Parallel waves in Einstein-nonlinear sigma models

Béatrice Bonga[1,*] and Gustavo Dotti[2,†]

[1] *Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, 6525 AJ Nijmegen, The Netherlands*

[2] *Facultad de Matemática, Astronomía, Física y Computación (FaMAF), Universidad Nacional de Córdoba and Instituto de Física Enrique Gaviola, CONICET. Ciudad Universitaria, (5000) Córdoba, Argentina*

(Received 4 January 2022; revised 13 January 2022; accepted 7 February 2022; published 23 February 2022)

We study a family of solutions of Einstein-nonlinear sigma models with S^2 and $SU(2) \sim S^3$ target manifolds. In the S^2 case, the solutions are smooth everywhere, are free of conical singularities, and approach asymptotically the metric of a cosmic string, with a mass per length that is proportional to the absolute value of the winding number from topological spheres onto the target S^2. This gives an interesting example of a relation between a mass and a topological charge. The case with target $SU(2)$ generalizes the stationary solution found in Canfora *et al.*Eur. Phys. J. C **81**, 55 (2021) to parallel waves with a nonplanar wave front $\mathcal{W}$. We prove that these $\mathcal{W}$-fronted parallel waves are subquadratic in the classification in Flores *et al.*Classical Quantum Gravity **20**, 2275 (2003) and thus are causally well behaved. These spacetimes have a nonvanishing baryon current and their geometry has many striking features.

DOI: 10.1103/PhysRevD.105.044049

I. INTRODUCTION: EINSTEIN-NONLINEAR SIGMA MODEL

Quantum chromodynamics (QCD), a non-Abelian gauge theory with $SU(3)$ gauge group, gives a description of hadrons in terms of their fundamental degrees of freedom: quarks and gluons. Hadrons, being composite particles, appear in the low energy limit of QCD, which corresponds to the nonperturbative, strongly coupled regime. At these energy scales it is found that effective theories become the most efficient tools to describe them. The leading term of the effective Lagrangian in Minkowski spacetime is the nonlinear sigma model (NLSM)

$$\mathcal{L} = \frac{K}{4}\mathrm{Tr}(L^a L_a), \tag{1}$$

where we neglected the quark masses (see, e.g., Sec. VII.1 in [1]) and

$$L_a = U^{-1}\partial_a U \tag{2}$$

is the Maurer-Cartan form for a field U with the target group $SU(2)$. This effective theory encodes the low energy dynamics of pions.

The NLSM in Eq. (2) cannot support static solitonic solutions in Minkowski spacetime. This was proved long ago by Derrick using an elegant scaling argument [2]. Solitons are of interest because they can be understood as baryons. Skyrme [3] introduced a modification to the NLSM to allow for such static, topologically stable solitonic solutions on Minkowski spacetime, by adding to the Lagrangian in Eq. (1) the term $\mathrm{Tr}([L_a, L_b][L^a, L^b])$, which is *part* of the subleading contributions to the QCD effective Lagrangian (see Sec. XI.4 in [1], as well as Chap. 9 in [4]).

Derrick's scaling argument, however, can also be circumvented in other ways, since it uses the symmetries of Minkowski spacetime and implicit boundary conditions. One such method is imposing periodic, crystal-like boundary conditions on flat spacetime [5,6]. Another, and the one explored in this paper, is coupling the NLSM to Einstein's gravity [7–10]. In fact, if we minimally couple the NLSM to gravity, it is possible to find solitonic solutions keeping only the low energy leading term Eq. (1), that is, working with the action

$$S = \int_M d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2\kappa} + \frac{K}{4}\mathrm{Tr}(L^a L_a)\right), \tag{3}$$

where L_a is the Maurer-Cartan form in Eq. (2) for a field $U\colon M \to SU(2)$, (M, g_{ab}) is the spacetime, κ is Newton's constant, and K is the coupling constant of the NLSM, which is proportional to the square root of the decay constant of pions. In geometrized units, we have [7]

$$0 < K\kappa \ll 1. \tag{4}$$

The NLSM term in (3) has an interesting geometric interpretation: as we prove in Sec. II, it is (proportional to) the

*bbonga@science.ru.nl
†gdotti@famaf.unc.edu.ar

trace of the pullback of the $S^3 = SU(2)$ metric onto spacetime. As such, it belongs to the family of Einstein-NLSM (ENLSM). In these theories, there is field $\Psi\colon M \to N$ with the target a compact boundaryless Riemannian manifold (N, G_{AB}), and the action is given by the Einstein-Hilbert term plus the trace of the pullback by Ψ of the metric G_{AB}:

$$S_{\mathrm{NLSM}} = \int d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2\kappa} + \frac{K}{2}g^{ab}\partial_a\Psi^A\partial_b\Psi^B G_{AB}(\Psi(x))\right). \tag{5}$$

Previous related work [8–10] uses the full Skyrmion model coupled to Einstein gravity instead of the simpler action in Eq. (3) [or equivalently Eq. (5)]. This simpler action was, however, recently considered in [7], where solutions with a stationary spacetime metric were found that describe solitonic matter.

In this work, we generalize those stationary spacetimes to be dynamical. Specifically, we find solutions to the action in Eq. (3) for which the spacetime metric has a Kerr-Schild character and describes a parallel wave with a nonplanar wave front with transverse metric $ds^2_{\mathcal{W}}$:

$$ds^2 = -dudv + H(u,\rho,\phi)du^2 + \underbrace{d\rho^2 + S(\rho)^2 d\phi^2}_{=ds^2_{\mathcal{W}}}. \tag{6}$$

Metrics of this form generalize plane-fronted waves with parallel propagation, or pp-waves for short. Such generalized pp-waves were studied in [11–13], where it was found that the rate of growth of H as a function of the distance d to a fixed point on $\mathcal{W}$, as $d \to \infty$, determines the causal behavior of the spacetime. We show below that our solutions correspond to the subquadratic case in the classification in [11–13], which has a much better causal behavior than the ordinary, plane fronted pp-waves [14]. These solutions are interesting not only given the physical model they are derived from but also because of their geometric properties as parallel waves traveling on a cylindrical background.

We also study a particular static solution of the theory (5) with target S^2. This solution has a metric with cylindrical symmetry, that is, of the form in Eq. (6) with $H = 0$. It is an interesting example of an *everywhere smooth* metric, which *asymptotically* looks like that of a cosmic string, since $S(\rho) \simeq \beta_+\rho$ for large ρ with $0 < \beta_+ < 1$. On the other hand, near $\rho = 0$ we find that $S \simeq \rho + \mathcal{O}(\rho^2)$, which ensures that $ds^2_{\mathcal{W}}$ is free of conical singularities. Most interesting, the mass per length $2\pi(1-\beta_+)/\kappa$ of the asymptotically apparent string is proportional to the absolute value of the winding number of—topologically—spacetime spheres onto the target S^2.

The paper is organized as follows. In Sec. II we review the derivation of the baryon charge conservation of the theory (5) and discuss some subtleties about integration on hypersurfaces that are not everywhere spacelike. In Sec. III, we present the field equations of the action Eq. (5) and derive the solution with the metric as in Eq. (6). A byproduct of these calculations gives a static solution of the theory (5) with target S^2. This is discussed in Sec. IV, where we elaborate on its geometry and establish the relationship between its mass per length and the topological winding number. In Sec. V we return to the $SU(2)$ NLSM. After exploring its geometry through the study of geodesics, we discuss the baryon charge and, for the particular case in [7] of a nonstatic U field leading to a static metric, we give different notions of mass per length and relate it to the baryon charge. The $SU(2)$ field configuration that we analyze does not carry a topological charge, and it can be regarded as a parallel wave propagating in an otherwise cylindrical spacetime. We end with Sec. VI by summarizing our main results and discussing their implications.

II. FIELDS AND CONSERVED CURRENTS

In this section, we derive the equation for the baryon conserved current thereby providing some further physical background to the QCD Einstein-NLSM with action (3). We also explain in some detail how to calculate the total baryon charge at an open spacelike surface Σ' that is computationally challenging to find, by using Stokes/Gauss theorem and replacing it with an integral on an asymptotically matching surface Σ.

