Canonical transformations generated by the boundary volume: Unimodular and non-Abelian teleparallel gravity

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Recently, a new choice of variables was identified to understand how the quantum group structure appeared in three-dimensional gravity [M. Dupuis, L. Freidel, F. Girelli, A. Osumanu, and J. Rennert, On the origin of the quantum group symmetry in 3d quantum gravity, arXiv:2006.10105]. These variables are introduced via a canonical transformation generated by a boundary term. We show that this boundary term can actually be taken to be the volume of the boundary and that the new variables can be defined in any dimension greater than three. In addition, we study the associated metric and teleparallel formalisms. The former is a variant of the Henneaux-Teitelboim model for unimodular gravity. The latter provides a non-Abelian generalization of the usual Abelian teleparallel formulation.

I. INTRODUCTION

Quantum states are built from the representations of the symmetries, identified at the classical level. The structure of the symmetries depends on the choice of variables used to describe the phase space of the classical theory. In the gravity case, moving from the metric to the frame field variables (Palatini formalism) enlarges the set of gauge symmetries since internal Lorentz rotations are absent in the usual metric formalism. On the other hand, treating gravity as a constrained topological field theory [1,2] has the opposite effect: the topological theory has a larger symmetry algebra. Gravity is recovered by imposing constraints that break the gauge symmetries down to those that appear in general relativity. Using the teleparallel formulation of gravity leads to yet another different type of symmetries, representing different representations at the quantum level (possibly related by dualities). Since both the metric and teleparallel formulations can be seen as a second-order formulation of gravity, it is interesting to study the relationship between the respective symmetries depending on the choice of variables. We refer to [3–11] for a recent discussion about these different issues in gravity.

In field theory, canonical transformations are typically generated by adding topological invariants or boundary terms to the action. Let us list some important examples in the context of gravity below (a more detailed discussion can be found in e.g., [3–11] and references therein).

  (i) In any dimension, the Gibbons-Hawking-York (boundary) term generates the canonical transformation changing the polarization from the superspace momentum representation, where the wave functionals depend on the Arnowitt-Deser-Misner (ADM) momentum $\tilde{\pi}^{ab} = \frac{1}{16\pi G}\sqrt{\det h}(K^{ab} - h^{ab}K)$, to the position representation, where the wave-functional $\Psi[h_{ab}]$ depends only on the intrinsic metric of the boundary, cf. [12–14].

 (ii) For gravity in four spacetime dimensions, the Holst term induces the Ashtekar-Barbero canonical transformation from the ADM-type of variables to the (complex) Ashtekar variables, which greatly simplifies the structure of the constraints, enabling the loop quantization of gravity [14–17].

(iii) In three dimensions, quantum groups play an important role in both the quantization of Chern-Simons theory [18] and the discrete path integral formulation [19] of gravity. Quantum groups appear as deformed gauge symmetries, where the quantum deformation is parametrized by the cosmological constant. This is in striking contrast to the classical formulation of three-dimensional gravity in the first-order formalism, where the gauge symmetries do *not*

depend on the cosmological constant. It was recently found that a canonical transformation, induced by a boundary term, could be used to have a (bulk) theory defined in terms of new variables, such that the gauge symmetries do depend *now* on the cosmological constant as well, see [20]. This provided a direct manner to retrieve the quantum group symmetries upon quantization.

It has been conjectured that quantum groups could also be relevant for quantum gravity in four dimensions [21–23], so one might wonder whether the canonical transformation or the boundary term that was identified in [20] can also be introduced in four dimensions. We show here that this is indeed the case and that there is a straightforward generalization of the boundary term in the first-order formalism. It is then natural to ask what happens to the second-order formalism and how the metric and teleparallel formulations of gravity could be expressed in terms of the new variables. Here, we answer these questions. In the metric formalism, we have a new notion of the Christoffel symbol. The volume term that is added to the action via the cosmological constant is replaced by the divergence of a vector field, whose norm depends on the cosmological constant. In fact, we recover a variant of the Henneaux-Teitelboim model for unimodular gravity [24]. On the other hand, in the teleparallel formalism [25], the new variables suggest a deformed notion of torsion, which defines a new and non-Abelian version of the standard teleparallel formulation.

The boundary term that generates the canonical transformation that was introduced in [20] depends on an internal vector r^I, whose norm sets the value of the cosmological constant. In order to not increase the number of degrees of freedom, it was assumed that the vector is an auxiliary background structure, whose field variations vanish on the covariant phase space, i.e., $\delta[r^I] = 0$. However, the boundary term [20] has a very strong resemblance with the volume term of the boundary. It seems natural, therefore, to relax the condition $\delta[r^I] = 0$. In fact, we will show that if we actually interpret the vector r^I as the *internal normal* n^I to the boundary, the generating functional [20] is exactly the volume of the boundary. In here, we show that given this interpretation, we can allow for $\delta[n^I] \neq 0$ and *still* perform the canonical transformation, *without* adding any new degrees of freedom in the bulk.

The new formulation has some interesting relations with already existing frameworks.

(i) The boundary term plays an important role in holographic renormalization [26,27] so that our new variables will also be relevant to this framework.

(ii) With the change of variables, the volume term that scales with the cosmological constant drops out of the action and is replaced by the divergence of a vector field and a constraint, which sets the cosmological constant as the norm of this vector field.

We will see below that the resulting action defines a variation of the Henneaux-Teitelboim model for unimodular gravity [24].

One can often see the first-order formalism for gravity as a gauge theory, so that the canonical variables are Lie algebra-valued differential forms. This is obviously realized in three dimension in the Chern-Simons formulation. In four dimensions, the McDowell-Mansouri formulation [28] provides a similar realization. Depending on the value of the cosmological constant and the signature of the metric, one usually deals with a (anti-)de Sitter or Poincaré (or Euclidian) gauge group, where the frame field has values in the boost or translation sector of the group. Accordingly, the (anti-)de Sitter or Poincaré curvature splits into the Lorentzian curvature sector, which depends on the cosmological constant, and the translational or boost sector, which represents torsion and is independent of Λ. Since the (anti-)de Sitter or Poincaré groups can be seen as the semicross product between the Lorentz transformations and the boosts or translations, there is a natural action of the Lorentz group on the frame field. On the other hand, there is no action of translations back onto rotations (the commutator of a translation and a rotation has no translational part).

