So the surface, which represents a time slice of the Friedmann static universe, is in fact given by two copies of a 3-dimensional hyperboloid. To visualize the embedding, one makes a $\theta = \frac{\pi}{2}$ slice, i.e., $z = 0$ in the Minkowski coordinates. In Fig. 7 we show an embedding for the static hyperbolic open Friedmann universe. Clearly there are two sheets,

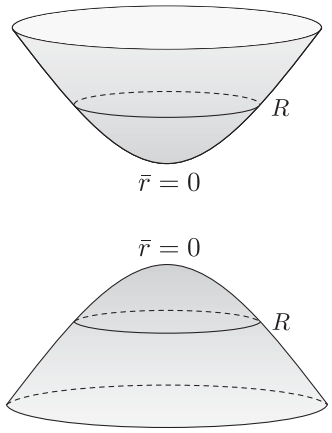

FIG. 7. Embedding diagram of a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Friedmann static open universe in 3-dimensional Minkowski space. The radial coordinate $\bar{r}$ related to the area defined by it, namely, $\bar{r} = R\sinh(\frac{r}{R})$, is the radial coordinate used in the diagram. This coordinate runs from 0 at one pole, to infinity, passing through R a some point, and the same at the other pole, it runs from 0, to infinity, passing through R a some point, with R being the characteristic radius of the Friedmann static universe. The poles are at the origin which is common to both branches, sheets, or branes of the spacetime, although this cannot be seen explicit in the embedding. The branches are separated, and the solution is a failed wormhole. Indeed, there are two disconnected spacetimes almost connected at $\bar{r} = 0$.

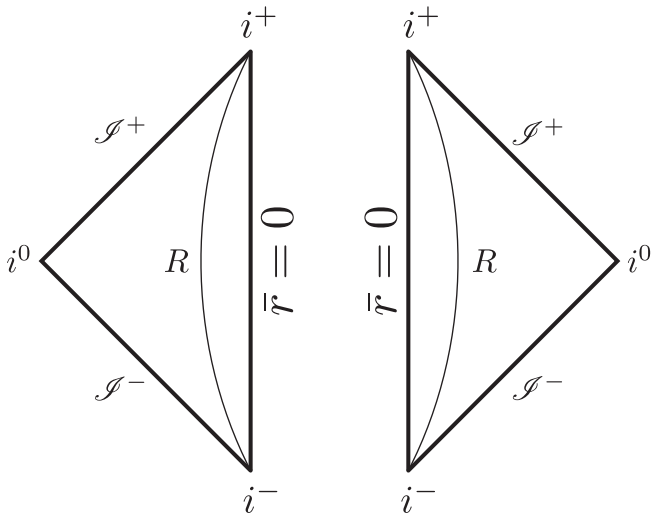

FIG. 8. Carter-Penrose diagram of the Friedmann static open universe. The vertical lines represent the poles of each hyperboloid branch, $\bar{r}=0$, where $\bar{r}$ is the radial coordinate related to the area defined by it, namely, $\bar{r}=R\sinh(\frac{r}{R})$. There is no hash symbol here because the two spacetimes are disconnected, no geodesic can pass from one spacetime to the other. The symbols i^-, i^0, and i^+ represent past timelike infinity, spatial infinity, and future timelike infinity, respectively, and the symbols $\mathscr{I}^-$ and $\mathscr{I}^+$, represent past and future null infinity, respectively. The timelike line R is drawn to call attention that the Friedmann static open universe has a characteristic intrinsic radius.

