Beta functions of (3 + 1)-dimensional projectable Hořava gravity

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We derive the full set of beta functions for the marginal essential couplings of projectable Hořava gravity in $(3+1)$-dimensional spacetime. To this end we compute the divergent part of the one-loop effective action in static background with an arbitrary spatial metric. The computation is done in several steps: reduction of the problem to three dimensions, extraction of an operator square root from the spatial part of the fluctuation operator, and evaluation of its trace using the method of universal functional traces. This provides us with the renormalization of couplings in the potential part of the action which we combine with the results for the kinetic part obtained previously. The calculation uses symbolic computer algebra and is performed in four different gauges yielding identical results for the essential beta functions. We additionally check the calculation by evaluating the effective action on a special background with spherical spatial slices using an alternative method of spectral summation. We conclude with a preliminary discussion of the properties of the beta functions and the resulting renormalization group flow, identifying several candidate asymptotically free fixed points.

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I. INTRODUCTION AND SUMMARY

A. State of the art

Hořava gravity (HG) has been suggested [1] as an approach to quantum gravity within the framework of renormalizable quantum field theory (see [2–6] for reviews). The key idea is borrowed from condensed matter physics and uses the notion of anisotropic (Lifshitz) scaling of time and space coordinates,[1]

$$t \mapsto b^{-d} t, \qquad x^i \mapsto b^{-1} x^i, \tag{1.1}$$

where d is the number of spatial dimensions and b is a positive scaling parameter. A theory whose bare (tree-level) action does not contain any irrelevant operators under this scaling is power-counting renormalizable, implying that it has fair chances to be perturbatively renormalizable in the strict sense; i.e., all divergences generated within perturbation theory can be absorbed into redefinition of the couplings in the action.

[1]Throughout the paper we use Latin indices to denote the spatial directions, $i = 1, \ldots, d$.

For any space dimension d bigger than 1, time and space in (1.1) scale differently. Clearly, a theory of gravity formulated in this setting cannot be Lorentz invariant nor completely diffeomorphism invariant. The special role of time restricts possible symmetries to foliation preserving diffeomorphisms (FDiff) which leave the constant-time slices invariant,

$$t \mapsto t'(t), \qquad x^i \mapsto x'^i(t, \mathbf{x}), \tag{1.2}$$

where $t'(t)$ is a monotonic function. In other words, we are left with time reparametrizations and time-dependent spatial diffeomorphisms.

To formulate the action of HG, the metric is expanded into time and space components as in the Arnowitt-Deser-Misner (ADM) decomposition,

$$\mathrm{d}s^2 = N^2 \mathrm{d}t^2 - \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t). \tag{1.3}$$

The lapse N, the shift N^i, and the spatial metric γ_{ij} transform in the usual way under FDiff,

$$N \mapsto N \frac{\mathrm{d}t}{\mathrm{d}t'}, \qquad N^i \mapsto \left(N^j \frac{\partial x'^i}{\partial x^j} - \frac{\partial x'^i}{\partial t} \right) \frac{\mathrm{d}t}{\mathrm{d}t'},$$
$$\gamma_{ij} \mapsto \gamma_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j}. \tag{1.4}$$

They are assigned the following dimensions under the scaling (1.1)[2]:

$$[N] = [\gamma_{ij}] = 0, \qquad [N^i] = d - 1. \tag{1.5}$$

The Lagrangian is then built out of all local FDiff-invariant operators that can be constructed from these fields and have dimension less than or equal to $2d$. The latter bound comes from the scaling dimension of the spacetime integration measure $[\mathrm{d}t\mathrm{d}^d x] = -2d$ and ensures that the terms in the action corresponding to relevant ($[\mathcal{O}] < 2d$) and marginal ($[\mathcal{O}] = 2d$) operators have nonpositive dimensions. This is a hallmark of power-counting renormalizability and is a necessary, but in general not sufficient, prerequisite of the true perturbative renormalizability.

The precise form of the action and its physical content depends on the assumption about the form of the lapse function N. In the *nonprojectable* version of HG N is assumed to be a full-fledged dynamical field that has dependence on both space and time coordinates. In this case the Lagrangian can depend on the spatial gradients of N through the FDiff-covariant vector $a_i = \partial_i \ln N$ [7], and the number of independent operators in the Lagrangian is very large [order $O(100)$] [8]. The action simplifies at low energies where it reduces to general relativity (GR) plus a sector describing the dynamics of the preferred foliation [4,9]. The latter sector is stable and, by appropriately choosing the couplings, its interactions with gravity and visible matter can be suppressed. In other words, despite the absence of Lorentz invariance or general covariance as fundamental principles in HG, it can reproduce the known phenomenology of GR at the scales at which it has been tested [10]. While the parameter space of the theory has been strongly constrained by the exquisite tests of Lorentz invariance in the matter sector [11] and in gravity [12], it still remains phenomenologically viable [13].

However, the question about renormalizability of the nonprojectable HG remains open. Although its Lagrangian is power-counting renormalizable, the dynamical lapse component of the metric induces an instantaneous interaction [4,14] which can potentially lead to nonlocal divergences [15]. Whether such divergences actually occur or not is presently unknown.

In this paper we focus on a different version of HG called *projectable* whose quantum properties are better understood. In projectable HG the lapse N is restricted to be a function of time only, $N = N(t)$. This assumption is compatible with the FDiff transformations (1.4). Further, by using the time reparametrization, N can be set to any given constant value, say $N = 1$. In this way lapse is completely eliminated from the model. The action reads [1]

$$S = \frac{1}{2G}\int \mathrm{d}t\,\mathrm{d}^d x\sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}), \tag{1.6}$$

where

$$K_{ij} = \frac{1}{2}(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i) \tag{1.7}$$

is the extrinsic curvature of the foliation and $K \equiv K_{ij}\gamma^{ij}$ is its trace. Here the dot denotes the time derivative and ∇_i is the covariant derivative associated with the spatial metric γ_{ij}; G and λ are dimensionless couplings. The potential part $\mathcal{V}$ does not involve time derivatives and is constructed out of the curvature tensor corresponding to γ_{ij}. Its form depends on the number of spatial dimensions. In $d = 3$ it reads [16]

$$\mathcal{V} = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij}R^{ij} + \nu_1 R^3 + \nu_2 R R_{ij}R^{ij} + \nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk}, \tag{1.8}$$

where we have used that in three dimensions the Riemann tensor is expressed in terms of Ricci R_{ij}. This expression includes all relevant and marginal terms that cannot be reduced to each other upon integration by parts and the use of Bianchi identities. It contains nine couplings $\Lambda, \eta, \mu_1, \mu_2$, and $\nu_a, a = 1, \ldots, 5$.

The spectrum of perturbations propagated by this action contains a transverse-traceless (tt) graviton and an additional scalar mode. Both modes have positive kinetic terms as long as G is positive and λ is either less than $1/3$ or greater than 1,

$$\lambda < 1/3 \quad \text{or} \quad \lambda > 1, \tag{1.9}$$

implying that the theory admits unitary quantization. Their dispersion relations around a flat background[3] are [4,15]

$$\omega_{tt}^2 = \eta k^2 + \mu_2 k^4 + \nu_5 k^6, \tag{1.10a}$$

$$\omega_s^2 = \frac{1-\lambda}{1-3\lambda}(-\eta k^2 + (8\mu_1 + 3\mu_2)k^4) + \nu_s k^6, \tag{1.10b}$$

where k is the spatial momentum and we have defined

$$\nu_s \equiv \frac{(1-\lambda)(8\nu_4 + 3\nu_5)}{1-3\lambda}. \tag{1.11}$$

These dispersion relations are problematic at low energies where they are dominated by the k^2-terms. Due to the negative sign in front of this term, either the scalar mode or the graviton behaves as a tachyon at low energies, implying

[2] We say that a field Φ has dimension $[\Phi]$ if it transforms under (1.1) as $\Phi \mapsto b^{[\Phi]}\Phi$.

[3] By this we mean the configuration $N^i = 0$, $\gamma_{ij} = \delta_{ij}$. It is a solution of equations following from (1.6) and (1.8) provided the cosmological constant Λ is set to zero.

that flat space is not a stable vacuum of the model. Attempts to suppress the instability by choosing λ close to 1 lead to the loss of perturbative control [4,9,17] (see, however, [2,18,19] for suggestions to restore the control by rearranging the perturbation theory). Alternatively, the instability can be eliminated by tuning $\eta = 0$ or by expanding around a curved vacuum. In both cases the theory does not appear to reproduce GR in the low-energy limit, as there is no regime where the dispersion relation of the tt-graviton would have the relativistic form $\omega_{tt}^2 \propto k^2$. This low-energy problem does not affect the ultraviolet (UV) behavior of the model: At high momenta both dispersion relations (1.10) are perfectly regular, as long as ν_5, $\nu_s > 0$.

Projectable HG has been proven to be perturbatively renormalizable [15,20] in any number of spacetime dimensions. Furthermore, for $d = 2$ its renormalization group (RG) flow was computed and shown to possess an asymptotically free UV fixed point, implying that the theory is UV complete [21]. Also in $d = 2$ Ref. [22] computed the renormalization of the cosmological constant.

Partial results about the RG flow of projectable HG in $d = 3$ were obtained in Ref. [23]. The UV behavior of the theory is parametrized by seven couplings in front of the marginal operators in (1.6) and (1.8): $G, \lambda, \nu_a, a = 1, \ldots, 5$. However, not all these couplings have physical meaning, which is reflected in the dependence of their β-functions on the choice of gauge used in the loop calculations. There are in total six *essential* couplings which enter into the on-shell effective action and whose β-functions are thus gauge invariant. These can be chosen as follows [23][4]:

$$\mathcal{G} = \frac{G}{\sqrt{\nu_5}}, \quad \lambda, \quad u_s = \sqrt{\frac{\nu_s}{\nu_5}}, \quad v_a = \frac{\nu_a}{\nu_5}, \quad a = 1,2,3, \qquad (1.12)$$

where ν_s is defined in (1.11). The one-loop β-function of λ depends only on the first three of these couplings and reads [23]

$$\beta_\lambda = \frac{\mathcal{G}}{120\pi^2(1-\lambda)(1+u_s)u_s} \times [27(1-\lambda)^2 + 3u_s(11-3\lambda)(1-\lambda) - 2u_s^2(1-3\lambda)^2]. \qquad (1.13)$$

The gauge-dependent β-function of G (not $\mathcal{G}$) was also computed for several gauge choices, and the results are summarized in the Appendix A 1.

The purpose of this work is to provide β-functions of the remaining couplings in the list (1.12).

[4]The coupling u_s is related to the coupling α used in [23] as $\alpha = u_s^2$.

B. Outline of the method and main results

The calculation in [23] was done using a combination of the background field approach followed by diagrammatic expansion around a special background. The latter was chosen to have flat spatial metric, $\gamma_{ij} = \delta_{ij}$, and arbitrary shift vector N^i. This allowed us to avoid the complicated vertices originating from the potential part of the action and profit from the relatively simple structure of the kinetic term, which for this background choice reduces to a linear combination of two quadratic invariants $(\partial_i N_j)^2$ and $(\partial_i N_i)^2$. Computing the one-loop effective action in this background we were able to extract the β-functions of G and λ.

However, application of this method to the renormalization of the couplings ν_a, $a = 1, \ldots, 5$ is infeasible. It would require evaluation of a huge number of Feynman diagrams with two and tree background metric insertions in the external legs. Instead, we use the powerful tools based on the Schwinger-DeWitt [24–29] or the Gilkey-Seeley [30–32] heat kernel method. They provide an effective resummation of the field perturbation series and allow one to obtain the UV divergences not as an expansion in powers of field perturbations, but as full nonlinear counterterms—local nonlinear functionals of the generic background field. The pioneering application of this method in quantum Einstein theory [33] proved to be very efficient and now underlies the majority of results on renormalization of (super)gravitational models. The basic tool of this method is the heat equation kernel whose proper-time expansion coefficients—the so-called HAMIDEW [34] or Gilkey-Seeley coefficients—carry full information about UV divergences and can be systematically calculated.

Despite powerful calculational advantages of the heat kernel method, its application to HG encounters two major difficulties. This method is most efficient for covariant operators in which all spacetime derivatives form covariant d'Alembertians or spatial Laplacians and are treated on equal footing. The existence of preferred time foliation obviously violates this property. Several approaches have been put forward to circumvent this problem and extend the heat kernel method to Lifshitz-type theories [35–39]. However, applications to HG models are marred by an extra difficulty—*nonminimal* operators arising in these models have higher-order derivative terms which are not exhausted by powers of the spacetime d'Alembertian $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ or spatial Laplacian $\Delta \equiv \gamma^{ij}\nabla_i\nabla_j$. The principal symbol term of these operators is nondiagonal in derivatives whose indices are contracted with the tensor field indices. To circumvent this difficulty we use the technique of the so-called *universal functional traces* (UFT) applicable to this class of higher-derivative nonminimal operators.

Originally this method was developed for spacetime covariant operators in [27,28] (see also [40]). Universal functional traces are the coincidence limits of the kernels of nonlocal (pseudodifferential) operators of the form

$$\nabla_{\mu_1}\cdots\nabla_{\mu_m}\frac{\hat{1}}{\square^n}\delta(x,y)\bigg|_{y=x}, \tag{1.14}$$

which are defined in curved spacetime with a generic metric $g_{\mu\nu}$ and with the covariant d'Alembertian operator $\square$ acting on a generic set of fields $\varphi = \varphi^A(x)$ (the hat symbol denotes the matrix structure of the operator kernel in the vector space of φ, $\hat{1}\varphi = \delta^A_B\varphi^B$, etc.). For appropriate values of parameters m and n these coincidence limits are UV divergent and contain all the information about one-loop UV divergences of the theory.

The latter property can easily be demonstrated on the example, say, of the higher-derivative theory with the inverse propagator $\square^N + P(\nabla)$, where $P(\nabla)$ is its lower-derivative part. By expanding the one-loop functional determinant $\mathrm{Tr}\,\ln(\square^N + P(\nabla)) = N\mathrm{Tr}\,\ln\square + \mathrm{Tr}\,\ln(1 + P(\nabla)/\square^N)$ in powers of the nonlocal term $P(\nabla)/\square^N$ and commuting to the left the powers of $P(\nabla)$ and to the right the inverse powers of $\square$, one finds that the result will be given by the infinite series of terms (1.14) multiplied by tensors of ever growing dimensionality. Only a finite number of these terms will be UV divergent, so that the overall one-loop divergent part will be known, provided one can calculate divergent parts of separate universal functional traces (1.14). One can easily and systematically do this by the Schwinger-DeWitt heat kernel technique, just as it can be done for the first term $N\mathrm{Tr}\,\ln\square$ of the expansion. This calculation is equally possible for any spacetime dimension and, moreover, can be extended to noninteger values of n in (1.14). As shown in [27], a similar technique applies to nonminimal operators.

This method admits the generalization to Lorentz violating Lifshitz theories with regular propagators (the class of HG models in which the proof of their perturbative renormalizability holds [15])—see, for instance, [41]. Fortunately, however, this generalization is not really needed in our case. Due to the properties of the projectable HG, the renormalization of its potential term can be done via a special three-dimensional reduction, upon which the one-loop effective action is represented as the trace of the square root of a certain sixth order operator fully covariant in the three-dimensional space. The operator is nonminimal, and bringing its square root to the form suitable for the application of the UFT technique presents a major computational challenge. We overcome it by the use of symbolic computer algebra. As a result, we obtain the one-loop effective action as a sum of UFTs (1.14) with half-integer n that are fully covariant in the three-dimensional space and, therefore, fully manageable by the technique of [27].

The divergent part of the one-loop background effective action provides us with the renormalized coupling constants ν_a, $a = 1,\ldots,5$, and allows us to determine the β-functions of $\mathcal{G}$ and the other four essential couplings collectively denoted by $\chi = (u_s, v_1, v_2, v_3,)$. The corresponding expressions have the form

$$\beta_{\mathcal{G}} = \frac{\mathcal{G}^2}{26880\pi^2(1-\lambda)^2(1-3\lambda)^2(1+u_s)^3u_s^3} \times\sum_{n=0}^{7}u_s^n\mathcal{P}_n^{\mathcal{G}}[\lambda, v_1, v_2, v_3], \tag{1.15a}$$

$$\beta_\chi = A_\chi\frac{\mathcal{G}}{26880\pi^2(1-\lambda)^3(1-3\lambda)^3(1+u_s)^3u_s^5} \times\sum_{n=0}^{9}u_s^n\mathcal{P}_n^{\chi}[\lambda, v_1, v_2, v_3], \tag{1.15b}$$

where the prefactor coefficients $A_\chi = (A_{u_s}, A_{v_1}, A_{v_2}, A_{v_3})$ equal

$$A_{u_s} = u_s(1-\lambda),\quad A_{v_1} = 1,\quad A_{v_2} = A_{v_3} = 2. \tag{1.16}$$

Note that the coupling $\mathcal{G}$ factorizes and its powers enter the β-functions only as overall coefficients. The functions $\mathcal{P}_n^{\mathcal{G}}[\lambda, v_1, v_2, v_3]$ and $\mathcal{P}_n^{\chi}[\lambda, v_1, v_2, v_3]$ are polynomials in λ and v_a, $a = 1, 2, 3$, with integer coefficients. $\mathcal{P}_n^{\mathcal{G}}$, $\mathcal{P}_n^{u_s}$, and $\mathcal{P}_n^{v_a}$ are, respectively, of fourth, fifth, and sixth order in λ. The maximum overall power of the couplings v_a is two for $\mathcal{P}_n^{\mathcal{G}}$, f$\mathcal{P}_n^{u_s}$ and three for $\mathcal{P}_n^{v_a}$. Explicit expressions for these polynomials are lengthy and are collected in Appendix A 2.

Equations (1.15), (1.16), and (A4)–(A8) represent the main results of this paper. Together with the β-function of λ, Eq. (1.13), they compose the full set of β-functions for essential coupling constants of projectable HG in $d = 3$. The files containing the β-functions in the *Mathematica* [42] format are available as Supplemental Material [43].

In the rest of the paper we describe in detail our calculation and various checks, to which we subject our results. In Sec. II we discuss the gauge-fixing procedure and the choice of the background and get the general expression for the one-loop effective action. In Sec. III we perform the reduction to three dimensions, reformulate the problem as an extraction of an operator square root, and describe an algorithm to perform this extraction. In Sec. IV we review the UFT technique and classify the UFTs required for our calculation. In Sec. V we extract the β-functions of the essential couplings and discuss their gauge invariance. In Sec. VI we perform an independent check of our results by computing the one-loop effective action on a sphere with a different method based on spectral decomposition. In Sec. VII we discuss our results and make several preliminary observations about the structure of the β-functions and the corresponding RG flow. Some lengthy formulas and technical details are relegated to the Appendixes.

II. GAUGE FIXING AND ONE-LOOP EFFECTIVE ACTION

A. The choice of the background and background covariant gauge fixing

We focus on the part of the action consisting of the marginal operators with respect to the scaling (1.1). They form a closed set under renormalization and determine the UV behavior of the theory. From now on we switch to the imaginary "Euclidean" time $\tau = it$. In this "signature" the tree-level action reads

$$S = \frac{1}{2G}\int d\tau\, d^3x \sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 + \nu_1 R^3 + \nu_2 R R_{ij} R^{ij} + \nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk}). \tag{2.1}$$

Renormalization of the theory implies the calculation of the UV divergent part of the effective action, which has the same covariant structure as the classical tree-level action (2.1) provided one works in the class of the so-called background covariant gauges [25] which were discussed in the context of HG in [15,20,23]. For the renormalization of the potential part of the action it is, therefore, sufficient to consider the metric background on which all its five tensor structures are nonvanishing and can be distinctly separated. This is the spacetime metric with a generic static three-dimensional part $g_{ij}(\mathbf{x})$ and vanishing shift functions $N^i = 0$. Static nature of g_{ij} and zero shift functions lead to the zero kinetic term of (2.1) whose contribution is not needed for the renormalization of couplings $\nu_1, \ldots, \nu_5$.[5]

Thus we perform the split of the full set of fields into this background and quantum fluctuations $h_{ij}(\tau, \mathbf{x})$ and $n^i(\tau, \mathbf{x})$,

$$\gamma_{ij}(\tau, \mathbf{x}) = g_{ij}(\mathbf{x}) + h_{ij}(\tau, \mathbf{x}), \qquad N^i(\tau, \mathbf{x}) = 0 + n^i(\tau, \mathbf{x}), \tag{2.2}$$

retain the relevant quadratic part of the full action on this background, and take the resulting Gaussian path integral over the fluctuations. We begin this procedure by considering first a special gauge-breaking part of the action which preserves the gauge invariance of the counterterms and is compatible with the anisotropic scaling (1.1) [15,20].

The gauge-fixing action is chosen as

$$S_{\rm gf} = \frac{\sigma}{2G}\int d\tau\, d^3x \sqrt{g} F^i \mathcal{O}_{ij} F^j, \tag{2.3}$$

which is the quadratic form in the gauge-condition functions F^i with the kernel $\mathcal{O}_{ij}$, both parametrically depending on the background fields in the way that this action is invariant under the simultaneous diffeomorphisms of both the full field (2.2) and the metric background—the so-called *background gauge transformations*. Notice that under these background diffeomorphisms the quantum fields n^i and h_{ij} transform as a vector and a rank-two tensor, respectively. For a static g_{ij} and vanishing background shifts the gauge conditions and the gauge-fixing matrix take the form

$$F^i = \dot{n}^i + \frac{1}{2\sigma}\mathcal{O}^{-1}_{ij}(\nabla_k h_{jk} - \lambda \nabla_j h), \tag{2.4a}$$

$$\mathcal{O}_{ij} = (g^{ij}\Delta^2 + \xi \nabla^i \Delta \nabla^j)^{-1}. \tag{2.4b}$$

Here and below the covariant derivatives are defined using the background metric g_{ij}. The gauge functions F^i are local linear combinations of quantum fields h_{ij} and n^i with operator coefficients. The gauge-fixing matrix $\mathcal{O}_{ij}$ is the nonlocal Green's function of the covariant fourth-order differential operator $\mathcal{O}^{-1}_{ij} = g^{ij}\Delta^2 + \xi \nabla^i \Delta \nabla^j$. This nonlocality can be resolved by integrating in an auxiliary field and does not spoil the locality of counterterms [15]. Under background gauge transformations F^i and $\mathcal{O}_{ij}$ transform, respectively, as the vector and the second rank tensor, so that the gauge-breaking action (2.3) is indeed invariant and provides explicit gauge invariance of quantum counterterms [20].[6] Further, this gauge fixing leads to a homogeneous falloff of all field propagators in UV and ensures that all counterterms are compatible with the naive power-counting arguments [15].

The gauge conditions (2.4) are parametrized by two constants ξ and σ. The gauge-breaking action (2.3) reads explicitly

$$S_{\rm gf} = \frac{1}{2G}\int d\tau\, d^3x \sqrt{g}\bigg(\sigma \dot{n}^i \mathcal{O}_{ij} \dot{n}^j + \dot{n}^i(\nabla^j h_{ij} - \lambda \nabla_i h) + \frac{1}{4\sigma}\nabla^k h_{ik}\mathcal{O}^{-1}_{ij}\nabla^l h_{jl} - \frac{\lambda}{2\sigma}\nabla_i h \mathcal{O}^{-1}_{ij}\nabla^k h_{jk} + \frac{\lambda^2}{4\sigma}\nabla_i h \mathcal{O}^{-1}_{ij}\nabla_j h\bigg). \tag{2.5}$$

The distinguished nature of this two-parameter family is that the cross terms between n^i and h_{ij} in $S_{\rm gf}$ completely cancel the analogous terms in the kinetic part of the classical action at the quadratic order,

[5] This is a strategy opposite to that of [23], where the renormalization of the kinetic term with its couplings G and λ was performed on a three-dimensional flat space background with nontrivial $N^i(\mathbf{x})$ treated by perturbations in external lines of Feynman diagrams.

[6] Strictly speaking, for F^i to be a vector under background transformations $\dot{n}^i$ should be replaced by the covariant time derivative $D_\tau n^i = \dot{n}^i - \bar{N}^k \partial_k n^i + \partial_k \bar{N}^i n^k$, where $\bar{N}^i$ is the background shift function [15], but on our background $\bar{N}^i = 0$.

$$S_{\rm kin} = \frac{1}{2G}\int d\tau\, d^3x\sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2)$$
$$= \frac{1}{2G}\int d\tau\, d^3x\sqrt{g}\bigg[-\frac{1}{4}h^{ij}\ddot{h}_{ij} + \frac{\lambda}{4}h\ddot{h}$$
$$-\dot{n}^i(\nabla^j h_{ij} - \lambda\nabla_i h) - \frac{1}{2}n_i\Delta n^i - \left(\frac{1}{2} - \lambda\right)n^i\nabla_i\nabla_j n^j$$
$$-\frac{1}{2}n^i R_{ij}n^j\bigg] + \cdots, \tag{2.6}$$

where dots mean higher-order terms of the expansion. In this expression we have integrated by parts in both space and time and used the staticity of the background 3-metric, $\dot{g}_{ij} = 0$. The cancellation of the cross terms between n^i and h_{ij} implies that the shift and metric sectors can be treated separately.

B. Shift and ghost parts of the action

From the sum of the kinetic action (2.6) and the gauge-breaking term (2.5) we obtain the quadratic in n^i part of the gauge-fixed action,

$$S_n = \frac{1}{2G}\int d\tau\, d^3x\sqrt{g}n^i\bigg[-\sigma\mathcal{O}_{ij}\partial_\tau^2 + \lambda\nabla_i\nabla_j$$
$$-\frac{1}{2}\nabla_j\nabla_i - \frac{1}{2}g_{ij}\Delta\bigg]n^j$$
$$= \frac{\sigma}{2G}\int d\tau\, d^3x\sqrt{g}n^i\mathcal{O}_{ij}[-\delta^j_k\partial_\tau^2 + \mathbb{B}^j{}_k(\nabla)]n^k, \tag{2.7}$$

where the differential operator $\mathbb{B}^i{}_j(\nabla)$ in spatial derivatives reads

$$\mathbb{B}^i{}_j(\nabla) = -\frac{1}{2\sigma}\delta^i_j\Delta^3 - \frac{1}{2\sigma}\Delta^2\nabla_j\nabla^i - \frac{\xi}{2\sigma}\nabla^i\Delta\nabla^k\nabla_j\nabla_k$$
$$-\frac{\xi}{2\sigma}\nabla^i\Delta\nabla_j\Delta + \frac{\lambda}{\sigma}\Delta^2\nabla^i\nabla_j + \frac{\lambda\xi}{\sigma}\nabla^i\Delta^2\nabla_j. \tag{2.8}$$

It is quite remarkable that the chosen two-parameter family of gauge conditions on a static background provides another very useful property—modulo the multiplication by the gauge-fixing matrix the corresponding Faddeev-Popov ghost operator coincides with the operator in the shift action (2.7). Indeed, the action of the ghost fields c^i and $\bar{c}_j$ reads

$$S_{\rm gh} = -\frac{1}{G}\int d\tau\, d^3x\sqrt{g}\bar{c}_i(\mathbf{s}F^i), \tag{2.9}$$

where $\mathbf{s}F^i$ is the Becchi-Rouet-Stora-Tyutin (BRST) transform of the gauge conditions. This is computed using the BRST transformations of the quantum fields h_{ij} and n^i which coincide with the infinitesimal diffeomorphisms of the full fields γ_{ij} and N^i with the gauge parameter replaced by the Grassmann ghost c^i,

$$\mathbf{s}h_{ij} = \nabla_i c_j + \nabla_j c_i + h_{ik}\nabla_j c^k + h_{jk}\nabla_i c^k + c^k\nabla_k h_{ij},$$
$$c_i = g_{ij}c^j, \tag{2.10a}$$

$$\mathbf{s}n^i = \dot{c}^i - n^j\nabla_j c^i + c^j\nabla_j n^i. \tag{2.10b}$$

After the substitution of (2.10) into (2.9), the ghost action in the quadratic order of all quantum fields takes the following form (bearing in mind zero background values of ghosts):

$$S_{\rm gh} = \frac{1}{G}\int d\tau\, d^3x\sqrt{g}\bar{c}_i(-\delta^i_j\partial_\tau^2 + \mathbb{B}^i{}_j(\nabla))c^j, \tag{2.11}$$

where the operator $\mathbb{B}^i{}_j$ exactly coincides with that of (2.7). This property is an artifact of the special choice of the gauge-fixing action and the static nature of the metric background, and it significantly simplifies further calculations, because the contributions of ghosts and shift functions are expressed through the functional determinant of one and the same operator.

