

Field sources for Simpson-Visser spacetimes

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Simpson-Visser (SV) spacetimes are the simplest globally regular modifications of the Schwarzschild, Reissner-Nordström and other black hole solutions of general relativity. They smoothly interpolate between these black holes and traversable wormholes. After a brief presentation of the Schwarzschild-like and Reissner-Nordström-like SV geometries, including their Carter-Penrose diagrams, we show that any static, spherically symmetric SV metric can be obtained as an exact solution to the Einstein field equations sourced by a combination of a minimally coupled phantom scalar field with a nonzero potential $V(\phi)$ and a magnetic field in the framework of nonlinear electrodynamics with the Lagrangian $\mathcal{L}(F)$, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ (in standard notations). Explicit forms of $V(\phi)$ and $\mathcal{L}(F)$ are presented for the cases of Schwarzschild-like and Reissner-Nordström-like SV metrics.

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I. INTRODUCTION

According to general relativity, under regular initial conditions, a sufficiently massive matter distribution must undergo a complete gravitational collapse, forming a zero proper volume end state with a curvature singularity, from which even light cannot escape [1,2]. At the singularity, physical quantities such as the spacetime curvature, tidal forces, energy density, and pressure, tend to infinity thus indicating a pathology of the classical theory of gravity. Since a singularity is an undesirable feature of the theory, there emerge strategies to conceal it from an observer. The cosmic censorship conjecture suggests that spacetime singularities are necessarily enclosed within a null hypersurface, causally disconnecting the interior from the exterior, known as horizons [3]. Such an end state is known as a black hole. However, under generalized initial conditions for varieties of matter fields, a number of possible end states have been reported where singularities are observable from null infinity, being called naked singularities [4–9].

Another way to avoid singularities while remaining in the framework of classical gravity is to find regular black hole models with finite spacetime curvature at the center (e.g., [10–15], and many others). One of the most popular

classes of such regular black holes are those sourced by nonlinear electrodynamics (NED), see, e.g., [12,16–18] and references therein. Singularity-free regular end states without a horizon are also possible, for example, in conformally flat collapsing star models of $f(R)$ gravity [19] or with the nonminimally coupled Gauss-Bonnet invariant [20]. Such solitonlike configurations also emerge with NED sources.

Quite a different kind of model shows pathways connecting distant parts of the same universe or two different universes—these are wormholes and certain classes of regular black holes. Among the latter, there are black holes containing an expanding cosmology beyond a horizon, the so-called black universes [21,22]. Such models are naturally obtained if one considers local concentrations of dark energy in the form of a phantom matter field as well as in some modified gravity theories [22–26]. Thus, globally regular static, spherically symmetric solutions with a magnetic field, describing both wormholes with flat or anti-de Sitter asymptotics and black universes, were obtained [27,28].

A discussion of wormhole geometries can be traced back to the papers by Flamm [29] and Einstein and Rosen [30], though the “tunnels” they describe only exist in some spatial sections of spacetime and are not traversable. As became clear much later [31,32], maintaining a static wormhole throat needs an amount of exotic matter that violates the null energy condition (NEC). Ellis [33] and

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Bronnikov [34] discussed such a static traversable wormhole on the basis of a solution of Einstein gravity coupled to a free phantom scalar field, and quite a lot of other wormhole solutions with various kinds of phantom matter were obtained and discussed afterwards. One can note that although phantom fields have always been a source of debate due to their negative kinetic energy, they are a strong candidate for dark energy and lead to some unique and interesting spacetimes [35,36]. Furthermore, phantom fields also appear in string theory in the form of negative tension branes, which play an important role in string dualities [37,38]. Interestingly, a minimally coupled scalar field can smoothly pass on from canonical to phantom form without creating any spacetime singularities leading to the “trapped ghost” concept [24,39,40]. Quite recently, examples of static, spherically symmetric wormhole models were obtained using classical spinor field sources [41–45], which can thus acquire exotic properties.

Concerning black holes, one simple approach to regularize a static, spherically symmetric geometry with a curvature singularity at $r = 0$ is to simply replace r with $\sqrt{x^2 + a^2}$ (and dr with dx) [46], where x is a new radial coordinate, and a is some positive constant. The result is a globally regular spacetime where the spherical (areal) radius r always remains positive, while the singularity at $r = 0$ turns into a regular minimum of r , a sphere of radius a . Due to regularity at $x = 0$, the metric can be extended to $x < 0$. Depending on the nature of the hypersurface $x = 0$, the resulting metric describes different geometries of interest: a timelike throat $x = 0$ corresponds to a reflection-symmetric traversable wormhole, while a spacelike hypersurface $x = 0$ corresponds to a bounce in the time-dependent quantity r , which is in fact one of the scale factors of a Kantowski-Sachs cosmological model inside a black hole. This phenomenon was named a *black bounce* [46]. (In fact, a black bounce is also a necessary feature of all black-universe models.) The intermediate case, a null throat, corresponds to what can be called a one-way wormhole but actually possesses black hole properties since there is a horizon.

