

## From nonextensive statistics and black hole entropy to the holographic dark universe

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We propose a new entropy construct that generalizes the Tsallis, Rényi, Sharma-Mittal, Barrow, Kaniadakis, and loop quantum gravity entropies and reduces to the Bekenstein-Hawking entropy in a certain limit. This proposal is applied to the Schwarzschild black hole and to spatially homogeneous and isotropic cosmology, where it is shown that it can potentially describe inflation and/or holographic dark energy.

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### I. INTRODUCTION

Since the golden days of classical thermodynamics, entropy has been viewed as a unique, universal, and fundamental quantity playing one of the most important roles in physics. However, with the development of quantum physics, quantum field theory, quantum gravity, and nonextensive thermodynamics, it progressively became clear that entropy is not unique and is not as fundamental. Somehow, it depends on the physical system under consideration and it changes across physical theories. Eventually, this property implies that we do not fundamentally understand what physical entropy is and that the basic principles underlying its construction should be revisited from a critical point of view.

Over the years, we have witnessed the appearance of a variety of entropies in many classical and quantum systems. Some of these entropy concepts were introduced starting from very different points of view, each one of which is certainly valid on its own, originating a legitimate proposal in the context in which it was conceived. These entropies highlight different aspects of natural phenomena or different approaches to physical theories. This proliferation of entropy notions makes it clear that entropy may not be a uniquely defined concept and that there may exist even more entropies than those already proposed.

One of the main surprises of theoretical physics in the 1970s was that black holes are not cold objects but have entropy and temperature. Bekenstein's association of

entropy with black holes, proportional to the black hole horizon area [1], remained odd and inconclusive until Hawking discovered that the Schwarzschild black hole (and, by extension, all black holes) radiate quanta of quantum fields living on that spacetime, emitting a black-body spectrum at a temperature  $T_H = \frac{1}{8\pi GM}$ , where  $M$  is the black hole mass [2]. The discovery of the Hawking temperature made sense of Bekenstein's black hole entropy and paved the way for the development of black hole thermodynamics ([3], see [4,5] for reviews).

One puzzling feature of the Bekenstein-Hawking entropy was, from its beginnings, that it is not proportional to the black hole volume, as familiar in classical thermodynamics, but rather it is proportional to the black hole horizon area. In classical thermodynamics, the entropy of a system is proportional to its mass and its volume and is an extensive and additive quantity; the fundamental reason why black hole entropy is instead nonextensive remains shrouded in mystery [6]. Given the elusive nature of the origin of this entropy, it is not surprising that recent literature [6–23] contemplates alternatives, replacing the Bekenstein-Hawking entropy with other constructs based on nonextensive statistics, including the Rényi [24] and Tsallis [25] nonextensive entropies (a better terminology would “nonadditive” entropies). Other notable notions of entropy which have been studied recently are the Barrow entropy arising from the attempt to take into account the quantum spacetime foam [26], and the Sharma-Mittal [27]

and Kaniadakis [28,29] proposals. Since entropy, temperature, internal energy, and heat transferred are related by the first law of thermodynamics, changing the notion of entropy entails changes in these other quantities, usually jeopardizing the first law, as discussed in [30].

What is more, horizons are not a prerogative of black holes but appear also in cosmology, hence horizon thermodynamics was extended to cosmological horizons. Incidentally, realistic black holes live in the universe and are not asymptotically flat: the simple addition of a cosmological constant to the Einstein equations change their black hole solutions and their thermodynamics becomes richer. However, cosmology itself poses several interesting questions, one of the most important being how to explain the present acceleration of the cosmic expansion discovered in 1998 with type Ia supernovae. For this purpose, many scenarios of dark energy and modified gravity have been proposed and are being tested and/or constrained as newer cosmological observations become available. Among the many scenarios advanced in the cosmology literature, the holographic dark energy proposal [31–64]) is directly related to entropy. Therefore, replacing the notion of entropy used in physics has a direct impact on this scenario.

Beginning from the realization that the various entropies alternative to the Bekenstein-Hawking one proposed in the literature share certain properties, including the fact that they reduce to the Bekenstein-Hawking entropy in a certain limit, we investigate two new generalized entropies that contain all these previous proposals as special cases. The first of these new entropies features six parameters, but we provide also a simplified version containing only three parameters, which we then apply to black holes and to holographic dark energy in cosmology. Already the simplified proposal has the potential of generating two vastly different energy scales associated with inflation or with the present acceleration.

In the next section, we review various entropies studied in the recent literature and introduce the generalized entropy that reproduces them for special parameter values. A simplified version of this generalized entropy is then applied to black holes in Sec. III and to holographic dark energy in Sec. IV. We mostly follow the notation of Ref. [65], using units in which the speed of light  $c$ , the Boltzmann constant  $K_B$ , and the reduced Planck constant  $\hbar$  are unity,  $G$  is Newton's constant,  $\kappa^2 \equiv 8\pi G$ , while the metric signature is  $(-+++)$ .

## II. POSSIBLE GENERALIZATIONS OF KNOWN ENTROPIES

Let us begin with the standard thermodynamical entropy of black hole physics, one of the most far-reaching applications of entropy that led to the development of black hole thermodynamics [3–5]. The Bekenstein-Hawking entropy is [1,2]

$$S = \frac{A}{4G}, \quad (1)$$

where  $A \equiv 4\pi r_h^2$  is the area of the horizon and  $r_h$  is the horizon radius (using the areal radius as the radial coordinate). This proposal, however, is not unique. Indeed, depending on the system under consideration, different entropies may be introduced. Let us recall some of the entropy concepts proposed thus far.