i.e., the universe has two branes, the two copies of the Friedmann static universe. Notice that we opted to map the two asymptotic flat regions to open sets of the future and past light cones to show both regions, although one should bear in mind that, first, this has no physical relevance as both regions are equivalent and, second, this has no relation with time reversal. Moreover, admitting it might not be clear from the embedding diagram, the vertices of the hyperbolas are identified as the same point hence, the static hyperbolic Friedmann universe can also be seen as a model of a failed wormhole where two asymptotic flat regions have a common point with circumferential radius $R\sinh\frac{r}{R}=0$, i.e., $r=0$, so that the wormhole's throat is a point, a zero measure set. Since the two hyperboloid branes are independent there is indeed no wormhole, it is a failed wormhole. By making appropriate identifications each of the two open infinite sheets turns into some closed 3-space, in which case the volume of such a universe would be finite. In Fig. 8 we show the causal structure of the resulting spacetime in a Carter-Penrose diagram. We use the hash symbol # to represent the connected sum of the spacetime manifolds in order to conserve the conformal structure in the Carter-Penrose diagram of the total spacetime. We see that it represents a universe in which the spatial sections are two copies of a 3-hyperboloid. The Friedmann hyperbolic universe is a spacetime composed of time times hyperbolic 3-space, actually two copies of it. A geodesic, or a free-falling particle, initially moving in the upper brane in the direction of decreasing radial coordinate would reach $r=0$ and would continue until it reaches infinity, without interacting with a mirror geodesic, or a mirror free-falling particle, initially moving in the lower brane in the direction of decreasing radial coordinate reaching the same $r=0$ and continuing until it reaches the infinity of its own brane.

The Friedmann static open universe is stable. Indeed, from Eq. (40) one has that for $k=-1$, $V''(R_0)>0$. This result confirms the expectations. For the static Friedmann universe, a small increase in the radius of the universe means less gravitational field due to matter with negative energy density, so less repulsion, and more cosmological tension field from Λ, so the universe oscillates around the original radius in a stable situation.

This open static universe proposed by Friedmann came after a suggestion by Fock, and was worked out by Friedmann before introducing, in the same paper, the new expanding time-dependent hyperbolic solutions. Friedmann's main motivation for presenting it was that the solution represented the other side of Einstein's static universe, the two solutions, Einstein's and Friedmann's, are indeed complementary to each other. It is a much forgotten universe. Since this static solution has a negative energy density and a negative cosmological constant, and violates all energy conditions, it seemed a physically inadmissible strange universe that could hardly captured any attention. This prejudice against solutions that violate the energy conditions came to an end when traversable wormholes, systems that violate several energy conditions, jumped into the limelight. We see here that the Friedmann static universe is a wormhole, albeit a failed one. Moreover, although the solution does not obey the energy conditions, Friedmann's static universe is interestingly stable. Thus, Friedmann had a prescient foresight in contemplating working out in detail the mathematics of this static solution.

We can now make a comparison between the Minkowski-Minkowski static open universe, or traversable wormhole, and the Friedmann static hyperbolic universe, or failed wormhole. Although the two universes are, of course, totally distinct solutions of the general relativistic field equations, there are both differences and similarities between them, although the similarities here are not so compelling. In relation to the matter properties, the Minkowski-Minkowski open universe, i.e., the traversable wormhole universe, is highly nonuniform, it is vacuum everywhere except at a thin shell throat with circumferential radius R, made of a perfect fluid with a negative energy density σ and a positive pressure p to hold it static. The Friedmann static open universe with characteristic radius R is uniform, permeated by a fluid with a repulsive negative energy density ρ and a negative, attractive, cosmological constant Λ to hold it static. Thus, both

spacetimes violate the energy condition, they have negative energy densities and have some form of pressure, positive tangential shell pressure in one case and negative cosmological constant pressure in the other case, to hold them static. In relation to the geometric and causal properties, one can compare the figures drawn, namely, a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Minkowski-Minkowski open universe, or traversable wormhole, and the corresponding Carter-Penrose diagram, shown in Figs. 3 and 4, respectively, and a $t =$ constant and $\theta = \frac{\pi}{2}$ slice of the Friedmann open universe, or failed wormhole, and the corresponding Carter-Penrose diagram shown in Figs. 7 and 8. The comparison leads to the conclusion that the two universes have some similarities. Both universes for large radii have two distinct open sheets, although the circumferential radius in the Minkowski-Minkowski open universe is finite not zero, and so composes a traversable wormhole, whereas in the Friedmann static open universe the circumferential radius goes to zero, and the wormhole fails to happen. The bare geometrical structure of the two universes is different, the Minkowski-Minkowski open universe has geometry $\mathbb{R} \times \Sigma$, where Σ is a 3-space with nontrivial topology, and the Friedmann static open universe has geometry $\mathbb{R} \times \mathbb{H}^3$, with negative curvature in the two copies of the spatial sections. The Minkowski-Minkowski open universe could be thought of as being the result of compressing, in a sense, the evenly distributed matter of the Friedmann static universe into a thin shell at some throat radius leaving the rest of the spacetime empty. The Minkowski-Minkowski open universe is stable, marginally, and the Friedmann open universe is stable. But here there are topological obstructions, one universe is connected, although not simply connected, the other universe is disconnected, it has two separate branches, and so one cannot pass from one universe to the other without changing the topology.