We parametrize $SU(2) \sim S^3$ using hyperspherical coordinates $z^A = (\alpha, \Theta, \Phi)$:

$$\mathbb{R}^4 \ni \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} \sin\alpha \sin\Theta \sin\Phi \\ \sin\alpha \sin\Theta \cos\Phi \\ \sin\alpha \cos\Theta \\ \cos\alpha \end{pmatrix}. \tag{7}$$

The normalized S^3 metric G_{AB} is

$$G_{AB}dz^A dz^B = \frac{1}{2\pi^2}(d\alpha^2 + \sin^2\alpha d\Theta^2 + \sin^2\alpha \sin^2\Theta d\Phi^2). \tag{8}$$

In terms of the coordinates (α, Θ, Φ), the $SU(2)$ matrices are given by

$$\begin{aligned} U^{\pm 1} &= \cos(\alpha)\mathbf{1}_2 \pm \sin(\alpha)\hat{n}\cdot\mathbf{t} = e^{\pm\alpha\hat{n}\cdot\mathbf{t}}, \\ \mathbf{t} &= (i\sigma_1, i\sigma_2, i\sigma_3) \end{aligned} \tag{9}$$

with σ_j the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

and

$$\hat{n} = (\sin(\Theta)\cos(\Phi), \sin(\Theta)\sin(\Phi), \cos(\Theta)). \quad (11)$$

The three pion fields are—mod normalization conventions—$\boldsymbol{\pi} = \alpha\hat{n}$ [1]. Inserting Eqs. (9)–(11) into Eq. (2) gives

$$L_a = [(\partial_a\alpha)\hat{n} + \sin(\alpha)\cos(\alpha)\partial_a\hat{n} + \sin^2(\alpha)(\hat{n}\times\partial_a\hat{n})]\cdot\mathbf{t}. \quad (12)$$

This implies, as anticipated, that

$$-\frac{1}{2}\mathrm{Tr}[L_aL_b] = \partial_a\alpha\partial_b\alpha + \sin^2\alpha\partial_a\Theta\partial_b\Theta + \sin^2\alpha\sin^2\Theta\partial_a\Phi\partial_b\Phi \quad (13)$$

is the pullback onto spacetime of the S^3 metric in Eq. (8) [compare (13) with (8)], whose trace $\mathrm{Tr}[L^aL_a]$ appears in the action [see Eq. (3)].

The vector field

$$J^a = \frac{1}{24\pi^2}\epsilon^{bcda}\mathrm{Tr}(L_bL_cL_d) \quad (14)$$

describes the baryon current and is dual to the 3-form (conventions as in [15])

$$\begin{aligned} {}^*J_{bcd} &= J^a\epsilon_{abcd} \\ &= \frac{1}{4\pi^2}\mathrm{Tr}(L_{[b}L_cL_{d]}) \\ &= \frac{3}{\pi^2}\sin^2(\alpha)\sin(\Theta)\partial_{[b}\alpha\partial_c\Theta\partial_{d]}\Phi. \end{aligned} \quad (15)$$

Index antisymmetrization is defined as a sum over signed permutations divided by the factorial of the number of antisymmetrized indices, so the above equation can be written in the language of forms as

$$\frac{1}{24\pi^2}\mathrm{Tr}(L\wedge L\wedge L) = \frac{1}{2\pi^2}\sin^2\alpha\sin^2\Theta d\alpha\wedge d\Theta\wedge d\Phi, \quad (16)$$

which is the pullback of the normalized S^3 volume form from Eq. (8). Since exterior derivatives commute with pullbacks and the dimension of S^3 is 3, the exterior derivative of this 3-form vanishes. In view of the duality in Eq. (15), this is equivalent to the condition of *baryon current conservation*:

$$\nabla_aJ^a = 0. \quad (17)$$

The total baryon charge $B_{\Sigma'}$ measured by the observers with velocities n^a normal to an (open oriented) spacelike hypersurface Σ' is

$$B_{\Sigma'} = \int_{\Sigma'} J_an^a\epsilon^{\Sigma'}_{bcd} \quad (\epsilon^{\Sigma'}_{bcd} = \epsilon_{abcd}n^a). \quad (18)$$

If $B_{\Sigma''}$ is a second such a surface, and either $\Sigma'' - \Sigma' = \partial V$, the boundary of an open subset of spacetime (the relative sign here indicates reversal of the normal), or V is topologically a cylinder with cups Σ'' and Σ' and the fields decay fast enough so that the flow through the lateral is null; then charge is conserved, meaning that $B_{\Sigma'} = B_{\Sigma''}$. This follows from the Gauss theorem and Eq. (17), or equivalently to the dual Stokes theorem and the fact that the 3-form dual to J_a, Eq. (15), is closed. It is important to recall how one switches from Stokes' to Gauss' version in the most general case: Consider a—possibly nonclosed—orientable hypersurface $\Sigma\subset M$, choose a normal smooth vector field N^a on Σ. We allow the case where Σ changes character from spacelike to timelike, as long as its normal N^a is null only on a zero measure set Σ_o. Let $n^a = N^a/\sqrt{|N^cN_c|}$. This vector field is smooth on $\tilde{\Sigma} = \Sigma\backslash\Sigma_o$ (a disconnected set if $\Sigma_o \neq \emptyset$), it is undefined on Σ_o, and it is normalized to $n^an_a = -1$ $(+1)$ on the spacelike (timelike) sectors of Σ. Let $\epsilon^{\tilde{\Sigma}}_{bcd} = \epsilon_{abcd}n^a$ be the volume form on $\tilde{\Sigma}$. Given a 3-form α_{abc} on M with dual $v_a = \frac{1}{6}\epsilon_{bcda}\alpha^{bcd}$ (that is, $\alpha_{abc} = \epsilon_{dabc}v^d$), we have

$$\alpha_{bcd}|_{\tilde{\Sigma}} = -v_an^a(n^kn_k)\epsilon^{\tilde{\Sigma}}_{bcd}|_{\tilde{\Sigma}}, \quad (19)$$

where on the left side we mean pullback. The above equality is used when proving Gauss theorem from Stokes theorem on manifolds with boundary (the $n^kn_k = \pm 1$ factor is the reason why we need to switch from outer to inner normal when leaving spacelike sectors of the boundary). Equation (19) is particularly useful when it is difficult to explicitly determine the timelike/spacelike sectors of Σ, since its left side is insensitive to these changes. From now on we will treat *integrals* on Σ; thus the distinction between Σ and $\tilde{\Sigma}$ is irrelevant. Suppose Σ is an open hypersurface that is asymptotically spacelike and that can be deformed onto a hypersurface Σ' that is spacelike everywhere and matches Σ in the asymptotic region. Suppose we are interested in the total charge $\int_{\Sigma'} v_an^a\epsilon_{\Sigma'}$ for a divergence-free vector field v^a, we do not know Σ' in detail, and, although we do know Σ, we would like to avoid determining the sectors where Σ is timelike/spacelike. In this case we can use the Gauss theorem and Eq. (19) and find that (the orientation is chosen such that, when timelike, n^a is future pointing)

$$\int_{\Sigma'} v_an^a\epsilon^{\Sigma'}_{bcd} = \int_{\Sigma} -v_an^a(n^kn_k)\epsilon^{\Sigma}_{bcd} = \int_{\Sigma}\epsilon_{abcd}v^a. \quad (20)$$

For $v^a = J^a$ given in Eq. (14) the above equation gives

$$B_{\Sigma'} = \int_{\Sigma'} J_a n^a \epsilon^{\Sigma'}_{bcd} = \frac{1}{4\pi^2}\int_{\Sigma} \mathrm{Tr}(L_{[a}L_b L_{c]}). \tag{21}$$

The integral on the left is the total baryon charge measured by observers with velocity n^a. The above equation shows that this can be calculated as an integral over the asymptotically matching surface Σ without knowing the sectors where Σ is not spacelike.

The baryon charge in Eq. (21) is a conserved quantity in the sense that the integral on the right is the same on surfaces in the same homology class. This quantity, however, does not necessarily have a *topological meaning*. If Σ is open and complete *and the field U tends to a constant* (say, the identity matrix) in the asymptotic region, we may regard $B_{\Sigma'}$ as the integral of the pullback of the normalized S^3 metric onto a manifold that is topologically equivalent to S^3 (the one point compactification of Σ'). In this case $B_{\Sigma'}$ will be an integer: the number of times this sphere wraps around $SU(2) = S^3$. In general, however, U does not have a common limit at infinity, and thus $B_{\Sigma'}$ is *not* an integer.

III. PARALLEL WAVE SOLUTIONS

In this section, we present the field equations of the QCD ENLSM (3) and a general class of parallel wave solutions that generalize the solutions with static metrics found in [7]. The field equations derived from the action in Eq. (3) are

$$\nabla^a L_a = 0 \tag{22}$$

and

$$G_{ab} = \kappa T_{ab}, \tag{23}$$

where the energy-momentum tensor of the $SU(2)$ field is

$$\begin{aligned} T_{ab} &= -\frac{K}{2}\mathrm{Tr}\left(L_a L_b - \frac{1}{2} g_{ab} L^c L_c\right) \\ &= K\bigg[\sin^2(\alpha)\sin^2(\Theta)\left(\partial_a\Phi\partial_b\Phi - \frac{1}{2}g_{ab}\partial^c\Phi\partial_c\Phi\right) \\ &\quad + \sin^2(\alpha)\left(\partial_a\Theta\partial_b\Theta - \frac{1}{2}g_{ab}\partial^c\Theta\partial_c\Theta\right) \\ &\quad + \left(\partial_a\alpha\partial_b\alpha - \frac{1}{2}g_{ab}\partial^\gamma\alpha\partial_\gamma\alpha\right)\bigg]. \end{aligned} \tag{24}$$

Equation (22) shows a minimal coupling to gravity of the pion field equation in Minkowski spacetime. Note that, since $L_a = U^{-1}\partial_a U$, this equation is indeed second order in the pion fields $\boldsymbol{\pi}$. Note also that it has the form of a conserved current equation. This is because the action in Eq. (3) is invariant under the $SU(2) \times SU(2)$ (global) transformation

$$U \to g_L U g_R^\dagger, \qquad (g_L, g_R) \in SU(2) \times SU(2), \tag{25}$$

and L_a is the Noether current under the left $SU(2)$ subgroup for which $g_R = \mathbf{I}$ [16]. The conservation of the additional conserved current from the subgroup $g_L = \mathbf{I}$ is trivially related to Eq. (22).