The canonical transformation that we will study in here amounts to choosing a different decomposition of the (anti-)de Sitter group (the Poincaré case can also be treated in the same way). Indeed, if the standard formulation amounts to use the Cartan decomposition, then the canonical transformation instead picks the Iwasawa decomposition. We have then a double cross product relation and a more symmetric treatment between the connection and the frame field: the connection acts on the frame field, and the frame field acts back on the connection. As a consequence the (anti-)de Sitter or Poincaré curvature can be split differently than before. We get a generalized notion of torsion, which can be viewed as a (non-Abelian) curvature in the frame field sector. This is especially relevant regarding the teleparallel formulation.

We finally note that the boundary term could also be added in the usual metric formulation of general relativity. We expect that this new boundary term will induce a canonical transformation from pure metric gravity to a version of unimodular gravity, similar to the one introduced in this paper. Since we are primarily interested in approaches to quantum gravity based on the first-order formalism, we do not explore that thread in this paper.

Outline: The article is organized as follows. In Sec. II, we review the gauge theory framework when dealing with a matched pair (double cross product) of Lie algebras, which will conveniently set up the conventions for the rest of the paper. In Sec. III, we discuss how the canonical

transformation can be implemented in any dimension by the boundary term given in terms of the volume of the boundary. In Sec. IV A, we determine the metric formulation in the new variables, while in Sec. IV B, we determine the dual teleparallel formulation.

II. GAUGE THEORY FOR A MATCHED PAIR OF LIE GROUPS

In the following, we develop the notion of a gauge theory for a matched pair of Lie algebras. This structure generalizes the more common semidirect product that can be found in the Poincaré case or in the decomposition of the Lorentz group in terms of rotations and boosts (although they do not form a Lie algebra). Matched pairs are the relevant structure which gives rise to the quantum group structure at the quantum level, for example, in the three-dimensional gravity case. A reader who would be more geometry inclined can skip this section, which nevertheless emphasizes that what is called torsion can also be viewed as a type of curvature.

The algebraic background behind the matched pairs can be found in [29]. We call a matched pair $\mathfrak{G} = \mathfrak{g} \bowtie \mathfrak{h}$ of Lie algebras the pair of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, with respective Lie brackets $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]_{\mathfrak{h}}$, equipped with two bilinear maps $\lhd : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{h}, (x, \alpha) \mapsto x \lhd \alpha$ and $\rhd : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}, (x, \alpha) \mapsto x \rhd \alpha$, such that $\mathfrak{h}$ is a right $\mathfrak{g}$ module, and $\mathfrak{g}$ is a left $\mathfrak{h}$ module. More intuitively, $\mathfrak{g}$ acts from the right on $\mathfrak{h}$ via $\lhd$, and there is a back action $\rhd$ of $\mathfrak{h}$ on $\mathfrak{g}$. To guarantee the Jacobian identity on $\mathfrak{G}$, the left and right actions must satisfy the following compatibility conditions: $\forall \alpha, \beta \in \mathfrak{g}, x, y \in \mathfrak{h}$,

$$x \rhd [\alpha, \beta]_{\mathfrak{g}} = [x \rhd \alpha, \beta]_{\mathfrak{g}} + [\alpha, x \rhd \beta]_{\mathfrak{g}}$$
$$+ (x \lhd \alpha) \rhd \beta - (x \lhd \beta) \rhd \alpha, \quad (2.1a)$$

$$[x, y]_{\mathfrak{h}} \lhd \alpha = [x \lhd \alpha, y]_{\mathfrak{h}} + [x, y \lhd \alpha]_{\mathfrak{h}}$$
$$+ x \lhd (y \rhd \alpha) - y \lhd (x \rhd \alpha). \quad (2.1b)$$

As a vector space, $\mathfrak{g} \bowtie \mathfrak{h}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$. If we introduce a matched basis, such that $(\alpha, x) \in \mathfrak{g} \oplus \mathfrak{h} \sim \mathfrak{g} \bowtie \mathfrak{h}$, we may express the Lie bracket of $\mathfrak{G}$ in terms of $[\cdot, \cdot]_{\mathfrak{g}}, [\cdot, \cdot]_{\mathfrak{h}}, \lhd$ and $\rhd$,

$$[(\alpha, x), (\beta, y)] \equiv ([\alpha, \beta]_{\mathfrak{g}} + x \rhd \beta - y \rhd \alpha, [x, y]_{\mathfrak{h}}$$
$$+ x \lhd \beta - y \lhd \alpha), \quad \forall \alpha, \beta \in \mathfrak{g}, \quad x, y \in \mathfrak{h}.$$
$$(2.2)$$

In particular, there is the mixed bracket

$$[(0, x), (\beta, 0)] \equiv [x, \beta] = x \lhd \beta + x \rhd \beta \in \mathfrak{g} \oplus \mathfrak{h}. \quad (2.3)$$

Notice that the left and right modules can be written in terms of the Lie bracket alone,

$$x \lhd \beta := [x, \beta]|_{\mathfrak{h}}, \qquad x \rhd \beta := [x, \beta]|_{\mathfrak{g}}, \quad (2.4)$$

where $[x, \beta]|_V$ is the projection of the Lie bracket $[\cdot, \cdot]$ on the vector space V. With this notation, the compatibility relations (2.1) become

$$x \rhd [\alpha, \beta]_{\mathfrak{g}} = [[x, \alpha]|_{\mathfrak{g}}, \beta]_{\mathfrak{g}} + [\alpha, [x, \beta]|_{\mathfrak{g}}]_{\mathfrak{g}}$$
$$+ [[x, \alpha]|_{\mathfrak{h}}, \beta]|_{\mathfrak{g}} - [[x, \beta]|_{\mathfrak{h}}, \alpha]|_{\mathfrak{g}}, \quad (2.5a)$$

$$[x, y]_{\mathfrak{h}} \lhd \alpha = [[x, \alpha]|_{\mathfrak{h}}, y]_{\mathfrak{h}} + [x, [y, \alpha]|_{\mathfrak{h}}]_{\mathfrak{h}}$$
$$+ [x, [y, \alpha]|_{\mathfrak{g}}]|_{\mathfrak{h}} - [y, [x, \alpha]|_{\mathfrak{g}}]|_{\mathfrak{h}}. \quad (2.5b)$$

Consider then a gauge connection A for such a Lie algebra, i.e., a $\mathfrak{G}$-valued one-form on a manifold $\mathcal{M}$, and consider its different components with respect to the matched Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$. This type of gauge theory was studied by Majid [29] at the discrete level of parallel propagators (holonomies). Below we will derive the infinitesimal picture.