- (i) The Tsallis entropy [25] (see also [20,66]) appears in the study of nonextensive statistics for systems with long range interactions, in which the partition function diverges and the standard Boltzmann-Gibbs entropy becomes inadequate; it is

$$S_T = \frac{A_0}{4G} \left( \frac{A}{A_0} \right)^\delta, \quad (2)$$

where  $A_0$  is a constant with the dimensions of an area and  $\delta$  is a dimensionless parameter that quantifies the nonextensivity. The standard Bekenstein-Hawking entropy (1) is recovered for  $\delta = 1$ .

- (ii) The Rényi entropy ([24], see also [11,17–19]) is defined as

$$S_R = \frac{1}{\alpha} \ln(1 + \alpha S) \quad (3)$$

where  $S$  is identified with the Bekenstein-Hawking entropy (1), and contains a parameter  $\alpha$ . The Rényi entropy was proposed as an index specifying the amount of information and, originally, had no relation with the statistics of physical systems.

- (iii) The Sharma-Mittal entropy is [27]

$$S_{SM} = \frac{1}{R} [(1 + \delta S_T)^{R/\delta} - 1] \quad (4)$$

where  $S_T$  is the Tsallis entropy, while  $R$  and  $\delta$  are free phenomenological parameters to be determined by the best-fitting of experimental data. The Sharma-Mittal entropy can be seen as a combination of the Rényi and Tsallis entropies.

- (iv) The Barrow entropy is [26]

$$S_B = \left( \frac{A}{A_{Pl}} \right)^{1+\Delta/2}; \quad (5)$$

here  $A$  is the usual black hole horizon area and  $A_{Pl} \equiv 4G$  is the Planck area. Formally, the Barrow entropy resembles the Tsallis nonextensive entropy but the physical principles underlying its introduction are radically different. The Barrow entropy was proposed as a toy model for the possible effects of quantum gravitational spacetime foam. The quantum-gravitational deformation is quantified by the

new exponent  $\Delta$ . The Barrow entropy reduces to the standard Bekenstein-Hawking entropy in the limit  $\Delta \rightarrow 0$ , while  $\Delta = 1$  corresponds to maximal deformation.

- (v) The Kaniadakis entropy [28,29]

$$\mathcal{S}_K = \frac{1}{K} \sinh(K\mathcal{S}), \tag{6}$$

reproduces the Bekenstein-Hawking entropy in the limit  $K \rightarrow 0$  of its parameter  $K$ . It can be regarded as a generalization of the Boltzmann-Gibbs entropy arising in relativistic statistical systems [28,29].

- (vi) Non-extensive statistical mechanics in loop quantum gravity gives the entropy [11,67–69]

$$\mathcal{S}_q = \frac{1}{1-q} [e^{(1-q)\Lambda(\gamma_0)\mathcal{S}} - 1], \tag{7}$$

where the entropic index  $q$  quantifies how the probability of frequent events is enhanced relatively to infrequent ones,

$$\Lambda(\gamma_0) = \frac{\ln 2}{\sqrt{3}\pi\gamma_0}, \tag{8}$$

and  $\gamma_0$  is the Barbero-Immirzi parameter, which is usually assumed to take one of the two values  $\frac{\ln 2}{\pi\sqrt{3}}$  or  $\frac{\ln 3}{2\pi\sqrt{2}}$ , depending on the gauge group used in loop quantum gravity. However,  $\gamma_0$  is a free parameter in scale-invariant gravity [70–72]. With the first choice of  $\gamma_0$ ,  $\Lambda(\gamma_0)$  becomes unity and the entropy (7) reduces to the Bekenstein-Hawking one in the limit  $q \rightarrow 1$ , which corresponds to extensive statistical mechanics. This loop quantum gravity entropy (7) was applied to black holes in [11,67,68] and to cosmology in [69].

The above entropies share the following properties:

- (1) *Generalized third law*: All these entropies vanish when the Bekenstein-Hawking entropy vanishes. In the third law of standard thermodynamics for closed systems in thermodynamic equilibrium, the quantity  $e^{\mathcal{S}}$  expresses the number of states, or the volume of these states, and therefore the entropy  $\mathcal{S}$  vanishes when the temperature does because the ground (vacuum) state should be unique. By contrast, the Bekenstein-Hawking entropy  $\mathcal{S}$  diverges when the temperature  $T$  vanishes and it goes to zero at infinite temperature. However, requiring any generalized entropy to vanish when the Bekenstein-Hawking entropy  $\mathcal{S}$  vanishes could be a natural requirement.

- (2) *Monotonically increasing functions*: All the above entropies are monotonically increasing functions of the Bekenstein-Hawking entropy  $\mathcal{S}$ .
- (3) *Positivity*: All the above entropies are positive, as is the Bekenstein-Hawking entropy (1). This is natural because  $e^{\mathcal{S}}$  corresponds to the number of states (or to the volume of these states), which is greater than unity.
- (4) *Bekenstein-Hawking limit*: All the above entropies reduce to the Bekenstein-Hawking entropy (1) in an appropriate limit.

In the preceding expressions, all entropies are functions of the Bekenstein-Hawking entropy (1). In this sense, the most general entropy  $\mathcal{S}_G$  would be a function of the Bekenstein-Hawking entropy  $\mathcal{S}$ ,

$$\mathcal{S}_G = \mathcal{S}_G(\mathcal{S}), \tag{9}$$

subject to certain natural requirements: we require the general entropy  $\mathcal{S}_G$  to possess the above properties.

