Covariant 3+1 correspondence of the spatially covariant gravity and the degeneracy conditions

Yu-Min Hu and Xian Gao[*]
School of Physics and Astronomy, Sun Yat-sen University, Guangzhou 510275, China

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A necessary condition for a generally covariant scalar-tensor theory to be ghostfree is that it contains no extra degrees of freedom in the unitary gauge, in which the Lagrangian corresponds to the spatially covariant gravity. Compared with analyzing the scalar-tensor theory directly, it is simpler to map the spatially covariant gravity to the generally covariant scalar-tensor theory using the gauge recovering procedures. In order to ensure the resulting scalar-tensor theory to be ghostfree absolutely, i.e., no matter if the unitary gauge is accessible, a further covariant degeneracy/constraint analysis is required. We develop a method of covariant $3+1$ correspondence, which maps the spatially covariant gravity to the scalar-tensor theory in $3+1$ decomposed form without fixing any coordinates. Then the degeneracy conditions to remove the extra degrees of freedom can be found easily. As an illustration of this approach, we show how the Horndeski theory is recovered from the spatially covariant gravity. This approach can be used to find more general ghostfree scalar-tensor theory.

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I. INTRODUCTION

Scalar-tensor theory is widely studied as one of the alternatives of general relativity (GR), which introduces additional scalar degrees of freedom (d.o.f.) other than the two-tensorial d.o.f. (i.e., the gravitational waves) of GR. In the theoretical aspect, one of the central problems in the development of scalar-tensor theory is to introduce only the healthy d.o.f. while evading the ghostlike (or simply the unwanted) d.o.f. that are associated with the Ostrogradsky instabilities [1,2].

The most straightforward approach is to construct a generally covariant Lagrangian, in which the scalar field (s) is (are) coupled to the spacetime metric covariantly. This is actually what the name "scalar-tensor theory" referred to originally. In the past decade, the successful construction of the higher-derivative single-field scalar-tensor theory with a single scalar d.o.f. has significantly enlarged our scope of the scalar-tensor theory [3–11]. Ghostfree generally covariant scalar-tensor theory with higher derivatives can be constructed by finely tuning the higher derivatives such that the higher derivatives are degenerate (see [12,13] for reviews and [14–18] for general discussions of the degeneracy conditions). Nevertheless, the generally covariant approach becomes more and more involved when going to higher orders, both in the derivatives of the scalar field and in the curvature.

From the point of view of d.o.f., scalar-tensor theory can be understood as any effective gravitational theory that propagates the tensor as well as the scalar d.o.f. In particular, a class of pure metric theories that respect only the spatial diffeomorphism was proposed and shown to have two tensor d.o.f. with an additional scalar d.o.f. [19,20]. In this sense, the ghost condensation [21], the effective theory of inflation [22,23] as well as the Hořava gravity [24,25] can be viewed as subclasses of spatially covariant gravity, which were proposed originally by different motivations. In particular, the degeneracy can be made easily, even trivially, in the spatially covariant gravity description, not only because the Lagrangian is built directly in a spacetime split manner, but also because the Lagrangian gets simplified dramatically when fixing the unitary gauge. In fact, one may try even ambitiously to build theories respecting only the spatial covariance at the level of the Hamiltonian instead of the Lagrangian [26–29].

These two apparently different approaches to scalar-tensor theory are related by "gauge fixing/recovering" procedures. If the gradient of the scalar field is timelike, we may fix the time coordinate as the scalar field $t=\phi$, such that the resulting theory appears to be a theory of spatially covariant gravity. Conversely, starting from a spatially covariant gravity, we may derive the corresponding generally covariant Lagrangian of the scalar field and spacetime metric by the so-called Stueckelberg trick.[1] A natural idea is thus; we first build the ghostfree spatially covariant gravity and then map it

[*]Corresponding author.
gaoxian@mail.sysu.edu.cn

[1]This is also to perform a broken time diffeomorphism.

to the generally covariant scalar-tensor theory, which yields a scalar-tensor theory that appears to be ghostfree at least in the unitary gauge. Based on this idea, both the generally covariant and spatially covariant monomials have been classified and their correspondence has been investigated in [30–32].

There are at least two subtleties in this correspondence. Firstly, the reversibility of this gauge fixing/recovering procedures relies on the assumption of a timelike scalar field. Secondly, even we assume that the scalar field is timelike, the generally covariant scalar-tensor theory got from the spatially covariant gravity appears arguably to have extra unwanted d.o.f. in coordinates that are not adapted to the unitary gauge [33,34].[2] This has also been reported in the study of mimetic gravity with couplings between the curvature and higher derivatives of the scalar field [38,39], which appears to propagate a lower number of d.o.f. in the unitary gauge. In order to construct a scalar-tensor theory that is ghostfree "absolutely", i.e., no matter whether the scalar field is timelike or not and in any coordinates, one needs to perform a further degeneracy or constraint analysis. Usually this is done by making a $3+1$ decomposition and performing the constraint analysis in the Hamiltonian formalism.

Compared to finding the degeneracy conditions for the most general scalar-tensor theory directly (e.g., the approach taken in [9–11]), starting from the spatially covariant gravity has already saved a lot of work. However, one still needs two steps; first finding the generally covariant scalar-tensor theory that corresponds to the ghostfree spatially covariant gravity, and then making a degeneracy analysis which needs a further covariant $3+1$ decomposition. One may wonder if we can derive the covariant $3+1$ correspondence of the spatially covariant gravity directly. This paper is devoted to this issue.

Generally, there are three apparently different formulations of the scalar-tensor theory. One is the generally covariant scalar-tensor theory (of which the Lagrangian is built) of the scalar field coupled to the metric through generally covariant derivatives. The second is the spatially covariant gravity, which corresponds to the generally covariant scalar-tensor theory in the coordinates adapted to the unitary gauge. The last one is the generally covariant $3+1$ decomposition of the scalar-tensor theory, which is convenient to use for the covariant degeneracy/constraint analysis. In this work, we shall develop a formalism, which we dub the "covariant $3+1$ correspondence", that can be used to derive the explicit generally covariant $3+1$ expressions from the spatially covariant gravity.

This work is organized as follows. In Sec. II we describe the three formulations of the scalar-tensor theory and their correspondences. In Sec. III we derive the explicit expressions of the covariant $3+1$ correspondence. We apply this correspondence in Sec. IV, in which we derive the covariant $3+1$ correspondence of the spatially covariant gravity of $d=2$ with d the total number of derivatives in spatially covariant gravity formulation. By canceling all the dangerous terms, we determine the degeneracy conditions easily. In Sec. V and Sec. VI, we further apply this method to spatially covariant gravity of $d=3$ without and with the acceleration, respectively. Not surprisingly, we can recover the whole Lagrangian of the Horndeski theory easily by this method. We summarize our results in Sec. VII.

[2]Such an extra mode is dubbed "instantaneous" or "shadowy" mode since it propagates with an infinite speed. See also [35–37] for early discussions.

II. THREE FACES OF THE SCALAR-TENSOR THEORY

A. Generally covariant formulations

The generally covariant scalar-tensor theory (GST) usually refers to the theory of scalar field(s) coupled to the spacetime metric. In the present work, we concentrate on the case of a single scale field. The action takes the general form

$$S_{\mathrm{GST}}=\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(\phi;g_{ab},\varepsilon_{abcd},{}^4R_{abcd};\nabla_a), \quad (1)$$

in which the Lagrangian is built of the scalar field ϕ, the spacetime metric g_{ab}, the spacetime curvature tensor ${}^4R_{abcd}$, as well as their covariant derivatives. The possible parity violation is encoded in the 4-dimension Levi-Civita tensor ε_{abcd}. It is the scalar-tensor theory in the form of (1), in which the general covariance is manifest, that is the subject in [3–11] and is also used in practical model building of cosmology and black holes, etc.

For the purpose of degeneracy/constraint analysis, splitting the 4-dimensional objects into their temporal and spatial parts, i.e., the so-called $3+1$ decomposition, is needed. The starting point of the $3+1$ decomposition is a timelike vector field n_a with normalization $n_a n^a=-1$. As usual, this timelike vector field is assumed to be hypersurface orthogonal, and thus the induced metric which projects any tensor field on the spatial hypersurface is

$$h_{ab}\equiv g_{ab}+n_a n_b. \quad (2)$$

All the 4-dimensional quantities are then split into parts that are orthogonal and tangent to the spatial hypersurface by projecting with n^a and h_{ab}, respectively. The decomposition of the 4-dimensional curvature tensor yields the Gauss-Codazzi-Ricci equations. For the scalar field, we have

$$\nabla_a\phi=-n_a\pounds_{\boldsymbol{n}}\phi+\mathrm{D}_a\phi, \quad (3)$$

where $\pounds_{\boldsymbol{n}}$ stands for the Lie derivative with respect to n^a, and D_a is the projected derivative defined by

$$\mathrm{D}_a\phi \coloneqq h_a^{a'}\nabla_{a'}\phi, \tag{4}$$

which is also the covariant derivative compatible with h_{ab}. The decompositions of the second and the third order derivatives of the scalar field with respect to a general normal vector n^a can be found in [31].