Let us denote by γ and h the components[1] of such a gauge connection with respect to the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The $\mathfrak{g} \bowtie \mathfrak{h}$ Lie algebra-valued gauge connection takes the form

$$A = \gamma + h \in (\mathfrak{g} \oplus \mathfrak{h}) \otimes \Lambda^1(M). \quad (2.6)$$

Due to the kick-back action between $\mathfrak{g}$ and $\mathfrak{h}$, there is a nontrivial mixing between the components of the $\mathfrak{g} \bowtie \mathfrak{h}$ basis, see (2.4). Consider, for example, a $\mathfrak{G}$-valued p-form $B = (\beta, b)$. Its exterior covariant derivative is

$$d_A B = d\beta + db + [\gamma + h, \beta + b]$$
$$= d\beta + db + [\gamma, \beta]_{\mathfrak{g}} + [\gamma, b] + [h, b]_{\mathfrak{h}} + [h, \beta]$$
$$= (d\beta + [\gamma, \beta]_{\mathfrak{g}} + [\gamma, b]|_{\mathfrak{g}} + [h, \beta]|_{\mathfrak{g}})$$
$$+ (db + [h, b]_{\mathfrak{h}} + [\gamma, b]|_{\mathfrak{h}} + [h, \beta]|_{\mathfrak{h}}). \quad (2.7)$$

For the individual component fields $(\beta, 0)$ and $(0, b)$, we have the covariant derivatives

$$D_{|\mathfrak{g}} \beta = d\beta + [\gamma, \beta]_{\mathfrak{g}} + [h, \beta]|_{\mathfrak{g}}, \quad (2.8)$$

$$D_{|\mathfrak{h}} b = db + [\gamma, b]|_{\mathfrak{h}} + [h, b]_{\mathfrak{h}}. \quad (2.9)$$

Due to the kick-back actions between the two Lie algebras, the covariant derivatives depend on both connection components γ and h. Consider then the components of the field strength F in the directions of $\mathfrak{g}$ and $\mathfrak{h}$,

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[1]We will relate (γ, h) to the spin connection ω and the frame field e.

$$\boldsymbol{F}[\boldsymbol{A}] = \mathrm{d}\boldsymbol{A} + \frac{1}{2}[\boldsymbol{A} \wedge \boldsymbol{A}] = \mathrm{d}\gamma + \mathrm{d}h + \frac{1}{2}[(\gamma + h) \wedge (\gamma + h)]$$

$$= \mathrm{d}\gamma + \mathrm{d}h + \frac{1}{2}[\gamma \wedge \gamma]_{\mathfrak{g}} + [\gamma \wedge h]|_{\mathfrak{g}} + [\gamma \wedge h]|_{\mathfrak{h}} + \frac{1}{2}[h \wedge h]_{\mathfrak{h}}$$

$$= \left(\mathrm{d}\gamma + \frac{1}{2}[\gamma \wedge \gamma]_{\mathfrak{g}} + [\gamma \wedge h]|_{\mathfrak{g}}\right) + \left(\mathrm{d}h + [\gamma \wedge h]|_{\mathfrak{h}} + \frac{1}{2}[h \wedge h]_{\mathfrak{h}}\right) \equiv \mathcal{F} + \mathcal{T}. \qquad (2.10)$$

The physical significance of splitting $\boldsymbol{F}$ into $\mathcal{F} \in \mathfrak{g} \otimes \Lambda^2(\mathcal{M})$ and $\mathcal{T} \in \mathfrak{h} \otimes \Lambda^2(\mathcal{M})$ will be clear below. We will identify, in fact, $\mathcal{F}$ and $\mathcal{T}$ with a deformed version of $\mathfrak{h}$-valued torsion and $\mathfrak{g}$-valued curvature,

$$\mathcal{F} = \mathrm{d}\gamma + \frac{1}{2}[\gamma \wedge \gamma]_{\mathfrak{g}} + [\gamma \wedge h]|_{\mathfrak{g}} \equiv F + [\gamma \wedge h]|_{\mathfrak{g}}, \qquad (2.11)$$

$$\mathcal{T} = \mathrm{d}h + [\gamma \wedge h]|_{\mathfrak{h}} + \frac{1}{2}[h \wedge h]_{\mathfrak{h}} \equiv T + \frac{1}{2}[h \wedge h]_{\mathfrak{h}}. \qquad (2.12)$$

Again, the curvature components depend on both connection types. The Bianchi identity for each component is

$$0 = \mathrm{d}_{\boldsymbol{A}}\boldsymbol{F} = \mathrm{d}\mathcal{F} + \mathrm{d}\mathcal{T} + [(\gamma + h) \wedge (\mathcal{F} + \mathcal{T})]$$

$$= (\mathrm{d}\mathcal{F} + [\gamma \wedge \mathcal{F}]_{\mathfrak{g}} + [h \wedge \mathcal{F}]|_{\mathfrak{g}} + [\gamma \wedge \mathcal{T}]|_{\mathfrak{g}})$$

$$+ (\mathrm{d}\mathcal{T} + [\gamma \wedge \mathcal{T}]_{\mathfrak{h}} + [h \wedge \mathcal{F}]|_{\mathfrak{h}} + [\gamma \wedge \mathcal{T}]|_{\mathfrak{h}}). \qquad (2.13)$$

Let us consider then the gauge transformations of such a $\mathfrak{G}$-valued connection. Let $\lambda = (\phi, t)$ be a $\mathfrak{G}$-valued scalar, which generates an infinitesimal transformation of the connection $\boldsymbol{A} = (\gamma, h)$,

$$\delta_{\lambda}[\boldsymbol{A}] = \mathrm{d}\lambda + [\boldsymbol{A}, \lambda] = (\mathrm{d}\phi + [\gamma, \phi]_{\mathfrak{g}} + [h, \phi]|_{\mathfrak{g}}$$

$$+ [\gamma, t]|_{\mathfrak{g}}) + (\mathrm{d}t + [h, t]_{\mathfrak{h}} + [\gamma, t]|_{\mathfrak{h}} + [h, \phi]|_{\mathfrak{h}}). \qquad (2.14)$$

For pure rotations ($t = 0$) and pure translations ($\phi = 0$), we find

$$\left. \begin{aligned} \delta_{\phi}[\gamma] &= \mathrm{d}\phi + [\gamma, \phi]_{\mathfrak{g}} + [h, \phi]|_{\mathfrak{g}} = \boldsymbol{D}_{|\mathfrak{g}}\phi, & \delta_{\phi}[h] &= [h, \phi]|_{\mathfrak{h}}, \\ \delta_t[\gamma] &= [\gamma, t]|_{\mathfrak{g}}, & \delta_t[h] &= \mathrm{d}t + [h, t]_{\mathfrak{h}} + [\gamma, t]|_{\mathfrak{h}} = \boldsymbol{D}_{|\mathfrak{h}}t. \end{aligned} \right\} \qquad (2.15)$$