C. Metric part of the action

The kinetic part for the metric perturbations has the form

$$-\frac{\sqrt{g}}{2G}h^A\mathbb{G}_{AB}\partial_\tau^2 h^B, \tag{2.12}$$

where we introduced a collective notation for the indices of a symmetric rank-2 tensor, $h^A \equiv h_{ij}$. The DeWitt metric in the space of such tensors and its inverse read

$$\mathbb{G}^{ij,kl} = \frac{1}{8}(g^{ik}g^{jl} + g^{il}g^{jk}) - \frac{\lambda}{4}g^{ij}g^{kl},$$
$$\mathbb{G}^{-1}_{ij,kl} = 2(g_{ik}g_{jl} + g_{il}g_{jk}) + \frac{4\lambda}{1-3\lambda}g_{ij}g_{kl}. \tag{2.13}$$

The part of the quadratic action with space derivatives of the metric is too lengthy (contains hundred of terms) to be written explicitly. We have obtained it using the tensor computer algebra package *xAct* [44–47] for *Mathematica* [42]. Schematically, it has the form

$$\mathcal{L}_{\rm pot,hh} + \mathcal{L}_{{\rm gf},hh} = \frac{\sqrt{g}}{2G}h^A\mathbb{D}_{AB}h^B, \tag{2.14}$$

where $\mathbb{D}_{AB}$ is a purely three-dimensional differential operator of sixth order. Note that in the indices A and B it is not an operator, but rather a quadratic form. In flat background (2.14) reduces to terms with exactly six derivatives [23],

$$\mathcal{L}_{\mathrm{pot},hh}+\mathcal{L}_{gf,hh}=\frac{1}{2G}\left[-\frac{\nu_5}{4}h^{ij}\Delta^3 h_{ij}+\left(\frac{\nu_5}{2}-\frac{1}{4\sigma}\right)h^{ik}\Delta^2\partial_i\partial_j h^{jk}+\left(-\nu_4-\frac{\nu_5}{2}-\frac{\xi}{4\sigma}\right)h^{ij}\Delta\partial_i\partial_j\partial_k\partial_l h^{kl}\right.$$
$$\left.+\left(2\nu_4+\frac{\nu_5}{2}+\frac{\lambda(1+\xi)}{2\sigma}\right)h\Delta^2\partial_k\partial_l h^{kl}+\left(-\nu_4-\frac{\nu_5}{4}-\frac{\lambda^2(1+\xi)}{4\sigma}\right)h\Delta^3 h\right]. \tag{2.15}$$

D. Total one-loop action

The one-loop effective action is given by the Gaussian path integral

$$\exp\left(-\Gamma^{1-\mathrm{loop}}\right)=\sqrt{\mathrm{Det}\,\mathcal{O}_{ij}}\int[dh^A dn^i dc^i d\bar{c}_j]\exp(-S^{(2)}[h^A,n^i,c^i,c_j]), \tag{2.16}$$

where the quadratic part of the full action consists of three contributions—metric, shift vector, and ghost ones,

$$S^{(2)}[h^A,n^i,c^i,\bar{c}_j]=\frac{1}{G}\int d\tau\, d^3x\sqrt{g}\left[\frac{1}{2}h^A(-\mathbb{G}_{AB}\partial_\tau^2+\mathbb{D}_{AB})h^B+\frac{1}{2}\sigma n^i\mathcal{O}_{ik}(-\delta^k_j\partial_\tau^2+\mathbb{B}^k{}_j)n^j+\bar{c}_i(-\delta^i_j\partial_\tau^2+\mathbb{B}^i{}_j)c^j\right]. \tag{2.17}$$

The normalization factor $\sqrt{\mathrm{Det}\,\mathcal{O}_{ij}}$ comes from the smearing of the gauge-fixing conditions with a Gaussian weight which leads to the gauge-breaking term (2.3) [15]. The result of the integration

$$\exp\left(-\Gamma^{1-\mathrm{loop}}\right)=\sqrt{\mathrm{Det}\,\mathcal{O}_{ij}}\frac{\mathrm{Det}\left(-\delta^i_j\partial_\tau^2+\mathbb{B}^i{}_j\right)}{\sqrt{\mathrm{Det}(-\mathbb{G}_{AB}\partial_\tau^2+\mathbb{D}_{AB})}\sqrt{\mathrm{Det}\left[\mathcal{O}_{ik}(-\delta^k_j\partial_\tau^2+\mathbb{B}^k{}_j)\right]}} \tag{2.18}$$

immediately shows that the contribution of the operator $\mathcal{O}_{ij}$ cancels out, while the shift and ghost parts reduce to the contribution of a single functional determinant. Factoring out and disregarding the ultralocal determinant of the DeWitt metric,[7] we write the effective action as the sum of two parts,

$$\Gamma^{1-\mathrm{loop}}=\frac{1}{2}\mathrm{Tr}\,\ln(-\delta^A_B\partial_\tau^2+\mathbb{D}^A{}_B)-\frac{1}{2}\mathrm{Tr}\,\ln\left(-\delta^i_j\partial_\tau^2+\mathbb{B}^i{}_j\right), \tag{2.19}$$

where

$$\mathbb{D}^A{}_B=(\mathbb{G}^{-1})^{AC}\mathbb{D}_{CB} \tag{2.20}$$

is now an *operator* in the condensed indices A and B. We presently turn to the computation of the functional traces entering in (2.19).

III. 3D REDUCTION: ONE-LOOP EFFECTIVE ACTION AS THE TRACE OF AN OPERATOR SQUARE ROOT

We begin by using the proper-time representation for the trace of the logarithm of an operator

$$\mathrm{Tr}\ \ln\mathbb{F}=-\int_0^\infty\frac{ds_6}{s_6}\mathrm{Tr}\,\mathrm{e}^{-s_6(-\partial_\tau^2+\mathbb{F})}, \tag{3.1}$$

where $\mathbb{F}$ is either $\mathbb{D}^A{}_B$ or $\mathbb{B}^i{}_j$. The subscript of the parameter s_6 emphasizes its scaling dimension, $[s_6]=-6$, which provides that the exponent is dimensionless (recall that the dimension of the operator $-\partial_\tau^2+\mathbb{F}$ is 6). Thus, we obtain the expression for the metric tensor part of the effective action,

[7]Which is actually canceled by the local measure $\sqrt{\mathrm{Det}\mathbb{G}_{AB}\delta(x,x')}$ arising in the Lagrangian path integral after the transition from the canonical one [48].

$$\Gamma^{1-\text{loop}}_{\text{metric}} = -\frac{1}{2}\int_0^\infty \frac{ds_6}{s_6}\,\text{Tr}\,e^{-s_6(-\delta^A_B\partial^2_\tau + \mathbb{D}^A{}_B)}$$
$$= -\frac{1}{2}\int d\tau d^3x \int \frac{ds_6}{s_6}\,\text{tr}\,e^{-s_6(-\delta^A_B\partial^2_\tau + \mathbb{D}^A{}_B)}\delta(\tau-\tau')\delta(\mathbf{x}-\mathbf{x}')|_{\tau=\tau',\mathbf{x}=\mathbf{x}'}, \quad (3.2)$$

where the operator acts on the first arguments of the δ-functions before taking the coincidence limit and "tr" stands for the simple trace over matrix indices $A = (ij)$. For a static (τ-independent) background it can be transformed by the following chain of relations:

$$\Gamma^{1-\text{loop}}_{\text{metric}} = -\frac{1}{2}\int d\tau\, d^3x \int \frac{ds_6}{s_6}\,\text{tr}\,e^{-s_6(-\delta^A_B\partial^2_\tau + \mathbb{D}^A{}_B)}\int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')}\delta(\mathbf{x}-\mathbf{x}')\Big|_{\tau-\tau',\mathbf{x}=\mathbf{x}'}$$
$$= -\frac{1}{2}\int d\tau\, d^3x \int \frac{ds_6}{s_6}\int \frac{d\omega}{2\pi} e^{-s_6\omega^2}\,\text{tr}\,e^{-s_6\mathbb{D}^A{}_B}\delta(\mathbf{x}-\mathbf{x}')|_{\mathbf{x}=\mathbf{x}'}$$
$$= -\frac{1}{4\sqrt{\pi}}\int d\tau\, d^3x \int \frac{ds_6}{s_6^{3/2}}\,\text{tr}\,e^{-s_6\mathbb{D}^A{}_B}\delta(\mathbf{x}-\mathbf{x}')|_{\mathbf{x}=\mathbf{x}'}$$
$$= -\frac{\Gamma(-1/2)}{4\sqrt{\pi}}\int d\tau\, d^3x\,\text{tr}\sqrt{\mathbb{D}^A{}_B}\delta(\mathbf{x}-\mathbf{x}')|_{\mathbf{x}=\mathbf{x}'}$$
$$= \frac{1}{2}\int d\tau\,\text{Tr}_3\sqrt{\mathbb{D}^A{}_B}. \quad (3.3)$$

Note that Tr_3 in the final formula is the functional trace in the three-dimensional sense (contrary to the four-dimensional one $\text{Tr} \equiv \text{Tr}_4$). Thus, we conclude that the calculation of the one-loop effective action boils down to the calculation of the functional trace of the square root of $\mathbb{D}^A{}_B$, which we denote by $Q_D{}^A_B \equiv \sqrt{\mathbb{D}^A{}_B}$. This is a purely three-dimensional problem. An analogous procedure can be carried out for the vector part of the trace. Introducing the notation $Q_B{}^i_j \equiv \sqrt{\mathbb{B}^i{}_j}$ the full one-loop action can be expressed as

$$\Gamma^{1-\text{loop}} = \frac{1}{2}\int d\tau[\text{Tr}_3 Q_D{}^A_B - \text{Tr}_3 Q_B{}^i_j]. \quad (3.4)$$

Let us outline the strategy for evaluation of the above operator traces. By commuting the covariant derivatives contracted with each other to the right and collapsing them into powers of the Laplacian the local operators $\mathbb{F} = (\mathbb{D}, \mathbb{B})$ can be brought into the following schematic form:

$$\mathbb{F} = \sum_{a=0}^{6}\mathcal{R}_{(a)}\sum_{6\geq 2k\geq a}\alpha_{a,k}\nabla_1\cdots\nabla_{2k-a}(-\Delta)^{3-k}. \quad (3.5)$$

Here $\mathcal{R}_{(a)}$ are background field tensors built of the curvature and its derivatives of the following dimensionality in units of inverse length:

$$\mathcal{R}_{(a)} = O\left(\frac{1}{l^a}\right). \quad (3.6)$$

We will refer to them as "coefficient functions." On the other hand, $\alpha_{a,k}$ are dimensionless scalar coefficients depending on the couplings λ, $\nu_1, \ldots, \nu_5$. Overall powers of derivatives and Laplacians are related to the dimensionality of the coefficient functions to maintain the total dimensionality of $\mathbb{F}$ which is six.

The square root of such operators can be obtained by the perturbation theory in powers of the background curvature and the derivatives of this curvature, that is again in powers of $1/l$. However, in contrast to $\mathbb{F}$ this is not a finite length polynomial, but rather a nonlocal *pseudodifferential* operator given by an infinite series in $\mathcal{R}_{(a)}$,

$$\sqrt{\mathbb{F}} = \sum_{a=0}^{\infty}\mathcal{R}_{(a)}\sum_{k\geq a/2}^{K_a}\tilde{\alpha}_{a,k}\nabla_1\cdots\nabla_{2k-a}\frac{1}{(-\Delta)^{k-3/2}}, \quad (3.7)$$

with some other coefficients $\tilde{\alpha}_{a,k}$ obtained from $\alpha_{a,k}$ above. At each dimensionality a the powers of derivatives are bounded from above by some finite number $2K_a - a$. Indeed, the number of free tensor indices K_{free} of the operator is fixed by the nature of the space it acts on: $K_{\text{free}} = 2$ for $\sqrt{\mathbb{B}}$ and $K_{\text{free}} = 4$ for $\sqrt{\mathbb{D}}$. All the indices of the derivatives that are not contracted with each other, minus the number of free indices, must be contracted with the indices of $\mathcal{R}_{(a)}$. The latter is bounded by a. Thus, we have

$$2k - a \leq K_{\text{free}} + a. \quad (3.8)$$

Altogether this means that the UV divergent part of (3.4) follows from the calculation of UFTs of the form

$$\int d^3x \mathcal{R}_{(a)}(\mathbf{x}) \nabla_1 \cdots \nabla_{2k-a} \frac{1}{(-\Delta)^{k-3/2}} \delta(\mathbf{x}, \mathbf{x}')|_{\mathbf{x}=\mathbf{x}'}. \quad (3.9)$$

Since the divergences of HG have at maximum the dimensionality $a = 6$, only a finite number of such traces will be needed. This method splits the problem into two steps—calculation of the operator square root (3.7) and the evaluation of UFTs (3.9)—which makes it computationally efficient.

The first step is the perturbative calculation of the square root (3.7). This calculation is based on the fact that in the lowest order approximation of the expansion in curvature the covariant derivatives commute. Thus, the procedure reduces to the extraction of the square root from a c-number matrix—the principal symbol of the operator, obtained by replacing the covariant derivatives with c-number momenta and neglecting all terms proportional to curvature. Going back in the resulting matrix from these momenta to covariant derivatives one gets the operator $\mathbb{Q}^{(0)}$. By denoting all curvature corrections in $\sqrt{\mathbb{F}}$ as $\mathbb{X}$,

$$\sqrt{\mathbb{F}} = \mathbb{Q}^{(0)} + \mathbb{X}, \quad (3.10)$$

one obtains the equation for this correction term

$$\mathbb{Q}^{(0)}\mathbb{X} + \mathbb{X}\mathbb{Q}^{(0)} = \mathbb{F} - (\mathbb{Q}^{(0)})^2 - \mathbb{X}^2. \quad (3.11)$$

This nonlinear equation can be solved by iterations because its right-hand side (RHS) is at least linear in curvature. Indeed, the difference $\mathbb{F} - (\mathbb{Q}^{(0)})^2 \propto R$ is nonzero entirely due to the commutation of covariant derivatives, proportional to the Riemann tensor R. At each stage of this iteration procedure one has to go from the operator $\mathbb{X}$ to its c-number symbol. Then one finds this symbol from the matrix equation (3.11) in which the right-hand side is known with a needed accuracy from the previous iteration stages. This is the so-called Sylvester equation, and its solution will be constructed below. In the meantime we focus on the square root of the principal symbol of $\mathbb{F}$.

A. Square root of the principal symbol and four gauge choices

1. Tensor sector

From Eq. (2.15) we read off the quadratic form $\mathbb{D}^{mn,kl}$ in flat space. Replacing the derivatives with the momenta, $\partial_i \mapsto i p_i$, and contracting with the inverse DeWitt metric $\mathbb{G}^{-1}_{ij,mn}$, we obtain the principal symbol of the operator $\mathbb{D}$,

$$\mathbb{D}(\mathbf{p})_{ij}{}^{kl} = p^6 \Bigg[\frac{\nu_5}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j) + \frac{4\nu_4(1-\lambda) + \nu_5}{1-3\lambda} \delta_{ij}\delta^{kl} + \left(-\frac{\nu_5}{2} + \frac{1}{4\sigma}\right)(\delta^k_i \hat{p}_j \hat{p}^l + \delta^l_i \hat{p}_j \hat{p}^k + \delta^k_j \hat{p}_i \hat{p}^l + \delta^l_j \hat{p}_i \hat{p}^k)$$
$$+ \left(\frac{-4\nu_4(1-\lambda) - \nu_5}{1-3\lambda}\right)\delta_{ij}\hat{p}^k \hat{p}^l + \left(-4\nu_4 - \nu_5 - \frac{\lambda(1+\xi)}{\sigma}\right)\hat{p}_i \hat{p}_j \delta^{kl} + \left(4\nu_4 + 2\nu_5 + \frac{\xi}{\sigma}\right)\hat{p}_i \hat{p}_j \hat{p}^k \hat{p}^l\Bigg], \quad (3.12)$$

where $\hat{\mathbf{p}} = \mathbf{p}/p$ is the unit vector along the momentum. This is a 6×6 matrix acting in the space of symmetric tensors h_{kl}. To extract its square root, we need to find its eigenvalues and eigenvectors. We do it by decomposing h_{kl} into a transverse-traceless, vector, and scalar parts. Namely, we write

$$h_{kl} = T_{(r)} e^{(r)}_{kl} + V_{(r)} \frac{1}{\sqrt{2}}(e^{(r)}_k \hat{p}_l + \hat{p}_k e^{(r)}_l) + \phi \frac{1}{\sqrt{2}}(\delta_{kl} - \hat{p}_k \hat{p}_l) + \psi \hat{p}_k \hat{p}_l, \quad (3.13)$$

where $e^{(r)}_k$, $r = 1, 2$ form the basis of unit vectors orthogonal to $\hat{\mathbf{p}}$, $e^{(r)}_{kl}$, $r = 1, 2$ are the two transverse traceless polarization tensors, and $T_{(r)}$, $V_{(r)}$, ϕ, and ψ are coefficients. It is straightforward to see that $e^{(r)}_{kl}$ are eigenvectors of (3.12) with the eigenvalue $\kappa_T = \nu_5 p^6$, whereas the vector polarizations $(e^{(r)}_k \hat{p}_l + \hat{p}_k e^{(r)}_l)/\sqrt{2}$ are eigenvectors with the eigenvalue $\kappa_V = p^6/2\sigma$. The projectors on the corresponding subspaces are

$$\mathbb{P}^{(T)kl}_{ij} \equiv \sum_{r=1,2} e^{(r)}_{ij} e^{(r)kl}$$
$$= \frac{1}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j) - \frac{1}{2}\delta_{ij}\delta^{kl} - \frac{1}{2}(\delta^k_i \hat{p}_j \hat{p}^l + \delta^l_i \hat{p}_j \hat{p}^k + \delta^k_j \hat{p}_i \hat{p}^l + \delta^l_j \hat{p}_i \hat{p}^k) + \frac{1}{2}(\delta_{ij}\hat{p}^k \hat{p}^l + \hat{p}_i \hat{p}_j \delta^{kl}) + \frac{1}{2}\hat{p}_i \hat{p}_j \hat{p}^k \hat{p}^l, \quad (3.14a)$$

$$\mathbb{P}^{(V)kl}_{ij} \equiv \sum_{r=1,2} \frac{1}{2}(e^{(r)}_i \hat{p}_j + \hat{p}_i e^{(r)}_j)(e^{(r)k} \hat{p}^l + \hat{p}^k e^{(r)l})$$
$$= \frac{1}{2}(\delta^k_i \hat{p}_j \hat{p}^l + \delta^l_i \hat{p}_j \hat{p}^k + \delta^k_j \hat{p}_i \hat{p}^l + \delta^l_j \hat{p}_i \hat{p}^k) - 2\hat{p}_i \hat{p}_j \hat{p}^k \hat{p}^l. \quad (3.14b)$$

In the scalar sector the situation is more subtle. Here we have two eigenvalues that in general are not degenerate. To see this we act with $\mathbb{D}(\mathbf{p})$ on the scalar part of (3.13) and find

$$\mathbb{D}(\mathbf{p})_{ij}{}^{kl}h_{kl}|_{\rm scalar} = p^6\left[\phi\frac{\nu_s}{\sqrt{2}}(\delta_{ij}-\hat{p}_i\hat{p}_j)+\left(\phi\sqrt{2}\lambda\left(\frac{\nu_s}{1-\lambda}-\frac{1+\xi}{\sigma}\right)+\psi\frac{(1-\lambda)(1+\xi)}{\sigma}\right)\hat{p}_i\hat{p}_j\right], \tag{3.15}$$

where ν_s is defined by Eq. (1.11). Thus, in the two-dimensional subspace of vectors $\Upsilon = (\phi,\psi)^{\rm T}$ the operator $\mathbb{D}(\mathbf{p})$ acts as a matrix

$$p^6\begin{pmatrix} \nu_s & 0 \\ \sqrt{2}\lambda(\frac{\nu_s}{1-\lambda}-\frac{1+\xi}{\sigma}) & \frac{(1-\lambda)(1+\xi)}{\sigma}\end{pmatrix}. \tag{3.16}$$

The corresponding eigenvalues and eigenvectors are

$$\kappa_{S1} = \nu_s p^6, \qquad \Upsilon_{S1} = \begin{pmatrix} 1 \\ \frac{\sqrt{2}\lambda}{1-\lambda}\end{pmatrix}, \tag{3.17a}$$

$$\kappa_{S2} = \frac{(1-\lambda)(1+\xi)}{\sigma}p^6, \qquad \Upsilon_{S2} = \begin{pmatrix} 0 \\ 1\end{pmatrix}. \tag{3.17b}$$

It is convenient to construct the operators $\mathbb{P}^{(S1)}$ and $\mathbb{P}^{(S2)}$ projecting on the vectors Υ_{S1} and Υ_{S2}, respectively. This is done using the linear forms conjugate to these vectors

$$\Upsilon_1^\dagger = (1\quad 0), \qquad \Upsilon_2^\dagger = \left(-\tfrac{\sqrt{2}\lambda}{1-\lambda}\quad 1\right), \tag{3.18}$$

that have the property $\Upsilon_r^\dagger\Upsilon_q = \delta_{rq}$, r, $q = 1, 2$. Then

$$\mathbb{P}^{(S1)} = \Upsilon_{S1}\otimes\Upsilon_{S1}^\dagger, \qquad \mathbb{P}^{(S2)} = \Upsilon_{S2}\otimes\Upsilon_{S2}^\dagger. \tag{3.19}$$

Restoring the spatial indices we have

$$\mathbb{P}_{ij}^{(1)kl} = \frac{1}{2}\left(\delta_{ij}-\frac{1-3\lambda}{1-\lambda}\hat{p}_i\hat{p}_j\right)(\delta^{kl}-\hat{p}^k\hat{p}^l),$$
$$\mathbb{P}_{ij}^{(2)kl} = \hat{p}_i\hat{p}_j\left(-\frac{\lambda}{1-\lambda}\delta^{kl}+\frac{1}{1-\lambda}\hat{p}^k\hat{p}^l\right). \tag{3.20}$$

It is now straightforward to verify that the principal symbol (3.12) decomposes into the sum of projectors,

$$\mathbb{D}(\mathbf{p})_{ij}{}^{kl} = \nu_5 p^6\mathbb{P}_{ij}^{(T)kl}+\frac{p^6}{2\sigma}\mathbb{P}_{ij}^{(V)kl}+\nu_s p^6\mathbb{P}_{ij}^{(S1)kl} + \frac{(1-\lambda)(1+\xi)}{\sigma}p^6\mathbb{P}_{ij}^{(S2)kl}. \tag{3.21}$$

Then its square root is obtained by taking the square roots of the coefficients,

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p}) = \sqrt{\nu_5}p^3\sum_\alpha u_\alpha\mathbb{P}^{(\alpha)}, \quad \alpha = T, V, S1, S2, \tag{3.22}$$

where

$$u_T = 1, \qquad u_V = \frac{1}{\sqrt{2\sigma\nu_5}}, \qquad u_{S1} = u_s,$$
$$u_{S2} = \sqrt{\frac{(1-\lambda)(1+\xi)}{\sigma\nu_5}}, \tag{3.23}$$

and u_s is defined in (1.12). Expanding the projectors, we finally arrive at

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p})_{ij}{}^{kl} = \sqrt{\nu_5}p^3\left[\frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)+\frac{u_s-1}{2}\delta_{ij}\delta^{kl}+\frac{u_V-1}{2}(\delta_i^k\hat{p}_j\hat{p}^l+\delta_i^l\hat{p}_j\hat{p}^k+\delta_j^k\hat{p}_i\hat{p}^l+\delta_j^l\hat{p}_i\hat{p}^k)\right.$$
$$\left.-\frac{u_s-1}{2}\delta_{ij}\hat{p}^k\hat{p}^l-\left(\frac{u_s}{2}\frac{1-3\lambda}{1-\lambda}-\frac{1}{2}+\frac{\lambda u_{S2}}{1-\lambda}\right)\hat{p}_i\hat{p}_j\delta^{kl}+\left(\frac{1}{2}-2u_V+\frac{u_s}{2}\frac{1-3\lambda}{1-\lambda}+\frac{u_{S2}}{1-\lambda}\right)\hat{p}_i\hat{p}_j\hat{p}^k\hat{p}^l\right]. \tag{3.24}$$

This principal symbol plays the central role in the perturbative calculation of the operator square root $\mathbb{Q}_{\mathbb{D}}$. For a general choice of the gauge parameters σ and ξ this calculation is prohibitively complex. Thus we restrict to four gauge choices that simplify expression (3.24).

Gauge (a) First, we consider a choice where the eigenvalues of the gauge modes coincide with those of the physical modes. Namely, we take

$$u_V = 1, \quad u_{S2} = u_s \Leftrightarrow \sigma = \frac{1}{2\nu_5}, \quad \xi = \frac{\nu_s}{2\nu_5(1-\lambda)}-1. \tag{3.25}$$

This yields

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p})_{ij}{}^{kl} = \sqrt{\nu_5}p^3\left[\frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)+\frac{u_s-1}{2}\delta_{ij}\delta^{kl} -\frac{u_s-1}{2}\delta_{ij}\hat{p}^k\hat{p}^l-\frac{u_s-1}{2}\hat{p}_i\hat{p}_j\delta^{kl} +\frac{3}{2}(u_s-1)\hat{p}_i\hat{p}_j\hat{p}^k\hat{p}^l\right]. \tag{3.26}$$

Importantly, this choice overlaps with the gauges considered in Ref. [23] (see also Appendix A 1) which allows us to use the results of this paper for the (gauge-dependent) β-function of the coupling G in this gauge.