With these settings, we can derive the covariant 3 + 1 decomposition (COD) of any 4-dimensional quantities. The GST action (1) can be recast in the form

$$S_{\mathrm{COD}} = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(\phi; n_a, h_{ab}, \varepsilon_{abcd}, {}^3R_{ab}; \mathrm{D}_a, \pounds_{\boldsymbol{n}}). \tag{5}$$

We emphasize that the action (5) is generally covariant since n_a is an arbitrary hypersurface orthogonal unit timelike vector field, and we have not yet chosen any specific coordinates. In particular, the familiar lapse function N and shift vector N^a do not appear in the Lagrangian.[3] In Eq. (5), ${}^3R_{ab}$ is the intrinsic curvature of the hypersurfaces. The projected derivative D_a and the Lie derivative $\pounds_{\boldsymbol{n}}$ can be viewed as the "intrinsic" and "extrinsic" derivatives, respectively. The Lie derivatives of n^a and h_{ab}

$$a_a = \pounds_{\boldsymbol{n}} n_a, \tag{6}$$

$$K_{ab} = \frac{1}{2}\pounds_{\boldsymbol{n}} h_{ab}, \tag{7}$$

define the acceleration and the extrinsic curvature as usual.

B. Spatially covariant formulation

In Eq. (5) n_a is an arbitrary unit timelike vector field that is hypersurface orthogonal. While the scalar field ϕ itself specifies a foliation of hypersurfaces with $\phi = \mathrm{const}$. In particular, when the gradient of the scalar field is also timelike, we are allowed to choose $n_a = u_a$, where

$$u_a \equiv -\frac{1}{\sqrt{2X}}\nabla_a\phi, \tag{8}$$

with the canonical kinetic term of the scalar field $X = -\frac{1}{2}\nabla_a\phi\nabla^a\phi$. u_a is nothing but the normal vector of the hypersurfaces with constant ϕ, which satisfies the normalization $u_a u^a = -1$. Choosing $n_a = u_a$ corresponds to the so-called unitary gauge in the literature.[4]

In the unitary gauge, i.e., when being decomposed with respect to the foliation specified by the scalar field ϕ itself, the decompositions of the derivatives of the scalar field get dramatically simplified. All the spatial derivatives of the scalar field drop out since

$$\overset{\mathrm{u}}{\mathrm{D}}_a\phi \equiv \overset{\mathrm{u}}{h}_a{}^{a'}\nabla_{a'}\phi = 0, \tag{9}$$

where $\overset{\mathrm{u}}{h}_{ab}$ is defined by

$$\overset{\mathrm{u}}{h}_{ab} \equiv g_{ab} + u_a u_b. \tag{10}$$

Here and throughout this paper, an overscript "u" denotes quantities defined with respect to u_a [31], which is related to the scalar field through (8). The first-order derivative of the scalar field (3) is thus written as $\nabla_a\phi = -u_a/N$, where we introduce

$$\frac{1}{N} = \sqrt{2X} = \pounds_{\boldsymbol{u}}\phi. \tag{11}$$

In Eq. (11) N is nothing but the lapse function, which arises since we have identified the "space" to be the hypersurfaces of constant ϕ. The decompositions of the second- and third-order derivatives of the scalar field in the unitary gauge can be found in [19,20,31]. Replacing n_a by u_a in Eq. (5) yields

$$S_{\mathrm{u.g.}} = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(\phi, u_a, \overset{\mathrm{u}}{h}_{ab}, \varepsilon_{abcd}, {}^3\overset{\mathrm{u}}{R}_{ab}; \overset{\mathrm{u}}{\mathrm{D}}_a, \pounds_{\boldsymbol{u}}). \tag{12}$$

At this point, all the ingredients are generally covariant. As a result, the unitary gauge Eq. (12) is generally covariant.

In the unitary gauge, since n_a is chosen to be u_a, the coordinates that are adapted to the foliation, i.e., the Arnowitt-Deser-Misner (ADM) coordinates, correspond to fixing $t = \phi$ (while spatial coordinates are left free). In these particular coordinates, we have $u_a = -N\delta_a^0$ and the time direction $t^a = \delta_0^a$. The unitary gauge action (12) is recast to

$$S_{\mathrm{SCG}} = \int \mathrm{d}t\mathrm{d}^3x\, N\sqrt{h}\mathcal{L}(t, N, h_{ij}, \varepsilon_{ijk}, {}^3R_{ij}; \nabla_i, \pounds_{\boldsymbol{u}}), \tag{13}$$

where $\pounds_{\boldsymbol{u}}$ is now understood to be $\frac{1}{N}(\partial_t - \pounds_{\vec{N}})$ with $\vec{N}$ the spatial component of $t^a - Nu^a = (0, N^i)$. Since the time coordinate t is fixed to be the value of ϕ, the general covariance is broken to the spatial diffeomorphism. Equation (13) appears to be a pure metric theory respecting spatial covariance, which we dub the spatially covariant gravity (SCG). The effective theory of inflation [22,23], the Hořava gravity [24,25] as well as the more general framework proposed in [19,20] can be viewed as subclasses of the general action of the SCG (13).

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[3] They merely encode the gauge freedom of choosing the time and space directions, i.e., fixing the coordinates.

[4] Usually the "unitary gauge" is referred to fixing the time coordinate $t = \phi$ in the literature. In this work, for the purpose of distinguishing the generally covariant and spatially covariant formulations, we use "unitary gauge" to denote choosing $n_a = u_a$. In particular, no specific coordinates have been fixed.

C. Theory triangle: Relations among different formulations

We have now three apparently different formulations of the theory. From the point of view of keeping the general covariance manifestly and/or of making the spacetime decomposition explicitly, different formulations have their own merits.

(a) The generally covariant scalar-tensor theory (1):

The general covariance is manifest in the action of GST, which is also convenient for model buildings in the cosmology and black hole physics. However, more calculations are needed to derive its spacetime decomposition in order to make the degeneracy/constraint analysis.

(b) The spatially covariant gravity (13):

The SCG is written in the already spacetime-decomposed manner, which is convenient for controlling the number of d.o.f. through a strict degeneracy/constraint analysis. In particular, comparing with the GST, the degenerate SCG Lagrangian with the desired number of d.o.f. can be constructed much easier. For example, the SCG [19,20] contains only the extrinsic curvature as the kinetic terms and thus is trivially degenerate. SCG with a dynamical lapse function has also been investigated in [40–42] (see also [43]). However, the general covariance is explicitly broken in SCG.

(c) The covariant 3 + 1 decomposition (5):

The COD Lagrangian can be viewed as the balance between GST and SCG. It is written in the spacetime-decomposed form and thus is convenient to perform the constraint analysis. On the other hand, it is generally covariant and has the exact equivalence to the GST. In other words, the Lagrangians of COD and GST are exactly the same, but merely written in different forms.

The relations among the three formulations are depicted in Fig. 1. Starting from the GST, we get the COD by performing a covariant 3 + 1 decomposition. Then we arrive at the SCG by choosing the unitary gauge and fixing the time coordinate. With this approach, the Lagrangian of the Horndeski theory in the unitary gauge was derived in [44]. Similar analysis was performed to get a geometric reformulation of the quadratic degenerate higher-order scalar-tensor theory [45]. For our purpose to use the SCG to generate GST theories, the inverse procedures of the 3 + 1 decomposition and the gauge fixing are required. To this end, we must determine the GST quantities that correspond to the SCG quantities. This procedure has been used in the covariant formulation of the Hořava gravity [36,46–48] (see also [49,50]), and is sometimes dubbed the Stueckelberg trick.

Since the SCG quantities are simply the unitary gauge quantities after fixing the time coordinate $t = \phi$, while the later are the GST quantities after choosing the unitary gauge $n_a = u_a$, the one-to-one correspondence between a SCG expression and a GST expression can be easily set up. For example, (8) and (11) can be viewed as the GST correspondences of u_a and N, respectively. The extrinsic curvature corresponds to

$$K_{ij} \to \overset{\mathrm{u}}{K}_{ab} = -\frac{1}{\sqrt{2X}} \overset{\mathrm{u}}{h}_{aa'} \overset{\mathrm{u}}{h}_{bb'} \nabla^{a'} \nabla^{b'} \phi, \tag{14}$$

where $\overset{\mathrm{u}}{h}_{ab}$ is defined in (10), which now should be understood as

$$\overset{\mathrm{u}}{h}_{ab} = g_{ab} + \frac{1}{2X} \nabla_a \phi \nabla_b \phi. \tag{15}$$

By plugging (15) in (14), we get the GST correspondence of K_{ij}. We refer to [31] for the more complete and detailed correspondences between the GST and SCG expressions.

As we have argued before, since the degenerate SCG Lagrangian can be constructed much easier than the GST, one may use the degenerate SCG as the "seed theory", and map it to the space of GST theories using the above correspondence. The resulting theory is the GST theory that is ghostfree, or propagates the correct number of d.o.f.,

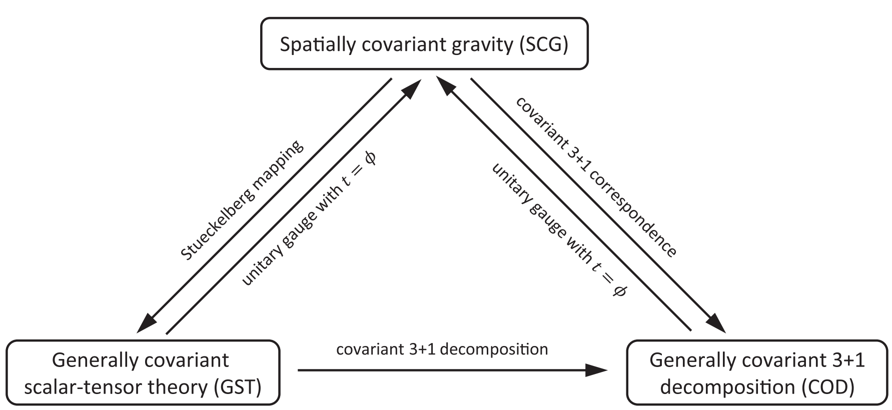

FIG. 1. Theory triangle: Three faces of the scalar-tensor theory.

when the unitary gauge is accessible.[5] In fact, this has already been performed for the GST and SCG polynomials [32] from the linear algebraic point of view.