What is nice about these transformations is that the variation of γ or h in each of the different components is either a commutator or a derivative. In other words, the two components of the curvature two-form behave completely analogous under gauge transformations. Since $\delta_{\lambda}[\boldsymbol{F}] = [\boldsymbol{F}, \lambda]$, we obtain

$$[\mathcal{F} + \mathcal{T}, \phi + t] = [\mathcal{F}, \phi]_{\mathfrak{g}} + [\mathcal{F}, t]|_{\mathfrak{g}} + [\mathcal{T}, \phi]|_{\mathfrak{g}}$$

$$+ [\mathcal{T}, t]_{\mathfrak{h}} + [\mathcal{F}, t]|_{\mathfrak{h}} + [\mathcal{T}, \phi]|_{\mathfrak{h}}. \qquad (2.16)$$

In other words,

$$\left. \begin{aligned} \delta_{\phi}[\mathcal{F}] &= [\mathcal{F}, \phi]_{\mathfrak{g}} + [\mathcal{T}, \phi]|_{\mathfrak{g}}, & \delta_{\phi}[\mathcal{T}] &= [\mathcal{T}, \phi]|_{\mathfrak{h}} \\ \delta_t[\mathcal{F}] &= [\mathcal{F}, t]|_{\mathfrak{g}}, & \delta_t[\mathcal{T}] &= [\mathcal{T}, t]_{\mathfrak{h}} + [\mathcal{F}, t]|_{\mathfrak{h}}. \end{aligned} \right\} \qquad (2.17)$$

A simple, but nontrivial, example of such a matched pair of Lie algebras is the Poincaré Lie algebra. It is a semicross product $\mathfrak{G} = \mathfrak{g} \rhd\!\!\lhd \mathfrak{h}$, where $\mathfrak{g}$ is either of the Lie algebra $\mathfrak{so}(n)$ or $\mathfrak{so}(n-1, 1)$, depending on the signature of spacetime, and $\mathfrak{h} \sim \mathbb{R}^n$ is the Abelian Lie algebra of translations on $\mathbb{R}^n$. The respective generators are the generators of (Lorentz) rotations $J_{MN} = -J_{NM}$ and translations P_M, which are labelled by spacetime indices $M, N, \ldots = 1, \ldots, n$.

In the Poincaré context, there is no kick-back action of $\mathfrak{h}$ on $\mathfrak{g}$ (recall $[J, P] \propto P$). Given (2.4), we thus have

$$(\phi, t) \in \mathfrak{g} \rhd\!\!\lhd \mathfrak{h}, \qquad t \lhd \phi := [t, \phi]|_{\mathfrak{h}}, \qquad t \rhd \phi := [t, \phi]|_{\mathfrak{g}} = 0. \qquad (2.18)$$

The translations commute, i.e., $[\cdot, \cdot]_{\mathfrak{h}} = 0$, and the covariant derivative for the component fields $(\phi, 0)$ and $(0, t)$ with values in $\mathfrak{g}$ or $\mathfrak{h}$ is given by

$$\boldsymbol{D}_{|\mathfrak{g}}\phi = \mathrm{d}\phi + [\gamma, \phi]_{\mathfrak{g}}, \qquad \boldsymbol{D}_{|\mathfrak{h}}t = \mathrm{d}t + [\gamma, t]_{\mathfrak{h}}. \qquad (2.19)$$

Consider then the components of the curvature two-form $\boldsymbol{F}$ in the rotational and translational directions $\mathfrak{g}$ and $\mathfrak{h}$ of the Poincaré Lie algebra,

$$\boldsymbol{F}[\boldsymbol{A}] = \left(\mathrm{d}\gamma + \frac{1}{2}[\gamma \wedge \gamma]_{\mathfrak{g}}\right) + (\mathrm{d}h + [\gamma \wedge h]|_{\mathfrak{h}}) = F + T. \qquad (2.20)$$

We recognize the usual notion of curvature F and torsion T, provided we identify γ with the spin connection ω and h with the frame field e.

The infinitesimal gauge transformations for pure rotations ($t = 0$) and pure translations ($\phi = 0$) are

$$\delta_\phi[\gamma] = \mathrm{d}\phi + [\gamma, \phi]_{\mathfrak{g}} = \boldsymbol{D}_{|\mathfrak{g}}\phi, \qquad \delta_\phi[h] = [h, \phi]|_{\mathfrak{h}},$$

$$\delta_t[\gamma] = 0, \qquad \delta_t[h] = \mathrm{d}t + [\gamma, t]|_{\mathfrak{h}} = \boldsymbol{D}_{|\mathfrak{h}}t, \qquad (2.21)$$

where $\delta_\phi[\gamma]$ is the $\mathfrak{so}$ gauge transformation for the spin connection $\omega = \gamma$, and $\delta_t[h]$ is an infinitesimal translation of the frame field $h = e$. Finally, there are the Poincaré transformations of the curvature two-form

$$\delta_\phi[F] = [F, \phi]_{\mathfrak{g}}, \qquad \delta_\phi[T] = [T, \phi]|_{\mathfrak{h}}$$

$$\delta_t[F] = 0, \qquad \delta_t[T] = [F, t]|_{\mathfrak{h}}. \qquad (2.22)$$

Although the translations $t \in \mathfrak{h}$ are Abelian, a general such translation will act nontrivially on the translational curvature (torsion). If we would choose, however, a connection γ such that $F[\gamma] = 0$, both the torsion component T and the rotational (Lorentz) component F of the Poincaré connection would be translational invariant. This choice underlies the teleparallel equivalent of general relativity.

In the following, we will consider the more complicated case of $\mathfrak{G} = \mathfrak{so}(n-1,1) \sim \mathfrak{so}(n-1) \bowtie \mathfrak{an}_{n-1}$, which is an example of a matched Lie algebra induced by the Iwasawa decomposition of the Lorentz Lie algebra $\mathfrak{so}(n,1)$.