Gauge (b) The second choice is similar, but now

$$u_V = u_{S2} = 1 \Leftrightarrow \sigma = \frac{1}{2\nu_5}, \qquad \xi = -\frac{1-2\lambda}{2(1-\lambda)}, \tag{3.27}$$

and we obtain

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p})_{ij}{}^{kl} = \sqrt{\nu_5}p^3\left[\frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) + \frac{u_s-1}{2}\delta_{ij}\delta^{kl} - \frac{u_s-1}{2}\delta_{ij}\hat{p}^k\hat{p}^l - \frac{1-3\lambda}{2(1-\lambda)}(u_s-1)\hat{p}_i\hat{p}_j\delta^{kl} + \frac{1-3\lambda}{2(1-\lambda)}(u_s-1)\hat{p}_i\hat{p}_j\hat{p}^k\hat{p}^l\right]. \tag{3.28}$$

This also overlaps with the choices considered in [23].

Gauge (c) Two other choices are adjusted to remove the term with four vectors $\hat{\mathbf{p}}$ in (3.24) which is challenging from the computational viewpoint.[8] The most simplifying choice is

$$u_V = 1, \qquad u_{S2} = \frac{3(1-\lambda)}{2} - \frac{(1-3\lambda)u_s}{2} \Leftrightarrow \sigma = \frac{1}{2\nu_5},$$
$$\xi = \frac{(3(1-\lambda)-(1-3\lambda)u_s)^2}{8(1-\lambda)} - 1. \tag{3.29}$$

The principal symbol becomes

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p})_{ij}{}^{kl} = \sqrt{\nu_5}p^3\left[\frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) + \frac{u_s-1}{2}\delta_{ij}\delta^{kl} - \frac{u_s-1}{2}\delta_{ij}\hat{p}^k\hat{p}^l - \frac{1-3\lambda}{2}(u_s-1)\hat{p}_i\hat{p}_j\delta^{kl}\right]. \tag{3.30}$$

A drawback of this choice is that it differs from the gauges studied in [23]. Therefore, in this gauge we cannot compute the running of the essential coupling $\mathcal{G}$ [see Eq. (1.12)] which requires the knowledge of the β-function for G. Nevertheless, we can still compute the running of u_s and v_a, $a = 1, 2, 3$.

Gauge (d) To remedy the above drawback of gauge (c) we also consider

$$u_V = u_{S2} = \frac{1-\lambda+(1-3\lambda)u_s}{2(1-2\lambda)} \Longleftrightarrow$$
$$\sigma = \frac{2(1-2\lambda)^2}{\nu_5(1-\lambda+(1-3\lambda)u_s)^2}, \quad \xi = -\frac{1-2\lambda}{2(1-\lambda)}. \tag{3.31}$$

Here the principal symbol takes the form

$$\mathbb{Q}_{\mathbb{D}}(\mathbf{p})_{ij}{}^{kl} = \sqrt{\nu_5}p^3\left[\frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) + \frac{u_s-1}{2}\delta_{ij}\delta^{kl} - \frac{u_s-1}{2}\delta_{ij}\hat{p}^k\hat{p}^l - \frac{(1-3\lambda)(u_s-1)}{2(1-2\lambda)}\hat{p}_i\hat{p}_j\delta^{kl} + \frac{(1-3\lambda)(u_s-1)}{4(1-2\lambda)}(\delta_i^k\hat{p}_j\hat{p}^l + \delta_i^l\hat{p}_j\hat{p}^k + \delta_j^k\hat{p}_i\hat{p}^l + \delta_j^l\hat{p}_i\hat{p}^k)\right]. \tag{3.32}$$

This gauge again overlaps with the gauges used in [23].

Comparison of the results obtained in four different gauges (a)–(d) provides a strong check of our calculation.

2. Vector sector

We now repeat the analysis for the vector operator $\mathbb{B}$ given by the expression (2.8). Its principal symbol reads

$$\mathbb{B}^i{}_j(\mathbf{p}) = p^6\left(\frac{1}{2\sigma}\delta_j^i + \frac{1-2\lambda+2\xi(1-\lambda)}{2\sigma}\hat{p}^i\hat{p}_j\right). \tag{3.33}$$

This can easily be written in terms of the transverse and longitudinal projectors,

$$\mathbb{B}^i{}_j(\mathbf{p}) = \frac{p^6}{2\sigma}\mathbb{P}^{(VT)i}{}_j + \frac{(1-\lambda)(1+\xi)}{\sigma}p^6\mathbb{P}^{(VL)i}{}_j, \tag{3.34}$$

where

$$\mathbb{P}^{(VT)i}{}_j = \delta_j^i - \hat{p}^i\hat{p}_j, \qquad \mathbb{P}^{(VL)i}{}_j = \hat{p}^i\hat{p}_j. \tag{3.35}$$

Then the square root reads

$$\mathbb{Q}_{\mathbb{B}}(\mathbf{p}) = \sqrt{\nu_5}p^3\sum_\alpha u_\alpha\mathbb{P}^{(\alpha)}, \qquad \alpha = VT, VL,$$
$$u_{VT} = u_V, u_{VL} = u_{S2}. \tag{3.36}$$

B. Canonical form of the pseudodifferential operators

The next step in the procedure outlined at the beginning of this section [see Eqs. (3.10) and (3.11)] consists of the recovery of the pseudo-differential operator $\mathbb{Q}^{(0)}$ from its symbol $\mathbb{Q}(\mathbf{p})$. The result of this procedure is the canonical form which we formulate as follows:

[8]When transformed back to configuration space, the four momenta become four covariant derivatives that must be commuted through the other operators in the course of the perturbative procedure, see below.

(1) All (fractional) powers of p^2 are replaced by covariant Laplacians $-\Delta$ and put to the right.
(2) Other occurrences of momenta are replaced by covariant derivatives, $p_i \mapsto -i\nabla_i$. The covariant derivatives whose indices are contracted with the tensor indices of the metric fluctuations or the shift vector are placed to the right.

As an example consider $\mathbb{Q}^{(0)}_{\mathbb{D}}$ in the gauge (a). The above prescription gives

$$\begin{aligned}\mathbb{Q}^{(0)kl}_{\mathbb{D}ij} = \sqrt{\nu_5}\bigg[\bigg(\frac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j) + \frac{u_s-1}{2}g_{ij}g^{kl}\bigg)(-\Delta)^{3/2} \\ + \frac{u_s-1}{2}g_{ij}\nabla^k\nabla^l(-\Delta)^{1/2} \\ + \frac{u_s-1}{2}g^{kl}\nabla_i\nabla_j(-\Delta)^{1/2} \\ + \frac{3(u_s-1)}{2}\nabla_i\nabla_j\nabla^k\nabla^l(-\Delta)^{-1/2}\bigg]. \end{aligned} \tag{3.37}$$

Note that the result of the action of this operator on a symmetric metric fluctuation $\mathbb{Q}^{(0)kl}_{\mathbb{D}ij}h_{kl}$ is automatically symmetric in the indices (i,j). In other words, this operator acts in the space of symmetric tensors, as it should. Similarly for the vector operator, its zeroth-order part in a generic (σ,ξ)-gauge takes the form

$$\mathbb{Q}^{(0)i}_{\mathbb{B}j} = \sqrt{\nu_5}[u_V\delta^i_j(-\Delta)^{3/2} + (u_V - u_{S2})\nabla^i\nabla_j(-\Delta)^{1/2}]. \tag{3.38}$$

In a more general case of the curvature dependent part $\mathbb{X}$ of the square root operator the number of derivatives is higher and more ordering ambiguities arise. Thus, we supplement this prescription by one more rule:

(3) The derivatives not covered by rules 1 and 2 are ordered by using the "SortCovDs" command of the *xAct* package [44].

C. Solution of the Sylvester equation

Perturbation theory for the square root operator $\mathbb{Q}$ implies solving the equation (3.11) for its curvature part $\mathbb{X}$. At each stage of the corresponding iteration procedure we will encounter the matrix equation of the following form:

$$\mathbb{Q}(\mathbf{p})\mathbb{X}(\mathbf{p}) + \mathbb{X}(\mathbf{p})\mathbb{Q}(\mathbf{p}) = \mathbb{Y}(\mathbf{p}). \tag{3.39}$$

Here $\mathbb{Q}(\mathbf{p})$ is the c-number symbol of $\mathbb{Q}^{(0)}$, the matrix $\mathbb{Y}(\mathbf{p})$, which is the symbol of the operator $\mathbb{Y} \equiv \mathbb{F} - (\mathbb{Q}^{(0)})^2 - \mathbb{X}^2$ in the right-hand side of Eq. (3.11), is assumed to be known, and we need to find $\mathbb{X}(\mathbf{p})$. All matrices depend on the three-dimensional wave number $\mathbf{p}$. Equation (3.39) is a special case of the Sylvester matrix equation, and its solution can be found with the general method of Ref. [49]. In the case at hand, however, it is easier to obtain the solution using the representation of $\mathbb{Q}(\mathbf{p})$ in terms of the projectors (3.22) and (3.35). The solution reads

$$\mathbb{X}(\mathbf{p}) = \frac{1}{\sqrt{\nu_5}p^3}\sum_{\alpha,\beta}\frac{1}{u_\alpha + u_\beta}\mathbb{P}^{(\alpha)}\mathbb{Y}(\mathbf{p})\mathbb{P}^{(\beta)}, \tag{3.40}$$

where the sum is taken over $\alpha,\beta = T, V, S1, S2$ (tensor sector) or $\alpha,\beta = VT, VL$ (vector sector). The proof goes by a direct substitution:

$$\begin{aligned}\mathbb{Q}(\mathbf{p})\mathbb{X}(\mathbf{p}) &= \sum_{\alpha,\beta}\frac{u_\alpha}{u_\alpha+u_\beta}\mathbb{P}^{(\alpha)}\mathbb{Y}(\mathbf{p})\mathbb{P}^{(\beta)}, \\ \mathbb{X}(\mathbf{p})\mathbb{Q}(\mathbf{p}) &= \sum_{\alpha,\beta}\frac{u_\beta}{u_\alpha+u_\beta}\mathbb{P}^{(\alpha)}\mathbb{Y}(\mathbf{p})\mathbb{P}^{(\beta)}.\end{aligned} \tag{3.41}$$

Here we have used the orthogonality of the projectors $\mathbb{P}^{(\alpha)}\mathbb{P}^{(\beta)} = \delta^{\alpha\beta}\mathbb{P}^{(\alpha)}$. Summing up the two expressions we obtain

$$\begin{aligned}\mathbb{Q}(\mathbf{p})\mathbb{X}(\mathbf{p}) + \mathbb{X}(\mathbf{p})\mathbb{Q}(\mathbf{p}) &= \sum_{\alpha,\beta}\mathbb{P}^{(\alpha)}\mathbb{Y}(\mathbf{p})\mathbb{P}^{(\beta)} \\ &= \bigg(\sum_\alpha\mathbb{P}^{(\alpha)}\bigg)\mathbb{Y}(\mathbf{p})\bigg(\sum_\beta\mathbb{P}^{(\beta)}\bigg) \\ &= \mathbb{Y}(\mathbf{p}),\end{aligned} \tag{3.42}$$

where in the last equality we used the completeness of the projector basis. In what follows we will denote the linear map from the RHS of the Sylvester equation to its solution by "Syl," so that we will write

$$\mathbb{X}(\mathbf{p}) = \mathrm{Syl}[\mathbb{Y}(\mathbf{p})]. \tag{3.43}$$

D. Perturbative scheme

As discussed in the beginning of this section, to find the one-loop renormalization of the action we need to construct an operator $\mathbb{Q}$ whose square coincides with the operator $\mathbb{F} = (\mathbb{D},\mathbb{B})$ entering the quadratic action for the fluctuations. We perform this construction perturbatively in the powers of the background curvature and its derivatives. Namely, we write

$$\mathbb{Q} = \mathbb{Q}^{(0)} + \mathbb{Q}^{(2)} + \mathbb{Q}^{(3)} + \mathbb{Q}^{(4)} + \mathbb{Q}^{(5)} + \mathbb{Q}^{(6)} + \cdots, \tag{3.44}$$

where the index in the brackets represents the order of the operator in powers of the inverse length scale characterizing the background curvature. Here the operator $\mathbb{Q}^{(0)}$ is given by (3.37) or (3.38), and it does not contain any background curvature. The operator $\mathbb{Q}^{(2)}$ is linear in curvature, and this is counted as second order, since the curvature contains two derivatives of the metric. The operator $\mathbb{Q}^{(3)}$ contains first derivatives of the curvature (three derivatives of the metric), and so on. Dots stand for higher-order terms that do not contribute into the divergent part of the action.

Substitution of this expansion into the defining relation

$$\mathbb{Q}^2 = \mathbb{F} \tag{3.45}$$

produces at each order an equation of the form,

$$\mathbb{Q}^{(0)}\mathbb{Q}^{(a)} + \mathbb{Q}^{(a)}\mathbb{Q}^{(0)} = \mathbb{Y}^{(a)}, \tag{3.46}$$

with the RHS

$$\mathbb{Y}^{(a)} = \mathbb{F} - \sum_{\substack{b,c<a \\ b+c\leq a}} \mathbb{Q}^{(b)}\mathbb{Q}^{(c)}. \tag{3.47}$$

The operator $\mathbb{Y}^{(a)}$ contains terms of order a and higher. Then $\mathbb{Q}^{(a)}$ is constructed by the following algorithm:

(1) Pick up the part of $\mathbb{Y}^{(a)}$ which is exactly of order a; let us denote it by $\mathbb{Y}^{(a)}_a$.

(2) Replace the covariant derivatives acting on the metric fluctuations in $\mathbb{Y}^{(a)}_a$ by the wave vector, $\nabla_i \mapsto ip_i$. This gives the c-matrix symbol $\mathbb{Y}^{(a)}_a(\mathbf{p})$.

(3) Solve the corresponding Sylvester equation and define a matrix

$$\mathbb{Q}^{(a)}(\mathbf{p}) = \mathrm{Syl}[\mathbb{Y}^{(a)}_a(\mathbf{p})]. \tag{3.48}$$

(4) Replace the wave vectors in $\mathbb{Q}^{(a)}(\mathbf{p})$ back by the covariant derivatives, ordering them in a canonical way (see Sec. III B). For tensor operators, symmetrize $\mathbb{Q}^{(a)kl}_{ij}$ in the indices (ij) and (kl).

(5) Construct the combination $\mathbb{Q}^{(0)}\mathbb{Q}^{(a)} + \mathbb{Q}^{(a)}\mathbb{Q}^{(0)}$ and bring it to the canonical form. By construction, this combination coincides with $\mathbb{Y}^{(a)}_a$, up to terms of order higher than a. Subtract it from $\mathbb{Y}^{(a)}$ to define a new operator $\mathbb{Z}^{(a+1)}$.

(6) Construct other products $\mathbb{Q}^{(b)}\mathbb{Q}^{(c)}$ with $b, c < a$, $b + c = a + 1$, bring them to the canonical form, and subtract from $\mathbb{Z}^{(a+1)}$. This determines $\mathbb{Y}^{(a+1)}$, according to Eq. (3.47).

In this way we arrive at an iterative procedure for a consecutive determination of $\mathbb{Q}^{(a)}$. According to (3.47) the right-hand side of (3.46) at different steps is given by

$$\mathbb{Y}^{(2)} = \mathbb{F} - (\mathbb{Q}^{(0)})^2, \tag{3.49a}$$

$$\mathbb{Y}^{(3)} = \mathbb{Y}^{(2)} - \mathbb{Q}^{(0)}\mathbb{Q}^{(2)} - \mathbb{Q}^{(2)}\mathbb{Q}^{(0)}, \tag{3.49b}$$

$$\mathbb{Y}^{(4)} = \mathbb{Y}^{(3)} - \mathbb{Q}^{(0)}\mathbb{Q}^{(3)} - \mathbb{Q}^{(3)}\mathbb{Q}^{(0)} - (\mathbb{Q}^{(2)})^2, \tag{3.49c}$$

$$\mathbb{Y}^{(5)} = \mathbb{Y}^{(4)} - \mathbb{Q}^{(0)}\mathbb{Q}^{(4)} - \mathbb{Q}^{(4)}\mathbb{Q}^{(0)} - \mathbb{Q}^{(2)}\mathbb{Q}^{(3)} - \mathbb{Q}^{(3)}\mathbb{Q}^{(2)}, \tag{3.49d}$$

$$\mathbb{Y}^{(6)} = \mathbb{Y}^{(5)} - \mathbb{Q}^{(0)}\mathbb{Q}^{(5)} - \mathbb{Q}^{(5)}\mathbb{Q}^{(0)} - \mathbb{Q}^{(2)}\mathbb{Q}^{(4)} - \mathbb{Q}^{(4)}\mathbb{Q}^{(2)} - (\mathbb{Q}^{(3)})^2. \tag{3.49e}$$

We have automated the algorithm described above using the *Mathematica* [42] package *xAct* [44].

A few comments are in order. First, the coefficient functions of the fifth order operator $\mathbb{Q}^{(5)}$ contain either a third derivative of curvature or a product of curvature with its first derivative. None of these combinations can give rise to a divergent counterterm in the one-loop action. Indeed, the renormalizability of the theory implies that the counterterms have the same structure as the terms in the bare action which have order 6 in our power counting (see [15,20]). To produce a sixth order contribution the coefficients of $\mathbb{Q}^{(5)}$ would have to be multiplied by a covariant object constructed from the metric using a single derivative. But such objects do not exist. Thus, we conclude that $\mathbb{Q}^{(5)}$ does not contribute into the beta functions and can be dropped. Then one can verify that the fifth order contributions can consistently be omitted in all $\mathbb{Y}^{(a)}$ at all stages of the calculation. In particular, instead of solving consecutively for $\mathbb{Q}^{(5)}$ and then for $\mathbb{Q}^{(6)}$ using the $\mathbb{Y}$ operators (3.49d) and (3.49e), we can directly construct $\mathbb{Q}^{(6)}$ in a single step by solving Eq. (3.46) with the RHS

$$\mathbb{Y}^{(6)} = \mathbb{Y}^{(4)} - \mathbb{Q}^{(0)}\mathbb{Q}^{(4)} - \mathbb{Q}^{(4)}\mathbb{Q}^{(0)} - \mathbb{Q}^{(2)}\mathbb{Q}^{(3)} - \mathbb{Q}^{(3)}\mathbb{Q}^{(2)} - \mathbb{Q}^{(2)}\mathbb{Q}^{(4)} - \mathbb{Q}^{(4)}\mathbb{Q}^{(2)} - (\mathbb{Q}^{(3)})^2. \tag{3.50}$$

Second, the most time-consuming part of the calculation are steps 5 and 6, which involve bringing various operators to the canonical form. In detail, this canonicalization proceeds as follows (for concreteness, we focus on the metric sector):

(i) All (fractional) powers of the Laplacian acting on the metric fluctuations are commuted through the coefficient functions and covariant derivatives to the right and collapsed to a single fractional Laplacian. The commutation is performed using the formula

$$[A^\alpha, B] = \sum_{n=1}^{\infty} C^n_\alpha \underbrace{[A, [A, \ldots, [A}_{n}, B]] \cdots] A^{\alpha - n}, \tag{3.51}$$

valid for arbitrary operators A and B. Here

$$C^n_\alpha = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \tag{3.52}$$

are the binomial coefficients. This formula is proved in Appendix B.

(ii) Next, we bring to the right all covariant derivatives contracted with the metric fluctuations. For example, an expression $\nabla^k\nabla\cdots\nabla(-\Delta)^\alpha h_{kl}$ after bringing it to the canonical form will read

$$\nabla\cdots\nabla\nabla^k(-\Delta)^\alpha h_{kl}+\cdots,$$

where dots stand for the terms with curvature that have been generated as the result of commutation.

(iii) The remaining derivatives (including possible derivatives acting on the curvature) are ordered using the "SortCovDs" command of the *xAct* package.

(iv) The Riemann tensors appearing from the commutations are replaced by their expressions in terms of the Ricci tensor and the scalar curvature. This step may generate additional Laplacians or contractions of derivatives with the metric fluctuations, so the ordering procedure is repeated iteratively until it converges.

When performing the commutation of fractional powers of Laplacian the formula (3.51) is truncated at the order relevant for a given step of the calculation. A single commutator of the Laplacian with a covariant derivative is proportional to curvature; thus it has second order in our counting. Every further commutator increases the order at least by one. In addition, it can be shown that the lowest-order coefficient function in the nested commutator

$$\underbrace{[\Delta,[\Delta,\ldots,[\Delta}_{n},\nabla]]\cdots] \qquad (3.53)$$

is a total derivative. Hence, it will not contribute into the effective action, unless it gets multiplied by another background tensor. The latter has at least dimension two, which further limits the number of nested commutators we need to consider at a given order. A straightforward analysis of possible cases tells us that in the commutator of the fractional Laplacian with a covariant derivative we need to go up to

(i) $n=4$ in the computation of $(\mathbb{Q}^{(0)})^2$,

(ii) $n=3$ in the computation of $\mathbb{Q}^{(0)}\mathbb{Q}^{(2)}$ and $\mathbb{Q}^{(2)}\mathbb{Q}^{(0)}$,

(iii) $n=2$ in $\mathbb{Q}^{(0)}\mathbb{Q}^{(3)}$ and $\mathbb{Q}^{(3)}\mathbb{Q}^{(0)}$,

(iv) $n=1$ in $\mathbb{Q}^{(0)}\mathbb{Q}^{(4)}$, $\mathbb{Q}^{(4)}\mathbb{Q}^{(0)}$, and $(\mathbb{Q}^{(2)})^2$,

(v) in the fifth order operators $\mathbb{Q}^{(2)}\mathbb{Q}^{(3)}$, $\mathbb{Q}^{(3)}\mathbb{Q}^{(2)}$ and sixth order operators $\mathbb{Q}^{(2)}\mathbb{Q}^{(4)}$, $\mathbb{Q}^{(4)}\mathbb{Q}^{(2)}$, $(\mathbb{Q}^{(3)})^2$ all derivatives can be treated as commutative.

Similarly, every commutator of the Laplacian with a coefficient function made of curvature increases the order at least by one, the lowest order term in

$$\underbrace{[\Delta,[\Delta,\ldots,[\Delta}_{n},\mathcal{R}_{(a)}]]\cdots] \qquad (3.54)$$

again being a total derivative. By considering possible cases we conclude that when commuting the fractional Laplacian with the coefficient functions we need to retain up to

(i) 3 nested commutators in $\mathbb{Q}^{(0)}\mathbb{Q}^{(2)}$,

(ii) 2 nested commutators in $\mathbb{Q}^{(0)}\mathbb{Q}^{(3)}$ and $(\mathbb{Q}^{(2)})^2$,

(iii) one commutator in $\mathbb{Q}^{(2)}\mathbb{Q}^{(3)}$, $\mathbb{Q}^{(3)}\mathbb{Q}^{(2)}$,

(iv) in $\mathbb{Q}^{(0)}\mathbb{Q}^{(4)}$ and $\mathbb{Q}^{(4)}\mathbb{Q}^{(2)}$, $\mathbb{Q}^{(2)}\mathbb{Q}^{(4)}$, $(\mathbb{Q}^{(3)})^2$ the commutator between fractional Laplacians and the coefficient functions can be omitted altogether.

The iterative algorithm of this section provides us with the square-root operator in the form (3.7) suitable for further processing with the technique of universal functional traces. We now describe this technique and draw the list of the required UFTs.

IV. UNIVERSAL FUNCTIONAL TRACES

A. Schwinger-DeWitt technique and the method of universal functional traces

The calculation of UFTs of the form (1.14) arising in (3.7) can be done by means of the heat kernel method and the Schwinger-DeWitt technique of the proper-time expansion on generic curved spacetime. The heat kernel method allows one to write down in (3.7) the integral representation for a generic power of the Laplacian,

$$\begin{aligned}\nabla\cdots\nabla\frac{\hat{1}}{(-\Delta)^\alpha}\delta(x,y)\big|^{\rm div}_{y=x}\\ =\frac{1}{\Gamma(\alpha)}\nabla\cdots\nabla\int_0^\infty ds s^{\alpha-1}{\rm e}^{s\Delta}\hat{\delta}(x,y)\big|^{\rm div}_{y=x},\end{aligned} \qquad (4.1)$$

in terms of the kernel of the heat equation $\hat{K}(s|x,y)=e^{s\Delta}\hat{\delta}(x,y)$ with the "Hamiltonian" $-\Delta$. Here $\Delta=g^{ij}\nabla_i\nabla_j$ is a covariant Laplacian acting on an arbitrary set of tensor fields $\phi^A(x)$ labeled by the index A, the hat denoting a matrix in their vector space, $\hat{1}=\delta^A{}_B$, and the matrix nature of the delta function $\hat{\delta}(x,y)\equiv\hat{1}\times\delta(x,y)$. Note that, in contrast to Eq. (3.1), the dimension of the proper-time parameter in this formula is $[s]=-2$ to match the dimensionality of the Laplacian.

Expansion of $\hat{K}(s|x,y)$ at small values of the proper time allows one to isolate in the coincidence limit $y=x$ the integrals diverging at the lower boundary $s=0$, which comprise UV divergences of the universal functional traces (1.14).[9] This expansion, in its turn, is based on the Schwinger-DeWitt technique [25,27]. In the most general

[9]For large positive α the operators can suffer from infrared (IR) divergences associated with the upper integration limit for s, but as we will be interested in UV divergences, which are clearly separated at one-loop order from the IR ones, we will disregard this issue.

setting this expansion is explicitly known for a *minimal* second-order operator of the form

$$\hat{F}(\nabla) = \Box + \hat{P} - \frac{\hat{1}}{6}R, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu, \quad (4.2)$$

"minimal" meaning that its second order derivatives form a covariant d'Alembertian determined with respect to the spacetime metric $g_{\mu\nu}$. The operator acts in the representation space of $\phi^A(x)$ in d-dimensional spacetime with coordinates x^μ, $\mu = 1,\ldots,d$ and is characterized by the set of "curvatures" $\mathfrak{R} = (\hat{P}, \widehat{\boldsymbol{R}}_{\mu\nu}, R_{\lambda\rho\mu\nu})$—the potential term $\hat{P} \equiv P^A{}_B$ (the term $-R\hat{1}/6$ is singled out from it for convenience), fiber bundle curvature $\widehat{\boldsymbol{R}}_{\mu\nu} \equiv \boldsymbol{R}^A{}_{B\mu\nu}$—commutator of covariant derivatives acting on the vector ϕ^B or a matrix $\hat{X} \equiv X^B_C$—and the Riemann tensor,

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu}V^\rho, \qquad [\nabla_\mu, \nabla_\nu]\phi = \widehat{\boldsymbol{R}}_{\mu\nu}\phi,$$
$$[\nabla_\mu, \nabla_\nu]\hat{X} = [\widehat{\boldsymbol{R}}_{\mu\nu}, \hat{X}]. \quad (4.3)$$

The heat kernel $\hat{K}(s|x,y) = e^{s\hat{F}(\nabla)}\delta(x,y)$ for the operator (4.2) has a small (or early) time asymptotic expansion at $s \to 0$,

$$\hat{K}(s|x,y) = \frac{\mathcal{D}^{1/2}(x,y)}{(4\pi s)^{d/2}} g^{1/2}(y) e^{-\frac{\sigma(x,y)}{2s}} \sum_{n=0}^{\infty} s^n \hat{a}_n(x,y), \quad (4.4)$$

where $\sigma(x,y)$ is the Synge world function—one-half of the square of geodetic distance between points x and y, and

$$\mathcal{D}(x,y) = g^{-1/2}(x)\left|\det\frac{\partial^2\sigma(x,y)}{\partial x^\mu \partial y^\nu}\right| g^{-1/2}(y) \quad (4.5)$$

is the (dedensitized) Pauli–Van Vleck–Morette determinant built of $\sigma(x,y)$. Both $\hat{\delta}(x,y)$ and $\hat{K}(s|x,y)$ are defined above as zero weight tensor densities with respect to x and tensor densities of weight one with respect to y, which explains the factor $g^{1/2}(y)$ in (4.4). The two-point matrix quantities $\hat{a}_n(x,y)$ bear the name of HAMIDEW [34] or Gilkey-Seely coefficients praising the efforts of mathematicians and physicists in heat kernel theory [25,30] (see review of physics implications of this theory in [27,31,32]).