The field equations (22) and (23) admit solutions in which the metric is a $\mathcal{W}$-fronted parallel wave metric (as defined in [17] and references therein):

$$\begin{aligned} ds^2 = &-dt^2 + dz^2 + H(t-z, r, \phi)(dt - dz)^2 \\ &+ \underbrace{\ell^2 e^{-2R(r)}(dr^2 + d\phi^2)}_{=ds^2_{\mathcal{W}}}. \end{aligned} \tag{26}$$

We may occasionally switch to null coordinates in $t-z$ space:

$$u = t - z, \qquad v = t + z, \tag{27}$$

and use an alternative radial variable ρ for the wave front cross section $\mathcal{W}$, which has $(+,+)$ metric

$$ds^2_{\mathcal{W}} = \ell^2 e^{-2R(r)}(dr^2 + d\phi^2) = d\rho^2 + S^2(\rho)d\phi^2. \tag{28}$$

In terms of r, ρ and $S(\rho)$ are given by

$$\frac{d\rho}{dr} = \pm S, \qquad S = \ell e^{-R(r)}. \tag{29}$$

Switching to (u, v, ρ, ϕ) coordinates, the metric in Eq. (26) becomes that of Eq. (6).

Note that H, R, r, and ϕ are dimensionless, ℓ, t, z, u, v, ρ have dimensions of length, and

$$-\infty < r, z, t, u, v < \infty, \qquad \phi \sim \phi + 2\pi. \tag{30}$$

The range of ρ depends on how S decays as $r \to \pm\infty$.

The wave vector k^a, given in $(u, v, *, *)$ coordinates by $k^a\partial_a = \partial_v$ is: (i) orthogonal to the wave fronts $u = \text{const}$, (ii) null, and (iii) covariantly constant. The latter property ensures that the spacetime is a member of the Kundt class, which are Lorentzian manifolds admitting a geodesic null congruence with vanishing optical scalars (expansion, shear, and twist) [18]. The wave vector k^a is used to select a time orientation by defining it to be future oriented. This choice implies that, in those regions where ∂_t in Eq. (26) is timelike, it is future oriented. We call these spacetimes $\mathcal{W}$-fronted parallel waves because the wave vector k is covariantly constant and the transverse metric on the wave fronts is $ds^2_{\mathcal{W}}$. These generalize pp-waves, which correspond to the particular case where $ds^2_{\mathcal{W}}$ is planar. We will prove below that the decay of H at large distances along $\mathcal{W}$ guarantees that our solutions fall, in the classification in [12], in the subquadratic type, making them causally well behaved, contrary to what happens with the flat fronted pp-waves [14].

Our $SU(2)$ field ansatz is

$$\alpha = \alpha(r), \qquad \Theta = q\phi, \qquad \Phi = F(t-z), \quad (31)$$

where F is a function that models the wave profile. It generalizes that in [7] allowing for nonstationary spacetime metrics [19].

For the ansatz (26)–(31), we find that Eqs. (22) and (23) reduce to three independent field equations:

$$(\alpha'(r))^2 = q^2\sin^2(\alpha(r)),$$
$$R''(r) = K\kappa q^2 \sin^2(\alpha(r)), \quad (32)$$

and [note the trivial way $F'(t-z)$ appears in this equation]

$$(\partial_r^2 + \partial_\phi^2)H$$
$$= -2K\kappa\ell^2 \exp(-2R(r))\sin^2(\alpha(r))\sin^2(q\phi)(F'(t-z))^2$$
$$= -2\left(\frac{\ell}{q}\right)^2 \exp(-2R(r))R''(r)\sin^2(q\phi)(F'(t-z))^2. \quad (33)$$

We find that the function F is constrained neither by the field equations nor by energy conditions. The dominant and strong energy conditions are satisfied in any case for this theory, as proved in [20]. Alternatively, the strong energy condition

$$R_{ab}\zeta^a\zeta^b \geq 0 \quad \text{for timelike } \zeta^a \quad (34)$$

follows from Proposition 2.2 in [11] and the facts that Eq. (33) implies that the $\mathcal{W}$-Laplacian of H is negative, whereas Eq. (32), together with $R^{\mathcal{W}}_{jk} = \mathrm{diag}(d^2R/dr^2, d^2R/dr^2)$ in (r, ϕ) coordinates, implies that $R^{\mathcal{W}}_{jk}$ is positive definite.

Discarding the uninteresting solution to Eq. (32) with $\alpha(r) = 0, \pi$ and $R(r) = Ar + b$, we are left with the solution

$$\alpha(r) = 2\arctan(\exp(\epsilon|q|r + C_1)),$$
$$R(r) = K\kappa \ln(\cosh(\epsilon|q|r + C_1)) + C_2 r + C_3, \quad (35)$$

with $\epsilon = \pm 1$ and C_1, C_2, and C_3 arbitrary constants. Interestingly, despite the fact that the field equations are nonlinear and coupled in an intricate way, *whether or not F' is a constant (and consequently whether the metric is static or dynamical), the field equations for the nonlinear sigma model together with the corresponding Einstein equations lead to the same solutions for $\alpha(r)$ and $R(r)$*. This is a remarkable property of the ansatz in Eqs. (26) and (31) that allows us to disentangle the physical effects introduced through $F(u)$.

We now analyze the constraints on the relevant integration constants. All algebraic (that is, nondifferential) curvature scalar fields made out of the Riemann tensor, the metric, and its inverse, and the volume form—for which a basis is given in [21]—are powers of the Ricci scalar $\mathcal{R}$,

$$\mathcal{R} = \frac{2}{\ell^2} e^{2R(r)} R''(r), \quad (36)$$

and thus are independent of H. Since

$$e^{2R(r)}R''(r) = K\kappa q^2 \cosh(\epsilon|q|r + C_1)^{2K\kappa-2} e^{2C_2 r + 2C_3}$$
$$\sim e^{|q||r|(2K\kappa-2)} e^{2C_2 r} \quad \text{as } |r| \to \infty, \quad (37)$$

in view of (4), requiring that the scalars of curvature be well behaved for $-\infty < r < \infty$ is equivalent to the condition found in [7]:

$$|C_2| < (1 - K\kappa)|q|. \quad (38)$$

The relation between the radial coordinates $-\infty < r < \infty$ and ρ is $d\rho/dr = \pm S$, with

$$S = \ell e^{-R(r)}$$
$$= \ell \cosh(\epsilon|q|r + C_1)^{-K\kappa} e^{-C_2 r - C_3}$$
$$\simeq \nu_\pm e^{-K\kappa|q||r|} e^{-C_2 r} =: \nu_\pm e^{\beta_\pm r} \quad \text{as } r \to \pm\infty, \quad (39)$$

where we assumed that the constants

$$\beta_+ = -K\kappa|q| - C_2, \qquad \beta_- = K\kappa|q| - C_2 \quad (40)$$

are nonzero. The positive constants $\nu_\pm$ depend on C_1, C_3, ϵ, κK, and ℓ. Note that

$$\beta_- - \beta_+ = 2K\kappa|q| \geq 0. \quad (41)$$

This inequality allows three out of four sign possibilities: (i) β_- and β_+ are positive, (ii) β_- is positive and β_+ negative, and (iii) β_- and β_+ are negative. The solutions of type (iii) are trivially related to those of type (i). This is a consequence of the symmetry of the metric [see Eq. (26)] under $r \to -r$: given a solution $\alpha(r)$, $R(r)$, and $H(t,z,r,\phi)$ of the field equations (32) and (33), the functions $\tilde{\alpha}(r) = \alpha(-r)$, $\tilde{R}(r) = R(-r)$, and $\tilde{H}(t,z,r,\phi) = H(t,z,-r,\phi)$ are also solutions, and the asymptotic behaviors of these two solutions are related by $(\tilde{\beta}_-, \tilde{\beta}_+) = (-\beta_+, -\beta_-)$. We will then assume from now, without loss of generality, that $\beta_- > 0$. This guarantees that $r = -\infty$ is *a point* at a finite distance from any other point in $\mathcal{W}$. We define ρ to be the $\mathcal{W}$-geodesic distance to this point:

$$\rho(r) = \ell \int_{-\infty}^{r} e^{-R(y)} dy, \quad (42)$$

so that $\rho \to 0$ as $r \to -\infty$, the upper sign choice holds in Eq. (29), that is $d\rho/dr = S = \ell e^{-R(r)}$, and

$$S(\rho) \simeq \beta_- \rho \quad \text{as } \rho \to 0. \tag{43}$$

To avoid conical singularities in $\mathcal{W}$, we further impose that $\beta_- = 1$. Adding also the regularity condition (38), the cases of interest narrow down to

(i) Case 1: $\beta_- = 1, \beta_+ > 0$.