III. CANONICAL TRANSFORMATION INDUCED BY THE BOUNDARY VOLUME TERM

A. Warm up: three dimensions

Consider the three-dimensional Einstein-Cartan action in first-order spin-connection variables with a cosmological constant Λ, together with the following boundary terms, whose significance will become clear below,

$$S_{\mathrm{EC}}[e, A|n] = \frac{1}{16\pi G}\left[\int_{\mathcal{M}} \varepsilon_{IJK} e^K \wedge \left(R^{IJ}[A] - \frac{\Lambda}{3} e^I \wedge e^J\right)\right.$$

$$\left. - 2\int_{\partial\mathcal{M}} \left(s\varepsilon_{IJK}e^K \wedge n^I \mathrm{d}_A n^J \right.\right.$$

$$\left.\left. + \frac{\sqrt{|\Lambda|}}{2}\varepsilon_{IJK}n^K e^I \wedge e^J\right)\right]. \qquad (3.1)$$

The action in the bulk is a functional of the Lorentz curvature $R^I{}_J[A] = \mathrm{d}A^I{}_J + A^I{}_K \wedge A^K{}_J$ and the triad e^I. The boundary term consists of two parts. The first term is the usual Gibbons-Hawking-York boundary term for first-order spin connection variables, where $\mathrm{d}_A[\cdot] = \mathrm{d}[\cdot] + [A, \cdot]$ is the covariant exterior derivative. The internal vector field n^I is constrained to lie orthogonal to the boundary,

i.e., $n_I\varphi^*_{\partial\mathcal{M}}e^I = 0$, where $\varphi^*_{\partial\mathcal{M}}: T^*\mathcal{M} \to T^*\partial\mathcal{M}$ is the pullback. In addition, n^I is normalized such that $s = \eta_{IJ}n^I n^J = n_I n^I = \{\pm 1, 0\}$, depending on whether the boundary is spacelike, timelike, or null.[2] Its orientation is such that $n^a := n^I e_I{}^a$ is the outwardly oriented normal to the boundary. If the torsionless equation is satisfied, then the first term is the integral of the trace of the extrinsic curvature. The second term is proportional to the induced volume of the boundary. The boundary conditions are

$$h^I{}_J\varphi^*_{\partial\mathcal{M}}\delta[e^J] = 0, \qquad \varphi^*_{\partial\mathcal{M}}\delta[n_I e^I] = 0, \qquad (3.2)$$

where $\delta[\cdot]$ is a variation on the infinite-dimensional space of kinematical histories, and $h_{IJ} = -sn_I n_J + \eta_{IJ}$ is the internal projector onto the boundary.

It is also useful to evaluate the action (3.1) for a spin connection $A^I{}_J$, which is torsionless (by going *half-shell*). We obtain

$$S_{\mathrm{EC}}[A, e|n]|_{\mathrm{d}_A e = 0} = S_{\mathrm{EH}}[g|n]$$

$$= \frac{1}{16\pi G}\left[\int_{\mathcal{M}} d^3v(R[g] - 2\Lambda)\right.$$

$$\left. + 2\int_{\partial\mathcal{M}} d^2v(sK - \sqrt{|\Lambda|})\right], \qquad (3.3)$$

where $R[g]$ is the Ricci scalar for the metric $g_{ab} = e_{Ia}e^I{}_b$, and $K = \nabla_a n^a$ is the trace of the extrinsic curvature. In addition, d^3v and d^2v are the canonical volume elements on $\mathcal{M}$ and $\partial\mathcal{M}$.[3] The volume term appears in the definition of the bulk plus boundary action (3.3) to cancel infrared divergencies that would otherwise appear when the boundary $\partial\mathcal{M}$ is sent to infinity [26].

Consider then a region $\Sigma \subset \partial\mathcal{M}$ within the boundary. The first variation of the action for given boundary conditions (3.2) determines the presymplectic potential $\Theta^{\mathrm{EC}}_\Sigma$ on the space of physical histories, i.e., the space of solutions to the field equations. A straightforward calculation gives

$$\Theta^{\mathrm{EC}}_\Sigma(\delta) = \frac{1}{8\pi G}\int_\Sigma (s\varepsilon_{IJ}\delta[e^I] \wedge K^J - \sqrt{|\Lambda|}\varepsilon_{IJ}e^I \wedge \delta[e^J])$$

$$- \frac{1}{8\pi G}\oint_{\partial\Sigma} s\varepsilon_{IJ}e^I\delta[n^J], \qquad (3.4)$$

where $K^I = \varphi^*_{\partial\mathcal{M}}\mathrm{d}_A n^I$ is the extrinsic curvature (a one-form along the boundary), and $\varepsilon_{IJ} = n^K\varepsilon_{KIJ}$ is the internal area

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[2] In the Lorentzian case, our metric signature is $(-++\ldots)$. Internal spacetime indices $I, J, K, \ldots$ are raised and lowered with the internal metric tensors η^{IJ} and η_{IJ}. Total antisymmetrization of indices $I_1, I_2, \ldots$ is obtained via $\omega_{[I_1 \ldots I_n]} = \frac{1}{n!}\sum_{\sigma \in S_n}(-1)^\sigma \omega_{I_{\sigma(1)}\ldots I_{\sigma(n)}}$.

[3] The volume elements are p-forms $d^3v = \frac{1}{6}\varepsilon_{IJK}e^I \wedge e^J \wedge e^K$ and $d^2v = \frac{1}{2}n^I\varepsilon_{IJK}e^J \wedge e^K$.

element at the boundary. Notice that the variation $\delta[n^I]$ of the internal normal only affects a corner term to the presymplectic potential (3.4). This is a consequence of the torsionless condition $\mathrm{d}_A e^I = 0$ and the boundary constraints $n_I n^I = s$ and $\varphi^*_{\partial\mathcal{M}}(n_I e^I) = 0$, hence $n_I \delta[n^I] = 0$ and $\varphi^*_{\partial\mathcal{M}}[\epsilon_{IJK}\delta[n^I]e^J \wedge K^K] = 0$.

Equation (3.4) is a manifestation of the well-known fact that the pullback of the triad and the extrinsic curvature are conjugate variables. From the perspective of the Chern-Simons, and Ponzano-Regge quantization of three-dimensional gravity, a connection representation is more appropriate. Following [14], we consider the canonical transformation, which is generated by the Gibbons-Hawking-York boundary term

$$\Theta_\Sigma(\delta) := \Theta^{\mathrm{EC}}_\Sigma(\delta) - \frac{1}{8\pi G}\delta\left[\int_\Sigma s\varepsilon_{IJ}e^I \wedge K^J\right]. \tag{3.5}$$