The substitution of expansion (4.4) in (4.1) expresses the UFTs in terms of the coincidence limits

$$\nabla_{\mu_1}\cdots\nabla_{\mu_k}\sigma(x,y)|_{y=x}, \qquad \nabla_{\mu_1}\cdots\nabla_{\mu_k}\mathcal{D}^{1/2}(x,y)|_{y=x},$$
$$\nabla_{\mu_1}\cdots\nabla_{\mu_k}\hat{a}_n(x,y)|_{y=x}. \quad (4.6)$$

Their remarkable property is that they are local functions of the curvatures and their covariant derivatives. These functions can be systematically calculated from the equation for the world function $g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma = 2\sigma$ and the recursive equations for $\hat{a}_n(x,y)$, which follow from the heat equation for $\hat{K}(s|x,y)$. For obvious dimensional reasons the general structure of these coincidence limits is the sum of various covariant monomials of curvatures and their covariant derivatives of relevant powers defined by k and n. Since $\sigma(x,y)$ and $\mathcal{D}^{1/2}(x,y)$ are determined solely by the spacetime metric, the first two sets in (4.6) are given by the sums of monomials of the form

$$\nabla_1\ldots\nabla_p\sigma(x,y)|_{y=x} \propto \overbrace{\nabla\cdots\nabla}^{k}\overbrace{R\cdots R}^{m}, \qquad 2m+k = p-2, \quad (4.7)$$

$$\nabla_1\ldots\nabla_p\mathcal{D}^{1/2}(x,y)|_{y=x} \propto \overbrace{\nabla\cdots\nabla}^{k}\overbrace{R\cdots R}^{m}, \qquad 2m+k = p, \quad (4.8)$$

in terms of purely metric curvatures R (we suppress the tensor indices for clarity), whereas the third set involves all the "curvatures" $\mathfrak{R} = (\hat{P}, \widehat{\boldsymbol{R}}_{\mu\nu}, R_{\mu\nu\alpha\beta})$ pertinent to the operator (4.2)

$$\nabla_1\cdots\nabla_p\hat{a}_n(x,y)|_{y=x} \propto \overbrace{\nabla\cdots\nabla}^{k}\overbrace{\mathfrak{R}\cdots\mathfrak{R}}^{m},$$
$$2m+k = p+2n. \quad (4.9)$$

In Appendix C 1 we briefly describe the recursive procedure of calculating all these coincidence limits.

By adjusting the general technique to our three-dimensional case, $d=3$, $\mu \mapsto i = 1,2,3$, with the operator $\hat{F} = \Delta$ ($\hat{P} = \frac{1}{6}R\hat{1}$) acting for the metric sector in the space of symmetric covariant tensors h_{kl} ($\hat{1} = \delta_{ij}{}^{kl}$) and in the space of vectors n^j and c^j ($\hat{1} = \delta^i_j$) for the shift and ghost sectors, respectively, we obtain

$$\nabla_{i_1}\cdots\nabla_{i_p}\frac{\hat{1}}{(-\Delta)^{N+1/2}}\delta(x,y)\bigg|_{y=x}$$
$$= \frac{1}{\Gamma(N+1/2)}\frac{1}{8\pi^{3/2}}\int_0^\infty ds s^{N-2}\nabla_{i_1}\cdots\nabla_{i_p}$$
$$\times\left[\mathcal{D}^{1/2}(x,y)e^{-\frac{\sigma(x,y)}{2s}}\sum_{n=0}^{\infty}s^n\hat{a}_n(x,y)\right]_{y=x}. \quad (4.10)$$

Here we have used that the UFTs needed for our calculation contain half-integer powers of the Laplacian, as implied by Eq. (3.7). These UFTs have UV divergences of degree $p-2N+2$ (recall that the delta function is three dimensional), which correspond to the proper-time integrals diverging at $s=0$. In view of the growing power of s in this expansion, only the few first terms will contribute to the UV divergences, which makes the method highly efficient. Among the divergences we will be interested only in the logarithmic ones of the form $\int_0^\infty ds/s$—the divergent coefficients of the counterterms of dimensionality six, which determine the beta functions of the theory.

B. Types of universal functional traces

Here we consider the types of universal functional traces arising in the trace of the operator $\mathbb{Q}$ regarding their number of derivatives and the powers of the Laplacian acted upon by these derivatives. We focus first on the tensor sector. As stated in Sec. III, the curvature expansion of $\mathbb{Q}_\mathbb{D}$ in the canonical form (3.7) reads

$$\mathbb{Q}_{\mathbb{D}ij}{}^{kl} = \left[\sum_{a,p}\mathcal{R}_{(a)}\tilde{\alpha}_{a,p}\nabla_1\cdots\nabla_p\frac{1}{(-\Delta)^{N+1/2}}\right]_{ij}^{kl}, \qquad (4.11)$$

where in each term we redefined the overall negative half-integer power of the Laplacian as $N+1/2$ and the number of derivatives as p. Recall that $\mathcal{R}_{(a)}$ are the background field tensors built of the curvature and its derivatives of the dimensionality a in units of inverse length [see Eq. (3.6)]. For $a=2$ this tensor is just the Ricci curvature $\mathcal{R}_{(2)}=R^{ij}$, for $a=3$ it is $\mathcal{R}_{(3)}=\nabla^k R^{ij}$, etc. Obviously, at any a the tensor quantity has *at maximum* a indices, $\mathcal{R}_{(a)}=\mathcal{R}_{(a)}^{i_1\cdots i_r}$, $r\le a$. Also, as mentioned in Sec. III, the parameter N is not independent but follows from the overall dimensionality of the operator $\mathbb{Q}_\mathbb{D}$ which is three, so that $a+p-2N-1=3$ or

$$2N = a+p-4, \qquad (4.12)$$

whence it follows, in particular, that $a+p$ is always even [denoted by $2k$ in (3.7)]. Thus,

$$\mathrm{Tr}\,\mathbb{Q}_\mathbb{D} = \int d^3x \sum_{a,p}\mathrm{tr}\left[\mathcal{R}_{(a)}\tilde{\alpha}_{a,p}\nabla_1\cdots \times\nabla_p\frac{1}{(-\Delta)^{N+1/2}}\delta_{ij}{}^{kl}(x,y)\big|_{y=x}\right], \qquad (4.13)$$

where tr is the trace which is taken over the multi-indices ij and kl *after* the action of every nonlocal operator on the tensor delta function has been enforced.

Another important point, also mentioned in Sec. III, is that for every a there is an upper bound on the number of derivatives p in these functional traces. Indeed, their p indices can be contracted at maximum with r indices of $\mathcal{R}_{(a)}$ and four indices of $\delta_{ij}{}^{kl}(x,y)$.[10] Therefore $p\le r+4$, and in view of $r\le a$ the upper bound on p is

$$p\le a+4, \qquad (4.14)$$

which coincides with (3.8) for $K_{\mathrm{free}}=4$. From (4.12) this leads to the upper bound on N,

$$N\le a. \qquad (4.15)$$

In (4.13) every UFT with p derivatives, which is conjugated to the background field tensor $\mathcal{R}_{(a)}$ of dimensionality a,

$$T_p^{(a)} \equiv \nabla_1\cdots\nabla_p\frac{1}{(-\Delta)^{N+1/2}}\delta_{ij}{}^{kl}(x,y)\big|_{y=x}, \quad N=\frac{a+p}{2}-2, \qquad (4.16)$$

is divergent when its degree of divergence $\Omega(T_p^{(a)})=p-2N+2=6-a$ is positive, or $a\le 6$. This, of course, corresponds to the logarithmically divergent counterterms of dimensionality six. Therefore, the set of logarithmically divergent terms in (4.13) is given by

$$a=0,2,3,4,6, \qquad (4.17)$$

where the contributions of $a=1$ and $a=5$ are absent because there are no background field tensors of dimensionality one. Thus we have the following five sets of universal functional traces which contribute to logarithmic divergences:

1. Traces with $a=0$, $p\le 4$, p-even, $N+\frac{1}{2}=\frac{p-3}{2}$

$$(-\Delta)^{3/2}\hat{\delta}(x,y)\big|_{y=x}, \qquad \nabla_{i_1}\nabla_{i_2}(-\Delta)^{1/2}\hat{\delta}(x,y)\big|_{y=x},$$
$$\nabla_{i_1}\nabla_{i_2}\nabla_{i_3}\nabla_{i_4}\frac{\hat{1}}{(-\Delta)^{1/2}}\delta(x,y)\big|_{y=x}. \qquad (4.18)$$

These are the most complicated ones, because they require the knowledge of coincidence limits up to $\nabla^8\sigma(x,y)|_{y=x}$, $\nabla^6\mathcal{D}^{1/2}(x,y)|_{y=x}$, $\nabla^6\hat{a}_0(x,y)|_{y=x}$, $\nabla^4\hat{a}_1(x,y)|_{y=x}$, $\nabla^2\hat{a}_2(x,y)|_{y=x}$, and $\hat{a}_3(x,x)$.

2. Traces with $a=2$, $p\le 6$, p-even, $N+\frac{1}{2}=\frac{p-1}{2}$

$$(-\Delta)^{1/2}\hat{\delta}(x,y)\big|_{y=x}, \quad \nabla^2\frac{\hat{1}}{(-\Delta)^{1/2}}\delta(x,y)\big|_{y=x},$$
$$\nabla^4\frac{\hat{1}}{(-\Delta)^{3/2}}\delta(x,y)\big|_{y=x}, \quad \nabla^6\frac{\hat{1}}{(-\Delta)^{5/2}}\delta(x,y)\big|_{y=x}. \qquad (4.19)$$

Here and in what follows we omit for brevity the indices of derivatives.

[10] If some of these $r+4$ indices are contracted with each other, then the possible number of derivatives is smaller because their indices cannot be contracted with anything else but those of the derivatives themselves. These contractions, however, do not count because they just shift the power N of the Laplacian.

3. Traces with $a=3$, $p\leq 7$, p-odd, $N+\frac{1}{2}=\frac{p}{2}$

$$\nabla\frac{\hat{\mathbf{1}}}{(-\Delta)^{1/2}}\delta(x,y)|_{y=x},\quad \nabla^3\frac{\hat{\mathbf{1}}}{(-\Delta)^{3/2}}\delta(x,y)|_{y=x},$$
$$\nabla^5\frac{\hat{\mathbf{1}}}{(-\Delta)^{5/2}}\delta(x,y)|_{y=x},\quad \nabla^7\frac{\hat{\mathbf{1}}}{(-\Delta)^{7/2}}\delta(x,y)|_{y=x}. \tag{4.20}$$

4. Traces with $a=4$, $p\leq 8$, p-even, $N+\frac{1}{2}=\frac{p+1}{2}$

$$\frac{\hat{\mathbf{1}}}{(-\Delta)^{1/2}}\delta(x,y)|_{y=x},\quad \nabla^2\frac{\hat{\mathbf{1}}}{(-\Delta)^{3/2}}\delta(x,y)|_{y=x},$$
$$\nabla^4\frac{\hat{\mathbf{1}}}{(-\Delta)^{5/2}}\delta(x,y)|_{y=x},$$
$$\nabla^6\frac{\hat{\mathbf{1}}}{(-\Delta)^{7/2}}\delta(x,y)|_{y=x},\quad \nabla^8\frac{\hat{\mathbf{1}}}{(-\Delta)^{9/2}}\delta(x,y)|_{y=x}. \tag{4.21}$$

5. Traces with $a=6$, $p\leq 10$, p-even, $N+\frac{1}{2}=\frac{p+3}{2}$

$$\frac{\hat{\mathbf{1}}}{(-\Delta)^{3/2}}\delta(x,y)|_{y=x},\quad \nabla^2\frac{\hat{\mathbf{1}}}{(-\Delta)^{5/2}}\delta(x,y)|_{y=x},$$
$$\nabla^4\frac{\hat{\mathbf{1}}}{(-\Delta)^{7/2}}\delta(x,y)|_{y=x}\quad \nabla^6\frac{\hat{\mathbf{1}}}{(-\Delta)^{9/2}}\delta(x,y)|_{y=x},$$
$$\nabla^8\frac{\hat{\mathbf{1}}}{(-\Delta)^{11/2}}\delta(x,y)|_{y=x},\quad \nabla^{10}\frac{\hat{\mathbf{1}}}{(-\Delta)^{13/2}}\delta(x,y)|_{y=x}. \tag{4.22}$$

In this fifth group the number of derivatives is high, but these traces are the simplest ones because they can be calculated in flat space and reduce to symmetrized products of the metric tensor.

The classification of the UFTs in the vector sector proceeds similarly. The only difference is the modification of the bounds (4.14) and (4.15) due to the different number of free indices of ${Q_B}^i_j$. We now have

$$p\leq a+2,\qquad N\leq a+1, \tag{4.23}$$

which removes the last entry at each order in the list of possible structures (4.18)–(4.22).

We have computed the divergences of all the needed tensor and vector UFTs using symbolic computer algebra [44]. The code is available at [50]. The full expressions are very long, so we do not write them explicitly. In Appendix C 2 we present the most laborious traces with $a=0$ and $a=2$. Some relations between the tensor, vector, and scalar functional traces, which can be obtained by integration by parts, are discussed in Appendix C 3. These relations can be used as a powerful check of the explicit results for their divergences.

V. BETA FUNCTIONS

In the last step of our calculation we combine the operator square root extracted with the procedure described in Sec. III D with the results for UFTs enumerated in Sec. IV B to obtain the divergent part of the one-loop effective action. Namely, we use Eq. (3.4), in which we substitute Eq. (4.13) (and a similar equation for $\mathrm{Tr}\,\mathbb{Q}_{\mathbb{B}}$). Upon collecting similar terms, integration by parts, and the use of Bianchi identities, the divergence takes the form

$$\Gamma^{1-\mathrm{loop}}|^{\mathrm{div}}=\ln L^2\int d\tau d^3x\sqrt{g}(C_{\nu_1}R^3+C_{\nu_2}RR_{ij}R^{ij}+C_{\nu_3}R^i_jR^j_kR^k_i+C_{\nu_4}\nabla_iR\nabla^iR+C_{\nu_5}\nabla_iR_{jk}\nabla^iR^{jk}). \tag{5.1}$$

Only the potential part of the action has logarithmic divergences on our static background. The coefficients C_{ν_a}, which are functions of the couplings $\lambda, \nu_1, \ldots, \nu_5$, represent the key result of the calculation.

The UV divergent factor $\ln L^2$ is related to the integral over the proper-time parameter,

$$\ln L^2=\int\frac{ds_2}{s_2}. \tag{5.2}$$

This integral comes from the heat-kernel representation of the powers of spatial Laplacian, Eq. (4.1). Hence, the dimension of the proper time here is $[s_2]=-2$, which we highlighted by the subscript.[11] This means that the divergent logarithm is related to the momentum renormalization scale k_* as

$$\ln L^2\simeq\ln\left(\frac{\Lambda^2_{\mathrm{UV}}}{k^2_*}\right), \tag{5.3}$$

where Λ_{UV} is a UV cutoff.

We are now ready to compute the β-functions of the couplings ν_a, $a=1,\ldots,5$. Comparing Eqs. (2.1) and (5.1), we read off the renormalized combinations of the coupling constants

$$\left(\frac{\nu_a}{2G}\right)_{\mathrm{ren}}=\frac{\nu_a}{2G}+C_{\nu_a}\ln L^2, \tag{5.4}$$

whence

$$\beta_{\nu_a}\equiv\frac{d\nu_{a,\mathrm{ren}}}{d\ln k_*}=-4GC_{\nu_a}+\nu_a\frac{\beta_G}{G},\quad a=1,2,\ldots,5. \tag{5.5}$$

[11] This is an important difference from the diagrammatic calculation of [23] which used the proper-time representation for the full field propagators in (3 + 1)-dimensional spacetime, and hence the proper-time dimension was −6.

Therefore, the potential term β-functions are expressed via constants C_{ν_a} and the β-function of G,

$$\beta_G \equiv \frac{dG_{\rm ren}}{d\ln k_*}, \tag{5.6}$$

which was previously obtained in [23] (see also Appendix A 1).

Neither β_G nor β_{ν_a} is gauge invariant. It is well-known that a change of gauge adds to the background effective action a linear combination of the equations of motion [51,52]. Such a contribution vanishes on-shell, but not for our off-shell background. This leads to gauge dependence of the one-loop effective action and the renormalized couplings. As shown in Ref. [23], this dependence amounts to a one-parameter family of transformations which, for an infinitesimal change of gauge, have the form

$$G \mapsto G - 2G^2\epsilon, \qquad \nu_a \mapsto \nu_a - 4G\nu_a\epsilon, \tag{5.7}$$

where ϵ is an infinitesimal parameter.

We can now construct combinations that are invariant under these transformations and whose β-functions, therefore, must be gauge invariant. In this way we arrive at the set of essential couplings (1.12). Their running is easily obtained from β_{ν_a}, β_G, and β_λ [see Eq. (1.13)]:

$$\beta_{\mathcal{G}} = \mathcal{G}\left(\frac{\beta_G}{G} - \frac{1}{2}\frac{\beta_{\nu_5}}{\nu_5}\right), \tag{5.8a}$$

$$\beta_{v_a} = \frac{1}{\nu_5}\left(\beta_{\nu_a} - \nu_a\frac{\beta_{\nu_5}}{\nu_5}\right), \quad a = 1,2,3, \tag{5.8b}$$

$$\beta_{u_s} = \frac{u_s\beta_\lambda}{(1-\lambda)(1-3\lambda)} + \frac{4(1-\lambda)\beta_{v_4}}{(1-3\lambda)u_s}, \tag{5.8c}$$

where $v_4 = \nu_4/\nu_5$ and its β-function is defined in the same way as in (5.8b) at $a = 4$. This leads us to our main results, Eqs. (1.15), (1.16), and (A4)–(A8).

We have calculated $\beta_{\mathcal{G}}$ in three different gauges $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and β_{u_s}, β_{v_a}, $a = 1,2,3$ in four gauges $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ from Sec. III A. We have found identical results. All steps of the calculation were performed by two independent codes—one for gauges $\mathbf{a}, \mathbf{b}$ and one for gauges $\mathbf{c}, \mathbf{d}$. Notice that though the final results agree, the intermediate expressions differ significantly in different gauges. In particular, the coupling-dependent coefficients in the square-root operator $\mathbb{Q}_{\mathbb{D}}$ (which contains a few thousands of distinct tensor structures) are dramatically different in gauges $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$. In general, they are rational functions with the denominator being a product of combinations $(u_\alpha + u_\beta)$, where u_α, $\alpha = T, V, S1, S2$, are the eigenvalues of the principal symbol $\mathbb{Q}_{\mathbb{D}}(\mathbf{p})$ defined in Eq. (3.23). This follows from the formula for the solution of the Sylvester equation (3.40) used at each iteration of the perturbative procedure to construct $\mathbb{Q}_{\mathbb{D}}$. In the gauges $\mathbf{a}, \mathbf{b}$ the eigenvalues corresponding to the gauge modes coincide with those of the physical modes, so that in the denominator of the coefficients we get only the powers of u_s and $(1 + u_s)$. On the other hand, in gauges $\mathbf{c}$ and $\mathbf{d}$ the gauge eigenvalues are different and we obtain multiple extra factors u_V, $(1 + u_V)$, u_{S2}, $(1 + u_{S2})$, etc. All these extra factors cancel in the essential β-functions, which provide a very powerful check of the correctness and consistency of our result.

Finally, let us make the following comment. Once the gauge invariance of the essential β-functions has been explicitly checked, we can invert the logic and derive the coefficients C_{ν_a} in the one-loop effective action for arbitrary values of the gauge parameters σ and ξ. Indeed, the gauge invariance of the β-functions for the ratios $v_a = \nu_a/\nu_5$, $a = 1,2,3,4$, implies that gauge-dependent parts of the coefficients C_{ν_a} in the one-loop effective action are proportional to the couplings ν_a themselves with a common proportionality factor,

$$C_{\nu_a}^{\rm gauge} = \nu_a \Xi(\lambda, \{\nu\}, \sigma, \xi), \quad a = 1, \ldots, 5. \tag{5.9}$$

Adding to this the invariance of the β-function for the essential coupling $\mathcal{G}$, one derives the gauge-dependent parts of the β-functions for G and ν_a,

$$\beta_G^{\rm gauge} = -4G^2\Xi, \qquad \beta_{\nu_a}^{\rm gauge} = -8G\nu_a\Xi. \tag{5.10}$$

The function Ξ can be fixed by a calculation of the effective action on a simple special background that can be carried out in a general (σ, ξ)-gauge. This task is performed in the next section and yields a remarkably simple result [see Eq. (6.35)]. In the Supplemental Material [43] we provide a *Mathematica* file with the coefficients C_{ν_a} in arbitrary (σ, ξ)-gauge, obtained by adding the gauge-dependent piece (5.9) to our explicit results in gauges $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

VI. ADDITIONAL CHECK: EFFECTIVE ACTION ON $R^1 \times S^3$

The complexity of our calculation for the full set of beta functions imposes the necessity of its efficient verification. It is based on the UFT method of [27–29] which, despite its power, is not commonly used in the literature and therefore requires detailed validation and caution. While the gauge independence of the essential β-functions discussed above already provides a strong argument in favor of the validity of our approach, we perform one more check using an alternative calculational scheme. Namely, we compute the divergence of the one-loop effective action of projectable HG on a static background with spherical spatial slices using spectral decomposition for the differential operators $\mathbb{D}$ and $\mathbb{B}$ entering the potential part of the action. The traces of their square roots are found by means of spectral summation, for which we use two different regularizations—the dimensional and the ζ-functional one. Due to the simplicity of the background, this calculation can be carried out in an arbitrary (σ,ξ)-gauge introduced in Sec. II A. As a by-product, it fixes the function Ξ from Eq. (5.9), and hence allows us to completely determine the dependence of the coefficients C_{ν_a} in Eq. (5.1) on the gauge choice. As another by-product, we derive the logarithmic dependence of the renormalized partition function of HG on $R^1\times S^3$ on the radius of the sphere.

Consider a static spacetime with spherical three-dimensional slices of inverse square radius κ. We have

$$R_{ij}=2\kappa g_{ij},\qquad R=6\kappa. \tag{6.1}$$

On this background, the general expression (5.1) for the divergent part of the effective action reduces to

$$\Gamma^{1-\mathrm{loop}}|^{\mathrm{div}}_{R^1\times S^3}=\ln L^2\sum_{a=1}^{3}C_{\nu_a}\int d\tau I_a, \tag{6.2}$$

where I_a are the following three nonvanishing invariants:

$$I_1=\int d^3x\sqrt{g}R^3|_{S^3}=9\times 48\pi^2\kappa^{3/2}, \tag{6.3a}$$

$$I_2=\int d^3x\sqrt{g}RR_{ij}R^{ij}|_{S^3}=3\times 48\pi^2\kappa^{3/2}, \tag{6.3b}$$

$$I_3=\int d^3x\sqrt{g}R^i_jR^j_kR^k_i|_{S^3}=48\pi^2\kappa^{3/2}. \tag{6.3c}$$

Therefore, an independent calculation of this divergent part of $\Gamma^{1-\mathrm{loop}}$ on $R^1\times S^3$ provides a check of the linear combination $9C_{\nu_1}+3C_{\nu_2}+C_{\nu_3}$. Notice that this combination is gauge-dependent, so the comparison between the general result and the calculation on the sphere must be performed in the same gauge.

A. Tensor and vector operators on S^3

Our starting point is the formula (3.4) for the one-loop effective action. On the homogeneous space—a sphere S^3—the tensor and vector operators can be explicitly diagonalized, and the functional traces of their square roots can be represented as spectral sums of square roots of their eigenvalues. Then the UV divergences can be obtained under appropriate (dimensional or ζ-functional) regularization of these spectral sums.