The values of the different constants are

$$\begin{aligned} \beta_- &= 1, \qquad \beta_+ = 1 - 2\kappa K|q|, \\ 1 < |q| &\le \frac{1}{2\kappa K}, \qquad C_2 = \kappa K|q| - 1. \end{aligned} \tag{44}$$

$\mathcal{W}$ has the manifold structure of a plane with (ρ, ϕ) polar coordinates. The metric $ds^2_{\mathcal{W}}$ is regular everywhere and asymptotically conical, with a deficit angle at infinity of $2\pi(1-\beta_+) = 4\pi K\kappa|q|$:

$$ds^2_{\mathcal{W}} \simeq \begin{cases} d\rho^2 + \rho^2 d\phi^2 & \text{as } \rho \to 0 \\ d\rho^2 + (1 - 2\kappa K|q|)^2 \rho^2 d\phi^2 & \text{as } \rho \to \infty. \end{cases} \tag{45}$$

The asymptotic formulas for the inverse of Eq. (42) are

$$r \simeq \begin{cases} \ln(\rho/\ell) & \text{as } \rho \to 0 \\ \frac{1}{\beta_+} \ln(\rho/\ell) & \text{as } \rho \to \infty. \end{cases} \tag{46}$$

(ii) Case 2: $\beta_- = 1$, $\beta_+ = -1$.

The values of the different constants are

$$\beta_- = 1, \quad \beta_+ = -1, \quad |q| = \frac{1}{\kappa K}, \quad C_2 = 0. \tag{47}$$

This case is of little interest, as it requires fine-tuning: $K\kappa|q| = 1$. Let

$$\rho_\infty = \ell \int_{-\infty}^{\infty} e^{-R(y)} dy, \tag{48}$$

then $\mathcal{W}$ has the manifold structure of S^2 with $(2\pi \frac{\rho}{\rho_\infty}, \phi)$ angular coordinates (respectively, colatitude and azimuth). The sphere is equipped with a smooth metric, smoothness at the poles follows from

$$ds^2_{\mathcal{W}} \simeq \begin{cases} d\rho^2 + \rho^2 d\phi^2 & \text{as } \rho \to 0 \\ d\tilde{\rho}^2 + \tilde{\rho}^2 d\phi^2 & \text{as } \tilde{\rho} \equiv \rho_\infty - \rho \to 0. \end{cases} \tag{49}$$

The asymptotic formulas for the inverse of Eq. (42) are

$$r \simeq \begin{cases} \ln(\rho/\ell) & \text{as } \rho \to 0 \\ -\ln(\frac{\rho_\infty - \rho}{\ell}) & \text{as } \rho \to \rho_\infty. \end{cases} \tag{50}$$

The solution presented here, with waves traveling along the positive z-direction, could have been taken as traveling oppositely by proposing $F(v)$ instead of $F(u)$ in Eq. (31). A linear superposition of such waves does not lead to solutions of the field equations.

Note that we have solved two out of three field equations, those in (32). We postpone the treatment of the nonhomogeneous linear equation (33) to Sec. V and consider, in the following section, the trivial case where $F = H = 0$.

IV. AN ENLSM WITH TARGET S^2

The field equations (32) and (33) admit the solution $F = 0, H = 0$, with $\alpha(r)$ and $R(r)$ as in Eq. (35). This may at first look as an uninteresting solution, since $\Phi = 0$ implies that the baryon current vanishes [see Eqs. (14) and (15)]. The U field wraps around the S^2 equator of S^3 defined by $(x^2)^2 + (x^3)^2 + (x^4)^2 = 1$ in Eq. (7):

$$\mathbb{R}^4 \ni \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\alpha \sin\Theta \\ \sin\alpha \cos\Theta \\ \cos\alpha \end{pmatrix}. \tag{51}$$

This static solution of the QCD ENLSM (3) is unstable since U can unwrap within S^3 [22]. The instability can readily be checked: if we linearly perturb this solution within the $SU(3)$ theory by setting $\alpha = \alpha(r) + \epsilon\alpha_1$, $\Theta = q\phi + \epsilon\Theta_1$, $\Phi = \epsilon\Phi_1$, and keeping only first order terms in ϵ, it readily follows from Eqs. (31) and (32) that a possible solution is $\alpha_1 = 0$, $\Theta_1 = 0$, $H = 0$, and Φ_1 an *arbitrary* function of $t - z$ [the lack of backreaction is due to the fact that the right side of Eq. (32) is order ϵ^2]. This certainly signals an instability, as the perturbation does not stay bounded in time, oscillating around the unperturbed static solution, as would be the case if this solution were stable.

However, $\alpha(r)$ and $R(r)$ as in Eq. (35) give a solution to a different theory: the ENLSM (5) with target S^2, the target 2-sphere being that defined in (51), parametrized with polar and azimuthal angles α and Θ, respectively. This follows from the fact that for $\Phi \equiv 0$, the matter field piece in (3) is the trace of the pullback of the S^2 metric, as follows from (13), so in particular (35) is a stationary point of the action

$$\tilde{S} = \int_M d^4x \sqrt{-g} \left(\frac{\mathcal{R}}{2\kappa} - \frac{K}{2} g^{ab} (\partial_a\alpha\partial_b\alpha + \sin^2\alpha \partial_a\Theta\partial_b\Theta) \right) \tag{52}$$

for the S^2 ENLSM.

The metric in this case has cylindrical symmetry,

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + S^2(\rho) d\phi^2, \tag{53}$$

and belongs to the class of Petrov type D spacetimes. The vector field $t^a\partial_a = \partial_t$ is a timelike global Killing vector field, orthogonal to the 3-Riemannian slices with metric $dz^2 + d\rho^2 + S^2(\rho)d\phi^2$. The results in Sec. V A show that this metric is geodesically complete. In Case 1, defined in Eq. (44), we get an everywhere smooth solution, free of conical singularities, which asymptotically looks like a cosmic string presenting a deficit angle sourced by regular matter fields. In Case 2, Eq. (47), the $t = \text{const}$ slices are cylinders $S^2_{(\rho,\phi)} \times \mathbb{R}_z$.

There is a topological number $q \in \pi_2(S^2) = \mathbb{Z}$ associated with these solutions, which guarantees their stability as solutions of the S^2 ENLSM. Its absolute value is proportional to the mass per length, as we now proceed to prove.

A. Topological number

In view of the first equation in (35), $\alpha(r)$ covers monotonically the interval $(0, \pi)$ as r goes from minus to plus infinity. This assures that (assuming q is an integer) the map from the $t = t_o$, $z = z_o$ submanifolds $\mathcal{W}$ onto S^2 are well defined in Case 2 (for which $\mathcal{W}$ is a sphere). Moreover, in Case 1, for which $\mathcal{W}$ is a plane, the limits at infinity are direction independent, so we get a map of the one point compactification of this plane, which is topologically a 2-sphere. As a consequence, in either case we have a topological number associated with this map. To compute it, we note that the canonical S^2 metric $d\alpha^2 + \sin^2\alpha d\Theta^2$ has normalized volume form $\frac{1}{4\pi}\sin\alpha d\alpha \wedge d\Theta$ which pulls back to the spacetime 2-form

$$\omega_q = \frac{q}{4\pi}\sin(\alpha(\rho))(d\alpha/d\rho)d\rho \wedge d\phi. \tag{54}$$

Since ω_q is closed, its integral on any two-surface $\mathcal{W}'$ in the same homology class as a $t = t_o$, $z = z_o$ two-surface $\mathcal{W}$ gives

$$\int_{\mathcal{W}'} \omega_q = -(q/2)[\Delta\cos(\alpha)] = \epsilon q. \tag{55}$$

This is the—signed—number of times that $\mathcal{W}'$ wraps around the target S^2 [that is, $\epsilon q \in \pi_2(S^2)$].

B. Mass per length

For the spacetime metric in Eq. (53), we find

$$T_{ab} = \frac{1}{\kappa}G_{ab} = \frac{1}{2\kappa}\mathrm{diag}(\mathcal{R}_{\mathcal{W}}, -\mathcal{R}_{\mathcal{W}}, 0, 0) \equiv (e, -e, 0, 0) \tag{56}$$

and

$$R^a{}_b = \frac{1}{2}\mathrm{diag}(0, 0, \mathcal{R}_{\mathcal{W}}, \mathcal{R}_{\mathcal{W}}), \tag{57}$$

where $\mathcal{R}_{\mathcal{W}} = -2S''(\rho)/S(\rho)$ is the Ricci scalar of $ds^2_{\mathcal{W}} = d\rho^2 + S^2(\rho)d\phi^2$.

As Minkowski spacetime, the metric (53) has a unit norm timelike covariantly constant vector field $t^a\partial_a = \partial_t$, orthogonal to $t = \text{const}$ hypersurfaces, which can be regarded as a velocity field of the congruence of privileged, "inertial" observers. The current $J^a = -T^a{}_b t^b$ (4-momentum density measured by these observers) is conserved: $\nabla_a J^a = 0$. Its flow through a $t = \text{const}$ surface Σ gives the total energy measured by these observers, and this is conserved in time. The volume form on Σ is $\epsilon_\Sigma = S(\rho)d\rho \wedge d\phi \wedge dz$, the normal is t^a, so that we need to integrate $e\epsilon_\Sigma = -\frac{1}{\kappa}S''(\rho)d\rho \wedge d\phi \wedge dz$ to obtain the total energy. The mass per unit length on Σ is obtained by omitting the integration on z and is found to be proportional to the absolute value of the topological charge (55):

$$\mu = -\frac{1}{\kappa}\int_{\mathcal{W}} S''(\rho)d\rho \wedge d\phi = \frac{2\pi}{\kappa}(1 - \beta_+) = 4\pi K|q|. \tag{58}$$

Note that $\mu = \frac{2\pi}{\kappa}(1 - \beta_+)$ is a standard result for cosmic strings [23].