C. Einstein and Friedmann static universes: One concept with two sides

The Einstein and Friedmann static universes can be seen as complementary to each other, i.e., they are two sides of the same concept. The concept, i.e., a collection of static constant curvature homogeneous universes, yields on one side of the concept a closed universe, a bubble universe, and on the other side of the concept an open universe which is a failed wormhole. The formalism presented in analyzing the two universes is well suited to show this point. From an algebraic point of view, one side of the concept is given by $k = 1$, the other side is given by $k = -1$, where k is a characteristic that gives how space curves, positively in one case, negatively in the other, respectively. This algebraic side appears clearly in the evaluation of the matter properties as displayed in Eqs. (30) and (31). More formally, to implement the idea of a closed universe, i.e., a bubble universe, and an open universe, i.e., a failed wormhole, one uses the equations of general relativity. For one universe one picks up $k = 1$, a 3-dimensional sphere. For the other universe one picks up $k = -1$, a 3-dimensional hyperboloid. From a matter point of view the two universes show a form of complementarity, as for $k = 1$ the matter obeys the energy conditions while for $k = -1$ the matter violates the energy conditions. From a geometrical point of view, the two sides of the concept appear when one picks up a manifold spacetime and at constant time imposes a space with constant curvature. One side is for positive curvature, a bubble universe, the other side for negative curvature, a failed wormhole. Comparison of Fig. 5 with Fig. 7 for a spatial geometrical representation of the bubble universe and the traversable wormhole, respectively, displays some complementarity of the two spaces, which can be further strengthened with the comparison of the spacetime drawings in the form of Carter-Penrose diagrams, given in Fig. 6 and in Fig. 8, respectively. From a stability point of view we have seen that one universe is stable and the other unstable, showing thus some form of complementarity.

So, that the two spacetimes, Einstein and Friedmann, demonstrate that they can be seen as complementary to each other, i.e., they are two sides of the same concept, is clear. It can be raised that the bubble universe has matter that obeys the energy conditions, whereas the failed wormhole has matter that does not obey. This is true, but again there is no real problem. In an early era of the Universe, when quantum gravity dominates, there is no necessity of obeyance to the classical energy conditions and the closed and open universes, created as bubble universes and failed wormholes. out of the spacetime foam they must coexist together. Some kind of inflation would grow these objects to macroscopic dimensions, making bubble universes and traversable wormholes distinct, but connected, objects, some obeying the energy conditions and being unstable, like the Einstein universe, others not obeying the energy conditions but being stable, like the Friedmann static universe.

IV. CONCLUSIONS: BUBBLE UNIVERSES AND TRAVERSABLE WORMHOLES, TWO SIDES OF ONE CONCEPT

We have analyzed the possible universes that can be built from a junction of two Minkowski spacetimes through a static, timelike thin matter shell. Taking aside the trivial Minkowski flat universe with no shell, there are two such universes. One is a static closed universe with a spherical thin shell with positive energy density and negative pressure that joins two Minkowski balls, i.e., it is the Minkowski-Minkowski closed universe, a bubble universe. The other universe is a static open universe with a spherical thin shell with negative energy density and positive pressure that joins two Minkowski asymptotic sheets, it is the Minkowski-Minkowski open universe, or traversable

wormhole. We have seen that they can be seen as complementary to each other. More specifically, they are two sides of one concept, the concept being the collection of nontrivial Minkowski-Minkovski spacetimes, with one side given by $\frac{1}{2}(\xi_{\mathrm{i}} - \xi_{\mathrm{e}}) = 1$, the other side given by $\frac{1}{2}(\xi_{\mathrm{i}} - \xi_{\mathrm{e}}) = -1$, where ξ_{i} and ξ_{e} characterize the relative direction of the interior and exterior normals to the shell, respectively.