Diagonalization of the tensor operator takes place in the complete orthonormal basis of tensor harmonics $H^{A(n)}_{ij}$ which we present, for the sake of dimensional regularization, on the d-dimensional sphere S^d. Here $A=t,v,s1,s2$ is the helicity index running over tensor, vector, and two scalar polarizations contained in the metric, whereas (n) enumerates all other quantum numbers at the level n. In the basis of these harmonics,

$$h_{ij}(x)=\sum_{A,(n)}h_{A(n)}H^{A(n)}_{ij}(x), \tag{6.4}$$

the operator $\mathbb{D}$, takes the following block-diagonal form:

$$\mathbb{D}|_{S^3}=\begin{bmatrix}\mathbb{D}_t&0&0&0\\0&\mathbb{D}_v&0&0\\0&0&\mathbb{A}_{11}&\mathbb{A}_{12}\\0&0&\mathbb{A}_{21}&\mathbb{A}_{22}\end{bmatrix}\equiv\mathrm{diag}[\mathbb{D}_t,\mathbb{D}_v,\mathbb{D}_s]. \tag{6.5}$$

The harmonics which provide this property can in their turn be expressed in terms of complete and orthonormal sets of transverse-traceless tensor $h^{TT(n)}_{ij}(x)$, transverse vector $\xi^{(n)}_i(x)$, and scalar $\phi^{(n)}(x)$ eigenfunctions of the covariant Laplacian $\Delta=g^{ij}\nabla_i\nabla_j$ (see Appendix D for details),

$$H^{t(n)}_{ij}(x)=h^{TT(n)}_{ij}(x),\qquad n\geq 2, \tag{6.6a}$$

$$H^{v(n)}_{ij}(x)=2\nabla_{(i}\frac{1}{\sqrt{2(-\Delta-\frac{1}{d}R)}}\xi^{(n)}_{j)}(x),\quad n\geq 2, \tag{6.6b}$$

$$H^{s1(n)}_{ij}(x)=\left(\nabla_i\nabla_j-\frac{1}{d}g_{ij}\Delta\right)\frac{1}{\sqrt{\frac{d-1}{d}(-\Delta)^2-\frac{1}{d}R(-\Delta)}}\phi^{(n)}(x),\quad n\geq 2, \tag{6.6c}$$

$$H^{s2(n)}_{ij}(x)=\frac{1}{\sqrt{d}}g_{ij}\phi^{(n)}(x),\qquad n\geq 0. \tag{6.6d}$$

We will denote their relevant eigenvalues as Δ^t_n, Δ^v_n, and Δ^s_n and write their orthonormality conditions in the form

$$\Delta h^{TT(n)}_{ij}(x)=\Delta^t_nh^{TT(n)}_{ij}(x),$$
$$\int d^dx\sqrt{g}h^{TT(n)}_{ij}(x)h^{ij}_{TT(m)}(x)=\delta^{(n)}_{(m)}, \tag{6.7a}$$

$$\Delta\xi^{(n)}_i(x)=\Delta^v_n\xi^{(n)}_i(x),\qquad\int d^dx\sqrt{g}\xi^{(n)}_i(x)\xi^i_{(m)}(x)=\delta^{(n)}_{(m)}, \tag{6.7b}$$

$$\Delta\phi^{(n)}(x)=\Delta^s_n\phi^{(n)}(x),\qquad\int d^dx\sqrt{g}\phi^{(n)}(x)\phi_{(m)}(x)=\delta^{(n)}_{(m)}. \tag{6.7c}$$

Integer quantum numbers (n) enumerating these eigenfunctions are of course different for tensor, vector, and scalar modes, but we will not introduce for them different notations, for in what follows we will need for each (n) only the eigenvalue Δ_n and its degeneracy D_n—the dimensionality of eigenvalue subspace. In generic dimension d, which we need for the sake of dimensional regularization, they read on the sphere of unit radius [53–56]

$$-\Delta_n^s = n(n+d-1),$$
$$D_n^s = \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}, \qquad n \geq 0, \tag{6.8a}$$
$$-\Delta_n^v = n(n+d-1) - 1,$$
$$D_n^v = \frac{n(n+d-1)(2n+d-1)(n+d-3)!}{(d-2)!(n+1)!}, \quad n \geq 1, \tag{6.8b}$$
$$-\Delta_n^t = n(n+d-1) - 2,$$
$$D_n^t = \frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!},$$
$$n \geq 2. \tag{6.8c}$$

In three dimensions the above complicated expressions for degeneracies simplify to

$$D_n^s = (n+1)^2, \quad D_n^v = 2n(n+2), \quad D_n^t = 2(n-1)(n+3). \tag{6.9}$$

The blocks of the matrix (6.5) for the nth level have the form of the functions of Δ_n times the relevant $D_n \times D_n$ unit matrices $\delta_{(n)}^{(m)}$,

$$\mathbb{D}_t(n)\delta_{(n)}^{(m)} = \int d^d x \sqrt{g} H_{t(n)}^{ij}(x) \mathbb{D}_{ij}{}^{kl} H_{kl}^{t(m)}(x), \tag{6.10a}$$

$$\mathbb{D}_v(n)\delta_{(n)}^{(m)} = \int d^d x \sqrt{g} H_{v(n)}^{ij}(x) \mathbb{D}_{ij}{}^{kl} H_{kl}^{v(m)}(x), \tag{6.10b}$$

$$\mathbb{A}_{ab}(n)\delta_{(n)}^{(m)} = \int d^d x \sqrt{g} H_{sa,(n)}^{ij}(x) \mathbb{D}_{ij}{}^{kl} H_{kl}^{sb,(m)}(x),$$
$$a = 1,2, \quad b = 1,2. \tag{6.10c}$$

They are calculated using the mode normalization and relations (D2) and (D3) of Appendix D. Their expressions, which are too lengthy to be presented explicitly, schematically read

$$\mathbb{D}_t(n) = \kappa^3 T_{(3)}(-\Delta_n^t), \tag{6.11a}$$

$$\mathbb{D}_v(n) = \kappa^3 V_{(3)}(-\Delta_n^v), \tag{6.11b}$$

$$\mathbb{A}_{11}(n) = \kappa^3 \frac{S_{(5)}(-\Delta_n^s)}{(\Delta_n^s)^2 + d\Delta_n^s},$$
$$\mathbb{A}_{12}(n) = \mathbb{A}_{21}(n) = \kappa^3 \frac{S_{(4)}(-\Delta_n^s)}{\sqrt{(\Delta_n^s)^2 + d\Delta_n^s}},$$
$$\mathbb{A}_{22}(n) = \kappa^3 S_{(3)}(-\Delta_n^s), \tag{6.11c}$$

where $T_{(q)}$, $V_{(q)}$, $S_{(q)}$, $q = 3, 4, 5$, are polynomials of qth order in their argument, and the denominators in $\mathbb{A}_{11}(n)$, $\mathbb{A}_{12}(n)$, and $\mathbb{A}_{21}(n)$ follow from the normalization factor in (6.6c). Cancellation of similar denominators in (6.11b) occurs due to Eqs. (D6) and (D2) of Appendix D.

The total 2×2 scalar block of the operator (6.5), $\mathbb{D}_s = \mathbb{D}_{s,ab}$, is still not diagonal, but in each nth eigenvalue subspace it can be diagonalized in the basis of finite-dimensional matrix eigenvectors $\Upsilon(n) = \Upsilon_{a(b)}(n)$ and $\Upsilon^\dagger(n) = \Upsilon^\dagger_{(b)a}(n)$, $(b) = \pm$, $a = 1, 2$,

$$\mathbb{A}(n) = \Upsilon(n) \begin{bmatrix} \Lambda_+(n) & 0 \\ 0 & \Lambda_-(n) \end{bmatrix} \Upsilon^\dagger(n), \qquad \Upsilon^\dagger(n)\Upsilon(n) \equiv \sum_c \Upsilon^\dagger_{(a)c}(n)\Upsilon_{c(b)}(n) = \delta_{(a)(b)}, \tag{6.12}$$

$$\Lambda_\pm(n) = \frac{1}{2}\left(\mathbb{A}_{11}(n) + \mathbb{A}_{22}(n) \pm \sqrt{\mathbb{A}_{11}^2(n) + \mathbb{A}_{22}^2(n) - 2\mathbb{A}_{11}(n)\mathbb{A}_{22}(n) + 4\mathbb{A}_{12}(n)\mathbb{A}_{21}(n)}\right). \tag{6.13}$$

As the result, the operator $\mathbb{D}$ becomes diagonal in all of its sectors, and the unregulated spectral sum representation of the functional trace of its square root, $\mathbb{Q} = \mathbb{D}^{1/2}$, takes the form

$$\mathrm{Tr}\,\mathbb{Q}_{\mathbb{D}}|_{S^3} = \int d^d x \sqrt{g} \sum_{A,(n)} H_{A(n)}^{kl}(x) (\sqrt{\mathbb{D}})_{kl}{}^{ij} H_{ij}^{A(n)}(x)$$
$$= \sum_{n=2}^{\infty} D_n^t \sqrt{\mathbb{D}_t(n)} + \sum_{n=2}^{\infty} D_n^v \sqrt{\mathbb{D}_v(n)} + \sum_{n=2}^{\infty} D_n^s \sqrt{\Lambda_+(n)} + \sum_{n=0}^{\infty} D_n^s \sqrt{\Lambda_-(n)}. \tag{6.14}$$

Summation in the tensor, vector, and scalar "+" sectors starts with $n = 2$, whereas in the scalar "−" sector it starts from $n = 0$, in accordance with the restrictions on n in (6.6).

The calculation of $\mathrm{Tr}\,\mathbb{Q}_{\mathbb{B}}$ on S^3 in the ghost and shift sectors proceeds along the same lines, except that we can, in view of the simplicity of these sectors, explicitly present the expressions for the corresponding operators. In particular, the operator $\mathbb{B}$ defined by Eq. (2.8), when converted to the canonical form, reads on S^3 as

$$\mathbb{B}^i{}_j|_{S^3} = \frac{1}{2\sigma}[\delta^i_j(-\Delta)^3 + (1-2(1-\lambda)(1+\xi))\nabla^i\nabla_j(-\Delta)^2 - 2\kappa\delta^i_j(-\Delta)^2 - 4(1-2\lambda)\xi\kappa\nabla^i\nabla_j(-\Delta) + 8\lambda\xi\kappa^2\nabla^i\nabla_j]. \quad (6.15)$$

In the basis of transverse and longitudinal vector modes,

$$c^j(x) = \sum_{(n)} c^T_{(n)}\xi^j_{(n)}(x) + \sum_{(n)} c^L_{(n)}\nabla^j\frac{1}{\sqrt{-\Delta}}\phi^{(n)}(x), \quad (6.16)$$

where $\xi^j_{(n)} \equiv g^{ji}\xi^{(n)}_i$ and $\phi^{(n)}$ are orthonormal sets of transverse vector and scalar Laplacian eigenfunctions (6.7b) and (6.7c) introduced above, this operator similar to (6.5) becomes diagonal $\mathbb{B} = \mathrm{diag}[\mathbb{B}_T, \mathbb{B}_L]$. Here, as it follows from (6.15),

$$\mathbb{B}_T = \frac{\kappa^3}{2\sigma}[(-\Delta^v_n)^3 - 2(-\Delta^v_n)^2]\delta^{(n)}_{(m)} \equiv \mathbb{B}_T(n)\delta^{(n)}_{(m)}, \quad (6.17a)$$

$$\mathbb{B}_L = \frac{\kappa^3}{2\sigma}[2(1-\lambda)(1+\xi)(-\Delta^s_n)^3 - 4(\xi-2\lambda+3)(-\Delta^s_n)^2 + 8(3-\lambda)(-\Delta^s_n) - 16] \equiv \mathbb{B}_L(n)\delta^{(n)}_{(m)}, \quad (6.17b)$$

and

$$\mathrm{Tr}\,\mathbb{Q}_{\mathbb{B}}|_{S^3} = \sum_{n=1}^{\infty} D^v_n\sqrt{\mathbb{B}_v(n)} + \sum_{n=1}^{\infty} D^s_n\sqrt{\mathbb{B}_s(n)}. \quad (6.18)$$

The vector modes sum starts with $n = 1$ because $D^v_0 = 0$, whereas the scalar modes sum begins with $n = 1$ because the expansion (6.16) does not include the zero mode of the scalar Laplacian.

B. Dimensional and ζ-functional regularization of spectral sums

Regularization and extraction of divergences by dimensional regularization consists in the extension of these sums to space dimensionality $d = 3 - \varepsilon$ with $\varepsilon \to 0$. In the ζ-functional regularization the square root power of the operator $\mathbb{F} = (\mathbb{D}, \mathbb{B})$ is analytically continued to $1/2 - \varepsilon'$.

$$\mathrm{Tr}\sqrt{\mathbb{F}}|_{\zeta\text{-reg.}} = \mathrm{Tr}\,\mathbb{F}^{\frac{1}{2}-\varepsilon'}, \quad (6.19)$$

which is provided by the replacement of all square roots in the above formula (6.14) by this power while keeping $d = 3$. We identify

$$\varepsilon' = \varepsilon/6, \quad (6.20)$$

as implied by the dimensionality of the operator.[12] One can show then that the resulting divergent parts of the trace—the pole terms in ε—coincide in both regularizations. Below we demonstrate this on the example of the tensor sector.

For the dimensionality $d = 3 - \varepsilon$ the multiplicity of the nth eigenvalue in the transverse-traceless tensor sector has for large $n \gg 1$ the following form:

$$D^t_n = \frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!} = 2n^{2-\varepsilon}\left(1 + \frac{2}{n} - \frac{3}{n^2} + O(\varepsilon)\right), \quad (6.21)$$

where we used that

[12] Indeed, in both dimensional and ζ-regularizations one has to introduce a dimensionful parameter k_* to keep the dimension of the operator trace equal to three. Equating the powers of this parameter in the two cases, we obtain the relation (6.20).

$$\frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+1)} = n^{-\varepsilon}(1+O(\varepsilon)), \qquad n\to\infty. \tag{6.22}$$

This yields

$$\mathrm{Tr}\sqrt{\mathbb{D}_t}|_{\mathrm{dim\,.reg.}} = \kappa^{3/2}\sum_{n=2}^{\infty} D_n^t\{T_{(3)}(n(n+d-1)-2)\}^{1/2} = \sum_{n=2}^{\infty} n^{5-\varepsilon} G_t\left(\frac{1}{n},\varepsilon\right). \tag{6.23}$$

Here we have explicitly disentangled the fractional power of the growing factor $n^{5-\varepsilon}$ as the coefficient of the function $G_t(\frac{1}{n},\varepsilon)$ which is regular at $n\to\infty$,

$$G_t\left(\frac{1}{n},\varepsilon\right) = 2\kappa^{3/2}\left(1+\frac{2}{n}-\frac{3}{n^2}\right)\left\{\frac{1}{n^6}T_{(3)}(n^2+2n-2)\right\}^{1/2} + O(\varepsilon) \tag{6.24}$$

[remember that $T_{(3)}(n^2+2n-2)$ is six order polynomial in n].

The divergent pole in ε of this series can be extracted using the Abel-Plana formula which expresses a discrete series $\sum_n f(n)$ in terms of the sum of the integral along the real axes of n and the integral of $f(z)$ on the imaginary axis,

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} dz f(z) + \frac{1}{2}f(0) + i\int_0^{\infty} dz\frac{f(iz)-f(-iz)}{\exp(2\pi z)-1}. \tag{6.25}$$

With $f(n) = (n+2)^{5-\varepsilon}G_t(1/(n+2),\varepsilon)$ the latter integral is convergent being exponentially damped at infinity, while the divergence of the integral over the real axes at $n\to\infty$ can be isolated by changing the integration variable, $n+2=1/y$, and integrating the needed number of times by parts. In the domain of convergence $\varepsilon>6$ integration by parts does not give extra terms at $y=0$, so that the analytic continuation to $\varepsilon=0$ yields UV divergences as a pole term

$$\begin{aligned}\mathrm{Tr}\sqrt{\mathbb{D}_t}|^{\mathrm{div}}_{\mathrm{dim\,.reg.}} &= \int_2^{\infty} dn n^{5-\varepsilon}G_t\left(\frac{1}{n},\varepsilon\right)|^{\mathrm{div}} = \int_0^{1/2}\frac{dy}{y^{7-\varepsilon}}G_t(y,\varepsilon)|^{\mathrm{div}} \\ &= \frac{1}{\varepsilon-6}\frac{1}{\varepsilon-5}\cdots\frac{1}{\varepsilon-1}\frac{1}{\varepsilon}\int_0^{1/2} dy y^{\varepsilon}\frac{d^6G_t(y,\varepsilon)}{dy^6}|^{\mathrm{div}} = \frac{1}{6!\varepsilon}\frac{d^6G_t(y,0)}{dy^6}|_{y=0}.\end{aligned} \tag{6.26}$$

This result agrees with the zeta-function regularization. Indeed, we have

$$\mathrm{Tr}\sqrt{\mathbb{D}_t}|_{\zeta-\mathrm{reg.}} = \kappa^{3/2-\varepsilon}\sum_{n=2}^{\infty}2(n-1)(n+3)\{T_{(3)}(n(n+2)-2)\}^{\frac{1}{2}-\frac{\varepsilon}{6}} = \sum_{n=2}^{\infty} n^{5-\varepsilon}F_t\left(\frac{1}{n},\varepsilon\right), \tag{6.27}$$

where $F_t(\frac{1}{n},\varepsilon)$ is a function different from $G_t(\frac{1}{n},\varepsilon)$, but coinciding with it at $\varepsilon=0$, $F_t(\frac{1}{n},0)=G_t(\frac{1}{n},0)$. Then, by expanding this function in Taylor series in $1/n$, one acquires a series of Riemannian zeta functions

$$\begin{aligned}\mathrm{Tr}\sqrt{\mathbb{D}_t}|^{\mathrm{div}}_{\zeta-\mathrm{reg.}} &= \sum_{n=2}^{\infty} n^{5-\varepsilon}\sum_{m=0}^{\infty}\frac{1}{m!}F_t^{(m)}(0,\varepsilon)\frac{1}{n^m}|^{\mathrm{div}} \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}F_t^{(m)}(0,\varepsilon)\zeta_R(m+\varepsilon-5)|^{\mathrm{div}} = \frac{1}{6!\varepsilon}\frac{d^6F_t(y,0)}{dy^6}|_{y=0},\end{aligned} \tag{6.28}$$

which coincides with (6.26). Here we used the fact that Riemann zeta-function $\zeta_R(z)$ has a simple pole only at $z=1$ with unit residue.

Analogous to the formula (6.23) we regularize the vector and scalar traces

$$\mathrm{Tr}\sqrt{\mathbb{D}_v} = \sum_{n=2}^{\infty} n^{5-\varepsilon}G_v\left(\frac{1}{n},\varepsilon\right), \qquad \mathrm{Tr}\sqrt{\mathbb{D}_s} = \sum_{n=2}^{\infty} n^{5-\varepsilon}G_+\left(\frac{1}{n},\varepsilon\right) + \sum_{n=0}^{\infty} n^{5-\varepsilon}G_-\left(\frac{1}{n},\varepsilon\right). \tag{6.29}$$

Then the total divergence reads

$$\mathrm{Tr}\,\mathbb{Q}|^{\mathrm{div}}_{S^3} = \frac{1}{6!\varepsilon}\frac{d^6}{dy^6}[G_t(y,0)+G_v(y,0)+G_+(y,0)+G_-(y,0)]|_{y=0}. \tag{6.30}$$

Full expressions for functions $G_t(y,0)$, $G_v(y,0)$, $G_\pm(y,0)$ can be obtained for an arbitrary gauge in the two-parameter family of Sec. II A, but are too lengthy to be presented here. The *Mathematica* code to calculate them can be found at [50].

The same procedure applies to the regularization of the trace of the vector operator square roots. The simplicity of this sector allows us to present the final result,

$$\mathrm{Tr}\,\mathbb{Q}_{\mathbb{B}}|^{\mathrm{div}}_{S^3} = \frac{\kappa^{3/2}}{\varepsilon}\sqrt{\nu_5}\left[-4u_V - \frac{(3-\lambda)^2(1+\lambda)u_{S2}}{16(1-\lambda)^3} + \frac{(5-6\lambda+5\lambda^2)u_V^2}{2(1-\lambda)u_{S2}} + \frac{4(1-\lambda)(5-3\lambda)u_V^4}{u_{S2}^3} - \frac{32(1-\lambda)^3u_V^6}{u_{S2}^5}\right], \tag{6.31}$$

where u_V and u_{S2} have been defined in Eq. (3.23). Combining this result with the tensor contribution, we finally find the divergent part of the one-loop effective action on the static spacetime with spherical 3-space in an arbitrary (σ,ξ)-gauge,

$$\Gamma^{1-\mathrm{loop}}|^{\mathrm{div}}_{R^1\times S^3} = \int d\tau\frac{\kappa^{3/2}}{\varepsilon}P(\lambda,u_s,v_1,v_2,v_3,\sigma,\xi), \tag{6.32a}$$

$$\begin{aligned}P(\lambda,u_s,v_1,v_2,v_3,\sigma,\xi) = {}& \frac{\sqrt{\nu_5}}{32(1-\lambda)^3(1-3\lambda)^3u_s^5}\{(1-\lambda)^6(144v_1+50v_2+18v_3-1)^3\\ &+2u_s^2(1-\lambda)^4(1-3\lambda)(144v_1+50v_2+18v_3-1)[72v_1(\lambda+1)\\ &+2v_2(17\lambda+8)+18v_3\lambda-5\lambda+4]\\ &+4u_s^4(1-\lambda)^2(1-3\lambda)^2[72v_1(16\lambda^2-9\lambda-3)+2v_2(200\lambda^2-120\lambda-33)\\ &+6v_3(24\lambda^2-16\lambda-3)-8\lambda^2+12\lambda-3]\\ &+6u_s^5(1-\lambda)^3(1-3\lambda)^3[12(v_2+v_3)(v_2(6v_2+25)+3v_3(4v_2+3)+6v_3^2)\\ &+432v_1(2v_2+2v_3+3)+430v_2+142v_3-11]+8u_s^6(1-3\lambda)^3\lambda(4\lambda^2-8\lambda+3)\}\\ &+3\nu_5(9v_1+3v_2+v_3)\left\{4\sqrt{2\sigma}+\frac{1}{(1-\lambda)}\sqrt{\frac{\sigma}{(1-\lambda)(1+\xi)}}\right\}.\end{aligned} \tag{6.32b}$$

Note that the $R^1\times S^3$ background for a generic radius $\kappa^{-1/2}$ of the 3-sphere is not a solution of equations of motion, so that this divergent part of the effective action is off-shell and, therefore, is gauge dependent. Equations of motion on static $R^1\times S^3$ imply that the derivative of the action with respect to κ should vanish and hold only at flat space geometry, $\kappa=0$, of infinitely large 3-sphere. The gauge dependence is described by the last term in (6.32b) and is remarkably simple.

In comparing this expression to our previous results, we need the relation between $1/\varepsilon$ and the divergent logarithm $\ln L^2$, Eq. (5.2). In dimensional regularization the latter is regularized as

$$\int\frac{ds_2}{s_2}\mapsto k_*^\varepsilon\int\frac{ds_2}{s_2^{(1-\varepsilon/2)}}\simeq\frac{2}{\varepsilon}, \tag{6.33}$$

where k_* is a parameter with units of momentum to keep the expression dimensionless (cf. footnote 12). This gives $\ln L^2 = 2/\varepsilon$. Comparing with Eqs. (6.2) and (6.3), we find

$$96\pi^2\kappa^{3/2}(9C_{\nu_1}+3C_{\nu_2}+C_{\nu_3}) = \kappa^{3/2}P(\lambda,u_s,v_1,v_2,v_3,\sigma,\xi). \tag{6.34}$$

We have checked that this equality is indeed satisfied in the four gauges a, b, c, d. This accomplishes the verification of our results on the static homogeneous spacetime.

As a corollary we obtain the expression for the function Ξ introduced in Eq. (5.9) which parametrizes the gauge dependence of the divergent coefficients in the effective action,

$$\Xi = \frac{1}{32\pi^2}\left\{4\sqrt{2\sigma}+\frac{1}{(1-\lambda)}\sqrt{\frac{\sigma}{(1-\lambda)(1+\xi)}}\right\}. \tag{6.35}$$

Its knowledge allows us to generalize our expressions for C_{ν_a}, $a=1,\ldots,5$, to arbitrary (σ,ξ)-gauge. The result is contained in the form of the *Mathematica* file in the Supplemental Material [43].

The final remark here is that the knowledge of the logarithmic divergences (6.32) allows one to extract the logarithmic dependence of the finite part of the effective action on the radius if the sphere $\kappa^{-1/2}$. This is easily seen within the ζ-functional regularization in which the overall scale of operators $\mathbb{F} = (\mathbb{D}, \mathbb{B}) \propto \kappa^3$ is raised to the fractional power in (6.19) and gives

$$\frac{1}{\varepsilon}\kappa^{\frac{3}{2}-\frac{\varepsilon}{2}} = \kappa^{\frac{3}{2}}\left(\frac{1}{\varepsilon} - \frac{1}{2}\ln\frac{\kappa}{k_*^2}\right), \tag{6.36}$$

where k_* is a normalization scale of the ζ-function regularization. This leads to the expression for the full effective action as a function of κ,

$$\Gamma^{1-\mathrm{loop}}|_{R^1\times S^3} = \int d\tau\left(\frac{\kappa^{3/2}}{\varepsilon} - \frac{\kappa^{3/2}}{2}\ln\frac{\kappa}{k_*^2}\right)P(\lambda, u_s, v_1, v_2, v_3, \sigma, \xi) + \int d\tau\kappa^{3/2}Q(\lambda, u_s, v_1, v_2, v_3, \sigma, \xi), \tag{6.37}$$

where the logarithmic term plays the role of the Coleman-Weinberg effective potential on the metric background of size $\kappa^{-1/2}$. In contrast to the logarithmic contribution, the second term is not controlled by the UV divergent coefficient and, contrary to the case of single-charge models, cannot be absorbed into the redefinition of the normalization k_*, because it carries a nontrivial dependence on multiple couplings.

VII. DISCUSSION

In this paper we have obtained the full set of one-loop β-functions for marginal essential coupling constants in projectable HG. The results underwent a number of very powerful checks that confirm gauge independence of these beta functions in a wide set of gauge conditions—the cornerstone of the physically invariant content of quantum gauge theories. These checks also provide a very deep verification and show high efficiency of the method of universal functional traces which replaces within the background field approach the standard Feynman diagrammatic technique. This method implicitly performs the summation of a humongous number of Feynman graphs and leads to a final result hardly achievable by standard momentum space methods in flat spacetime. As a by-product of our calculation we derived an expression for the divergence of the one-loop effective action of HG on static background in a two-parameter family of gauges.

The complexity of the expressions (1.15) for the β-functions with the polynomials $\mathcal{P}_n^\chi$ collected in Eqs. (A4)–(A8) is high, and we postpone a comprehensive analysis of the resulting RG flow for the future. At this point, we content ourselves with a few preliminary observations.

Clearly, the β-functions (1.15) are in general singular at $\lambda \to 1/3$, $\lambda \to 1$, or $u_s \to 0$.[13] This is not surprising, since the two first limits correspond to the boundaries of the unitarity domain (1.9), whereas in the last limit the dispersion relation of the scalar mode becomes degenerate [see Eq. (1.10b)]. Remarkably, however, for a special choice of the values

$$\{v^*\}: v_1 = 1/2, \qquad v_2 = -5/2, \qquad v_3 = 3 \tag{7.1}$$

the limit $u_s \to 0$ of the β-functions (1.15) becomes regular for any λ in the unitary domain:

$$\beta_{v_a}|_{\{v^*\},u_s\to 0} = 0, \quad a = 1, 2, 3, \tag{7.2a}$$

$$\beta_{u_s}|_{\{v^*\},u_s\to 0} = \frac{1893\lambda^2 - 6720\lambda + 4576}{6720\pi^2(1-\lambda)(1-3\lambda)}\mathcal{G}, \tag{7.2b}$$

$$\beta_{\mathcal{G}}|_{\{v^*\},u_s\to 0} = -\frac{159}{80\pi^2}\mathcal{G}^2. \tag{7.2c}$$

The point $\{v^*\}$, $u_s \to 0$ is special since it corresponds to the version of HG, in which the potential term is a square of the Cotton tensor C_{ij},

$$S = \frac{1}{2G}\int d\tau\, d^3x\sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 + \nu_5 C^{ij}C_{ij}) = \frac{2}{G}\int d\tau\, d^3x\sqrt{\gamma}(K_{ij} + \sqrt{\nu_5}C_{ij})\mathbb{G}^{ij,kl}(K_{kl} + \sqrt{\nu_5}C_{kl}), \tag{7.3}$$

$$C^{ij} = \varepsilon^{ikl}\nabla_k\left(R_l^j - \frac{1}{4}R\delta_l^j\right) = \varepsilon^{kl(i}\nabla_k R_l^{j)}, \tag{7.4}$$

where $\varepsilon^{ikl} = \epsilon^{ikl}/\sqrt{g}$, $\epsilon^{123} = 1$. In the second equality in (7.3) we used the tracelessness of the Cotton tensor and integration by parts.