Simpson and Visser (SV) [46] considered such a metric on the basis of the Schwarzschild one, and the result is a family of metrics that seamlessly interpolates between a Schwarzschild black hole, a black-bounce geometry, and a traversable wormhole. The SV metric is interesting in the sense that it is a minimal one-parameter extension of the Schwarzschild metric. In the same spirit, a black-bounce extension of the Reissner-Nordström spacetime was also reported [47]. The resulting SV metric generalizes the Schwarzschild-SV spacetime and interpolates between a Reissner-Nordström black hole and a wormhole. Lobo *et al.* [48] used this approach to construct a large family of globally regular black-bounce spacetimes that generalize the original SV model. Owing to their rich features, SV spacetimes have been a subject of great interest over

recent years, and their rotating extensions have also been obtained [49–51]. Interestingly, the rotating SV metric belongs to a special case of the parametrized non-Kerr metric constructed by Johannsen [52]. The geometric properties of black hole/black-bounce spacetimes are studied by using gravitational wave echo signals [12,53,54]. Their optical appearance has been investigated, including the effects of a surrounding accretion disk and strong gravitational lensing [55–61].

It should be noted that Lagrangian formulations for SV-like black bounce spacetimes seem to be still lacking. In this paper, we show that both uncharged and charged SV metrics as well as their possible modifications and extensions can be obtained as exact solutions to Einstein’s field equations minimally coupled with a self-interacting phantom scalar field combined with a NED field. It is shown that a scalar field and NED taken separately cannot create an SV black bounce spacetime. The need for a phantom scalar field was expected due to the presence of a wormhole as a solution to the field equations, while a NED ingredient accounts for adjusting the stress-energy tensor (SET) to a form relevant to particular SV spacetimes.

The paper is organized as follows. In Sec. II we discuss the general properties of uncharged and charged SV spacetimes, including their global features characterized by Carter-Penrose diagrams. In Sec. III we present the scalar-NED fields model for SV spacetimes. Our concluding remarks are contained in Sec. IV. We adopt the metric signature $(+, -, -, -)$ and work in the geometrized units $8\pi G = c = 1$.

II. SIMPSON-VISSER SPACETIMES

The SV spacetime metric [46] is a one-parameter modification of the Schwarzschild metric (to be called the Schwarzschild-SV, or S-SV metric), in which the spherical radius r is replaced by the expression $\sqrt{x^2 + a^2}$, where $x \in \mathbb{R}$ is a new radial coordinate, and $a = \text{const} > 0$ is a new parameter, so that the metric has the form

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$r(x) \equiv \sqrt{x^2 + a^2}, \quad (1)$$

with $A(x) = 1 - 2M/r(x)$, and $M > 0$ has the meaning of the Schwarzschild mass. Unlike the Schwarzschild metric (restored in the case $a = 0$), at $a > 0$ this metric is manifestly globally regular and twice (as $x \rightarrow \pm\infty$) asymptotically flat. The new parameter a controls interpolation between the Schwarzschild black hole metric and that of a Morris-Thorne-like traversable wormhole, see Fig. 1, left panel. Thus we obtain the following spacetime geometries:

- (i) $2M > a$: a regular black hole with two horizons at $x_h^\pm = \pm\sqrt{4M^2 - a^2}$ (curve 1 in Fig. 1).

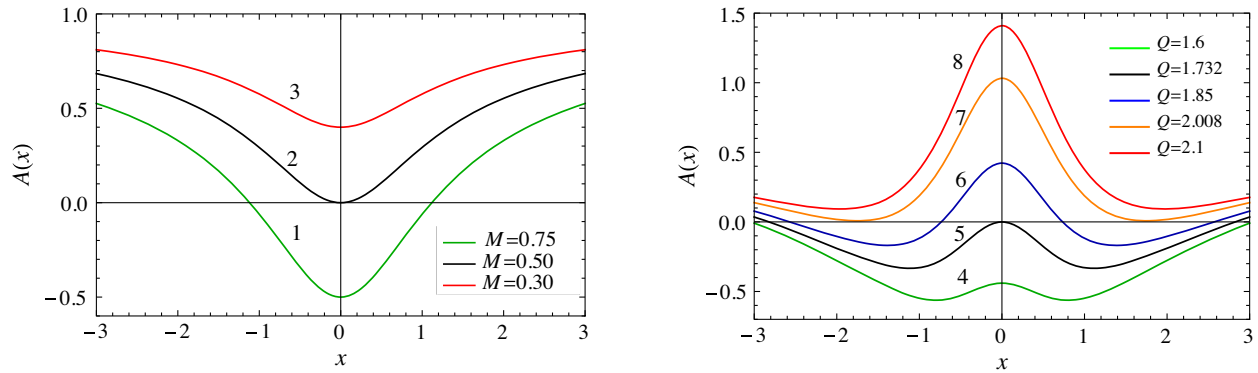


FIG. 1. The metric function $A(x)$ of the SV metric (1), with $a = 1$ taken as the length scale. Left panel: $A(x) = 1 - 2M/r(x)$, curves 1–3 correspond to $M = 0.75, 0.5, 0.3$, respectively. Right panel: $A(x) = 1 - 2M/r(x) + Q^2/r^2(x)$, curves 4–8 correspond to $M = 2$ and $Q = 1.6, 1.732, 1.85, 2.008, 2.1$, respectively.

- (ii) $2M = a$: a regular black hole with a single extremal (double) horizon at $x = 0$ (curve 2).
- (iii) $2M < a$: a symmetric traversable wormhole with a throat at $x = 0$ (curve 3). In particular, at $M = 0$ we obtain the Ellis wormhole, free from static gravitational forces as well as tidal forces [33,34].

Thus the SV geometries are not only globally regular but are also richer in their causal structure than the Schwarzschild geometry, as can be illustrated by their Carter-Penrose diagrams, see Fig. 2. A regular minimum of the spherical radius $r(x)$ occurring in a nonstatic ($A(x) < 0$) spacetime region, also often called a T region, was named a “black bounce” [46]. One can note that geometries containing a black bounce, described by solutions to the Einstein equations with phantom scalar fields, were considered earlier, in particular, in [21–24,27,28], and

in many cases led to so-called “black universes” [21,22], i.e., geometries where the T region of a black hole ultimately evolves into a de Sitter-like expanding universe.