We conclude that this simple solution of the S^2 ENLSM theory is: (i) smooth everywhere, (ii) geodesically complete, (iii) free of conical singularities, (iv) asymptotically conical in Case 1, with a mass per length sourced on the NLSM and proportional to the (absolute value) of its topological charge.

Remark. For electromagnetic fields, there is a direct link between the vanishing of the magnetic part of the Weyl tensor and the vanishing of the vorticity tensor ω_{ab} of the time translation Killing vector field (i.e., $\omega_{ab} = -\nabla_{[a}t_{b]} + a_{[a}t_{b]}$ with the acceleration given by $a_a = t^b\nabla_b t_a$) [24]. There are no such general results known for the ENLSM, but this example illustrates that this link in the electromagnetic case might be more general, as we find that the electric and magnetic part of the Weyl tensor in (t, z, ρ, ϕ) coordinates are

$$\begin{aligned}\mathcal{E}_{ab} &\coloneqq C_{acbd}t^c t^d \\ &= \mathrm{diag}\left(0, \frac{S''(\rho)}{3S(\rho)}, -\frac{S''(\rho)}{6S(\rho)}, -\frac{1}{6}S''(\rho)S(\rho)\right),\end{aligned} \tag{59}$$

$$\mathcal{B}_{ab} \coloneqq {}^*C_{acbd}t^c t^d = 0. \tag{60}$$

V. SOLUTIONS OF THE $SU(2)$ ENLSM

This section describes the dynamical spacetimes that are solutions to the full Einstein-$SU(2)$ NLSM in Eq. (3) with a nonvanishing baryon current. The backreaction of the nontrivial $\Phi = F(u)$ is the piece $H(u, \rho, \phi)$ that makes the metric a parallel wave. We present the general solution of Eq. (33) and single out a unique preferred one. For this, we study its asymptotic behavior, which is used throughout the rest of this section. Next, we probe the spacetime geometry through the study of geodesics in Sec. V A. The baryon charge is discussed in Sec. V B. Finally, in Sec. V C we review for the static case $F' = \omega$ different notions of mass per length and analyze its connection to the baryon charge.

The metric is Eq. (26) with $H \neq 0$, the $SU(2)$ field has α and R as in Eq. (35), and

$$\Theta = q\phi, \qquad \Phi = F(t-z) = F(u) \not\equiv 0. \tag{61}$$

The field equation (33) for H is, in view of $F \not\equiv 0$, nontrivial and admits a solution of the form

$$H(u, r, \phi) = -(F'(u))^2[h(r) + \psi(r)\cos(2q\phi)], \tag{62}$$

where

$$h''(r) = \left(\frac{\ell}{q}\right)^2 R''(r)\exp(-2R(r)),$$
$$\psi''(r) - 4q^2\psi(r) = -\left(\frac{\ell}{q}\right)^2 R''(r)\exp(-2R(r)). \tag{63}$$

Particular solutions for these equations are

$$h(r) = \left(\frac{\ell}{q}\right)^2 \int_r^\infty dz \int_z^\infty e^{-2R(y)}R''(y)dy \tag{64}$$

and

$$\psi(r) = \frac{\ell^2}{4|q|^3}\left[e^{2|q|r}\left(\int_r^\infty e^{-2|q|y-2R(y)}R''(y)dy\right) + e^{-2|q|r}\left(\int_{-\infty}^r e^{2|q|y-2R(y)}R''(y)dy\right)\right] =: \psi_1(r) + \psi_2(r). \tag{65}$$

Note that, since $R''(r) = K\kappa q^2 \sin^2(\alpha(r)) > 0$, both $h(r)$ and $\psi(r)$ (and indeed ψ_1 and ψ_2) are positive definite. To estimate the asymptotic form of H for the particular solution (62)–(65) we notice that

$$e^{-2R(r)}R''(r) \simeq \alpha_\pm e^{2\beta_\pm r} e^{\mp 2|q|r} \quad \text{as } r \to \pm\infty, \tag{66}$$

where $\alpha_\pm$ are positive constants involving $C1$, C_3, q, κK, and ϵ.

From (66) follows that, for Case 1 [Eq. (44)],

$$h(r) \simeq \begin{cases} \frac{\alpha_+\ell^2}{4q^2(\beta_+-|q|)^2} e^{2(\beta_+-|q|)r} & \text{as } r \to \infty \\ -Jr & \text{as } r \to -\infty, \end{cases} \tag{67}$$

and

$$\psi_1(r) \simeq \begin{cases} \frac{\alpha_+\ell^2}{8|q|^3(2|q|-\beta_+)} e^{2(\beta_+-|q|)r} & \text{as } r \to \infty \\ J_1 e^{2|q|r} & \text{as } r \to -\infty, \end{cases} \tag{68}$$

$$\psi_2(r) \simeq \begin{cases} \frac{\alpha_+\ell^2}{8|q|^3\beta_+} e^{2(\beta_+-|q|)r} & \text{as } r \to \infty \\ \frac{\alpha_-\ell^2}{8|q|^3(\beta_-+2|q|)} e^{2(\beta_-+|q|)r} & \text{as } r \to -\infty, \end{cases} \tag{69}$$

where J and J_1 are positive constants:

$$J = \left(\frac{\ell}{q}\right)^2 \int_{-\infty}^\infty e^{-2R(y)}R''(y)dy,$$
$$J_1 = \frac{\ell^2}{4|q|^3}\int_{-\infty}^\infty e^{-2|q|y-2R(y)}R''(y)dy. \tag{70}$$

The above formulas are also valid in Case 2, with the exception of Eq. (69):

$$\psi_2(r) \simeq \begin{cases} J_2 e^{-2|q|r} & \text{as } r \to \infty \\ \frac{\alpha_-\ell^2}{8|q|^3(2|q|-1)} e^{2(|q|-1)r} & \text{as } r \to -\infty \end{cases} \quad \text{(Case 2 only)}, \tag{71}$$

where

$$J_2 = \frac{\ell^2}{4|q|^3}\int_{-\infty}^\infty e^{2|q|y-2R(y)}R''(y)dy. \tag{72}$$

Now let us discuss the general solution of Eq. (33). The general solution of the associated homogeneous equation is

$$H_h(u, r, \phi) = A_0(u) + A_1(u)r + \sum_{n=1}^\infty [(C_n(u)e^{nr} + D_n(u)e^{-nr})\cos(n\phi) + (E_n(u)e^{nr} + F_n(u)e^{-nr})\sin(n\phi)]. \tag{73}$$

Thus, the general solution of (33) is H given in (62)–(65) plus a general solution H_h above. The only addition from (73) to (62) that does not worsen the general behavior as $|r| \to \infty$ is of the form $H_h = XF'(u)^2 r$. A suitable choice of X moves the linear (in $|r|$) growth as $r \to -\infty$ to a linear in r growth as $r \to \infty$. For this reason, in what follows we will stick to the particular solution in Eq. (62).

Collecting our results we find the following behavior of H in terms of ρ:

In Case 2, Eq. (47), we obtain

$$H \simeq -F'(u)^2 \begin{cases} -I\ln(\rho/\ell) & \text{as } \rho \to 0 \\ C\cos(2q\phi)[(\rho_\infty - \rho)/\ell]^{2|q|} & \text{as } \rho \to \rho_\infty, \end{cases} \tag{74}$$

where C is a positive constant. This behavior is singular in both poles of the sphere. We therefore disregard this case from now on.

In Case 1, Eq. (44), H has the asymptotic forms

$$H \simeq -F'(u)^2 \begin{cases} (-J/\beta_-)\ln(\rho/\ell) & \text{as } \rho \to 0 \\ [A + B\cos(2q\phi)](\rho/\ell)^{\frac{2(\beta_+ - |q|)}{\beta_+}} & \text{as } \rho \to \infty, \end{cases} \tag{75}$$

where A, B, J are positive constants and $\beta_+ - |q|$ is negative in view of (44). Note that H is bounded from above (assuming, as we do, that F' is bounded), and that it is negative definite if $B < A$. Since $A = \frac{\alpha_+ \ell^2}{4q^2(\beta_+ - |q|)^2}$ and $B = \frac{\alpha_+ \ell^2}{4q^2\beta_+(2|q| - \beta_+)}$, this is the case as long as $|q|$ is not too large. Specifically, if $1 < |q| < \frac{2+\sqrt{2}+4K\kappa}{1+8K\kappa+8(K\kappa)^2}$, H is negative everywhere [this constraint on $|q|$ uses that $K\kappa < 1/(2\sqrt{2})$].