This version of HG was originally suggested in [1] and its quantum properties were studied in [57]. It is known as HG with detailed balance and is interesting because the Cotton tensor can be rewritten as a variational derivative of the three-dimensional gravitational Chern-Simons theory,

$$C^{ij} = -\frac{1}{\sqrt{g}}\frac{\delta W_{\mathrm{CS}}[g]}{\delta g_{ij}(x)}, \tag{7.5}$$

$$W_{\mathrm{CS}}[g] = \frac{1}{2}\int d^3x\epsilon^{ijk}\left(\Gamma_{il}^m\partial_j\Gamma_{km}^l + \frac{2}{3}\Gamma_{il}^n\Gamma_{jm}^l\Gamma_{kn}^m\right), \tag{7.6}$$

defined in terms of the metric Christoffel symbol as a functional of g_{ij}. Further, there exists a deformation of the

[13]We do not consider the singularity at $u_s = -1$, because u_s is assumed to be positive by construction [see Eq. (1.12)].

TABLE I. Solutions of the system (7.8). The sixth column gives the value of the β-function for $\mathcal{G}$ at the respective solution and the seventh column indicates whether it corresponds to an asymptotically free fixed point. The eighth column tells if the fixed point is UV attractive along the λ-direction.

λ	u_s	v_1	v_2	v_3	$\beta_{\mathcal{G}}/\mathcal{G}^2$	Asymptotically free?	UV attractive along λ?
0.1787	60.57	−928.4	−6.206	−1.711	−0.1416	Yes	No
0.2773	390.6	−19.88	−12.45	2.341	−0.2180	Yes	No
0.3288	54533	3.798×10^8	−48.66	4.736	−0.8484	Yes	No
0.3289	57317	-4.125×10^8	−49.17	4.734	−0.8784	Yes	No

action (7.3) by relevant operators which preserves the detailed balance structure and is related to the topological massive gravity [58–60]. The detailed balance relation between d and $(d+1)$-dimensional theories appears in the context of stochastic quantization [61,62] and establishes a nontrivial connection between the renormalization properties of the two theories [63]. In our case this suggests an intriguing connection between the $(3+1)$-dimensional projectable HG and the three-dimensional gravitational Chern-Simons/topological massive gravity [57].

It is important to emphasize, however, that the point $\{v^*\}, u_s \to 0$ is *not* a fully regular point of the RG flow in HG, because the β-function (1.13) of the remaining essential coupling λ diverges in this limit,

$$\beta_\lambda|_{\{v^*\},u_s\to 0} \sim \frac{9}{40\pi^2}\frac{1-\lambda}{u_s}\mathcal{G}. \qquad (7.7)$$

Thus, the physical significance of the result (7.2) is unclear at the moment. It will be interesting to understand if the inclusion of fermionic degrees of freedom appearing in the stochastic quantization framework [57] can change the picture.

An important question is the existence and nature of fixed points of the RG flow. As already observed, the dependence of the β-functions on the coupling $\mathcal{G}$ factorizes. This coupling determines the overall strength of interactions in HG and must be small for the validity of the perturbative expansion. Its UV behavior determines whether the model is asymptotically free ($\mathcal{G} \to 0$) or has a Landau pole ($\mathcal{G} \to \infty$). On the other hand, the rest of the couplings λ, u_s, v_a are ratios of the coefficients in the action and need not be small. The search for fixed points of the RG flow thus splits into two steps. One first identifies the fixed points of the flow in the subspace of the couplings λ, u_s, v_a by solving the system,

$$\beta_\lambda/\mathcal{G} = 0, \qquad (7.8a)$$

$$\beta_\chi/\mathcal{G} = 0, \qquad \chi = u_s, v_1, v_2, v_3. \qquad (7.8b)$$

In the full parameter space, these solutions correspond to flow lines along the $\mathcal{G}$-direction. One then evaluates $\beta_{\mathcal{G}}$ at a given solution, whose sign determines whether the flow line goes to a Gaussian fixed point or a Landau pole.

Omitting the denominators in the expressions (1.13) and (1.15b), the system (7.8) becomes a system of five polynomial equations for five unknowns λ, u_s, v_a, $a = 1, 2, 3$. We have studied it numerically with the following results:

(i) We have found no solutions in the right part of the unitary domain, $\lambda > 1$. In this respect $(3+1)$-dimensional HG appears to be different from its $(2+1)$-dimensional counterpart, which possesses an asymptotically free fixed point at $\lambda = 15/14$ [21].

(ii) In the left part of the unitary domain, $\lambda < 1/3$, we found four solutions summarized in Table I. All these fixed points turn out to be asymptotically free. Note that the two last points correspond to very large values of v_1 and their validity requires further investigation. As discussed in [23], the fixed points at $\lambda < 1/3$ are UV repulsive along the λ-direction. We indicate this in the last column of Table I. We do not know if these fixed points are attractive or not along the other directions.

It is worth stressing that the results above should be taken with a grain of salt. Currently we do not have a firm proof of the absence of fixed points at $\lambda > 1$, nor do we claim that the list of fixed points at $\lambda < 1/3$ in Table I is exhaustive.

It was conjectured in [64] that the UV fixed points of HG can lie at infinite λ and that the limit $\lambda \to \infty$ is well-defined. We find that all β-functions (1.15) are finite at $\lambda \to \infty$, whereas β_λ is proportional to λ,[14]

$$\beta_\lambda = -\frac{3(3-2u_s)}{40\pi^2 u_s}\lambda\mathcal{G}, \qquad \lambda \to \infty. \qquad (7.9)$$

This behavior is compatible with the conjecture of [64]. Notice that the point $\lambda = \infty$ is UV attractive (repulsive) for $u_s > 3/2$ ($u_s < 3/2$). To identify fixed points of the RG flow at $\lambda = \infty$, we looked for solutions of the system

$$\beta_\chi/\mathcal{G}|_{\lambda=\infty} = 0, \qquad \chi = u_s, v_1, v_2, v_3. \qquad (7.10)$$

We have found eight solutions listed in Table II. Three among them are UV attractive along the λ-direction and correspond to asymptotically free fixed points. Clearly, the

[14]The directionality of the limit is not important: the resulting expressions for the β-functions are the same at $\lambda = \pm\infty$.

TABLE II. Solutions of the system (7.10) corresponding to fixed points of Hořava gravity at $\lambda = \infty$. The fourth column lists the value of the β-function for the coupling $\mathcal{G}$ at each solution, whose sign determines whether the flow is asymptotically free or runs into strong coupling, as indicated in the fifth column. The last column tells if the point is UV attractive along the λ-direction in the space of all couplings.

u_s	v_1	v_2	v_3	$\beta_{\mathcal{G}}/\mathcal{G}^2$	Asymptotically free?	UV attractive along λ?
0.01950	0.4994	−2.498	2.999	−0.2004	Yes	No
0.04180	−0.01237	−0.4204	1.321	−1.144	Yes	No
0.05530	−0.2266	0.4136	0.7177	−1.079	Yes	No
12.28	−215.1	−6.007	−2.210	−0.1267	Yes	Yes
21.60	−17.22	−11.43	1.855	−0.1936	Yes	Yes
440.4	−13566	−2.467	2.967	0.05822	No	Yes
571.9	−9.401	13.50	−18.25	−0.07454	Yes	Yes
950.6	−61.35	11.86	3.064	0.4237	No	Yes

structure of the RG flow around these points deserves further investigation. More generally, this strongly motivates a detailed study of the $\lambda \to \infty$ limit of HG. It is worth mentioning that a similar limit naturally arises in connection with nonrelativistic gravity to Perelman-Ricci flows [65].

Let us stress again that presently we do not know if Table II is exhaustive. We plan to return to a systematic classification of fixed points of HG and its RG flow in our future work.

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APPENDIX A: EXPLICIT EXPRESSIONS

1. Beta-function of G

The coupling G of HG is not *essential*; i.e., it is not defined using the on-shell quantities. Hence its β-function depends on the gauge choice. Reference [23] obtained this β-function for the $(3+1)$-dimensional projectable model in a subset of the two-parameter family of regular gauges described in Sec. II A. The results are as follows:

(i) σ-arbitrary, $\xi = -\frac{1-2\lambda}{2(1-\lambda)}$

$$\beta_G = \sqrt{\nu_5}\frac{\mathcal{G}^2}{40\pi^2(1-\lambda)(1-3\lambda)(1+u_s)u_s}[-27+74\lambda-57\lambda^2-u_s(5(1-3\lambda)(5-4\lambda)\sqrt{2\sigma\nu_5}+53-142\lambda+99\lambda^2) - u_s^2(1-3\lambda)(5(5-4\lambda)\sqrt{2\sigma\nu_5}+18-14\lambda)], \tag{A1}$$

(ii) $\sigma = \frac{1}{2\nu_5}$, $\xi = \frac{\nu_s}{2\nu_5(1-\lambda)} - 1$

$$\beta_G = -\sqrt{\nu_5}\frac{\mathcal{G}^2}{40\pi^2(1-\lambda)(1-3\lambda)(1+u_s)u_s}[32-89\lambda+57\lambda^2+3u_s(26-79\lambda+53\lambda^2)+2u_s^2(19-74\lambda+51\lambda^2)], \tag{A2}$$

(iii) $\sigma = \frac{1}{2\nu_s}$, $\xi = \frac{\nu_5}{2\nu_s(1-\lambda)} - 1$

$$\beta_G = -\sqrt{\nu_5}\frac{\mathcal{G}^2}{40\pi^2(1-\lambda)(1-3\lambda)(1+u_s)u_s}[47-154\lambda+117\lambda^2+3u_s(26-79\lambda+53\lambda^2)+u_s^2(23-83\lambda+42\lambda^2)]. \tag{A3}$$

This set of gauges overlaps with the gauges used in the present work. Thus, the gauge (ii) coincides with the gauge (**a**), Eq. (3.25), whereas the gauge (i) reduces to the gauges (**b**) and (**d**) for the appropriate choices of σ [see Eqs. (3.27) and (3.31)]. The expressions (A1) and (A2) are used in Sec. V to derive the β-function of the essential coupling $\mathcal{G}$.

2. Polynomials in the β-functions of essential couplings

In this Appendix we collect the expressions for the polynomials appearing in Eqs. (1.15). For the β-function of the coupling $\mathcal{G}$ the polynomials read,

$$\mathcal{P}_0^{\mathcal{G}} = (1-\lambda)^4(1809v_3^2 + 832v_2^2 + 16v_2(159v_3 - 217) - 4494v_3 + 2401), \tag{A4a}$$

$$\mathcal{P}_1^{\mathcal{G}} = 3\mathcal{P}_0^{\mathcal{G}}, \tag{A4b}$$

$$\begin{aligned}\mathcal{P}_2^{\mathcal{G}} = &-(1-\lambda)^2[3(15779\lambda^2 - 20362\lambda + 3967) + 64v_2^2(81\lambda^2 - 82\lambda + 1)\\ &+ 27v_3^2(279\lambda^2 - 238\lambda - 41) - 6v_3(8823\lambda^2 - 10620\lambda + 1561)\\ &+ 16v_2(v_3(675\lambda^2 - 582\lambda - 93) - 2307\lambda^2 + 2732\lambda - 365)],\end{aligned} \tag{A4c}$$

$$\begin{aligned}\mathcal{P}_3^{\mathcal{G}} = &(1-\lambda)^2[27v_3^2(961\lambda^2 - 1434\lambda + 401) + 64v_2^2(1717\lambda^2 - 2298\lambda + 581)\\ &+ 16v_2(19409\lambda^2 - 26004\lambda + 6415 + 3v_3(2741\lambda^2 - 3690\lambda + 949))\\ &+ 6v_3(39331\lambda^2 - 58728\lambda + 14873) - 345977\lambda^2 + 276750\lambda - 52741],\end{aligned} \tag{A4d}$$

$$\begin{aligned}\mathcal{P}_4^{\mathcal{G}} = &2(1-3\lambda)\{138545\lambda^3 - 328263\lambda^2 - 5888(1-\lambda)^3v_2^2\\ &- (1-\lambda)^2[16v_2(3119\lambda + 840v_3(1-\lambda) - 2396)\\ &- 3v_3(9v_3(353\lambda - 299) - 5012\lambda + 8210)] + 239597\lambda - 49947\},\end{aligned} \tag{A4e}$$

$$\begin{aligned}\mathcal{P}_5^{\mathcal{G}} = &2(1-3\lambda)\{159709\lambda^3 - 378471\lambda^2 + (1-\lambda)^2[16v_2(1243\lambda - 412)\\ &- 3v_3(243v_3(1-3\lambda) - 13280\lambda + 4366)] + 273933\lambda - 55375\},\end{aligned} \tag{A4f}$$

$$\mathcal{P}_6^{\mathcal{G}} = -6(1-3\lambda)^2(8465\lambda^2 - 16310\lambda + 3(1-\lambda)^2v_3(254 + 27v_3) + 7811), \tag{A4g}$$

$$\mathcal{P}_7^{\mathcal{G}} = 4(1-3\lambda)^2(48\lambda^2 - 38\lambda + 7). \tag{A4h}$$

Polynomials in the β-function of u_s are the following:

$$\begin{aligned}\mathcal{P}_0^{u_s} = &-3(1-\lambda)^5[537600v_1^2 + 78992v_2^2 + 14205v_3^2 + 2688v_1(154v_2 + 67v_3 - 16)\\ &+ 16v_2(4236v_3 - 959) - 5838v_3 + 329],\end{aligned} \tag{A5a}$$

$$\mathcal{P}_1^{u_s} = 3\mathcal{P}_0^{u_s}, \tag{A5b}$$

$$\begin{aligned}\mathcal{P}_2^{u_s} = &-2(1-\lambda)^3[2419200v_1^2(1-\lambda)^2 + 8v_2^2(42645\lambda^2 - 86482\lambda + 43837)\\ &+ v_3^2(58698 - 106947\lambda + 48249\lambda^2) + 4032v_1(462v_2(1-\lambda)^2 + 201v_3(1-\lambda)^2 + 30\lambda^2 - 44\lambda - 10)\\ &+ 8v_2(6252\lambda^2 - 9188\lambda - 1468) + 8v_2v_3(34335\lambda^2 - 71196\lambda + 36861)\\ &+ v_3(20556\lambda^2 - 30792\lambda - 3696) + 4533\lambda^2 - 3881\lambda + 1448],\end{aligned} \tag{A5c}$$

$$\begin{aligned}\mathcal{P}_3^{u_s} = &-2(1-\lambda)^3[806400v_1^2(1-\lambda)^2 + 8v_2^2(20709\lambda^2 - 32026\lambda + 14957) + v_3^2(61686\lambda^2\\ &- 52875\lambda + 20241) + 4032v_1(98 + 154v_2(1-\lambda)^2 + 67v_3(1-\lambda)^2 + 218\lambda^2 - 388\lambda)\\ &+ 8v_2(3v_3(7833\lambda^2 - 9656\lambda + 4231) + 4(8658\lambda^2 - 16817\lambda + 4324))\\ &+ v_3(81594\lambda^2 - 189660\lambda + 50262) - 2970\lambda^2 - 1529 + 6235\lambda],\end{aligned} \tag{A5d}$$

$$\mathcal{P}_4^{u_s} = (1-\lambda)(1-3\lambda)[32v_2^2(1-\lambda)^2(4081\lambda-1191) + v_3^2(133083\lambda^3 - 303453\lambda^2 + 207657\lambda - 37287) + 48384v_1(1-\lambda)^2(13\lambda-19) - 16v_2(1-\lambda)(7873\lambda^2 - 25922\lambda + 18109 + 3v_3(5419\lambda^2 - 6970\lambda + 1551)) - v_3(31938\lambda^3 + 15042\lambda^2 - 127314\lambda + 80334) + 10415\lambda^3 + 11815\lambda^2 - 30239\lambda + 10017], \tag{A5e}$$

$$\mathcal{P}_5^{u_s} = (1-\lambda)(1-3\lambda)[32v_2^2(1-\lambda)^2(661\lambda-203) + v_3^2(104787\lambda^3 - 240381\lambda^2 + 168345\lambda - 32751) + 16128v_1(1-\lambda)^2(13\lambda-19) + 16v_2(1-\lambda)(12761\lambda^2 - 14690\lambda + 69 - 3v_3(2677\lambda^2 - 3534\lambda + 857)) - v_3(178962\lambda^3 - 468990\lambda^2 + 347070\lambda - 57042) + 379967\lambda^3 - 512385\lambda^2 + 126609\lambda + 1081], \tag{A5f}$$

$$\mathcal{P}_6^{u_s} = -4(1-3\lambda)^2[6584v_2^2(1-\lambda)^3 - 27v_3^2(1-\lambda)^2(311\lambda-284) + 24v_2(1-\lambda)^2(405\lambda - 584 + 581v_3(1-\lambda)) - 3v_3(1-\lambda)^2(2507\lambda+2452) - 92671\lambda^3 + 205653\lambda^2 - 130039\lambda + 17539], \tag{A5g}$$

$$\mathcal{P}_7^{u_s} = -2(1-3\lambda)^2[(1-\lambda)^2(729v_3^2(1-3\lambda) - 16v_2(3133\lambda-1042) - 6v_3(11680\lambda-3863)) - 212947\lambda^3 + 494301\lambda^2 - 341005\lambda + 61647], \tag{A5h}$$

$$\mathcal{P}_8^{u_s} = -2(1-3\lambda)^3((1-\lambda)^2(243v_3^2 + 3360v_2 + 5646v_3) + 31443\lambda^2 - 61026\lambda + 29033), \tag{A5i}$$

$$\mathcal{P}_9^{u_s} = 4(1-3\lambda)^3(48\lambda^2 - 38\lambda + 7). \tag{A5j}$$

Polynomials in the β-function of v_1 are the following:

$$\mathcal{P}_0^{v_1} = -(1-\lambda)^6[11612160v_1^3 + 472088v_2^3 + 241920v_1^2(50v_2 + 18v_3 - 1) + 12v_2^2(40758v_3 - 427) + 1008v_1(4124v_2^2 + 4v_2(726v_3 - 23) + 6v_3(81v_3 + 4) - 31) + 78v_2(6v_3(345v_3 + 28) - 119) + 18v_3(3v_3(318v_3 + 77) - 119) - 385], \tag{A6a}$$

$$\mathcal{P}_1^{v_1} = 3\mathcal{P}_0^{v_1}, \tag{A6b}$$

$$\mathcal{P}_2^{v_1} = -(1-\lambda)^4\{34836480v_1^3(1-\lambda)^2 + 24v_2^3(54595\lambda^2 - 112134\lambda + 57539) + 108v_3^3(213\lambda^2 - 602\lambda + 389) + 161280v_1^2[225v_2(1-\lambda)^2 + 81v_3(1-\lambda)^2 - 6\lambda^2 + 8\lambda - 4] + 4v_2^2[6v_3(52401\lambda^2 - 110626\lambda + 58225) - 72285\lambda^2 + 86204\lambda - 22411] - 36v_3^2(1947\lambda^2 - 2236\lambda + 375) + 2v_1[32v_2^2(190749\lambda^2 - 384238\lambda + 193489) - 224v_2(2613\lambda^2 - 3196\lambda + 1051) + 528v_2v_3(7935\lambda^2 - 16124\lambda + 8189) + 243v_3^2(2703\lambda^2 - 5620\lambda + 2917) - 42v_3(8535\lambda^2 - 9652\lambda + 1885) + 7(22587\lambda^2 - 26516\lambda + 3353)] + 2v_2[18v_3^2(9687\lambda^2 - 21886\lambda + 12199) - 24v_3(6447\lambda^2 - 7402\lambda + 1387) + 52401\lambda^2 - 62686\lambda + 8885] + 18v_3(1245\lambda^2 - 1506\lambda + 205) + 14805\lambda^2 - 18928\lambda + 4151\}, \tag{A6c}$$

$$\begin{aligned}\mathcal{P}_3^{v_1} = &-(1-\lambda)^4\{11612160v_1^3(1-\lambda)^2 + 8v_2^3(19267\lambda^2 - 65030\lambda + 45763)\\ &- 324v_3^3(211\lambda^2 - 246\lambda + 35) + 483840v_1^2[25v_2(1-\lambda)^2 + 9v_3(1-\lambda)^2 - 2\lambda^2]\\ &- 12v_2^2[6v_3(1943\lambda^2 + 1938\lambda - 3881) + 68869\lambda^2 - 79372\lambda + 18995]\\ &- 108v_3^2(2255\lambda^2 - 2852\lambda + 683) + 6v_1[32v_2^2(17541\lambda^2 - 37822\lambda + 20281)\\ &- 224v_2(2061\lambda^2 - 2092\lambda + 499) + 528v_2v_3(543\lambda^2 - 1340\lambda + 797)\\ &+ 243v_3^2(15\lambda^2 - 244\lambda + 229) - 42v_3(9303\lambda^2 - 11188\lambda + 2653)\\ &+ 7(28539\lambda^2 - 38420\lambda + 9305)] - 6v_2[18v_3^2(2273\lambda^2 - 2034\lambda - 239)\\ &+ 24v_3(7175\lambda^2 - 8858\lambda + 2115) - 64777\lambda^2 + 87438\lambda - 21261]\\ &+ 18v_3(4687\lambda^2 - 6422\lambda + 1567) + 7(6785\lambda^2 - 8992\lambda + 2219)\},\end{aligned} \tag{A6d}$$

$$\begin{aligned}\mathcal{P}_4^{v_1} = &-2(1-\lambda)^2(1-3\lambda)\{1024v_2^3(1-\lambda)^2(45\lambda - 38) + 1728v_3^3(1-\lambda)^2(7\lambda - 6)\\ &+ 120960v_1^2(1-\lambda)^2(\lambda + 1) - 4v_2^2(1-\lambda)[384(59\lambda^2 - 109\lambda + 50)v_3\\ &+ 29133\lambda^2 - 55225\lambda + 25452] - 9v_3^2(877\lambda^3 + 871\lambda^2 - 4213\lambda + 2465)\\ &+ v_1[64v_2^2(1-\lambda)^2(1263\lambda - 1343) - 16v_2(1-\lambda)(3v_3(2463\lambda^2 - 5182\lambda + 2719)\\ &+ 33534\lambda^2 - 52670\lambda + 19076) + 3(9v_3^2(1-\lambda)^2(1513\lambda - 1833)\\ &+ 2v_3(82239\lambda^3 - 226251\lambda^2 + 205549\lambda - 61537) - 64219\lambda^3 + 203973\lambda^2\\ &- 210641\lambda + 71335)] + 4v_2[144v_3^2(1-\lambda)^2(101\lambda - 86) + 12v_3(1961\lambda^3 - 6699\lambda^2\\ &+ 7435\lambda - 2697) + 8085\lambda^3 - 7434\lambda^2 - 9300\lambda + 8755] + 6v_3(4487\lambda^3\\ &- 9281\lambda^2 + 4807\lambda + 7) + 55452\lambda^3 - 123853\lambda^2 + 81624\lambda - 13195\},\end{aligned} \tag{A6e}$$

$$\begin{aligned}\mathcal{P}_5^{v_1} = &-2(1-\lambda)^2(1-3\lambda)\{168v_2^3(51\lambda^3 - 149\lambda^2 + 125\lambda - 27) - 108v_3^3(9\lambda^3 + 9\lambda^2\\ &- 25\lambda + 7) - 4v_2^2(1-\lambda)[18v_3(117\lambda^2 - 366\lambda + 109) - 284\lambda^2 - 7265\lambda + 5425]\\ &+ 40320v_1^2(1-\lambda)^2(\lambda + 1) - 9v_3^2(3467\lambda^3 - 8839\lambda^2 + 6237\lambda - 865)\\ &+ v_1[64v_2^2(1-\lambda)^2(1717\lambda - 581) - 16v_2(1-\lambda)(3v_3(2741\lambda^2 - 3690\lambda + 949)\\ &+ 25940\lambda^2 - 40662\lambda + 12022) + 27v_3^2(961\lambda^3 - 2395\lambda^2 + 1835\lambda - 401)\\ &+ 6v_3(52267\lambda^3 - 148963\lambda^2 + 129881\lambda - 33185) - 288353\lambda^3 + 542255\lambda^2\\ &- 333355\lambda + 83485] - 2v_2[162v_3^2(3\lambda^3 + 35\lambda^2 - 51\lambda + 13) + 24v_3(1265\lambda^3\\ &- 2191\lambda^2 + 691\lambda + 235) + 30971\lambda^3 - 40323\lambda^2 + 13167\lambda - 4451] - 12v_3(6551\lambda^3\\ &- 11593\lambda^2 + 6124\lambda - 1112) + 109519\lambda^3 - 252396\lambda^2 + 177357\lambda - 34396\},\end{aligned} \tag{A6f}$$

$$\begin{aligned}\mathcal{P}_6^{v_1} = &\,2(1-3\lambda)^2\{56v_2^3(1-\lambda)^3(103\lambda - 13) + 108v_3^3(1-\lambda)^3(41\lambda - 11)\\ &+ 4v_2^2(1-\lambda)^3(2315\lambda + 54(89\lambda - 19)v_3 + 807) - 36v_3^2(1-\lambda)^3(657\lambda - 239)\\ &- 2v_1(1-\lambda)[5888v_2^2(1-\lambda)^3 + 16v_2(1-\lambda)^2(840v_3(1-\lambda) + 284\lambda - 1451)\\ &- 27v_3^2(1-\lambda)^2(353\lambda - 299) - 6v_3(1-\lambda)^2(5054\lambda + 1585) - 146609\lambda^3\\ &+ 330783\lambda^2 - 220781\lambda + 36675] - 2v_2(1-\lambda)^2[54v_3^2(169\lambda^2 - 212\lambda + 43)\\ &+ 96v_3(\lambda^2 + 29\lambda - 30) + 49685\lambda^2 - 66892\lambda + 16249] - 6v_3(1-\lambda)^2(7601\lambda^2\\ &- 11994\lambda + 4203) - 15115\lambda^4 + 38758\lambda^3 - 23950\lambda^2 - 8038\lambda + 8337\},\end{aligned} \tag{A6g}$$

$$\begin{aligned}\mathcal{P}_7^{v_1} = &-2(1-3\lambda)^2\{420v_2^2(6v_2+18v_3+17)(1-\lambda)^3(1-3\lambda)\\&+108v_3^2(15v_3-67)(1-\lambda)^3(1-3\lambda)-2v_1(1-\lambda)[16v_2(1-\lambda)^2(4078\lambda-1357)\\&-729v_3^2(1-\lambda)^2(1-3\lambda)+6v_3(1-\lambda)^2(14200\lambda-4703)+193645\lambda^3-448191\lambda^2\\&+313917\lambda-59575]+2v_2(1-\lambda)^2[3402v_3^2(3\lambda^2-4\lambda+1)+2016v_3(3\lambda^2-4\lambda+1)\\&+39661\lambda^2-53228\lambda+13045]+6v_3(1-\lambda)^2(1021\lambda^2-1958\lambda+799)\\&+25751\lambda^4-95078\lambda^3+122898\lambda^2-63194\lambda+9647\},\end{aligned}\tag{A6h}$$

$$\begin{aligned}\mathcal{P}_8^{v_1} = &-2(1-3\lambda)^3\{4(1-\lambda)^3[210v_2^3+9v_3^2(15v_3-67)+35v_2^2(18v_3+17)]\\&+6v_1(1-\lambda)[1680v_2(1-\lambda)^2+81v_3^2(1-\lambda)^2+2442v_3(1-\lambda)^2+10201\lambda^2\\&-19558\lambda+9323]+14v_2(1-\lambda)^2[v_3^2162(1-\lambda)+96v_3(1-\lambda)-951\lambda+935]\\&-6v_3(1-\lambda)^2(515\lambda-499)+3349\lambda^3-5135\lambda^2-105\lambda+1879\},\end{aligned}\tag{A6i}$$

$$\mathcal{P}_9^{v_1} = -4(1-3\lambda)^3[2v_1(48\lambda^3-86\lambda^2+45\lambda-7)+163\lambda^3-537\lambda^2+477\lambda-105].\tag{A6j}$$