In the case $2M = a$, such that the minimum of r coincides with a single horizon separating two R regions, the geometry has also been called “a one-way wormhole” [46], but it has definitely a black hole nature and looks from outside quite similarly to an extremal Reissner-Nordström black hole. A proper name for such a horizon may be a “black throat.”

Later on, Franzin *et al.* [47] considered more general black-bounce spacetimes on the basis of the Reissner-Nordström metric in the same minimal manner, leading to a number of geometries including regular black holes and traversable wormholes. The RN-SV metric has the form (1) with

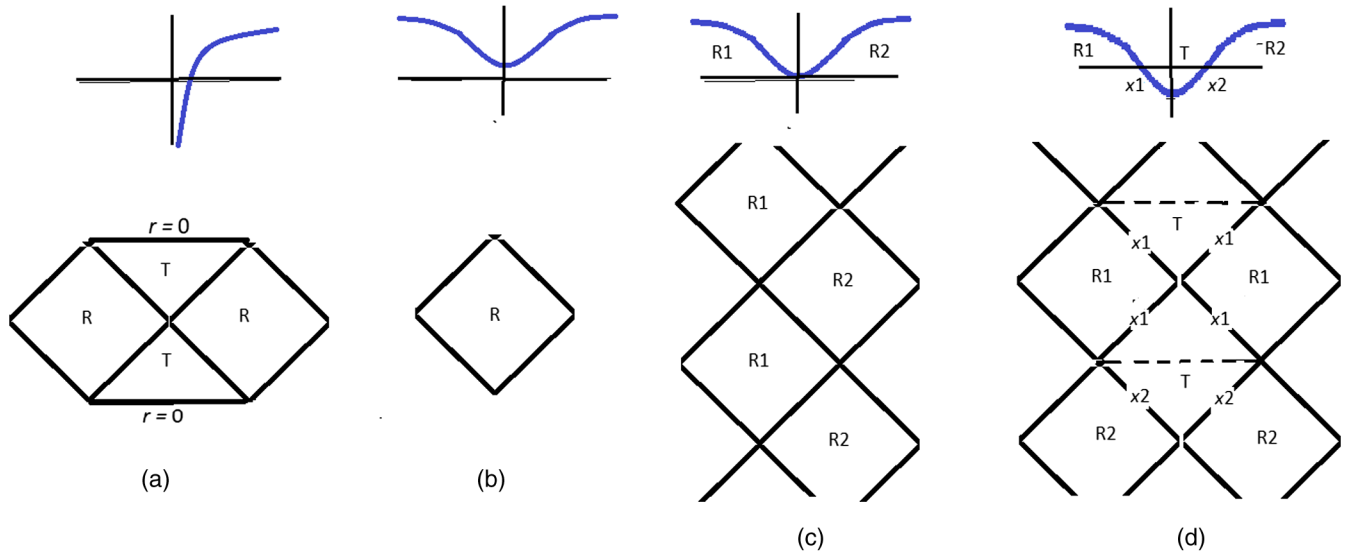


FIG. 2. Carter-Penrose diagrams (a) for the Schwarzschild geometry (for comparison) and (b),(c),(d) for S-SV geometries. Over the diagrams, the corresponding behavior of $A(x)$ is schematically shown. All tilted boundary lines in the diagrams correspond to $r = \infty$, all tilted internal lines depict horizons. The letters R and T mark R and T regions, enumerated where necessary, x_1 and x_2 mark the positions of horizons. The black bounce at $x = 0$ is shown by dashed lines in the T region. The diagrams (c) and (d) are infinitely continued upward and downward.

$$A(x) = 1 - \frac{2M}{r(x)} + \frac{Q^2}{r^2(x)} \equiv 1 - \frac{2M}{\sqrt{x^2 + a^2}} + \frac{Q^2}{x^2 + a^2}, \quad (2)$$

where Q is the electric charge parameter, such that at $Q = 0$ we return to the S-SV metric. At $a = Q = 0$, the RN-SV metric reduces to the Schwarzschild metric, while at $a \neq 0$ it describes a regular black hole or a traversable wormhole.

$$\begin{aligned} R &= \frac{2(-3aM + a^2 + Q^2)}{a^4}, \\ R_{\mu\nu}R^{\mu\nu} &= \frac{2(2a^4 + a^2(9M^2 + 2Q^2) - 2aM(3a^2 + 4Q^2) + 2Q^4)}{a^8}, \\ \mathcal{K} &= \frac{4(3a^4 + a^2(9M^2 + 4Q^2) - 2aM(4a^2 + 5Q^2) + 3Q^4)}{a^8}. \end{aligned} \quad (3)$$

All these expressions diverge as $a \rightarrow 0$. The curvature scalars are shown in Fig. 3.

The causal structures of RN-SV spacetimes are more diverse than those of the Reissner-Nordström geometries. Depending on the values of a , M and Q , the following kinds of geometry are obtained:

- (i) a regular black hole with two horizons (curves 1 and 4 in Fig. 1);
- (ii) a regular black hole with a single extremal (double) horizon at $x = 0$ (curve 2 in Fig. 1), such geometries exist with both zero and nonzero Q ;
- (iii) a regular black hole with two simple horizons and an extremal (double) horizon separating two T-regions (curve 5 in Fig. 1);
- (iv) a regular black hole with four horizons (curve 6 in Fig. 1);
- (v) a regular black hole with two extremal (double) horizons separating R regions (curve 7 in Fig. 1);
- (vi) a traversable wormhole (curves 3 and 8).