An example of such an entropy construct containing six parameters  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  could be

$$\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}) = \frac{1}{\alpha_+ + \alpha_-} \left[ \left( 1 + \frac{\alpha_+}{\beta_+} \mathcal{S}^{\gamma_+} \right)^{\beta_+} - \left( 1 + \frac{\alpha_-}{\beta_-} \mathcal{S}^{\gamma_-} \right)^{-\beta_-} \right], \tag{10}$$

where we assume all the parameters  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  to be positive. First, we show that the entropy  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  reduces to the entropies (2), (3), (4), (5), (6), and (7) already presented for appropriate choices of the parameter values.

- (i) In the limit  $\alpha_+ = \alpha_- \rightarrow 0$ , the choice  $\gamma_+ = \gamma_- \equiv \gamma$  gives

$$\mathcal{S}_G(\alpha_{\pm} \rightarrow 0, \beta_{\pm}, \gamma) \rightarrow \mathcal{S}^{\gamma}. \tag{11}$$

If we further choose  $\gamma = \delta$  or  $\gamma = 1 + \Delta/2$ , the Tsallis entropy (2) or the Barrow entropy (5) are reproduced, respectively.

- (ii) The parameter choice  $\alpha_- = 0$  yields

$$\begin{aligned} \mathcal{S}_G(\alpha_+, \alpha_- = 0, \beta_{\pm}, \gamma_+ = 1, \gamma_-) \\ = \frac{1}{\alpha_+} \left[ \left( 1 + \frac{\alpha_+}{\beta_+} \mathcal{S}^{\gamma_+} \right)^{\beta_+} - 1 \right]. \end{aligned} \tag{12}$$

Then, writing  $\alpha_+ = R$ ,  $\beta_+ = R/\delta$ , and  $\gamma_+ = \delta$ , one obtains the Sharma-Mittal entropy (4).

- (iii) In Eq. (12), if we further take the limit  $\alpha_+ \rightarrow 0$  simultaneously with  $\beta_+ \rightarrow 0$  keeping  $\alpha \equiv \alpha_+/\beta_+$  finite, and we choose  $\gamma_+ = 1$ , we obtain

$$\begin{aligned}
& \mathcal{S}_G \left( \alpha_+ \rightarrow 0, \alpha_- = 0, \beta_+ \rightarrow 0, \beta_-, \gamma_+ = 1, \gamma_-; \alpha \equiv \frac{\alpha_+}{\beta_+} \text{finite} \right) \\
& \rightarrow \frac{1}{\alpha_+} [e^{\beta_+ \ln(1 + \frac{\alpha_+}{\beta_+} \mathcal{S})} - 1] \simeq \frac{1}{\alpha_+} \left[ 1 + \beta_+ \ln \left( 1 + \frac{\alpha_+}{\beta_+} \mathcal{S} \right) - 1 \right] = \frac{\beta_+}{\alpha_+} \ln \left( 1 + \frac{\alpha_+}{\beta_+} \mathcal{S} \right) \\
& \equiv \frac{1}{\alpha} \ln(1 + \alpha \mathcal{S}), \tag{13}
\end{aligned}$$

which reproduces the Rényi entropy (3).

- (iv) Taking the limit  $\beta_{\pm} \rightarrow 0$ , choosing  $\gamma_{\pm} = 1$ , and writing  $\alpha_{\pm} = K$ , the general entropy (10) reduces to the Kaniadakis one (6),

$$\begin{aligned}
& \mathcal{S}_G(\alpha_{\pm} = K, \beta_{\pm} \rightarrow 0, \gamma_{\pm} = 1) \\
& \rightarrow \frac{1}{2K} (e^{K\mathcal{S}} - e^{-K\mathcal{S}}) = \frac{1}{K} \sinh(K\mathcal{S}). \tag{14}
\end{aligned}$$

- (v) Finally, taking  $\alpha_- = 0$  and  $\gamma_+ = 1$  in the generalized entropy (10), one obtains

$$\mathcal{S}_G = \frac{1}{\alpha_+} [e^{\beta_+ \ln(1 + \frac{\alpha_+}{\beta_+} \mathcal{S})} - 1] \tag{15}$$

and the further limit  $\beta_+ \rightarrow +\infty$  in conjunction with  $\alpha = 1 - q$  yields

$$\mathcal{S}_G \approx \frac{1}{1 - q} [e^{(1-q)\mathcal{S}} - 1] \tag{16}$$

corresponding to  $\Lambda(\gamma_0) = 1$  in the loop quantum gravity entropy (7), and which reduces to the Bekenstein-Hawking entropy  $\mathcal{S}$  as  $q \rightarrow 1$ .

It is straightforward to check that the entropy  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  in Eq. (10) satisfies the generalized third law, that is,  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}) \rightarrow 0$  when  $\mathcal{S} \rightarrow 0$ . The entropy  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  is a monotonically increasing function of  $\mathcal{S}$  because both  $(1 + \frac{\alpha_{\pm}}{\beta_{\pm}} \mathcal{S}^{\gamma_{\pm}})^{\beta_{\pm}}$  and  $-(1 + \frac{\alpha_{\pm}}{\beta_{\pm}} \mathcal{S}^{\gamma_{\pm}})^{-\beta_{\pm}}$  are monotonically increasing functions of  $\mathcal{S}$ , given that all the parameters  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  are assumed to be positive, and their sum is also monotonically increasing. Positivity is satisfied because  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}) = 0$  when  $\mathcal{S} = 0$  and  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  is a strictly increasing function of  $\mathcal{S}$ .