The behavior of the function $-H$ as a function of ρ for large ρ determines the causal behavior of the spacetime [12]. In our case we find from (75) that, for large ρ,

$$-H < -F'(u)^2[A + B](\rho/\ell)^{\frac{2(\beta_+ - |q|)}{\beta_+}}. \tag{76}$$

Since $\frac{2(\beta_+ - |q|)}{\beta_+} < 0$ this behavior falls well in the subquadratic case [$-H \sim \rho^p$, $p < 2$ for large ρ and fixed (u, ϕ)] in the classification in [12]. This guarantees that the spacetime is strongly causal (Theorem 3.1 in [12]).

A. Geometry of the spacetime

The class of spacetimes of the form (6) was studied in [11–13]. In the most interesting case where $F' \neq 0$ (and consequently $H \neq 0$), however, our case deviates slightly from the one studied in the above references, because the singular behavior of H as $\rho \to 0$ [see Eq. (75)] implies that the spacetime manifold is not $\mathbb{R}^2_{(u,v)} \times \mathcal{W}$ but

$$(\mathbb{R}^2_{(u,v)} \times \mathcal{W}) - ([u_1, u_2] \times \mathbb{R}_v \times \{p\}), \tag{77}$$

where $p \in \mathcal{W}$ is the point $\rho = 0$ and $[u_1, u_2]$ is the closure of the support of F'. We will see below, however, that a large family of geodesics is indeed well defined in the entire $\mathbb{R}^2_{(u,v)} \times \mathcal{W}$, as H simply drops from the geodesic equation: the singularity introduced by H is rather mild. For a metric of the form in Eq. (6), H does not contribute to any of the algebraic invariant scalar fields made out of the Riemann tensor, the metric, its inverse, and its volume form. The metric (6), however, which for $H = 0$ is type D in the Petrov classification, is generically type II if $H \neq 0$ (requiring that it be of type D imposes a partial differential equation for H which is incompatible with the field equations). As remarked above, the dominant and strong energy conditions are satisfied, and the spacetime is causally well behaved.

We proceed now to the study geodesics, for which we recall that we choose a time orientation such that the null vector field $k^a\partial_a = \partial_v$, which is covariantly constant and normal to the wave fronts $u = \text{const}$, is future oriented. The affine geodesics are obtained from the Euler-Lagrange equations of

$$\mathcal{L} = -\dot{u}\,\dot{v} + H(u,\rho,\phi)\dot{u}^2 + \underbrace{\dot{\rho}^2 + S(\rho)^2\dot{\phi}^2}_{=\mathcal{L}_\mathcal{W}}, \tag{78}$$

where a dot denotes the derivative with respect to the affine parameter s, which is chosen such that

$$\mathcal{L} = \kappa = \begin{cases} 1 & \text{if spacelike,} \\ 0 & \text{if null,} \\ -1 & \text{if timelike.} \end{cases} \tag{79}$$

Given the selected time orientation, future oriented causal curves must satisfy

$$\dot{u} \geq 0. \tag{80}$$

Now let $(x^1, x^2) = (\rho, \phi)$, $g^{ij}_\mathcal{W}$ and $\Gamma_{\mathcal{W}}{}^i_{jk}$, i, j, $k = 1, 2$ the metric inverse and Christoffel symbols for $ds^2_\mathcal{W}$. The geodesic equations from Eq. (78) are

$$\ddot{x}^i + \Gamma_{\mathcal{W}}{}^i_{jk}\dot{x}^j\dot{x}^k + \Gamma^i_{uu}\dot{u}^2 = 0, \tag{81}$$

$$\ddot{v} + 2\Gamma^v_{ju}\dot{x}^j\dot{u} + \Gamma^v_{uu}\dot{u}^2 = 0, \tag{82}$$

$$\ddot{u} = 0, \tag{83}$$

with

$$\Gamma^i_{uu} = -\frac{1}{2}g^{ij}_\mathcal{W}\partial_j H, \quad \Gamma^v_{uu} = -\partial_u H, \quad \Gamma^v_{ju} = -\partial_j H. \tag{84}$$

From these equations follows that $\Gamma^*_{*v} = 0$, justifying our assertion above that k^a is covariantly constant. From Eq. (83), we obtain

$$u(s) = \dot{u}_o s + u_o, \tag{85}$$

where u_0 and $\dot{u}_0$ are constants and represent the initial "position" and "velocity," respectively, of u at $s = 0$. This naturally leads us to consider two different types of geodesics:

(i) $\dot{u}_o = 0$, then $u(s) = u_o$ for all s.

For these geodesics, since $\dot{u} = 0$, H decouples from the geodesic equations (81)–(84), which then have smooth coefficients and can cross the origin at

$\rho = 0$ even if u_o in Eq. (86) is within the support of F. From Eqs. (81)–(83), we obtain

$$(u, v, x^j) = (u_o, v = \dot{v}_o s + v_o, x^j(s)), \quad (86)$$

where $x^j(s)$ is a geodesic of $\mathcal{W}$, that is, a solution of the Euler-Lagrange equations for the Lagrangian $\mathcal{L}_{\mathcal{W}}$ in Eq. (78).

The only *future causal geodesics* of this type are those with constant x^j, that is, null geodesics with tangent k^a:

$$(u_o, v = \dot{v}_o s + v_o, x^j(s) = x^j_o), \qquad \dot{v}_o > 0. \quad (87)$$

This shows, in passing, that no causal closed geodesics exist in this family, since $s \to (u(s), v(s), x^j(s))$ is injective. The geodesics in this class with nonconstant $x^j(s)$ are spacelike and, if $\dot{v}_o = 0$, they are contained in a $(u = u_o, v = v_o)$ submanifold $\mathcal{W}$. These submanifolds are then totally geodesic. In particular, if $\mathcal{W}$ were incomplete (which is not our case since we have chosen $\beta_- = 1$), there would be incomplete spacetime geodesics of the form (86).

(ii) $\dot{u}_o \neq 0$, $u(s) = \dot{u}_o s + u_o$ (since for future causal geodesics $\dot{u}_o \geq 0$, and the orientation of spacelike geodesics is irrelevant, we will assume $\dot{u}_o > 0$).

In this case, u is given by Eq. (85). Equation (81) for the x^j follows from a Lagrangian obtained from $\mathcal{L}_{\mathcal{W}}$ by adding a time-dependent (that is, s-dependent) potential:

$$\hat{\mathcal{L}}_{\mathcal{W}} = \dot{\rho}^2 + S(\rho)^2\dot{\phi}^2 + H(\dot{u}_o s + u_o, \rho, \phi)\dot{u}_o{}^2. \quad (88)$$

The Euler-Lagrange equations from $\hat{\mathcal{L}}_{\mathcal{W}}$ in Eq. (88), using $H(\dot{u}_o s + u_o, \rho, \phi) = -F'^2(\dot{u}_o s + u_o)[h(\rho) + \psi(\rho)\cos(2|q|\phi)]$ are (a prime on functions of a single variable denotes a derivative)

$$\begin{aligned} 2\ddot{\rho} &= 2S(\rho)S'(\rho)\dot{\phi}^2 - \dot{u}_o^2 F'^2(\dot{u}_o s + u_o) \\ &\quad \times [h'(\rho) + \psi'(\rho)\cos(2|q|\phi)], \\ \frac{d}{ds}(2S(\rho)^2\dot{\phi}) &= 2|q|\dot{u}_o^2 F'^2(\dot{u}_o s + u_o)\psi(\rho)\sin(2|q|\phi). \end{aligned} \quad (89)$$

The solutions $x^j(s) = (\rho(s), \phi(s))$ of Eq. (89) can be obtained from the simpler, particular solutions $(\tilde{\rho}(s), \tilde{\phi}(s))$ that correspond to the case with $\dot{u}_o = 1$ and $u_o = 0$, via the mapping (see Theorem 3.2 in [11]):

$$\begin{aligned} \rho(s) &= \tilde{\rho}((s - u_o)/\dot{u}_o), \\ \phi(s) &= \tilde{\phi}((s - u_o)/\dot{u}_o). \end{aligned} \quad (90)$$

After solving the equations for $x^j(s)$, v can be obtained as a final step using the first integral $\mathcal{L} = \kappa$:

$$\begin{aligned} v(s) = v_o + \frac{1}{\dot{u}_o}\int_{s_o}^{s} &[-\kappa + H(\dot{u}_o\tilde{s} + u_o, x(\tilde{s}))\dot{u}_o{}^2 \\ &+ \mathcal{L}_{\mathcal{W}}(x(\tilde{s}), \dot{x}(\tilde{s}))]d\tilde{s}. \end{aligned} \quad (91)$$

In the particular case where $F'(u) = \omega \neq 0$ is a nonzero constant, the metric is stationary, and consequently there is an additional constant of motion. This is reflected in the fact that the potential in Eq. (88) is time-independent, so that the energy

$$E = \dot{\rho}^2 + S(\rho)^2\dot{\phi}^2 - H(u_o, \rho, \phi)\dot{u}_o{}^2 \quad (92)$$

is conserved. Given the behavior of H as $\rho \to 0$ [see Eq. (75)], the potential energy becomes infinite as $\rho \to 0$, and thus $\rho = 0$ is *unreachable*.