One can verify that this list exhausts all possible horizon allocations at $a > 0$, $M > 0$ and any values of Q . In Figs. 4 and 5, we have depicted the Carter-Penrose diagrams for RN-SV spacetimes with two, three or four horizons. The horizons are located at regular zeros of the function $A(x)$, hence at

$$x = x_h = \pm \sqrt{2M^2 - a^2 - Q^2 \pm 2M\sqrt{M^2 - Q^2}}. \quad (4)$$

As is clear from (1) with $a > 0$ (to be called *general SV spacetimes*), we have everywhere $r \geq a$ with a minimum at $x = 0$, which is a throat if $A(0) \geq 0$, a black throat in the case $A(0) = 0$, and a black bounce if $A(0) < 0$. In all cases, spacetimes with the metric (1) have no center.¹

¹A center is understood as a location in space where the coordinate spheres shrink to a point, hence, it means $r = 0$ while $A > 0$. Note that in a region where $A < 0$, that is, in a T region, the parameter r is only time dependent, and a possible limit $r \rightarrow 0$ is actually a cosmological singularity occurring at a particular time instant. This happens, for example, at the Schwarzschild singularity $r \rightarrow 0$.

At $a \neq 0$, the RN-SV metric is manifestly globally regular since all metric coefficients are finite and smooth, hence all its curvature invariants are finite in the whole range $x \in \mathbb{R}$. For example, the Ricci scalar R , the quadratic Ricci invariant $R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar $\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ have the following values at $x = 0$:

Thus regular black hole spacetimes with the metric (1) are basically different from the great number of well-known regular black holes which contain either Minkowsky [62] or de Sitter regular centers [11–16,18], see also references therein. Other regular spherically symmetric black holes without a center were obtained in [63] as examples of vacuum brane-world configurations and in [21,22,27] as black-universe spacetimes.

Assuming the validity of Einstein's field equations $G_\mu^\nu = -T_\mu^\nu$ for SV spacetimes, let us calculate the components of the corresponding SET (the prime stands for d/dx):

$$\begin{aligned} T_t^t &= -G_t^t = \frac{1}{r^2} [1 - A(2rr'' + r'^2)] + A'rr' \\ &= -\frac{1}{r^7} [a^2(r(2Q^2 + x^2) - 4Mx^2) - Q^2x^2r + a^4(r - 4M)], \end{aligned} \quad (5)$$

$$\begin{aligned} T_x^x &= -G_x^x = -\frac{1}{r^2} [1 - A'rr' - Ar'^2] \\ &= \frac{1}{r^4} [a^4 + x^2(a^2 + Q^2)], \end{aligned} \quad (6)$$

$$\begin{aligned} T_\theta^\theta &= -G_\theta^\theta = -\frac{1}{r} [Ar'' + rA''/2 + A'r'] \\ &= -\frac{1}{r^7} [a^2x^2(r - M) + Q^2x^2r + a^4(r - M)] = T_\phi^\phi. \end{aligned} \quad (7)$$

Here, we first present the general form of nonzero G_μ^ν for the metric (1) with arbitrary $r(x)$ and $A(x)$, and only then give particular expressions for T_μ^ν with $r \equiv \sqrt{x^2 + a^2}$ and $A(x)$ given by (2).

There is an important subtle point about T_μ^ν , not always correctly taken into account in descriptions of black-hole studies. The point is that $T_t^t = \rho$, the energy density, and $-T_x^x = P_r$, the radial pressure, only under the condition $A > 0$, in other words, in static regions (R regions).

In regions where $A < 0$ (T regions), x is a temporal coordinate, t is a spatial one, therefore, it is T_x^x that should be identified as ρ whereas $T_t^t = -P_r$.

Concerning the NEC that in all cases requires $\rho + P_r \geq 0$, we can notice that $\rho + P_r = (T_t^t - T_x^x)\text{sign}A(x)$. On the other hand, due to the Einstein equations we have the relation

$$T_t^t - T_x^x = -2Ar''/r \quad (8)$$

for the general metric (1) with any $A(x)$ and $r(x)$. We see that with both positive and negative $A(x)$ it holds $\rho + P_r \propto -r''/r$. But we have necessarily $r'' > 0$ near a minimum of r , be it a wormhole throat or a black bounce. An important conclusion for all static, spherically symmetric spacetimes is that [48,64].

The NEC is necessarily violated in a neighborhood of any wormhole throat or any black bounce.

With $r(x) = \sqrt{x^2 + a^2}$ we have $r''/r = a^2/r^4 > 0$, hence the NEC is violated in the whole spacetime (except for horizons where $T_t^t - T_x^x = 0$ due to $A = 0$). The weak energy condition (it requires $\rho \geq 0$ in addition to the NEC) is thus also violated. The strong and dominant energy conditions are in such cases most often but not necessarily violated.