It is clear that there exists a limit of  $\mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  to the Bekenstein-Hawking entropy because  $\mathcal{S}_G$  reduces to the entropies (2), (3), (4), (5), (6), and (7), which have the required limiting behavior. More explicitly, we have

$$\lim_{\alpha_{\pm} \rightarrow 0} \mathcal{S}_G(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}) = \mathcal{S}. \tag{17}$$

We may also consider the three-parameter entropy-like quantity

$$\mathcal{S}_G(\alpha, \beta, \gamma) = \frac{1}{\gamma} \left[ \left( 1 + \frac{\alpha}{\beta} \mathcal{S} \right)^{\beta} - 1 \right], \tag{18}$$

where we assume again the parameters  $(\alpha, \beta, \gamma)$  to be positive. When  $\gamma$  and  $\alpha$  coincide, the expression (18) reduces to the Sharma-Mittal entropy (4) with  $\mathcal{S}_T = \mathcal{S}$ , that is,  $\delta = 1$ . By writing  $\gamma = (\alpha/\beta)^{\beta}$ , the limit  $\alpha \rightarrow \infty$  yields

$$\lim_{\alpha \rightarrow \infty} \mathcal{S}_G \left( \alpha, \beta, \gamma = \left( \frac{\alpha}{\beta} \right)^{\beta} \right) = \mathcal{S}^{\beta}. \tag{19}$$

The choices  $\beta = \delta$  and  $\beta = 1 + \Delta/2$  give the Tsallis entropy (2) and the Barrow entropy (5), respectively. If, instead, we consider the limit in which  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$  simultaneously while keeping  $\alpha/\beta$  finite, as in Eq. (13), we obtain the Rényi entropy (3) by replacing  $\alpha/\beta$  with  $\alpha$  and choosing  $\gamma = \alpha$ ,

$$\begin{aligned}
& \mathcal{S}_G \left( \alpha \rightarrow 0, \beta \rightarrow 0, \gamma; \frac{\alpha}{\beta} \text{finite} \right) \\
& \rightarrow \frac{1}{\gamma} \ln \left( 1 + \frac{\alpha}{\beta} \mathcal{S} \right) = \frac{1}{\alpha} \ln(1 + \alpha \mathcal{S}). \tag{20}
\end{aligned}$$

Another possibility consists of taking the limit  $\beta \rightarrow \infty$  in conjunction with  $\gamma = \alpha$ , which leads to the new type of expression

$$\mathcal{S}_G(\alpha, \beta \rightarrow \infty, \gamma) \rightarrow \frac{1}{\gamma} (e^{\alpha \mathcal{S}} - 1). \tag{21}$$

It is again straightforward to check that (18) satisfies all the conditions characterizing the generalized third law: monotonically increasing function of  $\mathcal{S}$ , positivity, and Bekenstein-Hawking limit.

To recap, we have proposed two new examples of entropy that may be valid for the description of certain physical systems, which we have not yet discussed. Eventually, several additional proposals for even more general entropies can be conceived. However, we still lack a physical principle selecting an entropy as unique and universal, perhaps containing many parameters depending on various quantities.

### III. BLACK HOLE THERMODYNAMICS WITH 3-PARAMETER GENERALIZED ENTROPY

It is interesting to see what happens when the generalized entropy (9) is ascribed to the prototypical black hole, given by the Schwarzschild geometry [65]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(2)}^2, \quad f(r) = 1 - \frac{2GM}{r}, \quad (22)$$

where  $M$  is the black hole mass and  $d\Omega_{(2)}^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$  is the line element on the unit two-sphere. The black hole event horizon is located at the Schwarzschild radius

$$r_H = 2GM. \quad (23)$$

Studying quantum field theory on the spacetime with this horizon, Hawking discovered that the Schwarzschild black hole radiates with a blackbody spectrum at the temperature [2]

$$T_H = \frac{1}{8\pi GM}. \quad (24)$$

In Ref. [30] we attempted to identify the Tsallis entropy (2) or the Rényi entropy (3) with the black hole entropy. As explained in general below, if we assume that the mass  $M$  coincides with the thermodynamical energy, then the temperature obtained from the thermodynamical law is different from the Hawking temperature, a contradiction for observers detecting Hawking radiation. Alternatively, if the Hawking temperature  $T_H$  is identified with the physical black hole temperature, the obtained thermodynamical energy differs from the Schwarzschild mass  $M$  even for the Tsallis entropy or the Rényi entropy, which seems to imply a breakdown of energy conservation. Below, we follow the same procedure employed in Ref. [30].

If the mass  $M$  coincides with the thermodynamical energy  $E$  of the system due to energy conservation, as in [11,17–19], in order for this system to be consistent with the thermodynamical equation  $dS_G = dE/T$  one needs to define the generalized temperature  $T_G$  as

$$\frac{1}{T_G} \equiv \frac{dS_G}{dM} \quad (25)$$

which is, in general, different from the Hawking temperature  $T_H$ . For example, in the case of the entropy (18), one has

$$\frac{1}{T_G} = \frac{\alpha}{\gamma} \left(1 + \frac{\alpha}{\beta} S\right)^{\beta-1} \frac{dS}{dM} = \frac{\alpha}{\gamma} \left(1 + \frac{\alpha}{\beta} S\right)^{\beta-1} \frac{1}{T_H}, \quad (26)$$

where

$$S = \frac{A}{4G} = 4\pi GM^2 = \frac{1}{16\pi GT_H^2}. \quad (27)$$

Because  $\frac{\alpha}{\gamma} \left(1 + \frac{\alpha}{\beta} S\right)^{\beta-1} \neq 1$ , it is necessarily  $T_G \neq T_H$ . Since the Hawking temperature (24) is the temperature perceived by observers detecting Hawking radiation, the generalized temperature  $T_G$  in (26) cannot be a physically meaningful temperature.