Polynomials in the β-function of v_2 are as follows:

$$\mathcal{P}_0^{v_2} = -3(1-\lambda)^6(8v_2+9v_3-7)^2(30v_3+106v_2+336v_1+7),\tag{A7a}$$

$$\mathcal{P}_1^{v_2} = 3\mathcal{P}_0^{v_2},\tag{A7b}$$

$$\begin{aligned}\mathcal{P}_2^{v_2} = &(1-\lambda)^4\{-192v_2^3(1197\lambda^2-1808\lambda+611)-16v_2^2[9v_3(3429\lambda^2-5288\lambda+1859)\\&+4(3921\lambda^2-4322\lambda+291)]-162v_3^3(447\lambda^2-686\lambda+239)-18v_3^2(4119\lambda^2-4628\lambda+371)\\&+2v_3(6957\lambda^2-8772\lambda+975)-336v_1[192v_2^2(9\lambda^2-14\lambda+5)\\&+8v_2(18v_3(21\lambda^2-34\lambda+13)+273\lambda^2-292\lambda+11)+27v_3^2(51\lambda^2-86\lambda+35)\\&+18v_3(117\lambda^2-120\lambda-1)-1371\lambda^2+1618\lambda-191]-v_2[9v_3^2(37983\lambda^2-59248\lambda+21265)\\&+6v_3(54417\lambda^2-59380\lambda+3275)-143457\lambda^2+173720\lambda-24327]+14(2481\lambda^2-3182\lambda+687)\},\end{aligned}\tag{A7c}$$

$$\begin{aligned}\mathcal{P}_3^{v_2} = &-3(1-\lambda)^4\{336v_1[64v_2^2(19\lambda^2-26\lambda+7)+8v_2(18v_3(13\lambda^2-18\lambda+5)+385\lambda^2-516\lambda+123)\\&+27v_3^2(27\lambda^2-38\lambda+11)+18v_3(173\lambda^2-232\lambda+55)-1763\lambda^2+2402\lambda-583]\\&+64v_2^3(2743\lambda^2-3728\lambda+985)+16v_2^2(3v_3(7423\lambda^2-10136\lambda+2713)\\&+4(5349\lambda^2-7178\lambda+1719))+v_2(9v_3^2(26511\lambda^2-36304\lambda+9793)\\&+6v_3(75361\lambda^2-101268\lambda+24219)-178737\lambda^2+244280\lambda-59607)\\&+162v_3^3(327\lambda^2-446\lambda+119)+18v_3^2(5547\lambda^2-7484\lambda+1799)\\&-6v_3(3103\lambda^2-4492\lambda+1109)-14(2677\lambda^2-3574\lambda+883)\},\end{aligned}\tag{A7d}$$

$$\begin{aligned}\mathcal{P}_4^{v_2} = {} & (1-\lambda)^2(1-3\lambda)\{64v_2^3(1-\lambda)^2(2369\lambda-1953)+1728v_3^3(1-\lambda)^2(21\lambda-20)\\ & -16(1-\lambda)v_2^2[3v_3(6145\lambda^2-11298\lambda+5153)+2(5124\lambda^2-14591\lambda+9193)]\\ & -9v_3^2(16909\lambda^3-30841\lambda^2+11563\lambda+2369)+1344v_1[-192v_2^2(1-\lambda)^3\\ & +6(1-\lambda)^2v_2(149\lambda-48v_3(1-\lambda)-137)-108v_3^2(1-\lambda)^3\\ & +9v_3(1-\lambda)^2(98\lambda-93)-372\lambda^3+1225\lambda^2-1317\lambda+462]\\ & +v_2[9v_3^2(1-\lambda)^2(20741\lambda-17925)-6v_3(7139\lambda^3+39053\lambda^2-97327\lambda+51135)\\ & +293769\lambda^3-562239\lambda^2+237523\lambda+29971]+6v_3(29825\lambda^3-68713\lambda^2+46923\lambda-8099)\\ & +245651\lambda^3-551007\lambda^2+363249\lambda-57837\},\end{aligned} \tag{A7e}$$

$$\begin{aligned}\mathcal{P}_5^{v_2} = {} & (1-\lambda)^2(1-3\lambda)\{64v_2^3(\lambda-1)^2(329-961\lambda)+324v_3^3(39\lambda^3-73\lambda^2+41\lambda-7)\\ & -16v_2^2(\lambda-1)[3v_3(1085\lambda^2-1482\lambda+397)+22990\lambda^2-27850\lambda+3776]\\ & -27v_3^2(7625\lambda^3-19677\lambda^2+14703\lambda-2651)+1344v_1[2v_2(1-\lambda)^2(181\lambda-137)\\ & +9v_3(1-\lambda)^2(38\lambda-31)+140\lambda^3-53\lambda^2-215\lambda+122]+v_2[27v_3^2(683\lambda^3-1273\lambda^2+681\lambda-91)\\ & -6v_3(102523\lambda^3-238935\lambda^2+159777\lambda-23365)+193117\lambda^3-427691\lambda^2+258807\lambda-27161]\\ & -6v_3(31359\lambda^3-58643\lambda^2+33565\lambda-6089)+499453\lambda^3-1131897\lambda^2+782671\lambda-150059\},\end{aligned} \tag{A7f}$$

$$\begin{aligned}\mathcal{P}_6^{v_2} = {} & -2(1-3\lambda)^2\{8576(\lambda-1)^4v_2^3-54v_3^3(1-\lambda)^3(85\lambda-31)-18v_3^2(1-\lambda)^3(3397\lambda-1203)\\ & +16v_2^2(1-\lambda)^3(1116v_3(1-\lambda)+634\lambda-709)+2016v_1(\lambda-1)^2(35\lambda^2-48\lambda+12)\\ & -v_2(1-\lambda)[9v_3^2(1-\lambda)^2(1669\lambda-1255)+6v_3(1-\lambda)^2(8860\lambda-1737)\\ & +14503\lambda^3-16053\lambda^2-9653\lambda+11135]-6v_3(1-\lambda)^2(2180\lambda^2-7623\lambda+5187)\\ & -29405\lambda^4+75256\lambda^3-46026\lambda^2-16152\lambda+16351\},\end{aligned} \tag{A7g}$$

$$\begin{aligned}\mathcal{P}_7^{v_2} = {} & -2(1-3\lambda)^2\{16v_2^2(1-\lambda)^3(727-2188\lambda)+1458v_3^3(1-\lambda)^3(1-3\lambda)\\ & +19602v_3^2(1-\lambda)^3(1-3\lambda)+672v_1(1-\lambda)^2(35\lambda^2-48\lambda+12)\\ & +v_2(1-\lambda)[1863v_3^2(1-\lambda)^2(1-3\lambda)-78v_3(1-\lambda)^2(1102\lambda-365)-111931\lambda^3\\ & +255965\lambda^2-176459\lambda+32629]+6v_3(1-\lambda)^2(14814\lambda^2-18371\lambda+3829)\\ & -47811\lambda^4+180304\lambda^3-236654\lambda^2+123472\lambda-19239\},\end{aligned} \tag{A7h}$$

$$\begin{aligned}\mathcal{P}_8^{v_2} = {} & -2(1-3\lambda)^3\{6(280v_2^2+9v_3^2(9v_3+121))(1-\lambda)^3-6v_3(1-\lambda)^2(707\lambda-715)\\ & +v_2(1-\lambda)[621v_3^2(1-\lambda)^2+7410v_3(1-\lambda)^2+10597\lambda^2-18998\lambda+8299]\\ & -9739\lambda^3+18073\lambda^2-6487\lambda-1883\},\end{aligned} \tag{A7i}$$

$$\mathcal{P}_9^{v_2} = -2(1-3\lambda)^3[2v_2(48\lambda^3-86\lambda^2+45\lambda-7)-295\lambda^3+1253\lambda^2-1271\lambda+301]. \tag{A7j}$$

Polynomials in the β-function of v_3 are as follows:

$$\mathcal{P}_0^{v_3} = -4(1-\lambda)^6(8v_2+9v_3-7)^3, \tag{A8a}$$

$$\mathcal{P}_1^{v_3} = 3\mathcal{P}_0^{v_3}, \tag{A8b}$$

$$\begin{aligned}\mathcal{P}_2^{v_3} = {} & -(1-\lambda)^4(8v_2+9v_3-7)\{768v_2^2(1-\lambda)(5\lambda-1)+3v_3^2(1-\lambda)(741\lambda-31)\\ & +8v_2(v_3(1-\lambda)(687\lambda-85)-306\lambda^2+496\lambda-206)+v_3(651\lambda^2-76\lambda-719)\\ & -5286\lambda^2+6824\lambda-1426\},\end{aligned} \tag{A8c}$$

$$\mathcal{P}_3^{v_3} = -(1-\lambda)^4(8v_2+9v_3-7)\{256v_2^2(1-\lambda)(53\lambda-17)+9v_3^2(1-\lambda)(1029\lambda-319) + 8v_2(3v_3(1-\lambda)(879\lambda-277)-470\lambda^2+592\lambda-170) + 3v_3(1995\lambda^2-2764\lambda+625)-17426\lambda^2+23608\lambda-5846\}, \tag{A8d}$$

$$\mathcal{P}_4^{v_3} = -(1-\lambda)^2(1-3\lambda)\{-64v_2^2(1-\lambda)[384v_2(1-\lambda)^2+v_3(495\lambda^2-1358\lambda+863) - 6(697\lambda^2-942\lambda+261)]+16v_2[3v_3^2(1-\lambda)^2(63\lambda-895) + v_3(1-\lambda)(38823\lambda^2-53248\lambda+15061)+31674\lambda^3-76106\lambda^2+57078\lambda-12630] - 27v_3^3(1-\lambda)^2(279\lambda+425)-12v_3^2(24879\lambda^3-59880\lambda^2+45529\lambda-10528) + 3v_3(87697\lambda^3-220167\lambda^2+178243\lambda-45677)+240090\lambda^3-542894\lambda^2+357966\lambda-55386\}, \tag{A8e}$$

$$\mathcal{P}_5^{v_3} = -(1-\lambda)^2(1-3\lambda)\{-64v_2^2(1-\lambda)[v_3(1069\lambda^2-1434\lambda+365)-6026\lambda^2+7932\lambda-2002] + 16v_2[3v_3^2(1-\lambda)^2(1445\lambda-517)-v_3(38317\lambda^3-88093\lambda^2+62567\lambda-12791) + 2(2335\lambda^3-9175\lambda^2+9161\lambda-2297)]+27v_3^3(169\lambda^3-499\lambda^2+451\lambda-121) - 12v_3^2(15961\lambda^3-36412\lambda^2+26015\lambda-5564)-v_3(216917\lambda^3-386819\lambda^2+188983\lambda-19945) + 2(264149\lambda^3-575591\lambda^2+382375\lambda-71269)\}, \tag{A8f}$$

$$\mathcal{P}_6^{v_3} = 2(1-3\lambda)^2\{(1-\lambda)^3[3v_3^2(45v_3(73\lambda-67)-2(6835\lambda-5644))-256v_2^2(32v_3-75)(1-\lambda)] - 16v_2(1-\lambda)^2[1056v_3^2(1-\lambda)^2-v_3(3911\lambda^2-7243\lambda+3332)+6(213\lambda^2-457\lambda+226)] - v_3(4877\lambda^4-40820\lambda^3+91880\lambda^2-80876\lambda+24939) - 4(3330\lambda^4-7283\lambda^3+1162\lambda^2+6195\lambda-3396)\}, \tag{A8g}$$

$$\mathcal{P}_7^{v_3} = -2(1-3\lambda)^2\{3v_3^2(1-\lambda)^3[135v_3(1-3\lambda)-14(289\lambda-92)] + 16v_2(1-\lambda)^2[v_3(1-\lambda)(412-1243\lambda)-5542\lambda^2+7230\lambda-1788] + v_3(1-\lambda)(93119\lambda^3-211785\lambda^2+147207\lambda-28337) + 4(4889\lambda^4-22219\lambda^3+32834\lambda^2-18597\lambda+3117)\}, \tag{A8h}$$

$$\mathcal{P}_8^{v_3} = -6(1-3\lambda)^3\{3v_3^2(15v_3-88)(1-\lambda)^3-v_3(1289\lambda^3-3119\lambda^2+2337\lambda-507) + 4(1297\lambda^3-2877\lambda^2+1858\lambda-282)\}, \tag{A8i}$$

$$\mathcal{P}_9^{v_3} = -4(1-3\lambda)^3[v_3(48\lambda^3-86\lambda^2+45\lambda-7)-151\lambda^3+257\lambda^2-135\lambda+21]. \tag{A8j}$$

APPENDIX B: COMMUTATOR OF AN OPERATOR IN A FRACTIONAL POWER

Here we derive the asymptotic series (3.51) for the commutator of an operator in a fractional power. The derivation starts with the representation

$$A^\alpha = \frac{1}{\Gamma(-\alpha)}\int_0^\infty ds s^{-\alpha-1}\mathrm{e}^{-sA}. \tag{B1}$$

Next, for $X(s) \equiv [\mathrm{e}^{-sA}, B]$ we have an equation,

$$\frac{dX}{ds} = -AX - [A,B]\mathrm{e}^{-sA}, \tag{B2}$$

with initial condition $X(0) = 0$. It is straightforward to verify that the solution has the form

$$X(s) = -\int_0^s ds_1 \mathrm{e}^{-(s-s_1)A}[A,B]\mathrm{e}^{-s_1 A}. \tag{B3}$$

Using Eqs. (B1) and (B3) we obtain

$$\begin{aligned}[A^\alpha, B] &= \frac{-1}{\Gamma(-\alpha)}\int_0^\infty ds s^{-\alpha-1}\int_0^s ds_1 \mathrm{e}^{-(s-s_1)A}[A,B]\mathrm{e}^{-s_1 A} \\ &= \frac{-1}{\Gamma(-\alpha)}\int_0^\infty ds s^{-\alpha-1}\left(s[A,B]\mathrm{e}^{-sA} + \int_0^s ds_1 [\mathrm{e}^{-(s-s_1)A}, [A,B]]\mathrm{e}^{-s_1 A}\right) \\ &= -\frac{\Gamma(-\alpha+1)}{\Gamma(-\alpha)}[A,B]A^{\alpha-1} - \frac{1}{\Gamma(-\alpha)}\int_0^\infty ds s^{-\alpha-1}\int_0^s ds_1 [\mathrm{e}^{-(s-s_1)A}, [A,B]]\mathrm{e}^{-s_1 A}. \end{aligned} \tag{B4}$$

In the last term we can again use the formula (B3) that will generate the second term in the expansion (3.51). Proceeding by induction we obtain the representation

$$[A^\alpha, B] = \sum_{n=1}^N C_\alpha^n \underbrace{[A,[A,\ldots[A}_{n}, B]\ldots]A^{\alpha-n} + \frac{(-1)^N}{\Gamma(-\alpha)}\int_0^\infty ds s^{-\alpha-1}\int_0^s ds_1 \frac{s_1^{N-1}}{(N-1)!}[\mathrm{e}^{-(s-s_1)A}, \underbrace{[A,[A,\ldots[A}_{N}, B]\ldots]\mathrm{e}^{-s_1 A}, \tag{B5}$$

which is valid for arbitrary N. This yields the formula (3.51).

APPENDIX C: MORE ON UNIVERSAL FUNCTIONAL TRACES

1. Coincidence limits of the heat kernel coefficients

The heat kernel $\hat{K}(s|x,y) = e^{s\hat{F}(\nabla)}\hat{\delta}(x,y)$ for the operator (4.2) satisfies the initial value problem for its heat equation

$$\partial_s \hat{K}(s|x,y) = \hat{F}(\nabla)K(s|x,y), \tag{C1a}$$

$$K(0|x,y) = \hat{\delta}(x,y). \tag{C1b}$$

The expansion (4.4) when substituted into Eq. (C1a) leads to the sequence of equations for the HAMIDEW coefficients $\hat{a}_n(x,y)$,

$$\sigma^\mu \nabla_\mu \hat{a}_0 = 0, \tag{C2a}$$

$$\begin{aligned}&(n+1)\hat{a}_{n+1} + \sigma^\mu\nabla_\mu \hat{a}_{n+1} \\ &\quad = \mathcal{D}^{-1/2}\Box(\mathcal{D}^{1/2}\hat{a}_n) + \left(\hat{P} - \frac{1}{6}R\hat{1}\right)\hat{a}_n, \quad n \geq 0,\end{aligned} \tag{C2b}$$

where $\sigma^\mu = \nabla^\mu\sigma(x,y)$ is the vector at the point x tangential to the geodetic curve connecting x and y. In deriving these recurrence equations we took into account that in the lowest two orders of the s-expansion the heat equation is satisfied in virtue of the equation for the Synge world function $\sigma(x,y)$,

$$\sigma = \frac{1}{2}\nabla^\mu\sigma\nabla_\mu\sigma, \tag{C3a}$$

and its corollary, the equation for the Van Vleck–Morette determinant (4.5)

$$\mathcal{D}^{-1}\nabla_\mu(\mathcal{D}\sigma^\mu) = d. \tag{C3b}$$

These equations in their turn should be amended by the initial conditions in the coordinate spacetime,

$$\sigma|_{y=x} = 0, \qquad \nabla_\mu\sigma|_{y=x} = 0, \qquad \hat{a}_0|_{y=x} = \hat{1}, \tag{C4}$$

the latter following from the initial condition (C1b).

By consecutively differentiating Eqs. (C2) and (C3) and taking their coincidence limits one can systematically derive in the closed form the expressions for the needed coincidence limits (4.6). The result of these calculations which basically reduce to the commutator algebra of covariant derivatives (4.3) begins with the following list [27][15]:

$$\begin{aligned}\sigma|_{y=x} &= 0, \qquad \nabla_\mu\nabla_\nu\sigma|_{y=x} = g_{\mu\nu}, \\ \nabla_\mu\nabla_\nu\nabla_\alpha\sigma|_{y=x} &= 0, \qquad \nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\sigma|_{y=x} = -\frac{2}{3}R_{\mu(\alpha\nu\beta)},\end{aligned} \tag{C5a}$$

$$\begin{aligned}&\nabla_\lambda\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\sigma|_{y=x} \\ &\quad = -\frac{1}{2}(\nabla_\lambda R_{(\alpha\mu\beta)\nu} + \nabla_\mu R_{(\alpha\nu\beta)\lambda} + \nabla_\nu R_{(\alpha\lambda\beta)\mu}),\end{aligned} \tag{C5b}$$

[15] The first and last indices in round brackets are symmetrized with the factor $1/2$.

$$\mathcal{D}^{1/2}|_{y=x}=1, \quad \nabla_\mu \mathcal{D}^{1/2}|_{y=x}=0, \quad \nabla_\mu\nabla_\nu \mathcal{D}^{1/2}|_{y=x}=\frac{1}{6}R_{\mu\nu}, \tag{C5c}$$

$$\nabla_\mu\nabla_\nu\nabla_\alpha \mathcal{D}^{1/2}|_{y=x}=\frac{1}{12}(\nabla_\mu R_{\nu\alpha}+\nabla_\nu R_{\alpha\mu}+\nabla_\alpha R_{\mu\nu}), \tag{C5d}$$

and

$$\hat{a}_0|_{y=x}=\hat{1}, \qquad \nabla_\mu \hat{a}_0|_{y=x}=0, \qquad \nabla_\mu\nabla_\nu \hat{a}_0|_{y=x}=\frac{1}{2}\hat{\boldsymbol{R}}_{\mu\nu},$$
$$\nabla_\mu\nabla_\nu\nabla_\alpha \hat{a}_0|_{y=x}=\frac{2}{3}\nabla_{(\mu}\hat{\boldsymbol{R}}_{\nu)\alpha}, \tag{C6a}$$

$$\hat{a}_1|_{y=x}=\hat{P}, \qquad \nabla_\mu \hat{a}_1|_{y=x}=\frac{1}{2}\nabla_\mu\hat{P}-\frac{1}{6}\nabla^\nu\hat{\boldsymbol{R}}_{\nu\mu}, \tag{C6b}$$

$$\begin{aligned}\nabla_\mu\nabla_\nu \hat{a}_1|_{y=x}=&\frac{1}{180}(2R^{\alpha\beta}R_{\alpha\mu\beta\nu}+2R_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda}{}_\nu-4R_{\mu\alpha}R^\alpha_\nu\\&+3\Box R_{\mu\nu}-\nabla_\mu\nabla_\nu R)\hat{1}-\frac{1}{6}\hat{\boldsymbol{R}}_{\alpha(\mu}\hat{\boldsymbol{R}}_{\nu)}{}^\alpha\\&+\frac{1}{6}\nabla_{(\mu}\nabla^\alpha\hat{\boldsymbol{R}}_{\nu)\alpha}+\frac{1}{2}\hat{P}\hat{\boldsymbol{R}}_{\mu\nu}+\frac{1}{3}[\hat{\boldsymbol{R}}_{\mu\nu},\hat{P}]\\&+\frac{1}{3}\nabla_\mu\nabla_\nu\hat{P},\end{aligned} \tag{C6c}$$

$$\begin{aligned}\hat{a}_2|_{y=x}=&\frac{1}{180}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-R_{\mu\nu}R^{\mu\nu}+\Box R)\hat{1}\\&+\frac{1}{12}\hat{\boldsymbol{R}}_{\mu\nu}\hat{\boldsymbol{R}}^{\mu\nu}+\frac{1}{2}\hat{P}^2+\frac{1}{6}\Box\hat{P}.\end{aligned} \tag{C6d}$$

Higher-order coincidence limits, which we need up to $\nabla^8\sigma|_{y=x}$, $\nabla^6\mathcal{D}^{1/2}|_{y=x}$, $\nabla^6\hat{a}_0|_{y=x}$, $\nabla^4\hat{a}_1|_{y=x}$, $\nabla^2\hat{a}_2|_{y=x}$, and $\hat{a}_3|_{y=x}$, take pages, so we do not present them here.

To obtain the needed list of traces (4.18)–(4.22), we apply this method in three spatial dimensions $d=3$, $\mu\mapsto i=1,2,3$ using the symbolic tensor algebra package *xAct* [44] for *Mathematica* [42]. For the tensor sector of the theory we have $\hat{1}=\delta_{ij}{}^{kl}\equiv\delta^k_{(i}\delta^l_{j)}$, $(\boldsymbol{R}_{mn})_{ij}{}^{kl}=-2\delta^{(k}_{(i}R^{l)}{}_{j)mn}$, whereas for the vector sector $\hat{1}=\delta^i_j$, $(\boldsymbol{R}_{mn})^i{}_j=R^i{}_{jmn}$.

2. Divergences of tensor and vector traces

To evaluate the one-loop effective action of HG, we compute the UFTs in the vector and tensor sectors using the method described in Sec. IV A. We keep only the logarithmically divergent part proportional to the infinite integral over the proper-time parameter,

$$\ln L^2\equiv\int\frac{ds}{s}. \tag{C7}$$

The UFTs are classified in Sec. IV B by the dimensionality a of the coefficient function that they multiply in the action; the dimension of the UFT itself is then $6-a$. Within each group we order the UFTs by the number of derivatives acting on the powers of Laplacian.

Let us start with the simplest traces of zero dimension ($a=6$). Their divergences cannot depend on curvature and are expressed purely in terms of the metric. They have the universal form

$$\begin{aligned}&\nabla_{i_1}\cdots\nabla_{i_{2N-2}}\frac{\hat{1}}{(-\Delta)^{N+1/2}}\delta(x,y)|^{\rm div}_{y=x}\\&=\frac{\ln L^2}{4\pi^2}\frac{(-1)^{N-1}}{(2N-1)!!}\sqrt{g}g^{(N-1)}_{i_1\cdots i_{2N-2}}\hat{1},\end{aligned} \tag{C8}$$

which is valid for both the tensor and the vector sectors. Here $g^{(N-1)}_{i_1\cdots i_{2N-2}}$ is the completely symmetric tensor built of the metric by the recurrence relations

$$g^{(1)}_{ij}=g_{ij}, \qquad g^{(2)}_{ijkl}=g_{ij}g_{kl}+g_{ik}g_{jl}+g_{il}g_{jk}, \tag{C9a}$$

$$g^{(N)}_{i_1\ldots i_{2N}}=\sum_{k=2}^{2N}g_{i_1i_k}g^{(N-1)}_{i_2\ldots i_{k-1}i_{k+1}\ldots i_{2N}}. \tag{C9b}$$

For other traces the full expressions are very lengthy because of the large number of free indices that they carry in general. To reduce the length, we can contract the indices in various combinations which appear in the effective action. Still, for the traces with $a=4$ and 3 (linear in curvature and in derivatives of curvature, respectively) the number of possible contractions is too high to be listed explicitly. On the other hand, these traces are obtained by a straightforward algebra from the lower heat-kernel coefficients listed in (C5) and (C6). Thus, we focus below on the most laborious traces with $a=2$ and $a=0$, which are quadratic and cubic in curvature. Full expressions for uncontracted UFTs for all $a=0,2,3,4,6$ can be obtained with the *Mathematica* code available at [50].

a. Traces with $a=2$, quadratic in curvature

These UFTs enter the divergent part of the action with the coefficient functions linear in curvature (to form the logarithmic divergences of the overall cubic power in the curvature), so that they represent local quantities quadratic in the curvature and having two free indices.