We have analyzed the possible universes that can be built from static homogeneous pressureless matter with a cosmological constant. There are two such universes. One is the static closed Einstein spherical universe. The other is the static open Friedmann hyperbolic universe. We have seen that they can be seen as complementary to each other, and, indeed, the idea of the construction of the static open universe by Friedmann was to find the complement to the Einstein universe. More specifically, they are two sides of one concept, the concept being the collection of constant curvature pressureless universes, with one side given by positive curvature, $k = 1$, the other side given by negative curvature, $k = -1$.

We have also seen that the Minkowski-Minkowski closed universe, a bubble universe that has positive energy density and pressure on the matching domain wall and thus being normal matter obeying the energy conditions, has resemblances with the static closed Einstein universe containing homogeneous normal matter with positive energy density, zero pressure, and positive cosmological constant, and thus also obeying the energy conditions, and that the static open Minkowski-Minkowski universe, a traversable wormhole that has negative energy density and positive pressure on the matching throat and thus being exotic matter violating the energy conditions, has resemblances with the static open Friedmann universe, a failed wormhole, containing homogeneous exotic matter with negative energy density, zero pressure, and negative cosmological constant, and thus also violating the energy conditions. The Minkowski-Minkowski universes both are linearly stable, marginally, and the Einstein and Friedmann static universes are linearly unstable and stable, respectively. One could think of the Minkowski-Minkowski universes as being a limit of the homogeneous universes when all the matter of the thin shell is spread evenly throughout the universes, or vice versa, in which case the homogeneous universes being a limit of the Minkowski-Minkowski universes when all the matter of the homogeneous universes is put somehow into thin shells. For the Einstein universe this would be possible classically, within general relativity, as the two universes have the same topology, and so, for constant universe radius, the Minkowski-Minkowski closed universe could be the end point of the Einstein universe. For the Friedmann static universe this could not be realized within general relativity, as the two universes have different topologies and so there is no way of changing classically, and so continuously, from one into the other, although quantum jumps of one to the other geometry might be conceivably possible.

The existence of universes and wormholes within the Universe is a tantalizing possibility allowed by the laws of physics. Indeed, in a early cosmic era when primordial scalar and gauge fields are dominant and symmetry breaking phase transitions naturally arise, universes may occur as bubbles within the Universe, and likewise, wormholes can exist in the form of traversable shortcuts for distant parts of the Universe or can even connect what would be distinct universes. Bubble universes and traversable wormholes are distinct objects. Normally, bubble universes are found as dynamic solutions, whereas, typically, traversable wormholes are studied as static structures, but of course they can both be static or dynamic. We have analyzed two static cases, the two Minkowski-Minkowski spacetimes and the two static homogeneous universes, and found that these spacetimes demonstrate, in the way of example, indeed two coupled examples that reinforce each other, that bubble universes and traversable wormholes can be seen as complementary to each other, i.e., they are two sides of some same concept. Dynamical cases where bubble universes and traversable wormhole are complementary to each other can also be found and studied. It is plausible that in a quantum gravity scenario or in a scenario in which quantum gravity is weak but non-negligible, both bubble universes and traversable wormholes are dynamically created alike, being as well two sides of the same concept. In addition, using this duality, one can infer that, arbitrarily advanced civilizations, with arbitrarily advanced technology to deal in practical terms with spacetime features, if they can build bubble universes, they can also build traversable wormholes, and conversely, if they are apt to build traversable wormholes, as has been often suggested, they should be apt to build bubble universes.

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