In what follows we analyze the more interesting case of a passing wave, that is, $F' \neq 0$ for $u_1 < u < u_2$ with u_1 and u_2 finite. In this case, the time-dependent potential is turned on only in the "time interval" $s_1 < s < s_2$, where

$$s_j = (u_j - u_o)/\dot{u}_o, \quad j = 1, 2. \quad (93)$$

In the nontrivial time interval $s_1 < s < s_2$, Eq. (89) admits radial solutions $\phi = \phi_o$ with $\sin(2|q|\phi_o) = 0$ and

$$\begin{aligned} 2\ddot{\rho} &= -F'^2(\dot{u}_o s + u_o)V'(\rho), \\ V(\rho) &\equiv h(\rho) + \psi(\rho)\cos(2|q|\phi_o). \end{aligned} \quad (94)$$

We would like to explore the possibility of reaching $\rho = 0$ along such a radial geodesic if the geodesic was approaching this point when the wave arrived [i.e., $\dot{\rho}(s_1) < 0$]. It is important to keep in mind Eq. (75), which implies that $V(\rho) \simeq -J\ln(\rho/\ell)$ as $\rho \to 0$ with J a positive constant. The asymptotic behavior of V as $\rho \to 0$ implies that $V' < 0$ in some interval $0 < \rho < \rho^*$. We assume, together with $\dot{\rho}(s_1) < 0$, that $\rho_1 = \rho(s_1) < \rho^*$. As a result, the right-hand side of Eq. (94) is nontrivial and positive for $s \in (s_1, s_2)$ so that the time-dependent potential tends to halt the approach to $\rho = 0$. To evaluate whether this happens or not, we use that F has compact support, and so does F'. Assuming F' is continuous, it is then necessarily bounded. In particular, there is a positive c such that $F'^2 < c$. This implies that the positive acceleration $\ddot{\rho}$ is bounded:

$$0 < 2\ddot{\rho} < -cV'(\rho), \quad (95)$$

and then, through the interval where $\dot{\rho} < 0$,

$$2\dot{\rho}\,\ddot{\rho} > -cV'(\rho)\dot{\rho}. \quad (96)$$

Assuming all these conditions hold for $s_1 < s < s_2' \le s_2$ and integrating the above inequality gives

$$\dot\rho(s_2')^2 > \dot\rho(s_1)^2 + c\underbrace{[V(\rho(s_1)) - V(\rho(s_2'))]}_{<0}. \tag{97}$$

This equation guarantees that $\rho = 0$ *cannot* be reached for $s \in (s_1, s_2)$, since $V(\rho) \to -\infty$ as $\rho \to 0$ and the available kinetic energy $\dot\rho^2$ would be entirely used up before this happens. Moreover, this analysis also allows us to show that, *for sufficiently large* $\dot u_o$, these radial geodesics can cross the wave without reversing the sign of $\dot\rho$; that is, $\rho(s_2) > 0$ and $\dot\rho(s_2) < 0$ are possible. This will be the case if the right-hand side of the inequality (97) is positive for $s_2' = s_2$. Since in view of Eq. (90),

$$\begin{aligned}\rho(s_1) &= \tilde\rho((s_1 - u_o)/\dot u_o)\\ &= \tilde\rho((u_1 - u_o)/\dot u_o^2 - u_o/\dot u_o),\\ \rho(s_2) &= \tilde\rho((s_2 - u_o)/\dot u_o)\\ &= \tilde\rho((u_2 - u_o)/\dot u_o^2 - u_o/\dot u_o),\end{aligned} \tag{98}$$

where the function $\tilde\rho$ does *not* depend on $\dot u_o$, neither on u_o, then it is clear from (98) that $\rho(s_2)$ can be made as close as we wish to $\rho(s_1)$, and the inequality

$$\dot\rho(s_1)^2 + c[V(\rho(s_1) - V(\rho(s_2))] > 0 \tag{99}$$

is satisfied by picking $\dot u_o$ large enough. Note that, in any case, the integral defining $v(s)$ in Eq. (91) is convergent. The conditions $\dot\rho(s_2) < 0$, $\dot\phi(s_2) = 0$ guarantee that the geodesic will reach $\rho = 0$ at $s = s_2 - \rho(s_2)/\dot\rho(s_2)$, since $\dot\rho$ is a constant for $s > s_2$.

In summary, for passing waves, we have found two kinds of future causal geodesics reaching (and crossing) $\rho = 0$: the null curves of the form (87), where u_o may or may not belong to the support of F', and the radial causal geodesics above. In the latter, $u \notin [u_1, u_2]$ when $\rho = 0$ is crossed, and the geodesic stays within the domain (77).

B. Baryon charge

The metric induced on a $t = t_o$ hypersurface Σ,

$$ds_\Sigma^2 = (1 + H)dz^2 + d\rho^2 + S^2(\rho)d\phi^2, \tag{100}$$

is, in view of Eq. (75), spacelike *for sufficiently large* ρ. Given any everywhere spacelike hypersurface Σ' that asymptotically matches Σ, we can use the results from Sec. II, specifically, Eq. (21), to calculate the baryon charge on Σ':

$$B = \int_{\Sigma'} J_a n^a \epsilon_{\Sigma'} = \frac{1}{2\pi^2}\int_\Sigma \sin^2(\alpha)\sin(\Theta)d\alpha \wedge d\Theta \wedge d\Phi. \tag{101}$$

What outcome should we expect for our field configuration? In the related Skyrme model on Minkowski spacetime, there are solutions for which the $SU(2)$ field U is time independent, $U(t, \vec x) = U(\vec x)$, and furthermore satisfies $\lim_{|\vec x|\to\infty} U(\vec x) = \boldsymbol{I}$, so that U can be regarded as a map from a one point compactification of $\mathbb{R}^3$ (which is topologically S^3) onto $SU(2) = S^3$, and these maps carry a topological invariant winding number in $\pi_3(S^3)$.

In our case, however, the ansatz in Eq. (31) *forbids* the possibility that U has a unique asymptotic limit on $t = t_o$ surfaces (except for the trivial vacuum configuration $\alpha = 0$): even if F in Eq. (31) has compact support, that is, it represents a passing wave, the limit of α at fixed ρ [equivalently, fixed r in Eq. (35)] and $|z| \to \infty$ will be a function of ρ, so the asymptotic values of U on Σ will not agree. As a consequence, the value of B—if finite—should not be expected to be an integer; it has no topological meaning since, although B is the integral on Σ of the pullback of the $SU(2) = S^3$ volume form, Σ cannot be regarded as a closed manifold.

While the baryon charge does not carry a topological meaning for this configuration, it remains an interesting conserved charge that describes the matter content of the solution. In particular, contrary to what happens for the stationary solutions in [7], the $\mathcal{W}$-fronted parallel waves have a finite baryon number whenever the z-integral below is finite:

$$\begin{aligned}B &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\sin^2(\alpha(r))\alpha'(r)dr\int_0^{2\pi}\sin(q\phi)q\,d\phi\\ &\quad\times\int_{-\infty}^{\infty}F'(t_o - z)dz\\ &= \epsilon\frac{\sin^2(q\pi)}{\pi}\int_{-\infty}^{\infty}F'(t_o - z)dz,\end{aligned} \tag{102}$$

where we have used that, as r grows $\alpha\colon 0 \to \pi$ for $\epsilon = 1$, and the reverse for $\epsilon = -1$. Note that $B = 0$ for integer q, but if we, following the arguments in [7], allow $q = n + \frac{1}{2}$, $n \in \mathbb{Z}$, then $B \neq 0$, and it is finite for a steplike function with finite ΔF.

In the stationary case $F' = \omega$ (a constant), if q is an integer plus one-half, we recover the infinite baryon charge in [7], with

$$\frac{dB}{dz} = \epsilon\frac{\omega}{\pi}, \qquad q \text{ half-integer}. \tag{103}$$

C. Mass per length in the static case

In the static case $F' = \omega$, besides having a notion of baryon charge per length, Eq. (103), we can also define mass per length. This is so because the asymptotically timelike vector field t^a given in $(t, z, *, *)$ coordinates by $t^a\partial_a = \partial_t$ is Killing (since $t^a\partial_a H = 0$). This implies that, for any constant x, the vector field (here $T = T_{cd}g^{cd}$)

$$T^a = \left(T^{ab} - \frac{1}{2}xTg^{ab}\right)t_b \tag{104}$$

satisfies $\nabla_a T^a = 0$. Once again, if Σ' is a timelike hypersurface that asymptotically agrees with a $t = \text{const}$ surface Σ, we can use Eq. (20) to calculate

$$\int_{\Sigma'} T_a n^a \epsilon^{\Sigma'}_{bcd} = \int_{\Sigma} \epsilon_{abcd} T^a. \tag{105}$$

The pullback onto Σ of the 3-form dual of T_a on the right-hand side above can be written, after using the first equation in (32), as

$$[K(1-x)q^2\sin^2(\alpha(r)) + K\omega^2\ell^2 e^{-2R(r)}\sin^2(\alpha(r))\sin^2(q\phi)]\, dr\wedge d\phi\wedge dz. \tag{106}$$

Note that, for either $q \in \mathbb{Z}$ or $q = n + \frac{1}{2}, n \in \mathbb{Z}$, $\int_0^{2\pi}\sin^2(q\theta)d\theta = \pi$, and that using again the first equation in Eq. (32) we can calculate

$$\int_{-\infty}^{\infty}\sin^2(\alpha(r))\,dr = \int_{-\infty}^{\infty}\sin(\alpha(r))\frac{|\alpha'(r)|}{|q|}dr = \frac{1}{|q|}\int_0^{\pi}\sin(\alpha)\,d\alpha = \frac{2}{|q|}. \tag{107}$$