Another NEC requirement, $\rho + P_\perp \geq 0$ (where $P_\perp = -T_\theta^\theta = -T_\phi^\phi$), does not lead to equally crucial conditions. For the RN-SV metric we have

$$\rho + P_\perp = \frac{3a^2M}{r^5} - \frac{2Q^2(a^2 - x^2)}{r^6}. \quad (9)$$

III. FIELD SOURCES OF THE SV METRICS

Let us consider the possible sources of SV geometries in the framework of general relativity, in the form of an uncharged scalar field $\phi(x)$ and a nonlinear electromagnetic field with the Lagrangian $\mathcal{L}(\mathcal{F})$, minimally coupled to gravity, so that the action may be written in the form

$$S = \int \sqrt{-g} d^4x (\mathcal{R} + 2\epsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) - \mathcal{L}(\mathcal{F})), \quad (10)$$

where $\epsilon = \pm 1$, so that $\epsilon = +1$ corresponds to a canonical scalar field with positive kinetic energy, and $\epsilon = -1$ to a phantom scalar field with negative kinetic energy; $\mathcal{L}(\mathcal{F})$ is a gauge-invariant NED Lagrangian density with $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}$ being the electromagnetic field tensor. Varying the action (10) with respect to the metric $g^{\mu\nu}$ leads to the Einstein equations

$$G_\mu^\nu = -T_\mu^\nu[\phi] - T_\mu^\nu[F], \quad (11)$$

where $T_\mu^\nu[\phi]$ and $T_\mu^\nu[F]$ are the SETs of the scalar and electromagnetic fields:

$$T_\mu^\nu[\phi] = 2\epsilon \partial_\mu \phi \partial^\nu \phi - \delta_\mu^\nu (\epsilon g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - V(\phi)), \quad (12)$$

$$T_\mu^\nu[F] = -2\mathcal{L}_\mathcal{F} F_{\mu\sigma} F^{\nu\sigma} + \frac{1}{2} \delta_\mu^\nu \mathcal{L}(\mathcal{F}), \quad (13)$$

where $\mathcal{L}_\mathcal{F} = d\mathcal{L}/d\mathcal{F}$; further on, varying the action in ϕ and $F_{\mu\nu}$, we obtain the field equations

$$2\epsilon \nabla_\mu \nabla^\mu \phi + dV(\phi)/d\phi = 0, \quad (14)$$

$$\nabla_\mu (\mathcal{L}_\mathcal{F} F^{\mu\nu}) = 0. \quad (15)$$

Considering the static, spherically symmetric metric (1), the corresponding assumptions for the scalar and electromagnetic fields are $\phi = \phi(x)$ and possible nonzero $F_{\mu\nu}$ components: $F_{tx} = -F_{xt}$ (a radial electric field) and $F_{\theta\phi} = -F_{\phi\theta}$ (a radial magnetic field). We will assume the existence of a magnetic field only, such that $F_{\theta\phi} = q \sin\theta$, where q is the magnetic monopole charge. Equation (15) is then trivially satisfied. The Faraday electromagnetic invariant takes the form $\mathcal{F} = 2q^2/r^4 \equiv 2q^2/(a^2 + x^2)^2$. Under these assumptions, the SETs (12) and (13) take the form

$$T_\mu^\nu[\phi] = \epsilon A(x) \phi'^2 \text{diag}(1, -1, 1, 1) + \delta_\mu^\nu V(\phi), \quad (16)$$

$$T_\mu^\nu[F] = \frac{1}{2} \text{diag}\left(\mathcal{L}, \mathcal{L}, \mathcal{L} - \frac{4q^2}{r^4} \mathcal{L}_\mathcal{F}, \mathcal{L} - \frac{4q^2}{r^4} \mathcal{L}_\mathcal{F}\right). \quad (17)$$

Let us now explain why a scalar field alone cannot be a source of SV metrics. To begin with, according to the most universal of the so-called global structure theorems [64,65], an asymptotically flat static, spherically symmetric Einstein-scalar configuration cannot contain more than one horizon, while the SV metrics with proper parameters contain at least two. But more specifically, the scalar field SET has the property $T_t^t[\phi] = T_\theta^\theta[\phi]$, and this is clearly not the case for (5) and (7), the SET components needed for SV metrics.

On the other hand, NED alone also cannot provide a source for SV metrics since, for the latter, the SET components (5) and (6) are not equal, whereas by Eq. (17), $T_t^t[F] = T_x^x[F]$. Moreover, an attempt to use a scalar field combined with a Maxwell field (which is a special case of NED corresponding to $\mathcal{L} = \mathcal{F}$) would also be a failure since for a Maxwell field $T_t^t - T_\theta^\theta = 4q^2/r^4$, while the corresponding difference for the SV metrics exhibits another x dependence.

It can, however, be shown that a combination of a scalar field and NED can provide a source for any SV metric (1) with an arbitrary function $A(x)$ and $r = \sqrt{x^2 + a^2}$.

Indeed, the difference of Eqs. (5) and (7) is free from the scalar field and leads to

$$\frac{1}{r^4} [2 - 2A(x) + r^2 A''(x)] = \frac{2q^2}{r^4} \mathcal{L}_\mathcal{F}. \quad (18)$$

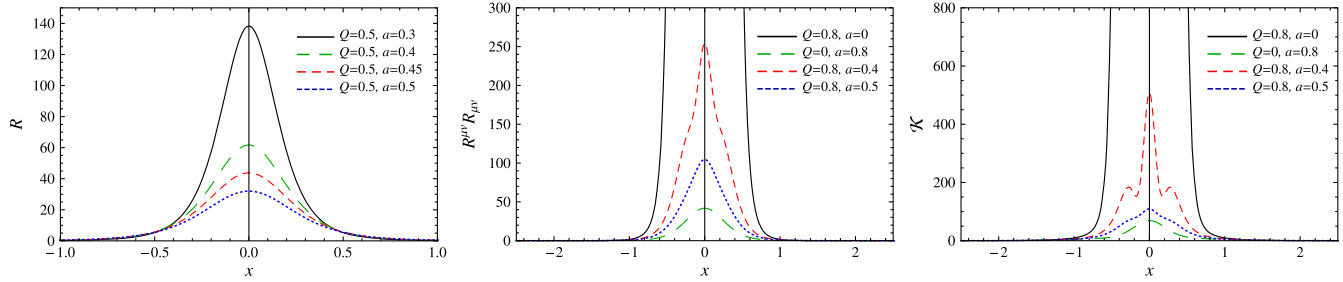


FIG. 3. The Ricci scalar, quadratic Ricci invariant and Kretschmann scalar behavior with different Q and a . It is straightforward to see that the spacetime has no singularity if $a \neq 0$.