Going back to (3.1), we now immediately have

$$\Theta_\Sigma(\delta) = -\frac{1}{16\pi G}\int_\Sigma \varepsilon_{IJK}e^I \wedge \delta[A^{JK} - 2\sqrt{|\Lambda|}n^{[J}e^{K]}]$$

$$\equiv -\frac{1}{16\pi G}\int_\Sigma \varepsilon_{IJK}e^I \wedge \delta[\Omega^{JK}], \tag{3.6}$$

where we introduced a new connection Ω^{IJ}. Notice that the variation $\delta[\Omega^{IJ}]$ contains a variation $\delta[n^I] \neq 0$. Since, however, the boundary conditions (3.2) must be satisfied, we obtain $\delta[n^I] \perp n^I$. In addition, the vector $n^a = n^I e_I{}^a$ lies orthogonal to the boundary, hence $\delta[n^I]e_I{}^a$ lies tangential to the boundary. Therefore, $\frac{1}{2}\varphi^*_{\partial\mathcal{M}}(\epsilon_{IJK}\delta[n^I]e^J \wedge e^K) = 0$, such that

$$\frac{1}{8\pi G}\int_\Sigma \varepsilon_{IJK}e^I \wedge \delta[n^J e^K] = \frac{1}{8\pi G}\int_\Sigma \varepsilon_{IJK}e^I \wedge n^J \delta[e^K]$$

$$= \frac{1}{16\pi G}\delta\left[\int_\Sigma \varepsilon_{IJK}e^I \wedge n^J e^K\right]. \tag{3.7}$$

To solve the equations of motion in terms of the new variables $(e^I{}_a, \Omega^{IJ}{}_a)$, it is then necessary to smoothly extend the internal vector n^I into the bulk. We thus write

$$\Omega^{IJ} = A^{IJ} + e^{[I}p^{J]} \equiv A^{IJ} - \mathcal{I}^{IJ}, \tag{3.8}$$

$$\mathcal{I}^{IJ} \equiv p^{[I}e^{J]} = \frac{1}{2}C^{IJ}{}_K e^K, \quad C^{IJ}{}_K = (p^I\delta^J_K - p^J\delta^I_K), \tag{3.9}$$

where p^I is an internal Lorentz vector that satisfies the constraints

$$p_I p^I = -4\Lambda, \qquad p_I|_{\partial\mathcal{M}} = 2\sqrt{|\Lambda|}n_I. \tag{3.10}$$

The next step ahead is to write the action in terms of the new connection Ω^{IJ}. Consider first the curvature,

$$R_{IJ}[A] = R_{IJ}[\Omega] + \mathrm{d}_\Omega\mathcal{I}_{IJ} + \mathcal{I}_{IK} \wedge \mathcal{I}^K{}_J$$

$$= R_{IJ}[\Omega] + \mathrm{d}_\Omega\mathcal{I}_{IJ} + \frac{1}{4}(p \wedge (p_I e_J - p_J e_I)$$

$$- p_K p^K e_I \wedge e_J), \tag{3.11}$$

where $p = p^I e_I$ and $\mathrm{d}_\Omega\mathcal{I}^I{}_J = \mathrm{d}\mathcal{I}^I{}_J + \Omega^I{}_K \wedge \mathcal{I}^K{}_J + \mathcal{I}^I{}_K \wedge \Omega^K{}_J = \mathrm{d}\mathcal{I}^I{}_J + [\Omega \wedge \mathcal{I}]^I{}_J$, and we used the fact that $e^I \wedge e_I = 0$. With the above expression, the first term of the action (3.1) is

$$\varepsilon_{JKI}e^I \wedge R^{JK}[A] = \varepsilon_{JKI}e^I \wedge \left(R^{JK}[\Omega] + \mathrm{d}_\Omega\mathcal{I}^{JK}\right.$$

$$+ \frac{1}{2}p^{[J}p \wedge e^{K]} - \frac{p_M p^M}{4}e^J \wedge e^K\right)$$

$$= \varepsilon_{JKI}e^I \wedge \left(R[\Omega]^{JK} + \mathrm{d}_\Omega\mathcal{I}^{JK}\right.$$

$$\left. - \frac{p_M p^M}{12}e^J \wedge e^K\right). \tag{3.12}$$

Going from the first line to the second line, we used the following identity

$$\frac{n_M n^M}{6}\varepsilon_{IJK}e^I \wedge e^J \wedge e^K = \frac{1}{2}n^I\varepsilon_{IJK}n \wedge e^J \wedge e^K, \tag{3.13}$$

for all $n^I : n^I n_I = \in \{0, \pm 1\}$. Returning to the definition of the action (3.1), we obtain

$$S_{\mathrm{EC}}[e, A|n] - \frac{1}{8\pi G}\int_{\partial\mathcal{M}}s\varepsilon_{JK}e^J \wedge K^K = \frac{1}{16\pi G}\left[\int_\mathcal{M}\varepsilon_{IJK}e^I \wedge (R^{JK}[\Omega] + \mathrm{d}_\Omega\mathcal{I}^{JK}) - \frac{1}{2}\int_{\partial\mathcal{M}}\varepsilon_{IJK}e^I \wedge e^J p^K\right]$$

$$= \frac{1}{16\pi G}\int_\mathcal{M}\varepsilon_{IJK}e^I \wedge \left(R[\Omega]^{JK} + \mathrm{d}_\Omega\mathcal{I}^{JK} + (\mathrm{d}_\Omega e^{[J})p^{K]} - \frac{1}{2}e^{[J} \wedge \mathrm{d}_\Omega p^{K]}\right)$$

$$= \frac{1}{16\pi G}\int_\mathcal{M}\varepsilon_{IJK}e^I \wedge \left(R^{JK}[\Omega] + \frac{1}{2}e^{[J} \wedge \mathrm{d}_\Omega p^{K]}\right). \tag{3.14}$$

Equation (3.14) suggests to introduce a new action,

$$S[e,\Omega,p] = \frac{1}{16\pi G}\int_{\mathcal{M}}\varepsilon_{IJK}e^I\wedge\left(R[\Omega]^{JK}+\frac{1}{2}e^{[J}\wedge\mathrm{d}_\Omega p^{K]}\right),$$
(3.15)

where p_I satisfies the mass shell condition $p_Ip^I=-4\Lambda$. To obtain the equations of motion, we vary the action for fixed boundary conditions. Given the action (3.15), the appropriate boundary conditions are

$$p_I|_{\partial M}=2\sqrt{|\Lambda|}n_I,\qquad h^{[I}{}_Kh^{J]}{}_L\varphi^*_{\partial\mathcal{M}}\delta[\Omega^{KL}]=0,\quad(3.16)$$

where $h^I{}_J$ is the projector $h^I{}_J=-sn^In_J+\delta^I_J$, and $\varphi^*_{\partial\mathcal{M}}:T^*M\to T^*(\partial\mathcal{M})$ is the pullback. The resulting equations of motion are

$$\mathcal{F}^{IJ}=R^{IJ}[\Omega]+e^{[I}\wedge\mathrm{d}_\Omega p^{J]}=0,\qquad(3.17)$$

$$\mathcal{T}^I=\mathrm{d}_\Omega e^I+\frac{1}{4}C_{JK}{}^Ie^J\wedge e^K=0.\qquad(3.18)$$

The variation with respect to p_I is redundant. Taking into account that $\delta[p_Ip^I]=2p_I\delta[p^I]=0$, i.e., $p^I\perp\delta[p^I]$, we obtain that the variation of the action with respect to p^I vanishes provided

$$\frac{1}{2}\varepsilon_{IJK}\mathrm{d}_\Omega(e^J\wedge e^K)\propto p_Id^3v.\qquad(3.19)$$

Given (3.18), this condition is always satisfied since $\frac{1}{2}\varepsilon_{IJK}\mathrm{d}_\Omega(e^J\wedge e^K)=-\frac{1}{2}\varepsilon_{IJK}p\wedge e^J\wedge e^K=-p_Id^3v$.