When the unitary gauge is not accessible, or at least when we do not fix the time coordinate to be the scalar field, apparently there arise extra d.o.f. which might be ghostlike. Our final purpose is to obtain the GST theory that is ghostfree "absolutely", which has the correct number of d.o.f. in the generally covariant sense no matter whether the scalar field is timelike so that the unitary gauge is accessible or not, and shows no extra d.o.f. in arbitrary coordinates. To this end, a further covariant $3+1$ decomposition is inevitable, which results in the COD formulation of the GST. This "two-step" approach, i.e., SCG → GST → COD, is correct and straightforward, is technically involved since both steps involve complicated correspondences among expressions in different formulations.

The main purpose of this work is to find a "one-step" approach, i.e., a method to derived the COD expressions from the SCG expressions directly, which we dub the covariant $3+1$ correspondence and shall explain in the next section.

III. COVARIANT 3+1 CORRESPONDENCE

The covariant $3+1$ correspondence is conceptually simple, which combines the above two steps together, but without expanding the intermediate GST in terms of the scalar field and 4-dimensional geometric quantities explicitly.

Firstly, we covariantize the SCG expressions by determining the corresponding unitary gauge expressions. For example, the spatial metric h_{ij}, although appears to be 3-dimension tensor, is actually the spatial component of a 4-dimension tensor

$$h_{ij} \to \overset{u}{h}_{ab} = g_{ab} + u_a u_b, \tag{16}$$

where u_a is nothing but the normalized gradient of the scalar field (8). Secondly, instead of recasting the unitary gauge expressions in terms of the scalar field and 4-dimension geometric quantities explicitly [e.g., (15)], we make a further $3+1$ decomposition with respect to a general spacelike foliation with normal vector n_a. For u_a, we write

$$u_a = -n_a \alpha + \beta_a, \tag{17}$$

and require that $n^a \beta_a \equiv 0$. Since both u_a and n_a are normalized (with sign -1), α and β_a are not independent, which satisfy

$$\alpha = -\sqrt{1+\beta^2}, \tag{18}$$

where $\beta^2 \equiv \beta_a \beta^a$. Since u_a is given in (8), α and β_a are related to the derivatives of the scalar field by

$$\alpha = -\frac{\pounds_{\boldsymbol{n}} \phi}{\sqrt{2X}}, \tag{19}$$

$$\beta_a = -\frac{\mathrm{D}_a \phi}{\sqrt{2X}}, \tag{20}$$

where the canonical kinetic term X is now decomposed to be

$$X = \frac{1}{2}(\pounds_{\boldsymbol{n}} \phi)^2 - \frac{1}{2}\mathrm{D}_a \phi \mathrm{D}^a \phi. \tag{21}$$

Throughout this paper, quantities without any overscript are defined with respect to a general normal vector field n_a. Therefore (17) becomes

$$u_a = n_a \sqrt{1+\beta^2} + \beta_a. \tag{22}$$

Equation (22) is the starting point of the following analysis, which is nothing but the covariant $3+1$ decomposition of the normalized gradient of the scalar field without fixing any coordinates. One can see from Eq. (22) that β_a encodes the deviation of the general foliation from the foliation specified by the scalar field. Therefore, the unitary gauge is simply defined to be

$$\text{unitary gauge:}\ \beta_a \to 0, \tag{23}$$

which implies $n_a \to u_a$ as expected.

The covariant $3+1$ correspondence of the spatial metric is[6]

$$\overset{u}{h}_{ab} = n_a n_b \overset{u}{h}_{\boldsymbol{nn}} - 2 n_{(a} \overset{u}{h}_{\hat{b})\boldsymbol{n}} + \overset{u}{h}_{\hat{a}\hat{b}}, \tag{24}$$

where

$$\overset{u}{h}_{\boldsymbol{nn}} = \beta^2, \tag{25}$$

$$\overset{u}{h}_{\hat{a}\boldsymbol{n}} = \alpha \beta_a, \tag{26}$$

$$\overset{u}{h}_{\hat{a}\hat{b}} = h_{ab} + \beta_a \beta_b. \tag{27}$$

Here h_{ab} is the induced metric associated with n_a, i.e., $h_{ab} \equiv g_{ab} + n_a n_b$. Here and in what follows, we use the notation in [51] that for a general spacetime tensor, an index

[5]Scalar-tensor theory with this property is also referred to be "U-degenerate", i.e., being degenerate in the unitary gauge [33].

[6]Throughout this paper, symmetrization is normalized, e.g., $A_{(a} B_{b)} \equiv \frac{1}{2}(A_a B_b + A_b B_a)$.

replaced by $\boldsymbol{n}$ denotes contraction with n_a, and indices with a hat denote projection with h_{ab}, i.e.,

$$T_{\cdots \boldsymbol{n} \cdots} = n^a T_{\cdots a \cdots}, \qquad T_{\cdots \hat{a} \cdots} = h_a{}^{a'} T_{\cdots a' \cdots}. \tag{28}$$

From (24) it is clear that the difference of $\overset{u}{h}_{ab}$ and h_{ab} is completely encoded in the nonvanishing β_a. Therefore $\overset{u}{h}_{ab}$ reduces to h_{ab} in the unitary gauge.

In the following, we derive the explicit expressions in the covariant $3+1$ correspondence. The fundamental objects are the covariant derivatives of u_a. For the first order derivative of u_a, we have

$$\nabla_a u_b = n_a n_b A - n_a B_b - \tilde{B}_a n_b + \Delta_{ab}, \tag{29}$$

with

$$A \equiv \nabla_{\boldsymbol{n}} u_{\boldsymbol{n}} = \dot{\alpha} - a^c \beta_c, \tag{30}$$

$$B_b \equiv \nabla_{\boldsymbol{n}} u_{\hat{b}} = -a_b \alpha + \dot{\beta}_b - K^c_b \beta_c, \tag{31}$$

$$\tilde{B}_a \equiv \nabla_{\hat{a}} u_{\boldsymbol{n}} = \mathrm{D}_a \alpha - K^c_a \beta_c, \tag{32}$$

$$\Delta_{ab} \equiv \nabla_{\hat{a}} u_{\hat{b}} = -K_{ab}\alpha + \mathrm{D}_a \beta_b. \tag{33}$$

Throughout this work, overdots on the spatial tensors with lower indices denote Lie derivatives with respect to the general normal vector n^a, e.g., $\dot{\alpha} = \pounds_{\boldsymbol{n}} \alpha$, $\dot{\beta}_a \equiv \pounds_{\boldsymbol{n}} \beta_a$, $\ddot{\beta}_a \equiv \pounds^2_{\boldsymbol{n}} \beta_a$, etc. Occasionally we also use dotted spatial tensors with upper indices for shorthand, in which the upper indices are raised by the inverse induced metric h^{ab}, e.g., $\dot{\beta}^a \equiv h^{ab}\dot{\beta}_b$, $\dot{K}^{ab} \equiv h^{aa'} h^{bb'} \dot{K}_{a'b'}$, etc.[7] Evaluating the Lie derivative of (20) explicitly yields

$$\dot{\beta}_a = -\frac{1}{2X}\beta_a \dot{X} - \frac{1}{\sqrt{2X}}(\mathrm{D}_a \dot{\phi} + a_a \dot{\phi}), \tag{34}$$

where

$$\dot{X} = \ddot{\phi}\dot{\phi} - \mathrm{D}^a\phi(\mathrm{D}_a\dot{\phi} + a_a\dot{\phi}) + K^{ab}\mathrm{D}_a\phi \mathrm{D}_b\phi. \tag{35}$$

From (35) it is transparent that $\dot{\beta}_a$ contains the second order Lie derivative of the scalar field $\ddot{\phi}$ through $\dot{X}$, which should be degenerate (with the extrinsic curvature) in order not to excite the unwanted d.o.f..

When considering the third-order derivative of the scalar field, the second-order derivative of u_a will arise. We have

$$\nabla_c \nabla_a u_b = -n_c n_a n_b U + n_c n_a V_b + n_c n_b \tilde{V}_a + n_a n_b W_c - n_c X_{ab} - n_a Y_{cb} - n_b \tilde{Y}_{ca} + Z_{cab}, \tag{36}$$

with

$$U = \dot{A} - a^d B_d - a^d \tilde{B}_d, \tag{37}$$

$$V_b = -a_b A + \dot{B}_b - B_d K^d_b - \Delta_{db} a^d, \tag{38}$$

$$\tilde{V}_a = -a_a A + \dot{\tilde{B}}_a - \tilde{B}_d K^d_a - \Delta_{ad} a^d, \tag{39}$$

$$W_c = \mathrm{D}_c A - K^d_c B_d - K^d_c \tilde{B}_d, \tag{40}$$

$$X_{ab} = -a_a B_b - \tilde{B}_a a_b + \dot{\Delta}_{ab} - \Delta_{ad} K^d_b - \Delta_{db} K^d_a, \tag{41}$$

$$Y_{cb} = -K_{cb} A + \mathrm{D}_c B_b - K^d_c \Delta_{db}, \tag{42}$$

$$\tilde{Y}_{ca} = -K_{ca} A + \mathrm{D}_c \tilde{B}_a - K^d_c \Delta_{ad}, \tag{43}$$

$$Z_{cab} = -K_{ca} B_b - \tilde{B}_a K_{cb} + \mathrm{D}_c \Delta_{ab}, \tag{44}$$

where $A, B_b, \tilde{B}_a, \Delta_{ab}$ are given in (30)–(33). For later convenience, we also evaluate the Lie derivatives of $A, B_b, \tilde{B}_a, \Delta_{ab}$ explicitly, which are given by

$$\dot{A} = \ddot{\alpha} - \beta^b \dot{a}_b - a^b \dot{\beta}_b + 2K^{ab} a_b \beta_a, \tag{45}$$

$$\dot{B}_b = -\alpha \dot{a}_b - a_b \dot{\alpha} + \ddot{\beta}_b - \beta^c \dot{K}_{bc} - K^d_b \dot{\beta}_d + 2K_{bc} K^{cd} \beta_d, \tag{46}$$

$$\dot{\tilde{B}}_a = \mathrm{D}_a \dot{\alpha} + a_a \dot{\alpha} - \beta^c \dot{K}_{ac} - K^d_a \dot{\beta}_d + 2K_{ac} K^{cd} \beta_d, \tag{47}$$

and

$$\begin{aligned} \dot{\Delta}_{ab} = &-\alpha \dot{K}_{ab} - K_{ab}\dot{\alpha} + \mathrm{D}_a \dot{\beta}_b + a_a \dot{\beta}_b \\ &- (a_a K_{bd} + a_b K_{da} - a_d K_{ab})\beta^d \\ &- (\mathrm{D}_a K_{bd} + \mathrm{D}_b K_{da} - \mathrm{D}_d K_{ba})\beta^d. \end{aligned} \tag{48}$$