In Eq. (25), assuming that the thermodynamical energy  $E$  is the black hole mass  $M$  leads to an unphysical result. As an alternative, assume that the thermodynamical temperature  $T$  coincides with the Hawking temperature  $T_H$  instead of assuming  $E = M$ . This assumption leads to

$$dE_G = T_H dS_G = \frac{dS_G}{dS} \frac{dS}{\sqrt{16\pi GS}} \quad (28)$$

which, in the case of Eq. (18), yields

$$\begin{aligned} dE_G &= \frac{\alpha}{\gamma} \left(1 + \frac{\alpha}{\beta} S\right)^{\beta-1} \frac{dS}{\sqrt{16\pi GS}} \\ &= \frac{\alpha}{\gamma \sqrt{16\pi G}} \left[ S^{-1/2} + \frac{\alpha(\beta-1)}{\beta} S^{1/2} + \mathcal{O}(S^{3/2}) \right]. \end{aligned} \quad (29)$$

The integration of Eq. (29) gives

$$\begin{aligned} E_G &= \frac{\alpha}{\gamma \sqrt{16\pi G}} \left[ 2S^{1/2} + \frac{2\alpha(\beta-1)}{3\beta} S^{3/2} + \mathcal{O}(S^{5/2}) \right] \\ &= \frac{\alpha}{\gamma} \left[ M + \frac{4\pi G\alpha(\beta-1)}{3\beta} M^3 + \mathcal{O}(M^5) \right], \end{aligned} \quad (30)$$

where the integration constant is determined by the condition that  $E_G = 0$  when  $M = 0$ . Even when  $\alpha = \gamma$ , due to the correction  $\frac{4\pi G\alpha(\beta-1)}{3\beta} M^3$ , the expression (30) of the thermodynamical energy  $E_R$  obtained differs from the black hole mass  $M$ ,  $E_G \neq E$ , which seems unphysical. In Einstein gravity, we always find  $E = M$  for the Schwarzschild black hole. We may consider a process in which the Schwarzschild black hole forms from the collapse of a sufficiently large spherically symmetric shell of dust with mass  $M$ . In this process, the thermodynamical energy  $E$  should initially be equal to the mass  $M$  of the dust,  $E = M$ . In Einstein gravity, the Jebsen-Birkhoff theorem [65] forces the spacetime outside the shell to be the Schwarzschild one (22), where  $M$  in Eq. (22) is the shell mass. Inside this shell, spacetime is empty and flat, due again to the Jebsen-Birkhoff theorem [65]. A black hole is formed by the collapse of the shell when the radius of the latter becomes smaller than its Schwarzschild radius (23). If energy is conserved because the geometry outside the shell is not changed, the thermodynamical energy  $E$  must be equal to  $M$  after the formation of the black hole event horizon. In general, the Jebsen-Birkhoff theorem does not hold in theories of gravity extending general relativity

[73,74], hence the geometry outside the shell is not always forced to be the Schwarzschild one and there might be emission of energy through the radiation of scalar modes. Therefore,  $E \neq M$  might signal a theory of gravity beyond Einstein gravity.

One can verify that the six-parameter generalized entropy (10), as well, seems inconsistent with the description of black hole thermodynamics.

#### IV. HOLOGRAPHIC COSMOLOGY WITH GENERALIZED ENTROPY

Dark energy models motivated by holography have been the subject of a considerable amount of literature (e.g., [31–60,75,76]). As a special case, a generalized (but still second order) Gauss-Bonnet gravity discussed in [75] and applied to inflation in [76] produces modified Friedmann equations not too dissimilar from those appearing in this section.

The density of the holographic dark energy (HDE) is proportional to the square of the inverse holographic cutoff  $L_{\text{IR}}$ ,

$$\rho_{\text{hol}} = \frac{3C^2}{\kappa^2 L_{\text{IR}}^2}, \quad (31)$$

where  $C$  is a free parameter. The holographic cutoff  $L_{\text{IR}}$  is usually assumed to be the same as the particle horizon  $L_p$  or the future horizon  $L_f$ . No compelling argument has been proposed thus far for choosing this quantity, hence the most general cutoff was proposed in Ref. [37]. In this proposal, the cutoff is assumed to depend upon  $L_{\text{IR}} = L_{\text{IR}}(L_p, \dot{L}_p, \ddot{L}_p, \dots, L_f, \dot{L}_f, \dots, a)$ , which in turn leads to the generalized version of HDE known as “generalized HDE” [37,77,78]. In the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe described by the line element

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2 \quad (32)$$

with scale factor  $a(t)$  in comoving coordinates  $(t, x, y, z)$ , one might speculate that the generalized HDE originates from one of several kinds of entropies associated with the cosmological horizon. In the FLRW spacetime (32), the particle horizon  $L_p$  and the future event horizon  $L_f$  are defined as

$$L_p \equiv a(t) \int_0^t \frac{dt'}{a(t')}, \quad L_f \equiv a(t) \int_t^\infty \frac{dt'}{a(t')}, \quad (33)$$

respectively, when these integrals converge. Differentiating both sides of these definitions leads to the expressions of the Hubble function in terms of  $L_p, \dot{L}_p$  or of  $L_f, \dot{L}_f$  (where an overdot denotes differentiation with respect to the comoving time  $t$ )

$$H(L_p, \dot{L}_p) = \frac{\dot{L}_p}{L_p} - \frac{1}{L_p}, \quad H(L_f, \dot{L}_f) = \frac{\dot{L}_f}{L_f} + \frac{1}{L_f}, \quad (34)$$

where the Hubble rate is  $H \equiv \dot{a}/a$ .