The field equations (3.17) and (3.18) have a simple geometric meaning. They can be rearranged into a single flatness constraint for a $\mathfrak{so}(1,2)\bowtie\mathfrak{an}_3$ matched connection. Consider the $\mathfrak{so}(1,2)\bowtie\mathfrak{an}_2$ Lie algebra-valued one-form

$$A_a=\frac{1}{2}\Omega^{IJ}{}_a\otimes J_{IJ}+e^I{}_a\otimes P_I,\qquad(3.20)$$

where J_{IJ} are the (Lorentz) generators of $\mathfrak{so}(1,2)$ and P_I are the generators of $\mathfrak{an}_2$. The commutation relations are

$$[J_{IJ},J_{I'J'}]=4\delta^{[R}_I\delta^{S]}_J\eta_{SS'}\delta^{[S'}_{I'}\delta^{R']}_{J'}J_{RR'},\qquad(3.21a)$$

$$[P^K,J_{IJ}]=2\delta^K_{[I}P_{J]}+J^K{}_{[I}p_{J]},\qquad(3.21b)$$

$$[P_I,P_J]=p_{[I}P_{J]}.\qquad(3.21c)$$

Going back to (2.3), we identify the action of the right (left) module,

$$P^K\triangleleft J_{IJ}=2\delta^K_{[I}P_{J]}\in\mathfrak{an}_3,\quad P^K\triangleright J_{IJ}=J^K{}_{[I}p_{J]}\in\mathfrak{so}(1,2).$$
(3.22)

To introduce the curvature of this connection, consider first the following covariant derivative, defined by its action on the basis elements of the $\mathfrak{so}(1,2)\bowtie\mathfrak{an}_2$ algebra,

$$\mathbf{d}J_{IJ}:=0,\qquad\mathbf{d}P_I:=\frac{1}{2}\mathrm{d}p^K\otimes J_{KI}.\qquad(3.23)$$

The definition (3.23) extends naturally to all $\mathfrak{so}(1,2)\bowtie\mathfrak{an}_2$ Lie algebra-valued p-forms $\boldsymbol{\omega}\in\Omega^p(\mathcal{M}:\mathfrak{so}(1,2)\bowtie\mathfrak{an}_2)$ via $\mathbf{d}(\boldsymbol{\omega}_1+f\boldsymbol{\omega}_2)=\mathbf{d}\boldsymbol{\omega}_1+f\mathbf{d}\boldsymbol{\omega}_2+\mathrm{d}f\wedge\boldsymbol{\omega}_2$ for all $f:\mathcal{M}\to\mathbb{R}$ and $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\in\Omega^p(\mathcal{M}:\mathfrak{so}(1,2)\bowtie\mathfrak{an}_2)$. Notice also that the derivative is flat, i.e., $\mathbf{d}^2=0$. Consider then a second such covariant derivative $\boldsymbol{D}$, which is defined by the deformation $\boldsymbol{D}=\mathbf{d}+[\boldsymbol{A},\cdot]$. Its curvature $\boldsymbol{D}^2=[\boldsymbol{F},\cdot]$ is given by

$$\boldsymbol{F}[A]=\mathrm{d}\boldsymbol{A}+\frac{1}{2}[\boldsymbol{A}\wedge\boldsymbol{A}]=\frac{1}{2}\mathcal{F}^{IJ}\otimes J_{IJ}+\mathcal{T}^I\otimes P_I=0,$$
(3.24)

where the deformed $\mathfrak{so}(1,2)$-valued curvature $\mathcal{F}$ and $\mathfrak{an}_2$-valued torsion $\mathcal{T}$ are defined as (3.17) and (3.18). Let us also stress that there always exists a gauge such that $\mathrm{d}p=0$, such that the notion of curvature and torsion $\mathcal{F}$, $\mathcal{T}$ would also coincide with (2.11) and (2.12). If the field equations (3.17) and (3.18) are satisfied, this derivative is flat, i.e., $\boldsymbol{F}[A]=0$.

The torsion two-form $T^I=\mathrm{d}_\Omega e^I$ is now sourced by the cosmological constant, see (3.18). On the other hand, the field equations for the $\mathfrak{so}(1,2)$ Lorentz part of the curvature two-form admit solutions where the spin connection $\Omega^I{}_J$ is flat and p^I is constant, i.e., $\mathrm{d}_\Omega p^I=0$. Hence, there is some Lorentz gauge element $\Lambda^I{}_J:\mathcal{M}\to SO(1,2)$ such that $\Omega^I{}_J=\Lambda_K{}^I\mathrm{d}\Lambda^K{}_J$.

Given some internal vector p^I on $\mathcal{M}$, such a gauge element $\Lambda^I{}_J$ can always be found (unless there are topological obstructions). In other words, the homogenous curvature of the underlying spacetime metric $g_{ab}=\eta_{IJ}e^I{}_ae^J{}_b$ has been encoded into a *flat* $\mathfrak{so}(1,2)$ connection with nonvanishing $\mathfrak{an}_2$-valued torsion $T^I=\mathrm{d}_\Omega e^I$. This was a key feature that simplified the theory at the discrete level [20], and this simplification will also be relevant for us to introduce a teleparallel equivalent of gravity in the case of homogeneously curved geometries that we will discuss below.