We are ready to use (29) and (36) to derive the covariant $3+1$ correspondences of various geometric quantities. For the extrinsic curvature, it is convenient to use the expression

$$\overset{u}{K}_{ab} = \overset{u}{h}_a{}^{a'} \overset{u}{h}_b{}^{b'} \nabla_{(a'} u_{b')}. \tag{49}$$

It immediately follows that

$$\overset{u}{K}_{ab} = n_a n_b \overset{u}{K}_{\boldsymbol{nn}} - 2n_{(a} \overset{u}{K}_{\hat{b})\boldsymbol{n}} + \overset{u}{K}_{\hat{a}\hat{b}}, \tag{50}$$

where

$$\overset{u}{K}_{\boldsymbol{nn}} = -\beta^2 \frac{1}{\alpha}\beta^c \dot{\beta}_c - \frac{1}{\alpha} K^{cd}\beta_c \beta_d + \beta^2 a^c \beta_c + \beta^c \beta^d \mathrm{D}_{(c}\beta_{d)}, \tag{51}$$

[7]Therefore $\dot{\beta}^a \equiv h^{aa'} \pounds_{\boldsymbol{n}} \beta_{a'} \neq \pounds_{\boldsymbol{n}} \beta^a$.

$$\overset{u}{K}_{\hat{a}\boldsymbol{n}} = \frac{1}{2}\beta_a\left(-\beta^c\dot{\beta}_c + \alpha\beta^c a_c + \frac{1}{\alpha}\beta^c\beta^d \mathrm{D}_{(c}\beta_{d)}\right) - \frac{1}{2}\beta^2\dot{\beta}_a + \frac{1}{2}\alpha\beta^2 a_a - K_{ad}\beta^d + \frac{1}{2\alpha}\beta^d \mathrm{D}_a\beta_d + \frac{1}{2}\alpha\beta^d \mathrm{D}_d\beta_a, \tag{52}$$

and

$$\overset{u}{K}_{\hat{a}\hat{b}} = -K_{ab}\alpha + \mathrm{D}_{(a}\beta_{b)} - \frac{1}{2}\beta_a(\alpha\dot{\beta}_b - a_b\alpha^2 - \beta^c \mathrm{D}_c\beta_b) - \frac{1}{2}\beta_b(\alpha\dot{\beta}_a - a_a\alpha^2 - \beta^c \mathrm{D}_c\beta_a). \tag{53}$$

For the acceleration, we shall use the expression

$$\overset{u}{a}_a \equiv u^b\nabla_b u_a. \tag{54}$$

It follows that

$$\overset{u}{a}_a = -n_a\overset{u}{a}_{\boldsymbol{n}} + \overset{u}{a}_{\hat{a}}, \tag{55}$$

where

$$\overset{u}{a}_{\boldsymbol{n}} = -\beta^c\dot{\beta}_c + \alpha a^c\beta_c + \frac{1}{\alpha}\beta^b\beta^c \mathrm{D}_b\beta_c, \tag{56}$$

and

$$\overset{u}{a}_{\hat{a}} = -\alpha\dot{\beta}_a + a_a\alpha^2 + \beta^b \mathrm{D}_b\beta_a. \tag{57}$$

For the spatial Ricci tensor we make use of

$${}^3\overset{u}{R}_{ab} = \overset{u}{h}_a{}^{a'}\overset{u}{h}_b{}^{b'}\overset{u}{h}{}^{cd}\overset{u}{\mathcal{R}}_{a'cb'd}, \tag{58}$$

where $\overset{u}{\mathcal{R}}_{acbd}$ is defined to be

$$\overset{u}{\mathcal{R}}_{acbd} = {}^4R_{acbd} - \nabla_{(a}u_{b)}\nabla_{(c}u_{d)} + \nabla_{(a}u_{d)}\nabla_{(c}u_{b)}. \tag{59}$$

Note $\overset{u}{\mathcal{R}}_{acbd}$ has exactly the same (anti)symmetries of the spacetime Riemann tensor. Therefore there are three independent projections with n_a and h_{ab}. By using the Gauss-Codazzi-Ricci equations of the Riemann tensor and (29), we find

$$\overset{u}{\mathcal{R}}_{\hat{c}\boldsymbol{n}\hat{d}\boldsymbol{n}} = -\dot{K}_{cd} + K_{ce}K^e_d + a_c a_d + \mathrm{D}_c a_d - (-K_{cd}\alpha + \mathrm{D}_{(c}\beta_{d)})(\dot{\alpha} - a^e\beta_e) + \frac{1}{4}(\dot{\beta}_c - a_c\alpha - 2K^e_c\beta_e + \mathrm{D}_c\alpha) \times (\dot{\beta}_d - a_d\alpha - 2K^f_d\beta_f + \mathrm{D}_d\alpha), \tag{60}$$

and

$$\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{d}\boldsymbol{n}} = \mathrm{D}_{a'}K_{cd} - \mathrm{D}_c K_{a'd} - \frac{1}{2}(-K_{a'd}\alpha + \mathrm{D}_{(a'}\beta_{d)})(\dot{\beta}_c - a_c\alpha - 2K^e_c\beta_e + \mathrm{D}_c\alpha) + \frac{1}{2}(-K_{cd}\alpha + \mathrm{D}_{(c}\beta_{d)})(\dot{\beta}_{a'} - a_{a'}\alpha - 2K^e_{a'}\beta_e + \mathrm{D}_{a'}\alpha), \tag{61}$$

and

$$\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{b}'\hat{d}} = {}^3R_{a'cb'd} + (K_{a'b'}K_{dc} - K_{a'd}K_{b'c}) - (-K_{a'b'}\alpha + \mathrm{D}_{(a'}\beta_{b')})(-K_{cd}\alpha + \mathrm{D}_{(c}\beta_{d)}) + (-K_{a'd}\alpha + \mathrm{D}_{(a'}\beta_{d)})(-K_{cb'}\alpha + \mathrm{D}_{(c}\beta_{b')}). \tag{62}$$

Plugging (59) together with the above projections in (58), after long and tedious manipulations, we find

$${}^3\overset{u}{R}_{ab} = n_a n_b\,{}^3\overset{u}{R}_{\boldsymbol{nn}} - 2n_{(a}\,{}^3\overset{u}{R}_{\hat{b})\boldsymbol{n}} + {}^3\overset{u}{R}_{\hat{a}\hat{b}}, \tag{63}$$

where

$${}^3\overset{u}{R}_{\boldsymbol{nn}} = \beta^2(\beta^2h^{cd} - \beta^c\beta^d)\overset{u}{\mathcal{R}}_{\hat{c}\boldsymbol{n}\hat{d}\boldsymbol{n}} + 2\beta^2\alpha\beta^{a'}h^{cd}\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{d}\boldsymbol{n}} + \alpha^2\beta^{a'}\beta^{b'}h^{cd}\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{b}'\hat{d}}, \tag{64}$$

and

$${}^3\overset{u}{R}_{\hat{b}\boldsymbol{n}} = \beta_b\alpha(\beta^2h^{cd} - \beta^c\beta^d)\overset{u}{\mathcal{R}}_{\hat{c}\boldsymbol{n}\hat{d}\boldsymbol{n}} + [h_b{}^{a'}(\beta^2h^{cd} - \beta^c\beta^d) + (1+2\beta^2)\beta_b\beta^{a'}h^{cd}]\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{d}\boldsymbol{n}} + (h_b{}^{b'} + \beta_b\beta^{b'})\alpha\beta^{a'}h^{cd}\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{b}'\hat{d}}, \tag{65}$$

and

$$
\begin{aligned}
{}^3\overset{u}{R}_{\hat{a}\hat{b}} = {} & [\beta_a\beta_b(\alpha^2 h^{cd} - \beta^c\beta^d) + \beta^2 h_a{}^c h_b{}^d - h_a{}^c\beta_b\beta^d - h_b{}^d\beta_a\beta^c]\overset{u}{\mathcal{R}}_{\hat{c}\boldsymbol{n}\hat{d}\boldsymbol{n}} \\
& - \alpha[h_a{}^{a'}h_b{}^d\beta^c + h_b{}^{a'}h_a{}^d\beta^c - (h_a{}^{a'}\beta_b + h_b{}^{a'}\beta_a)h^{cd} - 2\beta_a\beta_b\beta^{a'}h^{cd}]\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{d}\boldsymbol{n}} \\
& + [h_a{}^{a'}h_b{}^{b'}(h^{cd} + \beta^c\beta^d) + (h_a{}^{a'}\beta_b + h_b{}^{a'}\beta_a)\beta^{b'}h^{cd} + \beta_a\beta_b\beta^{a'}\beta^{b'}h^{cd}]\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{b}'\hat{d}}, \qquad (66)
\end{aligned}
$$

where $\overset{u}{\mathcal{R}}_{\hat{c}\boldsymbol{n}\hat{d}\boldsymbol{n}}$, $\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{d}\boldsymbol{n}}$, and $\overset{u}{\mathcal{R}}_{\hat{a}'\hat{c}\hat{b}'\hat{d}}$ are given in (60)–(62), respectively.