Tensor traces can have an even number of derivatives running from zero to six. For the trace without derivatives there are three possibilities of index contractions:

$$g^{ij}(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = -\frac{\ln L^2}{16\pi^2}\sqrt{g}g^{kl}\frac{1}{30}\left(\frac{1}{2}R^{mn}R_{mn}+\frac{1}{4}R^2+\Delta R\right), \tag{C10a}$$

$$g_{kl}(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = -\frac{\ln L^2}{16\pi^2}\sqrt{g}g_{ij}\frac{1}{30}\left(\frac{1}{2}R^{mn}R_{mn}+\frac{1}{4}R^2+\Delta R\right), \tag{C10b}$$

$$\delta^i_k(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = -\frac{1}{16\pi^2}\ln L^2\times\sqrt{g}\left(-\frac{43}{60}\delta^l_jR^{mn}R_{mn}+\frac{7}{12}R_{mj}R^{ml}-\frac{7}{12}RR^l_j+\frac{7}{20}\delta^l_jR^2+\frac{1}{15}\delta^l_j\Delta R\right). \tag{C10c}$$

For the two-derivative case there are nine contractions,

$$\delta_{mn}{}^{kl}\nabla_i\nabla_j\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{mn}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{\ln L^2}{8\pi^2}\sqrt{g}\left(\frac{19}{15}R_{ik}R^k_j-\frac{41}{60}g_{ij}R_{mn}R^{mn}-\frac{19}{15}RR_{ij}+\frac{1}{2}g_{ij}R^2+\frac{3}{10}\nabla_i\nabla_jR+\frac{1}{10}\Delta R_{ij}-\frac{1}{10}g_{ij}\Delta R\right), \tag{C11a}$$

$$g_{mn}g^{kl}\nabla_i\nabla_j\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{mn}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(-\frac{1}{5}R_{ik}R^k_j+\frac{3}{40}g_{ij}R_{mn}R^{mn}+\frac{1}{5}RR_{ij}-\frac{1}{16}g_{ij}R^2+\frac{3}{20}\nabla_i\nabla_jR+\frac{1}{20}\Delta R_{ij}-\frac{1}{20}g_{ij}\Delta R\right), \tag{C11b}$$

$$g_{mn}\nabla^i\nabla^k\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{mn}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(-\frac{1}{15}R^{ik}R_{jk}+\frac{1}{40}\delta^i_jR_{mn}R^{mn}+\frac{1}{15}RR^i_j-\frac{1}{48}\delta^i_jR^2+\frac{1}{20}\nabla^i\nabla_jR+\frac{1}{60}\Delta R^i_j-\frac{1}{60}\delta^i_j\Delta R\right), \tag{C11c}$$

$$\delta^l_n\nabla^i\nabla^k\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{jn}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(\frac{149}{120}R^i_kR^{kj}-\frac{8}{15}g^{ij}R_{mn}R^{mn}-\frac{119}{120}RR^{ij}+\frac{13}{48}g^{ij}R^2+\frac{37}{120}\nabla^i\nabla^jR-\frac{7}{40}\Delta R^{ij}-\frac{1}{30}g^{ij}\Delta R\right), \tag{C11d}$$

$$g^{kl}\nabla^i\nabla_m\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{mj}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(-\frac{1}{15}R^{ik}R^j_k+\frac{1}{40}g^{ij}R_{mn}R^{mn}+\frac{1}{15}RR^{ij}-\frac{1}{48}g^{ij}R^2+\frac{1}{20}\nabla^i\nabla^jR+\frac{1}{60}\Delta R^{ij}-\frac{1}{60}g^{ij}\Delta R\right), \tag{C11e}$$

$$\delta^l_n\nabla^i\nabla_m\frac{1}{(-\Delta)^{1/2}}\delta_{jl}{}^{mn}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(-\frac{1}{120}R^{ik}R_{jk}-\frac{7}{60}\delta^i_jR_{mn}R^{mn}+\frac{1}{20}RR^i_j+\frac{1}{16}\delta^i_jR^2-\frac{13}{120}\nabla^i\nabla_jR+\frac{29}{120}\Delta R^i_j-\frac{1}{30}\delta^i_j\Delta R\right), \tag{C11f}$$

$$\nabla^k\nabla^l\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{ij}(x,y)|^{\rm div}_{y=x}$$
$$=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(\frac{14}{15}R^i_kR^{kj}-\frac{37}{120}g^{ij}R_{mn}R^{mn}-\frac{13}{30}RR^{ij}+\frac{7}{48}g^{ij}R^2+\frac{13}{60}\nabla^i\nabla^jR-\frac{3}{20}\Delta R^{ij}-\frac{1}{60}g^{ij}\Delta R\right), \tag{C11g}$$

$$\nabla_m\nabla^k\frac{1}{(-\Delta)^{1/2}}\delta_{kl}{}^{mn}(x,y)|^{\rm div}_{y=x}=\frac{1}{8\pi^2}\ln L^2\sqrt{g}\left(\frac{11}{120}R_{kl}R^{kn}-\frac{3}{20}\delta^n_lR_{mn}R^{mn}-\frac{13}{60}RR^n_l+\frac{1}{80}\delta^n_lR^2+\frac{1}{40}\nabla_l\nabla^nR+\frac{1}{120}\Delta R^n_l\right), \tag{C11h}$$

$$\nabla_m\nabla_n \frac{1}{(-\Delta)^{1/2}}\delta_{ij}{}^{mn}(x,y)|^{\rm div}_{y=x} = \frac{1}{8\pi^2}\ln L^2\sqrt{g}\bigg(-\frac{1}{15}R^k_iR_{jk}+\frac{1}{40}g_{ij}R_{mn}R^{mn}+\frac{1}{15}RR_{ij}-\frac{1}{48}g_{ij}R^2-\frac{7}{60}\nabla_i\nabla_jR+\frac{11}{60}\Delta R_{ij}-\frac{1}{60}g_{ij}\Delta R\bigg). \tag{C11i}$$

For the four-derivative trace there are five ways to contract indices:

$$g_{mn}\nabla_i\nabla_j\nabla^k\nabla^l \frac{1}{(-\Delta)^{3/2}}\delta_{kl}{}^{mn}(x,y)|^{\rm div}_{y=x} = \frac{1}{4\pi^2}\ln L^2\sqrt{g}\bigg(\frac{1}{30}R_{ik}R^k_j-\frac{1}{80}g_{ij}R_{mn}R^{mn}-\frac{1}{30}RR_{ij}+\frac{1}{96}g_{ij}R^2-\frac{1}{40}\nabla_i\nabla_jR-\frac{1}{120}\Delta R_{ij}+\frac{1}{120}g_{ij}\Delta R\bigg), \tag{C12a}$$

$$\delta^m_n\nabla_i\nabla_j\nabla_k\nabla^l \frac{1}{(-\Delta)^{3/2}}\delta_{\rm lm}{}^{kn}(x,y)|^{\rm div}_{y=x} = \frac{1}{4\pi^2}\ln L^2\sqrt{g}\bigg(-\frac{247}{120}R_{ik}R^k_j+\frac{179}{240}g_{ij}R_{mn}R^{mn}+\frac{41}{60}RR_{ij}-\frac{19}{96}g_{ij}R^2-\frac{13}{10}\nabla_i\nabla_jR-\frac{1}{60}\Delta R_{ij}+\frac{1}{60}g_{ij}\Delta R\bigg), \tag{C12b}$$

$$g^{mn}\nabla_i\nabla_j\nabla_k\nabla_l \frac{1}{(-\Delta)^{3/2}}\delta_{mn}{}^{kl}(x,y)|^{\rm div}_{y=x} = \frac{1}{4\pi^2}\ln L^2\sqrt{g}\bigg(\frac{1}{30}R_{ik}R^k_j-\frac{1}{80}g_{ij}R_{mn}R^{mn}-\frac{1}{30}RR_{ij}+\frac{1}{96}g_{ij}R^2-\frac{1}{40}\nabla_i\nabla_jR-\frac{1}{120}\Delta R_{ij}+\frac{1}{120}g_{ij}\Delta R\bigg), \tag{C12c}$$

$$\nabla^i\nabla_m\nabla^k\nabla^l \frac{1}{(-\Delta)^{3/2}}\delta_{kl}{}^{mj}(x,y)|^{\rm div}_{y=x} = \frac{1}{4\pi^2}\ln L^2\sqrt{g}\bigg(-\frac{77}{40}R^{ik}R^j_k+\frac{307}{240}g^{ij}R_{mn}R^{mn}+\frac{113}{60}RR^{ij}-\frac{55}{96}g^{ij}R^2-\frac{17}{30}\nabla^i\nabla^jR+\frac{1}{30}\Delta R^{ij}+\frac{1}{120}g^{ij}\Delta R\bigg), \tag{C12d}$$

$$\nabla_i\nabla_k\nabla_l\nabla^m \frac{1}{(-\Delta)^{3/2}}\delta_{mj}{}^{kl}(x,y)|^{\rm div}_{y=x} = \frac{1}{4\pi^2}\ln L^2\sqrt{g}\bigg(-\frac{17}{40}R_{ik}R^k_j-\frac{13}{240}g_{ij}R_{mn}R^{mn}-\frac{9}{20}RR_{ij}+\frac{3}{32}g_{ij}R^2-\frac{29}{60}\nabla_i\nabla_jR-\frac{1}{20}\Delta R_{ij}+\frac{1}{120}g_{ij}\Delta R\bigg). \tag{C12e}$$

There is only one way to contract indices for the trace with six derivatives:

$$\nabla_i\nabla_j\nabla_k\nabla_l\nabla^m\nabla^n \frac{1}{(-\Delta)^{3/2}}\delta_{mn}{}^{kl}(x,y)|^{\rm div}_{y=x} = \frac{1}{6\pi^2}\ln L^2\sqrt{g}\bigg(\frac{57}{10}R_{ik}R^k_j-\frac{457}{160}g_{ij}R_{mn}R^{mn}-\frac{49}{20}RR_{ij}+\frac{47}{64}g_{ij}R^2+\frac{283}{80}\nabla_i\nabla_jR+\frac{1}{80}\Delta R_{ij}+\frac{19}{80}g_{ij}\Delta R\bigg). \tag{C13}$$

Vector traces In these groups there is one trace without derivatives:

$$(-\Delta)^{1/2}\delta_j{}^i(x,y)|^{\rm div}_{y=x} = -\frac{\ln L^2}{16\pi^2}\sqrt{g}\bigg(-\frac{3}{20}\delta^i_jR^{kl}R_{kl}+\frac{1}{6}R^{ik}R_{kj}-\frac{1}{6}R^i_jR+\frac{11}{120}\delta^i_jR^2+\frac{1}{30}\delta^i_j\Delta R\bigg), \tag{C14}$$

three traces with two derivatives:

$$\delta^l_k\nabla_i\nabla_j \frac{1}{(-\Delta)^{1/2}}\delta_k{}^l(x,y)|^{\rm div}_{y=x} = \frac{\ln L^2}{8\pi^2}\sqrt{g}\bigg(\frac{2}{15}R_{ik}R^k_j-\frac{11}{120}g_{ij}R_{kl}R^{kl}-\frac{2}{15}RR_{ij}+\frac{1}{16}g_{ij}R^2+\frac{3}{20}\nabla_i\nabla_jR+\frac{1}{20}\Delta R_{ij}-\frac{1}{20}g_{ij}\Delta R\bigg), \tag{C15a}$$

$$\nabla_i\nabla_k \frac{1}{(-\Delta)^{1/2}}\delta_j{}^k(x,y)|^{\rm div}_{y=x} = \frac{\ln L^2}{8\pi^2}\sqrt{g}\left(\frac{7}{20}R_{ik}R^k_j - \frac{17}{120}g_{ij}R_{kl}R^{kl} - \frac{4}{15}RR_{ij} + \frac{1}{16}g_{ij}R^2 + \frac{2}{15}\nabla_i\nabla_j R - \frac{1}{15}\Delta R_{ij} - \frac{1}{60}g_{ij}\Delta R\right), \tag{C15b}$$

$$\nabla_i\nabla^j \frac{1}{(-\Delta)^{1/2}}\delta_j{}^k(x,y)|^{\rm div}_{y=x} = \frac{\ln L^2}{8\pi^2}\sqrt{g}\left(-\frac{3}{20}R_{ij}R^{jk} + \frac{1}{40}\delta^k_i R_{jl}R^{jl} + \frac{3}{20}RR^k_i - \frac{1}{48}\delta^k_i R^2 - \frac{1}{30}\nabla_i\nabla^k R + \frac{1}{10}\Delta R^k_i - \frac{1}{60}\delta^k_i\Delta R\right), \tag{C15c}$$

and a single trace with four derivatives:

$$\nabla_i\nabla_j\nabla^l\nabla_k \frac{1}{(-\Delta)^{3/2}}\delta^k{}_l(x,y)|^{\rm div}_{y=x}$$
$$= \frac{\ln L^2}{4\pi^2}\sqrt{g}\left(-\frac{11}{20}R_{ik}R^k_j + \frac{47}{240}g_{ij}R_{kl}R^{kl} + \frac{2}{15}RR_{ij} - \frac{1}{32}g_{ij}R^2 - \frac{21}{40}\nabla_i\nabla_j R - \frac{1}{120}\Delta R_{ij} + \frac{1}{120}g_{ij}\Delta R\right). \tag{C16}$$

b. Traces with $a = 0$, cubic in curvature

These traces have dimension six and thus enter the divergent part of the action without extra curvature coefficients. Hence, for our purposes, it is sufficient to calculate them with all indices contracted and integrated over the whole three-dimensional space. In the integrals we freely integrate by parts in order to convert them to the sum of basic curvature invariants of Eq. (2.1).

Tensor traces can have no derivatives, two derivatives, or four derivatives. There are two possible index contractions in the traces without derivatives, which are expressed directly in terms of the coincidence limit of the third Schwinger-DeWitt coefficient $a_{3_{ij}{}^{kl}}(x,x)$,

$$\int d^3x\delta_{kl}{}^{ij}(-\Delta)^{3/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = \frac{3\ln L^2}{32\pi^2}\int d^3x\sqrt{g}\delta_{kl}{}^{ij}a_{3_{ij}{}^{kl}}(x,x)$$
$$= \frac{3\ln L^2}{32\pi^2}\int d^3x\sqrt{g}\left(\frac{31}{45}R^i_jR^j_kR^k_i - \frac{233}{210}R_{ij}R^{ij}R + \frac{673}{2520}R^3 + \frac{5}{84}R\Delta R - \frac{67}{420}R_{ij}\Delta R^{ij}\right), \tag{C17a}$$

$$\int d^3x g_{kl}g^{ij}(-\Delta)^{3/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = \frac{3\ln L^2}{32\pi^2}\int d^3x\sqrt{g}g_{kl}g^{ij}a_{3_{ij}{}^{kl}}(x,x)$$
$$= \frac{3\ln L^2}{32\pi^2}\int d^3x\sqrt{g}\left(-\frac{1}{60}R^i_jR^j_kR^k_i + \frac{1}{35}R_{ij}R^{ij}R - \frac{3}{560}R^3 + \frac{1}{112}R\Delta R + \frac{1}{280}R_{ij}\Delta R^{ij}\right). \tag{C17b}$$

The traces with two derivatives admit three possible contractions of indices,

$$\int d^3x g_{ij}\nabla^k\nabla^l(-\Delta)^{1/2}\delta_{kl}{}^{ij}(x,y)|^{\rm div}_{y=x} = \int d^3x g_{kl}\nabla^i\nabla^j(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x}$$
$$= -\frac{\ln L^2}{16\pi^2}\int d^3x\sqrt{g}\left(-\frac{1}{120}R^i_jR^j_kR^k_i + \frac{1}{70}R_{ij}R^{ij}R - \frac{3}{1120}R^3 + \frac{1}{224}R\Delta R + \frac{1}{560}R_{ij}\Delta R^{ij}\right), \tag{C18a}$$

$$\int d^3x\delta^j_l\nabla_k\nabla^i(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|^{\rm div}_{y=x} = -\frac{\ln L^2}{16\pi^2}\int d^3x\sqrt{g}\left(-\frac{23}{80}R^i_jR^j_kR^k_i + \frac{753}{1120}R_{ij}R^{ij}R - \frac{22}{105}R^3 - \frac{1}{84}R\Delta R - \frac{61}{560}R_{ij}\Delta R^{ij}\right). \tag{C18b}$$

The trace with four derivatives admits only one type of index contractions and reads

$$\int d^3x \nabla_k\nabla_l\nabla^i\nabla^j \frac{1}{(-\Delta)^{1/2}}\delta_{ij}{}^{kl}(x,y)\big|^{\rm div}_{y=x} = \frac{\ln L^2}{8\pi^2}\int d^3x\sqrt{g}\left(-\frac{211}{240}R^i_jR^j_kR^k_i + \frac{2987}{1680}R_{ij}R^{ij}R - \frac{703}{2240}R^3 + \frac{31}{1344}R\Delta R + \frac{141}{1120}R_{ij}\Delta R^{ij}\right). \tag{C19}$$

Vector traces: There are two traces in this group,

$$\int d^3x \delta^j_i(-\Delta)^{3/2}\delta^i{}_j(x,y)\big|^{\rm div}_{y=x} = \frac{3\ln L^2}{32\pi^2}\int d^3x\sqrt{g}\left(\frac{23}{180}R^i_jR^j_kR^k_i - \frac{43}{210}R_{ij}R^{ij}R + \frac{253}{5040}R^3 + \frac{29}{1680}R\Delta R - \frac{5}{168}R_{ij}\Delta R^{ij}\right), \tag{C20a}$$

$$\int d^3x \nabla^j\nabla_i(-\Delta)^{1/2}\delta^i{}_j(x,y)\big|^{\rm div}_{y=x} = -\frac{\ln L^2}{16\pi^2}\int d^3x\sqrt{g}\left(\frac{1}{20}R^i_jR^j_kR^k_i - \frac{67}{1680}R_{ij}R^{ij}R - \frac{3}{1120}R^3 - \frac{9}{1120}R\Delta R - \frac{13}{560}R_{ij}\Delta R^{ij}\right). \tag{C20b}$$

3. Relations between universal functional traces of different tensor ranks

Tensor traces with $a = 0$, $p \le 4$, $N + 1/2 = (p-3)/2$ listed in Eqs. (C17)–(C19) are the most complicated ones, because they require the knowledge of $\nabla^8\sigma(x,y)|_{y=x}$, $\nabla^6\mathcal{D}^{1/2}(x,y)|_{y=x}$, $\nabla^6\hat{a}_0(x,y)|_{y=x}$, $\nabla^4\hat{a}_1(x,y)|_{y=x}$, $\nabla^2\hat{a}_2(x,y)|_{y=x}$, and $\hat{a}_3(x,x)$. When used in the calculation of the effective action, the indices of derivatives in them are necessarily contracted with indices of the tensor delta function. Such traces can be simplified to the case of lower tensor rank—to vector and scalar ones.

Consider first the four-derivative trace. Since this trace is already cubic in curvature, it is not multiplied by any dimensionful quantity and enters as an integral (C19). We use the general formula valid for an arbitrary kernel $\hat{G}(x,y)$,

$$\nabla^x_k\hat{G}(x,y)|_{y=x} + \nabla^y_k\hat{G}(x,y)|_{y=x} = \nabla^x_k\hat{G}(x,x), \tag{C21}$$

and denote

$$\hat{G}(x,y)\overleftarrow{\nabla}_k \equiv \nabla^y_k\hat{G}(x,y), \quad \hat{G}(x,y)\overleftarrow{\nabla_k\nabla_l} \equiv \nabla^y_l\nabla^y_k\hat{G}(x,y). \tag{C22}$$

Note the order of derivatives is important and gets reversed with this notation. Neglecting the total-derivative terms, we obtain

$$\int d^3x\nabla_k\nabla_l\nabla^i\nabla^j\frac{1}{(-\Delta)^{1/2}}\delta_{ij}{}^{kl}(x,y)|_{y=x} = \int d^3x\nabla^i\nabla^j\frac{1}{(-\Delta)^{1/2}}\delta_{ij}{}^{kl}(x,y)\overleftarrow{\nabla_k\nabla_l}|_{y=x}. \tag{C23}$$

By the definition of the tensor delta function we have

$$\delta_{ij}{}^{kl}(x,y)\overleftarrow{\nabla_k\nabla_l} = \nabla_i\nabla_j\delta_{\rm scalar}(x,y), \tag{C24}$$

because $\int d^3y\delta_{ij}{}^{kl}(x,y)\overleftarrow{\nabla_k\nabla_l}\varphi(y) \equiv \int d^3y\delta_{ij}{}^{kl}(x,y)\times \nabla_k\nabla_l\varphi(y) = \nabla_i\nabla_j\varphi(x)$ for any scalar test function $\varphi(x)$, and $\delta_{\rm scalar}(x,y)$ means the scalar delta function which is a scalar with respect to the argument x and the scalar *density* with respect to the second argument y. Thus (C23) can be rewritten as

$$\begin{aligned}\int d^3x\nabla_k\nabla_l\nabla^i\nabla^j\frac{1}{(-\Delta)^{1/2}}\delta_{ij}{}^{kl}(x,y)|_{y=x} &= \int d^3x\left((\nabla_i\nabla_j)^2\frac{1}{(-\Delta_{\rm scalar})^{1/2}} + \nabla_i\nabla_j[(-\Delta)^{-1/2},\nabla^i\nabla^j]\right)\delta_{\rm scalar}(x,y)|_{y=x}\\ &= \int d^3x((-\Delta)^{3/2} + \nabla^iR_{ij}\nabla^j(-\Delta)^{-1/2} + \nabla_i\nabla_j[(-\Delta)^{-1/2},\nabla^i\nabla^j])\delta_{\rm scalar}(x,y)|_{y=x}.\end{aligned} \tag{C25}$$

The commutator in the last term can be calculated using Eq. (3.51). We have used this identity as a check of the symbolic computation result and verified that an independent evaluation of the left-hand side (purely tensorial) coincides with the scalar type functional trace on the right-hand side.

The two-derivative traces contain three possible index contractions. Two of them, Eq. (C18a), trivially reduce to the purely scalar case,

$$\begin{aligned}&\int d^3x g^{ij}\nabla_k\nabla_l(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|_{y=x}\\&\quad=\int d^3x\nabla_k\nabla_l(-\Delta)^{1/2}g^{kl}\delta_{\rm scalar}(x,y)|_{y=x}\\&\quad=-\int d^3x(-\Delta)^{3/2}\delta_{\rm scalar}(x,y)|_{y=x}.\end{aligned}\tag{C26}$$

For the third trace (C18b) we have

$$\begin{aligned}&\int d^3x\delta^i_k\nabla_l\nabla^j(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)|_{y=x}\\&\quad=-\int d^3x\delta^i_k\nabla^j(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)\overleftarrow{\nabla}_l|_{y=x}\\&\quad=\int d^3x\delta^i_k\nabla^j(-\Delta)^{1/2}\nabla_{(i}\delta_{j)}{}^k(x,y)|_{y=x},\end{aligned}\tag{C27}$$

where we took into account that $\delta_{ij}{}^{kl}(x,y)\overleftarrow{\nabla}_l=-\nabla_{(i}\delta_{j)}{}^k(x,y)$ from the definition of the vector delta function[16] $\delta_j{}^k(x,y)$. Commuting the Laplacian to the right, we arrive at

$$\begin{aligned}&\int d^3x\delta^i_k\nabla^j(-\Delta)^{1/2}\nabla_{(i}\delta_{j)}{}^k(x,y)|_{y=x}\\&\quad=\frac{1}{2}\int d^3x(\nabla^j\nabla_k(-\Delta)^{1/2}\delta_j{}^k(x,y)-\delta^i_k(-\Delta)^{3/2}\delta_i{}^k(x,y)\\&\qquad+\nabla^j[(-\Delta)^{1/2},\nabla_k]\delta_j{}^k(x,y)\\&\qquad+\delta^i_k\nabla^j[(-\Delta)^{1/2},\nabla_j]\delta_i{}^k(x,y))|_{y=x}.\end{aligned}\tag{C28}$$

The first term can be further reduced to scalar traces, but we do not do it here. This relation has also been used as a check by independent calculation of the left- and right-hand sides.

APPENDIX D: METRIC DECOMPOSITION ON A SPHERE S^d

Orthonormal basis of harmonics (6.6) on the d-dimensional sphere, $H^{A(n)}_{ij}$, is motivated by the tensor decomposition into the transverse-traceless tensor, the vector and scalar parts

$$h_{ij}=h^{TT}_{ij}+\nabla_i\xi_j+\nabla_j\xi_i+\left(\nabla_i\nabla_j-\frac{1}{d}g_{ij}\Delta\right)E+\frac{1}{d}g_{ij}\phi,\tag{D1a}$$

$$g^{ij}h^{TT}_{ij}=0,\qquad\nabla^ih^{TT}_{ij}=0,\qquad\nabla_i\xi^i=0,\tag{D1b}$$

in which their ingredients $h^{TT}_{ij}(x)$, $\xi_i(x)$, $E(x)$, and $\phi(x)$ are in their turn expanded in the complete basis of irreducible tensor, vector, and scalar eigenmodes of the covariant Laplacian (6.7). The square-root normalization factors of the latter easily follow from the relations which are valid for generic transverse vectors ξ_i, ζ_i and scalars Φ, Ψ on S^d,

$$4\int d^dx\sqrt{g}\nabla_{(i}\xi_{j)}\nabla^{(i}\zeta^{j)}=2\int d^dx\sqrt{g}\xi_i\left(-\Delta-\frac{1}{d}R\right)\zeta^i,\tag{D2}$$

$$\begin{aligned}&\int d^dx\sqrt{g}\left(\nabla^i\nabla^j-\frac{1}{d}g^{ij}\Delta\right)\Phi(x)\left(\nabla_i\nabla_j-\frac{1}{d}g_{ij}\Delta\right)\Psi(x)\\&\quad=\int d^dx\sqrt{g}\Phi(x)\left(\frac{d-1}{d}\Delta^2+\frac{1}{d}R\Delta\right)\Psi(x).\end{aligned}\tag{D3}$$

This provides orthonormality and completeness relations for $H^{A(m)}_{ij}(x)$,

$$\int d^dx\sqrt{g}H^{A(m)}_{ij}(x)H^{ij}_{B(n)}(x)=\delta^{(m)}_{(n)}\delta^A_B,\tag{D4}$$

$$\sum_{A,(n)}H^{A(n)}_{ij}(x)H^{kl}_{A(n)}(y)=\frac{1}{\sqrt{g}}\delta_{ij}{}^{kl}(x,y).\tag{D5}$$

On S^d the operator $\mathbb{D}_{ij}{}^{kl}$ is composed only of covariant derivatives and covariantly conserved metric. In the calculation of the matrix elements (6.10) all covariant derivatives, including those in the definition of the spherical harmonics (6.6), can be grouped into powers of the Laplacian Δ. Then the replacement of Δ by its eigenvalues tells us that the matrix elements in each sector are functions of the eigenvalues of the Laplacian in this sector. This leads to Eqs. (6.11).

Clearly, in the tensor sector the function $\mathbb{D}_t(n)$ is polynomial in the Laplacian eigenvalues and has cubic order, according to the dimensionality of $\mathbb{D}$. In the vector and scalar sectors this need not be the case *a priori*, because of the normalization factors in (6.6b) and (6.6c) with the inverse powers of the Laplacian. In the vector sector, however, the normalization factors cancel, due to the identity

$$\mathbb{D}_{ij}{}^{kl}\nabla_{(k}\xi_{l)}=\nabla_{(i}\mathbb{D}_v\xi_{j)},\tag{D6}$$

valid for any transverse vector ξ_i. Equation (D6) is just the statement that $\mathbb{D}$ transforms a vector polarization of the metric into a vector polarization.

[16]Note that unlike in the rest of the text, $\delta_j{}^k(x,y)$ here acts on vectors with *lower* indices.

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