As argued, e.g., in Ref. [79], the standard Einstein-Friedmann equations can be derived from the Bekenstein-Hawking entropy (1). The physical radius of the cosmological horizon in spatially flat FLRW universes is

$$r_{\text{H}} = \frac{1}{H}, \quad (35)$$

which tells us that the entropy inside this horizon can be given by the Bekenstein-Hawking entropy (1) with the identification  $A \equiv 4\pi r_{\text{h}}^2 = 4\pi r_{\text{H}}^2$ . Because the incremental change of the energy  $E$ , or the increase of the heat  $Q$ , contained in this region is given by

$$dQ = -dE = -\frac{4\pi}{3} r_{\text{H}}^3 \dot{\rho} dt = -\frac{4\pi}{3H^3} \dot{\rho} dt = \frac{4\pi}{H^2} (\rho + P) dt \quad (36)$$

(where we used the conservation law  $\dot{\rho} + 3H(\rho + P) = 0$ ), by using the Gibbons-Hawking temperature [80]

$$T = \frac{1}{2\pi r_{\text{H}}} = \frac{H}{2\pi} \quad (37)$$

and the first law of thermodynamics  $TdS = dQ$ , we obtain

$$\dot{H} = -4\pi G(\rho + P). \quad (38)$$

The integration of Eq. (38) leads to the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, \quad (39)$$

where the integration constant corresponds to the cosmological constant  $\Lambda$ .

It is possible to derive the black hole entropy from holography. As shown below, if we replace the Bekenstein-Hawking entropy (1) with another entropy and we apply the procedure illustrated between Eqs. (36) and (39), the Friedmann equation (39) is modified and extra contributions, which can be seen as holographic dark energy, arise from the nonstandard entropy. For example, if we use the Tsallis entropy (2) instead of the Bekenstein-Hawking entropy (1), Eq. (38) is modified to

$$\delta \left( \frac{H}{H_1} \right)^{2(1-\delta)} \dot{H} = -4\pi G(\rho + P), \quad (40)$$

where  $H_1^2 \equiv 4\pi/A_0$ . The integration of Eq. (40) yields

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_T) + \frac{\Lambda}{3},$$

$$\rho_T = \frac{3}{8\pi G} \left[ H^2 - \frac{\delta}{2-\delta} H_1^2 \left( \frac{H}{H_1} \right)^{2(2-\delta)} \right]. \quad (41)$$

If we interpret  $\rho_T$  as the holographic dark energy due to the holographic infrared cutoff  $L_{\text{IR},T}$ ,  $\rho_T = \frac{3C^2}{\kappa^2 L_{\text{IR},T}^2}$ , then the holographic infrared cutoff  $L_{\text{IR},T}$  can be identified with

$$L_{\text{IR},T} = \frac{1}{C \sqrt{H^2 - \frac{\delta}{2-\delta} H_1^2 \left( \frac{H}{H_1} \right)^{2(2-\delta)}}}$$

$$= \frac{1}{C \sqrt{\left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2 - \frac{\delta}{2-\delta} H_1^2 \left( \frac{\dot{L}_p + \frac{1}{L_p}}{H_1} \right)^{2(2-\delta)}}}$$

$$= \frac{1}{C \sqrt{\left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2 - \frac{\delta}{2-\delta} H_1^2 \left( \frac{\dot{L}_f + \frac{1}{L_f}}{H_1} \right)^{2(2-\delta)}}}. \quad (42)$$

Equivalently, such a FLRW equation can always be rewritten in terms of a generalized cosmological dark fluid (see [81] for a review). A similar procedure for the Rényi entropy (3) gives

$$\rho_R = \frac{3\pi\alpha}{8G^2} \ln \left( 1 + \frac{GH^2}{\pi\alpha} \right). \quad (43)$$

In the case of the Sharma-Mittal entropy (4), if we simplify the situation by replacing the Tsallis entropy  $\mathcal{S}_T$  in Eq. (4) with the Bekenstein-Hawking entropy  $\mathcal{S}$  in (1) contained in it as a limit, we obtain

$$\rho_{\text{SM}} = \frac{3}{8\pi G} \left[ H^2 - \frac{\pi}{G(2-R/\delta)} \left( \frac{GH^2}{\pi} \right)^{2-R/\delta} \right. \\ \left. \times {}_2F_1 \left( 1 - \frac{R}{\delta}, 2 - \frac{R}{\delta}, 3 - \frac{R}{\delta}; -\frac{GH^2}{\pi} \right) \right], \quad (44)$$

where  ${}_2F_1(a, b, c; z)$  is the hypergeometric function. For the Barrow entropy (5), one obtains instead

$$\rho_B = \frac{3}{8\pi G} \left[ H^2 - \left( \frac{1+\Delta/2}{1-\Delta/2} \right) \frac{16\pi G}{A_{\text{Pl}}^2} \left( \frac{H^2}{4\pi A_{\text{Pl}}} \right)^{1-\Delta/2} \right]. \quad (45)$$