For the purpose to analyze the scalar-tensor theory involving the third order derivative of the scalar field, we also need the covariant 3 + 1 correspondence of the spatial derivatives of the extrinsic curvature and of the acceleration. It is convenient to employ the expression

$$\overset{u}{\mathrm{D}}_c\overset{u}{K}_{ab} = \overset{u}{h}_c{}^{c'}\overset{u}{h}_a{}^{a'}\overset{u}{h}_b{}^{b'}\overset{u}{\mathcal{K}}_{c'a'b'}, \qquad (67)$$

with

$$\overset{u}{\mathcal{K}}_{cab} = \nabla_c\nabla_{(a}u_{b)} + \nabla_c u_{(a|}u^d\nabla_d u_{|b)}. \qquad (68)$$

Together with (29) and (36), we can get the covariant 3 + 1 correspondence of $\overset{u}{\mathrm{D}}_c\overset{u}{K}_{ab}$ explicitly. Similarly, we make use of

$$\overset{u}{\mathrm{D}}_a\overset{u}{a}_b = \overset{u}{h}_a{}^{a'}\overset{u}{h}_b{}^{b'}\overset{u}{\mathcal{A}}_{a'b'}, \qquad (69)$$

with

$$\overset{u}{\mathcal{A}}_{ab} = u^c\nabla_a\nabla_c u_b + \nabla_a u^c\nabla_c u_b. \qquad (70)$$

Together with (29) and (36), we then get the covariant 3 + 1 correspondence of $\overset{u}{\mathrm{D}}_a\overset{u}{a}_b$ explicitly.

Before proceeding, let us take the trace of the extrinsic curvature K as an illustrative example. From (50) one finds

$$
\begin{aligned}
K \to \overset{u}{K} &\equiv g^{ab}\overset{u}{K}_{ab} \\
&= \sqrt{1+\beta^2}K - \frac{K^{ab}\beta_a\beta_b}{\sqrt{1+\beta^2}} \\
&\quad + \frac{1}{\sqrt{1+\beta^2}}\beta^a\dot{\beta}_a + a^a\beta_a + \mathrm{D}^a\beta_a, \qquad (71)
\end{aligned}
$$

which is the covariant 3 + 1 correspondence of K. Clearly in the unitary gauge $n_a \to u_a$, i.e., in the limit $\beta_a \to 0$, the above reduces to K. On the other hand, generally $\dot{\beta}_a$ arises, which signals the extra d.o.f. when deviating from the unitary gauge.

IV. DEGENERATE ANALYSIS: $d = 2$

In the above we have derived the explicit covariant 3 + 1 correspondences of various SCG quantities. When deviating from the unitary gauge, there arise extra Lie derivatives of β_a and/or K_{ab} (with coefficients proportional to β_a), which correspond to higher temporal derivatives of the scalar field and/or the metric. This also explains the apparent appearance of extra modes for the SCG theory in general coordinates [33,52]. It is possible, however, that such "dangerous" terms can get cancelled by combining several SCG terms. In other words, there might exist particular SCG combinations, of which the COD formulation is also degenerate. Since the COD and GST are exactly equivalent, this means the corresponding GST is degenerate.

As a simple example, in this section we consider the linear combination

$$\mathcal{L}^{(2)}_{\mathrm{SCG}} = c_1 K_{ij}K^{ij} + c_2 K^2 + c_3{}^3\mathrm{R} + c_4 a_i a^i, \qquad (72)$$

where the coefficients c_i's are functions of t and N. The Lagrangian in (72) is the combination of four SCG monomials with $d = 2$, where d is the total number of the derivatives (temporal or spatial) in each monomial. We refer to [31] for more details on the classification of SCG monomials according to the derivatives. The unitary gauge correspondence of (72) reads

$$\mathcal{L}^{(2)}_{\mathrm{u.g.}} = c_1\overset{u}{K}_{ab}\overset{u}{K}{}^{ab} + c_2\overset{u}{K}{}^2 + c_3{}^3\overset{u}{R} + c_4\overset{u}{a}_a\overset{u}{a}{}^a. \qquad (73)$$

In (72), the coefficients c_i's are understood as functions of the scalar field ϕ as well as its canonical kinetic term X.

In the spatially covariant formulation, only the spatial metric acquires kinetic term through the extrinsic curvature. In the covariant correspondence, extra terms carrying temporal derivative arise. In the current case, these are $\dot{\beta}_a$ (i.e., $\dot{X}$) and $\dot{K}_{ab}$. Therefore, it is convenient to group terms according to the orders of temporal derivatives of each term. After some manipulations, the full covariant 3 + 1 correspondence can be written as

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\mathrm{COD}} = {} & \mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}^2} + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}K} + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{K}} + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{K^2} \\
& + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}} + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{K} + \mathcal{L}^{(2)}_{\mathrm{COD}}|_{0}. \qquad (74)
\end{aligned}
$$

There are four kinds of terms that are of the second order in temporal derivatives, which are

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}^2} = \dot{\beta}_a\dot{\beta}^a\left[c_4 + \frac{1}{2}(c_1+c_3+2c_4)\beta^2\right] + (\dot{\beta}_a\beta^a)^2\left[-\frac{1}{2}(c_1+c_3+2c_4) + \frac{c_1+c_2}{1+\beta^2}\right], \quad (75)$$

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}K} = 2(c_1+c_3)\dot{\beta}_a\beta_b K^{ab} - \frac{2(c_1+c_2)}{1+\beta^2}(\dot{\beta}_a\beta^a)(K_{cd}\beta^c\beta^d) + 2(c_2-c_3)(\dot{\beta}_a\beta^a)K, \quad (76)$$

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{K}} = 2c_3\,(\beta^a\beta^b - h^{ab}\beta^2)\dot{K}_{ab}, \quad (77)$$

and

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{K^2} = [c_1 + (c_1+3c_3)\beta^2]K_{ab}K^{ab} + [c_2 + (c_2-c_3)\beta^2]K^2 - 2(c_2-2c_3)KK_{ab}\beta^a\beta^b - 2(c_1+3c_3)K^c_aK_{bc}\beta^a\beta^b + \frac{c_1+c_2}{1+\beta^2}(K_{ab}\beta^a\beta^b)^2. \quad (78)$$

The terms of the first order in temporal derivatives are

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{\dot{\beta}} = \frac{(c_1+c_3)}{\sqrt{1+\beta^2}}(\dot{\beta}_a\beta_b\mathrm{D}^a\beta^b) + (c_1+c_3+2c_4)\sqrt{1+\beta^2}(\dot{\beta}_a\beta_b\mathrm{D}^b\beta^a) + \frac{1}{\sqrt{1+\beta^2}}[2(c_2-c_3)(\mathrm{D}_c\beta^c) - (c_1+c_3+2c_4)(\beta^c\beta^d\mathrm{D}_c\beta_d)](\dot{\beta}_a\beta^a) + \sqrt{1+\beta^2}[2c_4 + (c_1+c_3+2c_4)\beta^2](a^a\dot{\beta}_a) + \frac{1}{\sqrt{1+\beta^2}}[c_1+2c_2-c_3-2c_4-(c_1+c_3+2c_4)\beta^2](a^c\beta_c)(\dot{\beta}_a\beta^a), \quad (79)$$

and

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{K} = -\frac{2(c_2-c_3)}{\sqrt{1+\beta^2}}(K_{ab}\beta^a\beta^b)(\mathrm{D}_c\beta^c) - \frac{2(c_1+c_3)}{\sqrt{1+\beta^2}}(K_{ab}\beta^c\beta^a\mathrm{D}^b\beta_c) + 2(c_1+c_3)\sqrt{1+\beta^2}(a^aK_{ab}\beta^b) - 4c_3\sqrt{1+\beta^2}(\beta^a\mathrm{D}_aK) + 4c_3\sqrt{1+\beta^2}(\beta^a\mathrm{D}_bK^b_a) + 2(c_1+c_3)\sqrt{1+\beta^2}(K_{ab}\mathrm{D}^b\beta^a) + 2(c_2-c_3)\sqrt{1+\beta^2}K(a^a\beta_a) + 2(c_2-c_3)\sqrt{1+\beta^2}K(\mathrm{D}_a\beta^a) - \frac{2(c_1+c_2)}{\sqrt{1+\beta^2}}(a^c\beta_c)(K_{ab}\beta^a\beta^b), \quad (80)$$