Thus, omitting the integration in z, the right side of Eq. (105) gives an "x-mass" per z-unit:

$$\mu_x = 4\pi K|q|(1-x) + K\ell^2\omega^2\pi\int_{-\infty}^{\infty} e^{-2R(r)}\sin^2(\alpha(r))\,dr. \tag{108}$$

For $x = 0$ and $\omega = 0$ (that is, $\Phi = F \equiv 0$, the case treated in Sec. IV), this calculation should reduce, in view of Eq. (104), to that in Sec. IV B. In fact, for $x = 0$ and arbitrary ω, the mass per z-unit (108) gives

$$\mu_{x=0} = 4\pi K|q| + K\ell^2\omega^2\pi\int_{-\infty}^{\infty} e^{-2R(r)}\sin^2(\alpha(r))\,dr, \tag{109}$$

which contains, in addition to the $\omega = 0$ stringlike mass $4\pi K|q|$ in Eq. (58), a positive contribution proportional to ω^2. It is interesting to analyze the origin of this splitting. The relation of the $F = 0$ stringlike mass with the winding number on the target S^2 was discussed in Sec. IV. The emergence of such a term in this case, where the target is $SU(2) = S^3$, can be traced back to the first term in Eq. (106), which using Eq. (32) as in (107), gives $\sim K(1-x)|q|\sin(\alpha(r))\alpha'(r)dr\wedge d\phi\wedge dz$. Since integration in z is omitted, we end up having an integral of the pullback of an S^2 volume form [that of the $\Phi = \text{const}$ 2-sphere in Eq. (7)].

Let us now analyze the $x = 1$ mass per length. Twice this mass gives

$$2\mu_{x=1} = \int_{\Sigma'}(2T_{ab} - Tg^{ab})t^a n^b \epsilon^{\Sigma'}_{pqr}, \tag{110}$$

which agrees with the Komar mass [see, e.g., Eq. (11.2.10) in [15]], since

$$2\mu_{x=1} = -\frac{1}{8\pi}\int_{\partial\Sigma'}\epsilon_{abcd}\nabla^b t^d = -\frac{1}{8\pi}\int_{\partial\Sigma}\epsilon_{abcd}\nabla^b t^d. \tag{111}$$

From (108), we find that

$$2\mu_{x=1} = 2K\ell^2\omega^2\pi\int_{-\infty}^{\infty} e^{-2R(r)}\sin^2(\alpha(r))\,dr. \tag{112}$$

When $\omega = 0$, this vanishes, as expected from (111), since $\nabla^b t^a = 0$ in this case.

VI. CONCLUSIONS

In this paper, we proved that the ENLSM in Eq. (3) that corresponds to the minimal coupling to gravity of the leading term of the low-energy effective QCD Lagrangian, admits parallel wave solutions of the form (26), with nonplanar wave fronts $ds^2_{\mathcal{W}}$. Asymptotically along the wave fronts [that is, as $\rho \to \infty$ in Eq. (28)], the H function in Eq. (26) decays with a negative power of ρ and the metric approaches that of a cosmic string. As noticed in [7], nonstationary matter fields are compatible with stationary metrics: this happens if $F(u) = \omega(t - z)$ in Eq. (31). In this particular case, different notions of mass per length were studied, one of them nicely splitting into a cosmic-string-like term and a contribution proportional to ω^2 [see Eq. (109)].

There is a subcase where the matter field $U: M \to SU(2)$ has target $S^2 \subset S^3 = SU(2)$. When regarded as a solution of the ENLSM (3) with target S^2, this static solution is stable. It carries a topological charge $q \in \pi_2(S^2) \simeq \mathbb{Z}$, and asymptotically looks like a string with mass per length $\mu = 4\pi K|q|$, thereby offering an interesting example of a connection between a topological charge and a mass. This solution is smooth everywhere, is free of conic singularities, and thus is an example of a regular source for an asymptotically stringlike metric with a mass per length related to a topological charge.

As explained in Sec. V B, since the U field in the ansatz (31) has a direction dependent limit, the conserved baryon charge has no direct topological interpretation. It would be interesting to see if there are solutions of the field equations with a uniform asymptotic limit, nontrivial $SU(2)$ configurations that somehow generalize the connection found in Sec. IV between topological charges and notions of mass for the ENLSM with target S^2. If there is a direct link between mass and a conserved topological charge in the form of a bound, gravitational radiation should naturally shut off when this bound is reached. Of course, besides the interest of the model (3) as the minimal coupling to Einstein gravity of lowest QCD effective action, finding relationships between topological charges and mass notions in generic ENLSM (5), that is, with arbitrary target manifolds, stands as an interesting problem by itself.

ACKNOWLEDGMENTS

We thank Fabrizio Canfora for many enlightening discussions throughout the development of this work. G. D. is partially supported by Grants Nos. PIP 11220080102479 (CONICET-Argentina) and No. 30720110 101569CB (Universidad Nacional de Córdoba).

[1] J. Donoghue, E. Golowich, and B. Holstein, *Dynamics of the Standard Model*, 2nd ed., Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (Cambridge University Press, Cambridge, England, 2014).
[2] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, J. Math. Phys. (N.Y.) **5**, 1252 (1964).
[3] T. Skyrme, A non-linear field theory, Proc. R. Soc. A **260**, 127 (1961); Particle states of a quantized meson field, Proc. R. Soc. A **262**, 233 (1961); A unified field theory of mesons and baryons, Nucl. Phys. **31**, 556 (1962).
[4] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2007).
[5] F. Canfora, M. Lagos, and A. Vera, Crystals of superconducting Baryonic tubes in the low energy limit of QCD at finite density, Eur. Phys. J. C **80**, 697 (2020).
[6] F. Canfora, S. H. Oh, and A. Vera, Analytic crystals of solitons in the four dimensional gauged non-linear sigma model, Eur. Phys. J. C **79**, 485 (2019).
[7] F. Canfora, A. Giacomini, M. Lagos, S. H. Oh, and A. Vera, Gravitating superconducting solitons in the $(3+1)$-dimensional Einstein gauged non-linear σ-model, Eur. Phys. J. C **81**, 55 (2021).
[8] E. Ayón-Beato, F. Canfora, M. Lagos, J. Oliva, and A. Vera, Analytic self-gravitating 4-Baryons, traversable NUT-AdS wormholes, flat space-time multi-Skyrmions at finite volume and a novel transition in the $SU(3)$-Skyrme model, Eur. Phys. J. C **80**, 384 (2020).
[9] S. L. Cacciatori, F. Canfora, M. Lagos, F. Muscolino, and A. Vera, Analytic multi-Baryonic solutions in the SU(N)-Skyrme model at finite density, J. High Energy Phys. 12 (2021) 150.
[10] F. Canfora, Ordered arrays of Baryonic tubes in the Skyrme model in $(3+1)$ dimensions at finite density, Eur. Phys. J. C **78**, 929 (2018).
[11] A. M. Candela, J. L. Flores, and M. Sanchez, On general plane fronted waves: Geodesics, Gen. Relativ. Gravit. **35**, 631 (2003).
[12] J. L. Flores and M. Sanchez, Causality and conjugate points in general plane waves, Classical Quantum Gravity **20**, 2275 (2003).
[13] J. L. Flores and M. Sanchez, On the geometry of PP-wave type spacetimes, Lect. Notes Phys. **692**, 79 (2006).
[14] R. Penrose, A remarkable property of plane waves in general relativity, Rev. Mod. Phys. **37**, 215 (1965).
[15] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[16] S. Scherer, Introduction to chiral perturbation theory, Adv. Nucl. Phys. **27**, 277 (2003).
[17] J. L. Flores and M. Sánchez, On the geometry of pp-wave type spacetimes, in *Analytical and Numerical Approaches to Mathematical Relativity*, edited by J. Frauendiener, D. J. Giulini, and V. Perlick, Lecture Notes in Physics Vol. 692 (Springer, Berlin, Heidelberg, 2006).
[18] That the metric is Lorentzian irrespective of the range of H, is clear from the fact that the determinant of (26) is $-\ell^4 \exp(-4R(r)) < 0$.
[19] The solutions of the system in Eq. (3) presented in [7] correspond to $F = \omega(t-z)$ and lead to a static metric with $H(t-z, r, \phi) = H(r, \phi)$ in (26).
[20] G. W. Gibbons, Causality and the Skyrme model, Phys. Lett. B **566**, 171 (2003).
[21] J. Carminati and R. G. McLenaghan, Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space, J. Math. Phys. (N.Y.) **32**, 3135 (1991).
[22] We thank an anonymous referee for this observation.
[23] B. Linet, The static metrics with cylindrical symmetry describing a model of cosmic strings, Classical Quantum Gravity **17**, 1109 (1985).
[24] L. Herrera, G. A. Gonzalez, L. A. Pachon, and J. A. Rueda, Frame dragging, vorticity and electromagnetic fields in axially symmetric stationary spacetimes, Classical Quantum Gravity **23**, 2395 (2006).