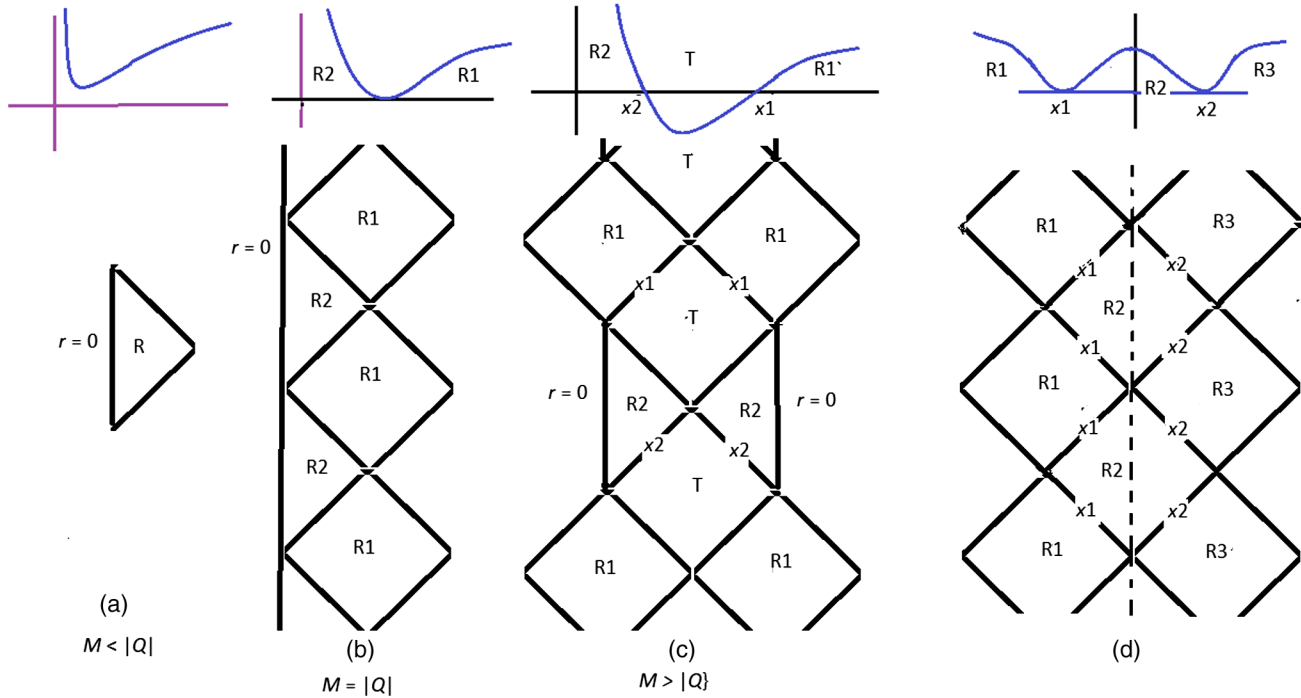


FIG. 4. Carter-Penrose diagrams for the Reissner-Nordström geometries (a),(b),(c), for comparison, and the RN-SV geometry with two extremal horizons (d). The corresponding behavior of $A(x)$ is shown over the diagrams. All tilted boundary lines in the diagrams correspond to $r = \infty$, all tilted internal lines to horizons. The letters R and T mark R and T regions, enumerated where necessary, x_1 and x_2 mark the positions of horizons. The dashed line in (d) shows the throat $x = 0$ in the region R2. The diagrams (b),(c),(d) are infinitely continued upward and downward.

The left-hand side of this equation is, by our assumptions, a known function of x , and since we have $\mathcal{F} = 2q^2/r^4 \equiv 2q^2/(a^2 + x^2)^2$, it is easy to express $d\mathcal{L}/dx = \mathcal{L}_{,\mathcal{F}}d\mathcal{F}/dx$ in terms of x . Integrating, we then obtain \mathcal{F} as a function of x and consequently of \mathcal{F} .

Next, the difference of (5) and (6) is free from a NED contribution and leads to

$$\begin{aligned} \phi^2 &= -\epsilon \frac{a^2}{x^2 + a^2} = \frac{a^2}{x^2 + a^2} \\ \Rightarrow \phi &= \pm \arctan \frac{x}{a} + \phi_0, \quad \phi_0 = \text{const.} \end{aligned} \quad (19)$$

It is evident here that the scalar field should be phantom, $\epsilon = -1$, and without loss of generality we can put $\phi_0 = 0$.

Then, using the already known expressions for \mathcal{L} and ϕ^2 as functions of x , we can use Eq. (6) to calculate the potential V as a function of x and consequently of ϕ :

$$V(x) = -A\phi^2 - \frac{1}{2}\mathcal{L} - G_x^x. \quad (20)$$

This completes the calculation. The scalar field equation (14) is a consequence of the Einstein equations, and it is not necessary to consider it, but it can be used in order to verify the correctness of the results.