The terms containing no temporal derivative are

$$\mathcal{L}^{(2)}_{\mathrm{COD}}|_{0} = c_3{}^3\mathrm{R} + 2c_3({}^3R_{ab}\beta^a\beta^b) + (c_2-c_3)(\mathrm{D}_a\beta^a)^2 + (c_1+c_3)(a^a\beta^b\mathrm{D}_a\beta_b) + 2(\mathrm{D}_aa^a)c_3\beta^2 - 2c_3(\beta^a\beta^b\mathrm{D}_ba_a) + (a^a\beta_a)[2(c_2-c_3)(\mathrm{D}_c\beta^c) - (c_1+c_3+2c_4)(\beta^c\beta^d\mathrm{D}_c\beta_d)] + \frac{1}{2}(c_1+c_3+2c_4)(\beta^a\beta^c\mathrm{D}_a\beta^b\mathrm{D}_c\beta_b) + \frac{1}{2}(c_1+c_3)(\mathrm{D}_a\beta_b\mathrm{D}^b\beta^a) + \frac{1}{2}(c_1+c_3)(\mathrm{D}_b\beta_a\mathrm{D}^b\beta^a) + (c_1+c_3+2c_4)(1+\beta^2)(a^a\beta^b\mathrm{D}_b\beta_a) - \frac{c_1+c_3+2c_4}{2(1+\beta^2)}(\beta^a\beta^b\mathrm{D}_b\beta_a)^2 - \frac{c_1+c_3}{2(1+\beta^2)}(\beta^a\beta^c\mathrm{D}_b\beta_c\mathrm{D}^b\beta_a) + \frac{1}{2}(a^a\beta_a)^2[c_1+2c_2-5c_3-2c_4-(c_1+c_3+2c_4)\beta^2] + \frac{1}{2}(a_aa^a)[2c_4 + (c_1+5c_3+4c_4)\beta^2 + (c_1+c_3+2c_4)(\beta^2)^2]. \quad (81)$$

The presence of $\dot{\beta}^2$, $\dot{\beta}K$, and $\dot{K}$ terms correspond to the higher temporal derivatives, and thus signal the possible propagation of extra mode(s). Our goal is thus to tune the coefficients $c_1, \ldots, c_4$ such that all these "dangerous" terms are suppressed. In the following, we replace β_a (and its spatial derivatives) in terms of the scalar field ϕ, its kinetic term X and their temporal and spatial derivatives.

In the rest part of this work, we suppress the subscript "COD" for simplicity. Schematically we write

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}|_{\dot{X}^2} + \mathcal{L}^{(2)}|_{\dot{X}K} + \mathcal{L}^{(2)}|_{\dot{K}} + \mathcal{L}^{(2)}|_{K^2} + \mathcal{L}^{(2)}|_{\dot{X}} + \mathcal{L}^{(2)}|_{K} + \mathcal{L}^{(2)}|_{0}, \tag{82}$$

where the first line are monomials of the second order in the Lie derivative, the second line are monomials of the first order in the Lie derivative and containing spatial derivatives only. For the terms of the second order in the Lie derivatives, we have

$$\mathcal{L}^{(2)}|_{\dot{X}^2} = \frac{\dot{X}^2(\mathrm{D}\phi)^2}{8X^3\dot{\phi}^2}[(c_1+c_2)(\mathrm{D}\phi)^2 + c_4\dot{\phi}^2], \tag{83}$$

$$\mathcal{L}^{(2)}|_{\dot{X}K} = -\frac{\dot{X}}{2X^2\dot{\phi}^2}\{K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi[(c_1+c_3)\dot{\phi}^2 - (c_1+c_2)(\mathrm{D}\phi)^2] + (c_2-c_3)K(\mathrm{D}\phi)^2\dot{\phi}^2\}, \tag{84}$$

$$\mathcal{L}^{(2)}|_{\dot{K}} = \frac{c_3}{X}\dot{K}_{ab}(\mathrm{D}^a\phi\mathrm{D}^b\phi - h^{ab}(\mathrm{D}\phi)^2), \tag{85}$$

and

$$\begin{aligned}\mathcal{L}^{(2)}|_{K^2} = -\frac{1}{2X\dot{\phi}^2}[&-(c_2\dot{\phi}^2 - c_3(\mathrm{D}\phi)^2)K^2\dot{\phi}^2 \\ &- (c_1\dot{\phi}^2 + 3c_3(\mathrm{D}\phi)^2)K_{ab}K^{ab}\dot{\phi}^2 \\ &+ 2(c_2-2c_3)KK_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi\dot{\phi}^2 \\ &+ 2(c_1+3c_3)K_a^cK_{bc}\mathrm{D}^a\phi\mathrm{D}^b\phi\dot{\phi}^2 \\ &- (c_1+c_2)(K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi)^2].\end{aligned} \tag{86}$$

We shall pay special attention to the terms involving $\dot{K}$, which should be reduced by the integrations by parts using

$$\mathcal{C}^{ab}\dot{K}_{ab} \simeq -K\mathcal{C}^{ab}K_{ab} - (\pounds_{\boldsymbol{n}}\mathcal{C}^{ab})K_{ab}. \tag{87}$$

After performing the integrations by parts, since the $\dot{K}$ terms have been reduced, there are three types of terms that are second order in the Lie derivatives. The $\dot{X}^2$ terms are not affected as in (83), while the $\dot{X}K$ and K^2 terms become

$$\begin{aligned}\mathcal{L}^{(2)}|_{\dot{X}K} = -\frac{1}{2X^2\dot{\phi}^2}\dot{X}\Bigg[&\left(c_1 - c_3 + 2X\frac{\partial c_3}{\partial X}\right)K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi\dot{\phi}^2 \\ &- (c_1+c_2)K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi(\mathrm{D}\phi)^2 + \left(c_2+c_3-2X\frac{\partial c_3}{\partial X}\right)K\,(\mathrm{D}\phi)^2\dot{\phi}^2\Bigg],\end{aligned} \tag{88}$$

and

$$\begin{aligned}\mathcal{L}^{(2)}|_{K^2} = \frac{1}{2X\dot{\phi}^2}[&(c_2\dot{\phi}^2 + c_3(\mathrm{D}\phi)^2)K^2\dot{\phi}^2 + (c_1\dot{\phi}^2 - c_3(\mathrm{D}\phi)^2)K_{ab}K^{ab}\dot{\phi}^2 \\ &- 2(c_2+c_3)KK_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi\dot{\phi}^2 - 2(c_1-c_3)K_a^cK_{bc}\mathrm{D}^a\phi\mathrm{D}^b\phi\dot{\phi}^2 \\ &+ (c_1+c_2)(K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi)^2].\end{aligned} \tag{89}$$

We are now ready to determine the coefficients in order make the COD Lagrangian degenerate.

(1) No $\dot{X}^2$ term: From (83) we must set

$$c_1 + c_2 = 0, \tag{90}$$

$$c_4 = 0. \tag{91}$$

(2) No $\dot{X}K$ terms: From (88) we must set

$$c_1 - c_3 + 2X\frac{\partial c_3}{\partial X} = 0, \tag{92}$$

$$c_2 + c_3 - 2X\frac{\partial c_3}{\partial X} = 0. \tag{93}$$

We have the unique solutions for the coefficients:

$$c_1 = -c_2 = c_3 - 2X\frac{\partial c_3}{\partial X}, \qquad c_4 = 0. \tag{94}$$

This is nothing but corresponds to the Horndeski Lagrangian in the unitary gauge [44]. In other words, the specific combination

$$\mathcal{L}^{(2)}_{\mathrm{SCG}} = \left(c_3 - 2X\frac{\partial c_3}{\partial X}\right)(K_{ij}K^{ij} - K^2) + c_3{}^3R, \tag{95}$$

represents the SCG Lagrangian of which the corresponding GST is degenerate.[8] Clearly the GR is a special case with c_3 being constant.

It is interesting to check, after applying the degeneracy conditions,

$$\begin{aligned}\mathcal{L}^{(2)}|_{K^2} \to &\left(c_3 - 2X\frac{\partial c_3}{\partial X}\right)(K_{ab}K^{ab} - K^2)\\ &+\frac{\partial c_3}{\partial X}(-2h^{cd}\mathrm{D}^a\phi\mathrm{D}^b\phi + 2h^{bc}\mathrm{D}^a\phi\mathrm{D}^d\phi\\ &-(h^{ac}h^{bd} - h^{ab}h^{cd})(\mathrm{D}\phi)^2)K_{cd}K_{ab}.\end{aligned} \tag{96}$$

The second line is proportional to $\mathrm{D}_a\phi$ and thus is vanishing in the unitary gauge.

After imposing the above conditions, at the linear order in the Lie derivatives, there are terms proportional to K and $\dot{X}$. For terms proportional to $\dot{X}$, we find

$$\mathcal{L}^{(2)}|_{\dot{X}} = \frac{\dot{X}}{X^2\dot{\phi}}\left(c_3 - X\frac{\partial c_3}{\partial X}\right)(\mathrm{D}^a\phi\mathrm{D}_a\mathrm{D}_b\phi\mathrm{D}^b\phi - \mathrm{D}^2\phi(\mathrm{D}\phi)^2). \tag{97}$$

These two types of terms are safe since they have nothing to do with the degeneracy, which can also be further reduced by the integrations by parts.