The three-parameter entropy (18) gives

$$\rho_G = \frac{3}{8\pi G} \left[ H^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)} \left( \frac{G\beta H^2}{\pi\alpha} \right)^{2-\beta} \right. \\ \left. \times {}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta H^2}{\pi\alpha} \right) \right], \quad (46)$$

which is expressed in terms of the particle horizon  $L_p$  or the future event horizon  $L_f$  by

$$\rho_G = \frac{3}{8\pi G} \left[ \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)} \left( \frac{G\beta \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2}{\pi\alpha} \right)^{2-\beta} \right. \\ \left. \times {}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta \left( \frac{\dot{L}_p}{L_p} - \frac{1}{L_p} \right)^2}{\pi\alpha} \right) \right] \\ = \frac{3}{8\pi G} \left[ \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)} \left( \frac{G\beta \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2}{\pi\alpha} \right)^{2-\beta} \right. \\ \left. \times {}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta \left( \frac{\dot{L}_f}{L_f} + \frac{1}{L_f} \right)^2}{\pi\alpha} \right) \right], \quad (47)$$

where the hypergeometric series terminates and reduces to a polynomial if  $\beta$  is an integer  $m \geq 1$ . One can define the pressure of the holographic dark energy  $P_G$  by means of the covariant conservation law

$$\dot{\rho}_G + 3H(\rho_G + P_G) = 0; \quad (48)$$

the equation of state parameter  $w_G$  can then be written as

$$w_G \equiv \frac{P_G}{\rho_G} = -1 - \frac{\dot{\rho}_G}{3H\rho_G} = -1 - \frac{2}{3} \dot{H} \left[ H^2 - \frac{\pi\alpha}{G\beta\gamma(1-\beta)} \left( \frac{G\beta H^2}{\pi\alpha} \right)^{2-\beta} {}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta H^2}{\pi\alpha} \right) \right]^{-1} \\ \times \left[ 1 - \frac{2-\beta}{\gamma(1-\beta)} \left( \frac{G\beta H^2}{\pi\alpha} \right)^{1-\beta} {}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta H^2}{\pi\alpha} \right) \right. \\ \left. + \frac{2-\beta}{\gamma(3-\beta)} \left( \frac{G\beta H^2}{\pi\alpha} \right)^{2-\beta} {}_2F_1 \left( 2 - \beta, 3 - \beta, 4 - \beta; -\frac{G\beta H^2}{\pi\alpha} \right) \right]. \quad (49)$$

When the matter contribution is negligible and the cosmological constant vanishes, the Friedmann equation reads

$$H^2 = \frac{8\pi G}{3} \rho_G \quad (50)$$

and then Eq. (46) gives

$${}_2F_1 \left( 1 - \beta, 2 - \beta, 3 - \beta; -\frac{G\beta H^2}{\pi\alpha} \right) = 0. \quad (51)$$

Therefore, the zeros  $Z_i$  of the hypergeometric function  ${}_2F_1(1-\beta, 2-\beta, 3-\beta; z)$  correspond to de Sitter universes with Hubble constant  $H$  given by

$$Z_i = -\frac{G\beta H^2}{\pi\alpha}. \quad (52)$$

Then, in spite of the absence of a true cosmological constant  $\Lambda$ , Eq. (52) gives the effective cosmological constant

$$\Lambda_{\text{eff}} = \frac{3\pi\alpha Z_i}{G\beta}. \quad (53)$$

Since  $H$  is constant ( $\dot{H} = 0$ ), if  $H$  is given by Eq. (52) the equation of state parameter  $w_G$  in (49) is almost  $-1$ ,  $w_G \sim -1$ . If  $\Lambda_{\text{eff}}$  in (53) is large, this effective cosmological constant may describe inflation. On the other hand, if  $\Lambda_{\text{eff}}$  is sufficiently small, the effective cosmological constant may describe the accelerated expansion of the present universe. If the effective cosmological constant is slightly larger than the present dark energy, this effective constant could potentially solve the Hubble tension problem.

Let us first consider the case in which  $Z_i$  (which we now write as  $Z_1$  for  $i = 1$ ) is sufficiently small. When  $\frac{G\beta H^2}{\pi\alpha}$  is small, the hypergeometric function  ${}_2F_1(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha})$  is expanded as

$$\begin{aligned} & {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right) \\ &= 1 - \frac{(1-\beta)(2-\beta)G\beta H^2}{3-\beta} \frac{1}{\pi\alpha} \\ &+ \frac{(1-\beta)(2-\beta)^2}{4-\beta} \left(\frac{G\beta H^2}{\pi\alpha}\right)^2 + \mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^3\right). \end{aligned} \quad (54)$$

Therefore, if we neglect the terms of order  $\mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^2\right)$  in Eq. (54) when  $H$  is small, Eqs. (54) and (51) give

$$Z_1 = -\frac{G\beta H^2}{\pi\alpha} \sim -\frac{(3-\beta)}{(1-\beta)(2-\beta)}, \quad (55)$$

that is,

$$H^2 \sim \frac{(3-\beta)\pi\alpha}{(1-\beta)(2-\beta)G\beta} \quad (56)$$

which becomes small when  $\beta \lesssim 3$  and the terms of order  $\mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^2\right)$  in Eq. (54) can be dropped. This conclusion hints at the idea that the solution (56) could explain dark energy in the present universe. We may assume

$$3-\beta \sim \mathcal{O}(10^{-2n}), \quad \alpha \sim \mathcal{O}(10^{-2m}), \quad (57)$$

and then Eq. (56) gives

$$H^2 \sim (10^{-n-m+28} \text{ eV})^2; \quad (58)$$

therefore, if  $n+m = 61$ , it is  $H \sim 10^{-33} \text{ eV}$ , which reproduces the present energy scale of the dark energy. If another zero  $Z_2$  exists with absolute value slightly smaller than  $Z_1$ , the effective cosmological constant can potentially solve the Hubble tension problem, i.e., the recent observational tension between the value of the Hubble constant inferred from small redshifts (as in the observations of Type Ia supernova calibrated by Cepheids [82]) and that from large redshifts inferred from the cosmic microwave background (CMB) [83]. This problem might be solved, or at least alleviated, if there is effectively dark energy just after the CMB was emitted. Our model admitting two zeros  $Z_{1,2}$  with  $|Z_2|$  slightly larger than  $|Z_1|$  might play the role of the effective dark energy just after the CMB.