Using this algorithm for the function $A(x)$ given by (2) (the RN-SV metric), we obtain

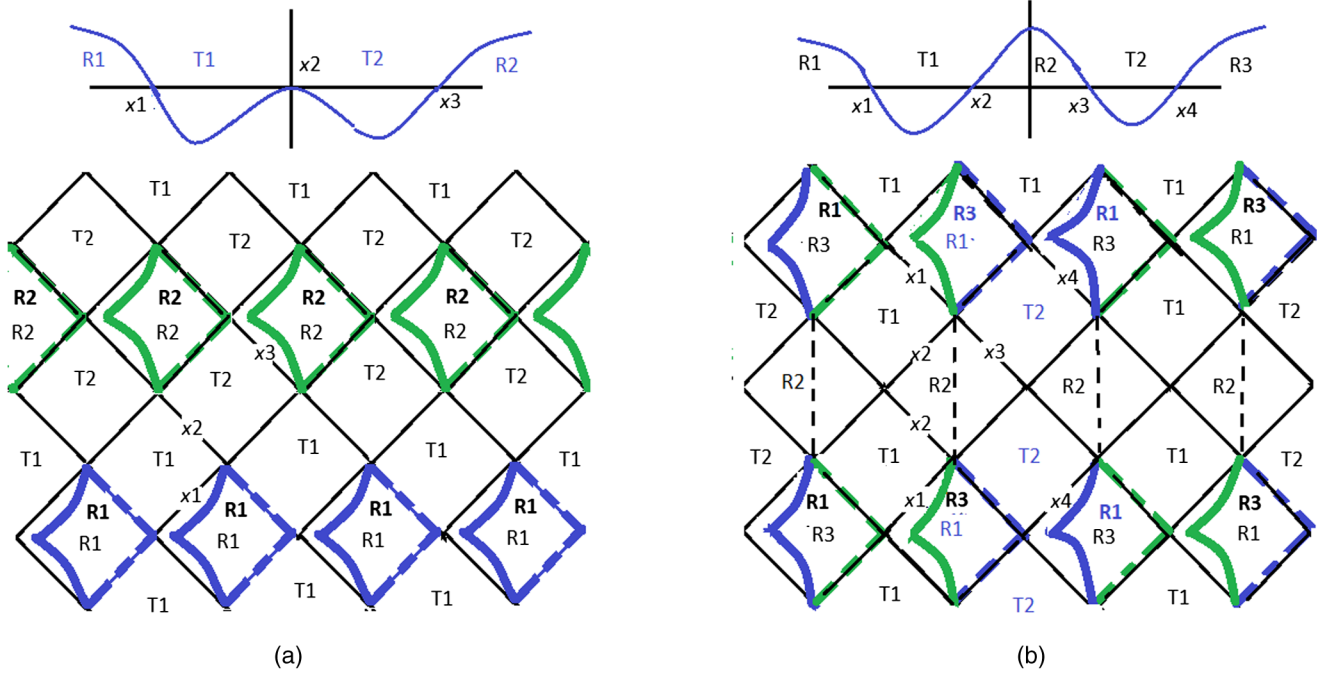


FIG. 5. Carter-Penrose diagrams for the RN-SV geometries with three (a) and four (b) horizons, with the qualitative behavior of $A(x)$ shown at the top. All thick lines in the diagrams correspond to $r = \infty$, all tilted thin lines to horizons. Dashed thick lines mean that the corresponding spatial infinity is placed on the lower, “directly invisible” layer of the diagram, under an overlapping horizon. The letters R and T mark R and T regions with appropriate numbers. Numbers of the horizons (x_1, x_2, \dots) are easily determined according to regions they separate; some of them are shown as examples. The sphere $x = 0$ coincides with the horizon x_2 between the regions T1 and T2 in the left diagram; in the right one, it is a throat in R2, shown by thin dashed lines. Each of these diagrams occupies the whole plane plus a countable set of overlappings.

$$\begin{aligned} \mathcal{L}(\mathcal{F}) &= \frac{12}{5} \frac{Ma^2}{(x^2 + a^2)^{5/2}} + \frac{2Q^2(3x^2 - a^2)}{3(x^2 + a^2)^3} \\ &= \frac{12Ma^2}{5(2q^2/\mathcal{F})^{5/4}} + \frac{2Q^2[3(2q^2/\mathcal{F})^{1/2} - 4a^2]}{3(2q^2/\mathcal{F})^{3/2}}, \end{aligned} \quad (21)$$

$$\begin{aligned} V(\phi) &= \frac{2a^2[6M\sqrt{x^2 + a^2} - 5Q^2]}{15(x^2 + a^2)^3} \\ &= \frac{2\cos^6\phi}{15a^4} (6Ma \sec\phi - 5Q^2). \end{aligned} \quad (22)$$

At $Q = 0$, we obtain the corresponding results for the S-SV spacetime.