V. DEGENERATE ANALYSIS: $d=3$ WITHOUT a_i

In this section we consider the SCG Lagrangian

$$\begin{aligned}\mathcal{L}^{(3)} = &c_1^{(0;3,0)}K_{ij}K^{jk}K^i_k + c_2^{(0;3,0)}K_{ij}a^ia^j\\ &+c_3^{(0;3,0)}K_{ij}K^{ij}K + c_4^{(0;3,0)}Ka_ia^i + c_5^{(0;3,0)}K^3\\ &+c_1^{(0;1,1)}K_{ij}\nabla^ia^j + c_2^{(0;1,1)}K\nabla_ia^i\\ &+c_1^{(1;1,0)}{}^3R^{ij}K_{ij} + c_2^{(1;1,0)}{}^3RK,\end{aligned} \tag{98}$$

which is the linear combination of SCG monomials of $d = 3$. In (98) all the coefficients $c_n^{(c_0;d_2,d_3)}$ are functions of t and N. We refer to [31] for details on the meaning of the superscripts. In this section, we turn off the terms involving the acceleration a_i, i.e., we set

$$c_2^{(0;3,0)} = c_4^{(0;3,0)} = c_1^{(0;1,1)} = c_2^{(0;1,1)} = 0. \tag{99}$$

A. The third order in the Lie derivative

At the third order in the Lie derivatives, schematically, there are in total six types of monomials, of which five are dangerous,

$$\dot{X}^3, \qquad \dot{X}^2K, \qquad \dot{X}\dot{K}, \qquad K\dot{K}, \qquad \dot{X}K^2, \tag{100}$$

and 1 is safe,

$$K^3. \tag{101}$$

At the third order in the Lie derivatives, $\dot{X}^3, \dot{X}^2K, \dot{X}\dot{K}$ terms cannot be reduced by integrations by parts.[9] Therefore, we must to suppress them by setting the corresponding coefficients to be vanishing identically. On the other hand, the terms involving $\dot{K}$ should be reduced by the integrations by parts. For the $K\dot{K}$ term, schematically we write

$$\begin{aligned}\mathcal{C}^{ab,cd}K_{cd}\dot{K}_{ab} \simeq &-\frac{1}{2}K\mathcal{C}^{ab,cd}K_{cd}K_{ab} - \frac{1}{2}(\pounds_{\boldsymbol{n}}\mathcal{C}^{ab,cd})K_{ab}K_{cd}\\ &+\frac{1}{2}(\mathcal{C}^{ab,cd} - \mathcal{C}^{cd,ab})K_{cd}\dot{K}_{ab},\end{aligned} \tag{102}$$

which cannot be reduced further. After performing the integration by parts, the $K\dot{K}$ terms should eliminated by tuning the coefficients, if not being vanishing identically.

After performing the integration by parts of $K\dot{K}$ terms using (102), for the $\dot{X}^3$ terms, we find

$$\mathcal{L}_3^{(3)}|_{\dot{X}^3} = -\frac{\dot{X}^3(\mathrm{D}\phi)^6}{(2X)^{9/2}\dot{\phi}^3}(c_1^{(0;3,0)} + c_3^{(0;3,0)} + c_5^{(0;3,0)}), \tag{103}$$

therefore we need to impose one condition,

$$c_1^{(0;3,0)} + c_3^{(0;3,0)} + c_5^{(0;3,0)} = 0, \tag{104}$$

from which we solve

$$c_5^{(0;3,0)} = -c_1^{(0;3,0)} - c_3^{(0;3,0)}. \tag{105}$$

[8] The corresponding GST is the Horndeski Lagrangian $\mathcal{L}_4$ (in the convention of [5,6]).

[9] Although the $\dot{X}\dot{K}$ term can also be transformed by the integration by parts: $\mathcal{F}\dot{X}\dot{K} \simeq -K\mathcal{F}\dot{X}K - \dot{\mathcal{F}}\dot{X}K - \mathcal{F}\ddot{X}K$, we find it is not necessary since the new term $\ddot{X}K$ will arise. Therefore we simply keep the $\dot{X}\dot{K}$ term in its original form.

For the $\dot{X}^2K$ terms, we have

$$\mathcal{L}_3^{(3)}|_{\dot{X}^2K} = \frac{\dot{X}^2(\mathrm{D}\phi)^2}{(2X)^{7/2}\dot{\phi}^3}[2XK_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi(3c_1^{(0;3,0)} + c_1^{(1;1,0)} + 2c_2^{(1;1,0)} + 2c_3^{(0;3,0)}) + (K(\mathrm{D}\phi)^2\dot{\phi}^2 - (\mathrm{D}\phi)^2K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi)(-c_1^{(1;1,0)} - 2c_2^{(1;1,0)} + c_3^{(0;3,0)} + 3c_5^{(0;3,0)})], \tag{106}$$

After applying the condition (104), the above is reduced to be

$$\mathcal{L}_3^{(3)}|_{\dot{X}^2K} \to -\frac{\dot{X}^2(\mathrm{D}\phi)^2}{(2X)^{7/2}\dot{\phi}}(K(\mathrm{D}\phi)^2 - K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi)(3c_1^{(0;3,0)} + c_1^{(1;1,0)} + 2c_2^{(1;1,0)} + 2c_3^{(0;3,0)}). \tag{107}$$

Thus we need to impose the second condition

$$3c_1^{(0;3,0)} + c_1^{(1;1,0)} + 2c_2^{(1;1,0)} + 2c_3^{(0;3,0)} = 0, \tag{108}$$

from which we solve

$$c_3^{(0;3,0)} = -\frac{3}{2}c_1^{(0;3,0)} - \frac{1}{2}c_1^{(1;1,0)} - c_2^{(1;1,0)}. \tag{109}$$

For the $\dot{X}\dot{K}$ terms, we find

$$\mathcal{L}_3^{(3)}|_{\dot{X}\dot{K}} = \frac{\dot{X}(\mathrm{D}\phi)^2}{(2X)^{5/2}\dot{\phi}}\dot{K}_{ab}(h^{ab}(\mathrm{D}\phi)^2 - \mathrm{D}^a\phi\mathrm{D}^b\phi)(c_1^{(1;1,0)} + 2c_2^{(1;1,0)}). \tag{110}$$

In deriving (110) we have not used the conditions (104) and (108). Therefore we need to impose the third condition

$$c_1^{(1;1,0)} + 2c_2^{(1;1,0)} = 0, \tag{111}$$

from which we solve

$$c_2^{(1;1,0)} = -\frac{1}{2}c_1^{(1;1,0)}. \tag{112}$$

Using (112), (109) is reduced to be

$$c_3^{(0;3,0)} = -\frac{3}{2}c_1^{(0;3,0)}. \tag{113}$$

Plugging (113) into (105) yields

$$c_5^{(0;3,0)} = \frac{1}{2}c_1^{(0;3,0)}. \tag{114}$$

For the $K\dot{K}$ terms, we find

$$\mathcal{L}_3^{(3)}|_{K\dot{K}} = -\frac{c_1^{(1;1,0)} + 2c_2^{(1;1,0)}}{2\sqrt{2X}\dot{\phi}}\dot{K}_{ab}K_{cd}(h^{ab}\mathrm{D}^c\phi\mathrm{D}^d\phi - h^{cd}\mathrm{D}^a\phi\mathrm{D}^b\phi). \tag{115}$$

Fortunately, this term gets cancelled exactly after imposing the condition (112). Therefore, after performing the integration by parts and imposing the condition (112), the $K\dot{K}$ terms are removed automatically.

Then we are left with only the $\dot{X}K^2$ terms, which have two origins. One corresponds to those already existing in the original expression, the other corresponds to those arising from $K\dot{K}$ terms after the integration by parts. The full expression of $\dot{X}K^2$ terms without the above degeneracy conditions are tedious and we do not present it in the current work. After applying all the above three conditions (104), (108), and (111), we find that

$$\mathcal{L}_3^{(3)}|_{\dot{X}K^2} = -\frac{\dot{X}\dot{\phi}}{2(2X)^{5/2}}\left(3c_1^{(0;3,0)} + 2X\frac{\partial c_1^{(1;1,0)}}{\partial X}\right) \times [(K^2 - K_{ab}K^{ab})(\mathrm{D}\phi)^2 - 2KK_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi + 2K_a^cK_{bc}\mathrm{D}^a\phi\mathrm{D}^b\phi]. \tag{116}$$

In order to remove this term, we need to impose the fourth condition

$$3c_1^{(0;3,0)} + 2X\frac{\partial c_1^{(1;1,0)}}{\partial X} = 0, \tag{117}$$

from which we solve

$$c_1^{(0;3,0)} = -\frac{2}{3} X \frac{\partial c_1^{(1;1,0)}}{\partial X}. \tag{118}$$

It is interesting that, at the third order in the Lie derivatives, we have already got the whole 4 conditions (114), (113), (112), and (118) in the Horndeski theory of $\mathcal{L}_5$ [44].

B. The second and the first orders in Lie derivatives

As a consistency check, in the following we shall show that all the dangerous terms at the second and first order in Lie derivatives are indeed removed.

At the second order in the Lie derivatives, schematically, there are in total 4 types of monomials, of which three are dangerous

$$\dot{X}^2, \qquad \dot{X}K, \qquad \dot{K}, \tag{119}$$

and one is safe,

$$K^2. \tag{120}$$

The terms involving $\dot{K}$ can be fully reduced by using

$$\mathcal{C}^{ab}\dot{K}_{ab} \simeq -K\mathcal{C}^{ab}K_{ab} - (\pounds_{\boldsymbol{n}}\mathcal{C}^{ab})K_{ab}, \tag{121}$$

where $\mathcal{C}^{ab}$ contains no Lie derivative. For the terms of the second order in the Lie derivatives, after imposing the four conditions (114), (113), (112), and (118), we have examined that all the "dangerous" terms (i.e., involving $\dot{X}^2$, $\dot{K}$, and $\dot{X}K$) get cancelled automatically. Therefore we do not need to impose any further condition.