In general, the hypergeometric function can have several or even an infinite number of zeros. If there are a root of order unity or a large and negative root  $Z_i$  of the equation  ${}_2F_1(1-\beta, 2-\beta, 3-\beta; Z_i) = 0$ , then Eq. (52) can give the large Hubble rate  $H$  corresponding to the inflationary epoch. The Hubble rate  $H$  and the effective cosmological constant  $\Lambda_{\text{eff}}$  are given by Eqs. (52) and (53), respectively. If, for the sake of illustration, we retain the first three terms in Eq. (54), the latter assumes the form

$$1 - \frac{(1-\beta)(2-\beta)G\beta H^2}{3-\beta} \frac{1}{\pi\alpha} + \frac{(1-\beta)(2-\beta)^2}{4-\beta} \left(\frac{G\beta H^2}{\pi\alpha}\right)^2 = 0 \quad (59)$$

with solutions

$$\begin{aligned} \frac{G\beta H^2}{\pi\alpha} = Z_{\pm} &\equiv -\frac{\frac{(1-\beta)(2-\beta)}{3-\beta} \pm \sqrt{\frac{(1-\beta)^2(2-\beta)^2}{(3-\beta)^2} - 4\frac{(1-\beta)(2-\beta)^2}{4-\beta}}}{\frac{2(1-\beta)(2-\beta)^2}{4-\beta}} \\ &= -\frac{4-\beta}{2(2-\beta)(3-\beta)} \left(1 \pm \sqrt{1 - \frac{4(3-\beta)^2}{(4-\beta)(1-\beta)}}\right). \end{aligned} \quad (60)$$

As in Eqs. (57) and (58) we assume  $\beta \lesssim 3$ , obtaining

$$Z_+ = -\frac{4-\beta}{2(2-\beta)(3-\beta)}, \quad Z_- = -\frac{3-\beta}{2(1-\beta)(2-\beta)} \quad (61)$$

[here  $Z_-$  corresponds to  $Z_1$  in Eq. (55)]. Therefore, if one writes  $\alpha$  and  $\beta$  as in Eq. (57) and chooses  $n+m = 61$  as done below Eq. (58), one finds again a Hubble constant  $H$  that reproduces the present value of the dark energy scale. If, instead,  $\frac{G\beta H^2}{\pi\alpha} = Z_+$ , one finds



$$H^2 \sim (10^{n-m+28} \text{ eV})^2 \quad (62)$$

and the choice  $n + m = 61$  gives

$$H^2 \sim (10^{-2m+89} \text{ eV})^2. \quad (63)$$

Assuming GUT scale ( $\sim 10^{16}$  GeV =  $10^{25}$  eV) inflation  $H \sim 10^{2 \times 25 - 28}$  eV =  $10^{22}$  eV, we obtain  $m \sim 33$  or  $34$ . Therefore  $Z_+$  may produce the inflationary epoch of the early universe.

Similarly, one can consider generalized HDE coming from our six-parameter entropy (10): then, there are many more possibilities to realize realistic cosmic histories by choosing appropriately the corresponding parameters.

## V. CONCLUSIONS

New and old definitions of entropy abound in the literature, mostly arising from nonextensive statistical mechanics and thermodynamics or from quantum gravity, and varying according to the physical theory or the physical systems considered. Not surprisingly, a recurrent feature is the presence of long-range forces, due to which the partition function diverges and the Boltzmann-Gibbs entropy fails. Here we have discussed black holes and the holographic universe as physical systems and we have proposed two generalized entropies that satisfy certain basic requirements: each of these generalized entropies  $\mathcal{S}_G$  must vanish when  $\mathcal{S}$  vanishes, must be positive-definite, and must reduce to the Bekenstein-Hawking entropy (1) in some limit. It is clear that some requirements must be imposed on generalized entropies to restrict the range of possible proposals, and the three conditions we impose

seem minimal requirements. It is quite possible that the spectrum of generalized entropies that they allow is still too wide and that it should be restricted further. In the meantime, we adopt two proposals for generalized entropies [given by Eqs. (10) and (18)], which are very general yet directly linked to the physics explored in many recent works. (One could add extra terms to (10), but that would mean adding extra parameters and these terms would have to be set to zero anyway to reproduce, e.g., the Kaniadakis entropy (6).)

Both generalized entropies (10) and (18) satisfy the three criteria above and reproduce a variety of entropy notions introduced in the literature, including the Rényi [24], Tsallis [25], Sharma-Mittal [27], Kaniadakis [28,29], Barrow [26], and Majhi's loop quantum gravity [11,67–69] entropy proposals. Although we have restricted our attention to gravitational systems such as black holes and cosmology, our prescriptions are potentially much more general and could be applied to many other systems of interest in statistical mechanics, information theory, and other areas of physics in which long-range interactions are present, but not only. These applications will be the subject of future research.

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