We notice a curious feature of the sources for RN-SV geometry. The metric itself contains the charge parameter Q , associated in the original Reissner-Nordström metric with an electric or magnetic charge in Maxwell’s electrodynamics. We have found, however, that our source of the RN-SV solution does not contain any Maxwell field but, instead, is sourced by a magnetic field obeying NED; moreover, according to Eq. (21), for a given value of the parameter Q in the metric, the source also contains a separate free parameter q , the NED magnetic charge, and the corresponding Lagrangian function $\mathcal{L}(\mathcal{F})$ depends on both. In Fig. 6, the behavior of NED field Lagrangian

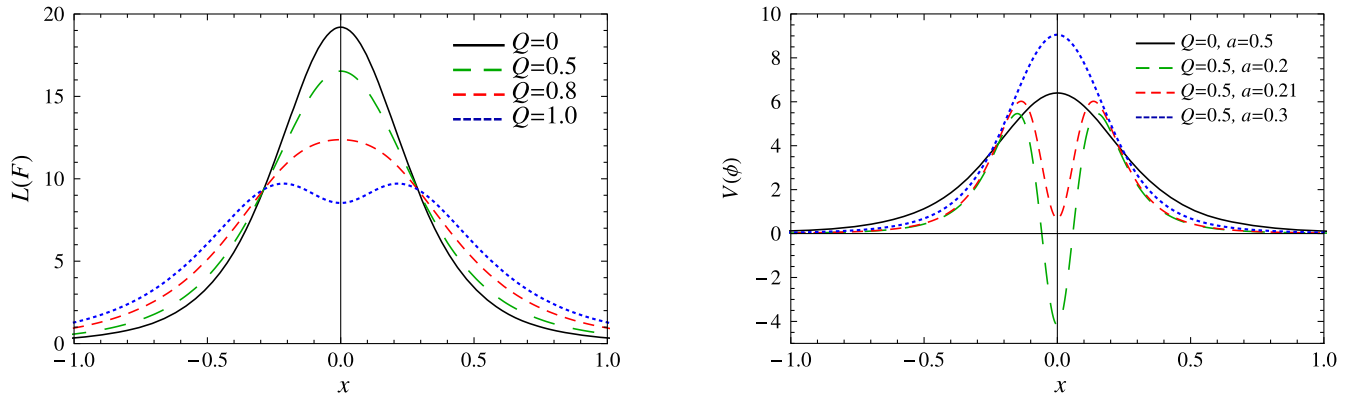


FIG. 6. The NED Lagrangian density $\mathcal{L}(\mathcal{F})$ and the field potential $V(\phi)$ behavior with different Q and a .

density and the scalar field potential is shown with varying Q and a . As expected, $\mathcal{L}(\mathcal{F})$ does not diverge at the point $x = 0$.

A certain shortcoming of this construction is that the function $\mathcal{L}(\mathcal{F})$ in (21) contains fractional powers of \mathcal{F} , which makes meaningless negative values of \mathcal{F} . This, in particular, explains why we use magnetic fields with $\mathcal{F} > 0$ rather than electric ones implying $\mathcal{F} < 0$. An attempt to create a similar construction with electric fields also leads to fractional powers of \mathcal{F} . If we try to employ $|\mathcal{F}|$ instead of \mathcal{F} , the function $\mathcal{L}(\mathcal{F})$ still loses analyticity at $\mathcal{F} = 0$.

IV. CONCLUDING REMARKS

Simpson and Visser [46] have considered a very simple and theoretically appealing spherically symmetric black bounce spacetime (SV spacetime), which modifies the Schwarzschild black hole metric in a precisely controlled and minimal manner and interpolates between regular black holes and traversable wormholes. The new length scale parameter a is responsible for the curvature singularity regularization and can potentially mimic the quantum gravity effects at a small-length scale. Further on, the charged (RN-SV) and rotating SV spacetimes have also been proposed. All of them are globally regular, and in particular, in S-SV and RN-SV regular black holes, their $r = 0$ singularities are replaced by a black bounce that ultimately leads to another asymptotically flat universe. This class of spacetimes generalizes and broadens the well-known variety of regular black holes and seems to be very intriguing owing to its simplicity and a unified treatment of distinct kinds of geometries.

As we saw in the present paper, these spacetimes have in general multiple horizons and complex global causal structures. Depending on the values of a , the $x = 0$ hypersurface can have a different nature: a wormhole throat, a black bounce, or a black throat in the intermediate case. The RN-SV spacetime as a whole describes a variety of geometries including traversable wormholes and different kinds of regular black holes.

With the radius $r(x)$ having a minimum, the corresponding SET, as expected, violates the NEC and other energy conditions. While canonical scalar fields cannot lead to

wormhole or regular black hole geometries [65], it is possible with phantom scalar fields [21,27]. Still we have seen here that SV-like geometries cannot be sourced by a scalar field or NED taken separately. However, we have shown that the RN-SV and S-SV spacetimes are exact solutions of the Einstein field equations with a combination of these two sources. This work is important in the light that it uplifts the status of SV-like space-times from ad-hoc mathematical models to a class of exact solution of gravitational field equations.

Let us also stress that the algorithm we have used makes it possible to find a similar combination of field sources for any static, spherically symmetric SV geometry with the metric (1). These can include, for example, the proper modifications of Reissner-Nordsröm-de Sitter space-times, with $A(x) = 1 - 2M/r + Q^2/r^2 - \Lambda r^2/3$, which will then possess up to six horizons and accordingly very complex causal structures. The sources of black-bounce spacetimes proposed in [48] can also be easily calculated. A task of interest for a future study can be to find a source for SV-like geometries with rotation [49–51,66].

Another issue of importance is the stability of SV spacetimes, which problem cannot be formulated until we know their precise dynamic context. In this regard, it can also be of interest to find possible alternative sources of the same SV spacetimes, because the stability properties of the same geometry may be different in the presence of different sources of this geometry. This problem was, in particular, discussed in [67,68] using as an example the simplest Ellis wormhole [33,34].

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