There are two types of terms of the first order in Lie derivatives, $\dot{X}$ and K. These two types of terms are always safe. Nevertheless, it is interesting to see that after imposing the four conditions (114), (113), (112), and (118)

$$\begin{aligned}\mathcal{L}_1^{(3)}|_{\dot{X}} = \frac{\dot{X}}{2(2X)^{5/2}\dot{\phi}} \Bigg\{ &-c_1^{(1;1,0)} 4X G_{ab} \mathrm{D}^a\phi \mathrm{D}^b\phi \\ &+ \left(3c_1^{(1;1,0)} - 2X\frac{\partial c_1^{(1;1,0)}}{\partial X}\right)[(\mathrm{D}\phi)^2 \mathrm{D}_a\mathrm{D}_b\phi \mathrm{D}^a\mathrm{D}^b\phi - (\mathrm{D}\phi)^2(\mathrm{D}^2\phi)^2 \\ &+ 2(\mathrm{D}^2\phi)\mathrm{D}^a\phi\mathrm{D}^b\phi\mathrm{D}_a\mathrm{D}_b\phi - 2\mathrm{D}^a\phi\mathrm{D}^b\phi\mathrm{D}_a\mathrm{D}_c\phi\mathrm{D}^c\mathrm{D}_b\phi]\Bigg\}.\end{aligned} \tag{122}$$

Moreover, after integration by parts, there also arise terms involving the Lie derivatives of the acceleration $\dot{a}_a$, which are possibly dangerous. We have checked that these terms are exactly cancelled out after imposing the four conditions (114), (113), (112), and (118).

VI. DEGENERATE ANALYSIS: $d = 3$ WITH a_i

In this section we consider the Lagrangian of $d = 3$ (98) with all the coefficients are present. As in the previous section, we first focus on the terms of the third order in Lie derivatives. Due to the presence of $\nabla_i a_j$ terms, there arise $\ddot{X}$ terms. Schematically, there are two types of terms

$$\ddot{X}\dot{X}, \qquad \ddot{X}K, \tag{123}$$

due to the presence of a_i. Nevertheless, by performing the integrations by parts

$$\mathcal{C}\ddot{X}\dot{X} \simeq -\frac{1}{2}(K\mathcal{C} + \dot{\mathcal{C}})\dot{X}^2, \tag{124}$$

and

$$\mathcal{C}^{ab}\ddot{X}K_{ab} \simeq -K\mathcal{C}^{ab}\dot{X}K_{ab} - (\pounds_{\boldsymbol{n}}\mathcal{C}^{ab})\dot{X}K_{ab} - \mathcal{C}^{ab}\dot{X}\dot{K}_{ab}, \tag{125}$$

the two terms $\ddot{X}\dot{X}$ and $\ddot{X}K$ can be reduced to the 6 types of terms in (100) and (101) that already exist in the case without a_i.

In the following, we first perform the integrations by parts to reduce the $\ddot{X}$ terms, then make a similar analysis as in Sec. V. For the $\dot{X}^3$ terms, we have

$$\begin{aligned}\mathcal{L}_3^{(3)}|_{\dot{X}^3} = -\frac{\dot{X}^3(\mathrm{D}\phi)^4}{2(2X)^{9/2}\dot{\phi}^3}\Bigg[&2X\dot{\phi}^2\frac{\partial\left(c_1^{(0;1,1)} + c_2^{(0;1,1)}\right)}{\partial X} - 4X\left(c_1^{(0;1,1)} + c_2^{(0;1,1)} - c_2^{(0;3,0)} - c_4^{(0;3,0)}\right) \\ &+ \left(-c_1^{(0;1,1)} - c_2^{(0;1,1)} + 2c_1^{(0;3,0)} + 2c_2^{(0;3,0)} + 2c_3^{(0;3,0)} + 2c_4^{(0;3,0)} + 2c_5^{(0;3,0)}\right)(\mathrm{D}\phi)^2\Bigg].\end{aligned} \tag{126}$$

We must set

$$\frac{\partial\left(c_1^{(0;1,1)}+c_2^{(0;1,1)}\right)}{\partial X}=0, \quad (127)$$

$$c_1^{(0;1,1)}+c_2^{(0;1,1)}-c_2^{(0;3,0)}-c_4^{(0;3,0)}=0, \quad (128)$$

$$-c_1^{(0;1,1)}-c_2^{(0;1,1)}+2c_1^{(0;3,0)}+2c_2^{(0;3,0)}+2c_3^{(0;3,0)} + 2c_4^{(0;3,0)}+2c_5^{(0;3,0)}=0. \quad (129)$$

We solve

$$c_4^{(0;3,0)}=-c_2^{(0;3,0)}+f_1(\phi), \quad (130)$$

$$c_2^{(0;1,1)}=-c_1^{(0;1,1)}+f_1(\phi), \quad (131)$$

$$c_5^{(0;3,0)}=-c_1^{(0;3,0)}-c_3^{(0;3,0)}-\frac{1}{2}f_1(\phi), \quad (132)$$

where $f_1(\phi)$ is an arbitrary function of ϕ only.

For the $\dot{X}^2K$ terms, after applying the above conditions (130)–(132), we have

$$\mathcal{L}_3^{(3)}|_{\dot{X}^2K}=-\frac{\dot{X}^2}{(2X)^{7/2}\dot{\phi}}(K(\mathrm{D}\phi)^2-K_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi) \times\left[\left(-c_1^{(0;1,1)}+3c_1^{(0;3,0)}+c_1^{(1;1,0)}+c_2^{(0;3,0)}+2c_2^{(1;1,0)}+2c_3^{(0;3,0)}+2f_1(\phi))(\mathrm{D}\phi)^2 + 2X\dot{\phi}^2\frac{\partial c_1^{(0;1,1)}}{\partial X}+X(-2c_1^{(0;1,1)}+2c_2^{(0;3,0)}\right)\right]. \quad (133)$$

We must have

$$-c_1^{(0;1,1)}+3c_1^{(0;3,0)}+c_1^{(1;1,0)}+c_2^{(0;3,0)}+2c_2^{(1;1,0)} + 2c_3^{(0;3,0)}+2f_1(\phi)=0, \quad (134)$$

$$\frac{\partial c_1^{(0;1,1)}}{\partial X}=0, \quad (135)$$

$$-2c_1^{(0;1,1)}+2c_2^{(0;3,0)}=0, \quad (136)$$

from which we solve

$$c_1^{(0;1,1)}=f_2(\phi), \quad (137)$$

$$c_2^{(0;3,0)}=f_2(\phi), \quad (138)$$

$$c_3^{(0;3,0)}=-\frac{3c_1^{(0;3,0)}}{2}-\frac{c_1^{(1;1,0)}}{2}-c_2^{(1;1,0)}-f_1(\phi). \quad (139)$$

Again, $f_2(\phi)$ is an arbitrary function of ϕ only.

For the $\dot{X}\dot{K}$ terms, after applying the above conditions (130)–(132) and (137)–(139), we have

$$\mathcal{L}_3^{(3)}|_{\dot{X}\dot{K}}=\frac{\dot{X}}{(2X)^{5/2}\dot{\phi}}[(c_1^{(1;1,0)}+2c_2^{(1;1,0)}+f_1(\phi)-f_2(\phi))(h^{ab}\dot{K}_{ab})(\mathrm{D}\phi)^4 -(c_1^{(1;1,0)}+2c_2^{(1;1,0)}+f_1(\phi)-f_2(\phi))(\mathrm{D}\phi)^2\dot{K}_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi + 2(f_1(\phi)-f_2(\phi))X(h^{ab}\dot{K}_{ab})(\mathrm{D}\phi)^2+2f_2(\phi)X\dot{K}_{ab}\mathrm{D}^a\phi\mathrm{D}^b\phi]. \quad (140)$$

We must have

$$c_1^{(1;1,0)}+2c_2^{(1;1,0)}+f_1(\phi)-f_2(\phi)=0, \quad (141)$$

$$f_1(\phi)-f_2(\phi)=0, \quad (142)$$

$$f_2(\phi)=0, \quad (143)$$

from which we solve

$$f_1(\phi)=f_2(\phi)=0, \quad (144)$$

and

$$c_1^{(1;1,0)} + 2c_2^{(1;1,0)} = 0. \tag{145}$$

At this point, we can already fix that

$$c_2^{(0;3,0)} = c_4^{(0;3,0)} = c_1^{(0;1,1)} = c_2^{(0;1,1)} = 0, \tag{146}$$

and therefore we have been back to the case without a_i. As a result, the subsequent analysis is exactly the same as the case without a_i in Sec. V.

VII. CONCLUSION

A necessary condition for a generally covariant scalar-tensor theory (GST) to be ghostfree is that it is ghostfree in the unitary gauge when the scalar field is timelike, in which the theory takes the form of the spatially covariant gravity (SCG). One may use the SCG as the starting point to search for the ghostfree GST. To this end, a further covariant $3+1$ decomposition (COD) of the GST without fixing any coordinates is also needed. Therefore, in principle, one needs "two steps" (SCG → GST → COD) to complete the analysis. In this work, we developed a "one step" method, which we dub the "covariant $3+1$ correspondence", to derive the corresponding COD from SCG directly. The resulting COD expressions can be used as the starting point of the further degeneracy/constraint analysis.

In Sec. III we derive the explicit expressions of this covariant $3+1$ correspondence. We take the SCG Lagrangians of $d=2$ and $d=3$ as simple illustrations of this method in the subsequent sections. By deriving the corresponding COD using this method, one can determine the degeneracy conditions easily. Not surprisingly, the resulting Lagrangians with these degeneracy conditions are nothing but correspond to the Horndeski theory in the unitary gauge. In other words, one could rediscover the Horndeski theory with this method in a quite simple manner. In this work, we only consider SCG Lagrangians in which the lapse function is nondynamical. If we start with more general degenerate SCG Lagrangians (e.g., with a dynamical lapse function [40–42]), the method in this work may be used to search for more general ghostfree scalar-tensor theory with higher-order derivatives and curvature terms. We shall report the progress in the future.

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