Before going to the general case, let us compare what we have just done with respect to [20]. There, the starting action is

$$S[e,A] \equiv \frac{1}{16\pi G}\left[\int_{\mathcal{M}} \varepsilon_{IJK}e^K \wedge \left(R^{IJ}[A] - \frac{\Lambda}{3}e^I \wedge e^J\right) - \frac{1}{2}\int_{\partial\mathcal{M}}(\varepsilon_{IJK}r^K e^I \wedge e^J)\right]$$

$$= \frac{1}{16\pi G}\int_{\mathcal{M}} \varepsilon_{IJK}e^I \wedge \left(R^{JK}[\Omega] + \frac{1}{2}e^{[J} \wedge d_\Omega r^{K]}\right), \tag{3.25}$$

but the internal Lorentz vector r^I is *not* interpreted as the normal of the boundary. In order to not increase the number of degrees of freedom, it is assumed that $\delta[r^I] = 0$. This ensures that the $\mathfrak{so}(1,3)$ connection $A^I{}_J$ is shifted to $\tilde{\Omega}^{IJ} = A^{IJ} - r^{[I}e^{J]}$ just as in (3.6). The vector r^I is also normalized as in (3.10) in order to cancel the volume term,

$$r_I r^I = -4\Lambda. \tag{3.26}$$

Without loss of generality, the vector r is assumed to have only one nonzero component, which is proportional to the square root of the (absolute value of) the cosmological constant. As a consequence, $dr^I = 0$ and the equations of motion then coincide with the generalized curvature and torsion being zero. We note that there is always a choice of gauge such that the normal p^I can be the constant vector r^I, so that the two approaches are the same.

B. Beyond three dimensions

In the previous section, we considered gravity in three dimensions. Our next step is to generalize the construction to arbitrary spacetime dimensions $d \geq 3$. Given the $\mathfrak{so}(1, d-1)$ spin connection $A^I{}_J$, its conjugate momentum is the gravitational B field, which is a bivector-valued $(d-2)$-form,

$$B_{IJ}[e] = \frac{1}{(d-2)!}\varepsilon_{IJK_1\cdots K_{d-2}}e^{K_1} \wedge \cdots \wedge e^{K_{d-2}}, \tag{3.27}$$

where e^I is the coframe that defines the metric $g_{ab} = \eta_{IJ}e^I{}_a e^J{}_b$. Consider then the usual Einstein-Cartan action with a boundary term proportional to the volume of the boundary

$$S_{EC}[e,A,p] = \frac{1}{16\pi G}\left[\int_{\mathcal{M}}\left(B_{IJ}[e] \wedge R^{IJ}[A] - \frac{2\Lambda}{d!}\varepsilon_{I_1\dots I_d}e^{I_1} \wedge \dots \wedge e^{I_d}\right) - \frac{1}{(d-1)!}\int_{\partial\mathcal{M}}p^J\varepsilon_{JI_1\dots I_{d-1}}e^{I_1} \wedge \dots \wedge e^{I_{d-1}}\right], \tag{3.28}$$

where $R^{IJ}[A]$ is the $\mathfrak{so}(1, d-1)$-valued curvature two-form $R^I{}_J[A] = dA^I{}_J + A^I{}_K \wedge A^K{}_J$, and p_I is an internal vector (a vector-valued 0-form). The critical points of the action are found by imposing the following boundary conditions,

$$h^I{}_K h^J{}_L \varphi^*_{\partial\mathcal{M}}\delta[A^{KL}] = 0, \quad \delta[p_I p^I] = 0, \quad p_I\varphi^*_{\partial\mathcal{M}}e^I = 0, \tag{3.29}$$

where $\varphi^*_{\partial\mathcal{M}}: T^*\mathcal{M} \to T^*(\partial\mathcal{M})$ is the pullback, and $h_{IJ} = sn_I n_J + h_{IJ}$ is the projector onto the boundary such that $n_I n^I = s \in \{\pm 1\}$, and $h^I{}_J \varphi^*_{\partial\mathcal{M}}e^J = \varphi^*_{\partial\mathcal{M}}e^I$. The resulting field equations are the d-dimensional Einstein equations for the metric $g_{ab} = e_{Ia}e^I{}_b$ with a cosmological constant Λ.

Consider then a region Σ within the boundary, i.e., $\Sigma \subset \partial\mathcal{M}$. The presymplectic potential is obtained from the first variation of the action (3.28). Taking into account the boundary conditions (3.29), we obtain

$$\Theta_\Sigma(\delta) = \frac{(-1)^{(d-2)}}{16\pi G}\int_\Sigma B_{IJ} \wedge \delta[A^{IJ} - p^{[I}e^{J]}]$$

$$\equiv \frac{(-1)^{(d-2)}}{16\pi G}\int_\Sigma B_{IJ} \wedge \delta[\Omega^{IJ}]. \tag{3.30}$$

Next, we need to express the field equations in terms of the new and shifted connection Ω. This requires smoothly extending the internal boundary vector p_I into the bulk such that $p_I\varphi^*_{\partial\mathcal{M}}e^I = 0$ is still satisfied. Given such an extension of p_I from the boundary into the bulk, we introduce the shifted connection

$$\Omega^{IJ} = A^{IJ} + e^{[I}p^{J]} \equiv A^{IJ} - \mathcal{I}^{IJ}, \tag{3.31}$$

$$\mathcal{I}^{IJ} \equiv p^{[I}e^{J]} = \frac{1}{2}C^{IJ}{}_K e^K, \quad C^{IJ}{}_K = (p^I\delta^J_K - p^J\delta^I_K). \tag{3.32}$$

Just as in three dimensions, we reformulate the action in terms of the new connection. Consider first the curvature scalar,

$$B_{IJ}[e] \wedge R^{IJ}[A] = B_{IJ}[e] \wedge (R^{IJ}[\Omega] + d_\Omega\mathcal{I}^{IJ} + \mathcal{I}^I{}_K \wedge \mathcal{I}^{KJ})$$

$$= B_{IJ}[e] \wedge \left(R^{IJ}[\Omega] + d_\Omega\mathcal{I}^{IJ} + \frac{1}{2}p^{[I}p \wedge e^{J]}\right.$$

$$\left. - \frac{p_K p^K}{4}e^I \wedge e^J\right), \tag{3.33}$$

where p denotes the one-form $p = p_I e^I$. We then also have the identity

$$p_M p^M \frac{1}{d!} \varepsilon_{I_1...I_d} e^{I_1} \wedge ... \wedge e^{I_d}$$
$$= \frac{1}{(d-1)!} p^J \varepsilon_{JI_1...I_{d-1}} p \wedge e^{I_1}...e^{I_{d-1}}. \qquad (3.34)$$

We thus have

$$S_{EC}[e, \Omega, p] = \frac{1}{16\pi G} \left[\int_{\mathcal{M}} B_{IJ} \wedge (R^{IJ}[\Omega] + d_{\Omega}\mathcal{I}^{IJ}) - \frac{(-1)^{(d-2)}}{d-1} \int_{\partial\mathcal{M}} B_{IJ} \wedge p^I e^J \right]$$
$$= \frac{1}{16\pi G} \int_{\mathcal{M}} \left[B_{IJ} \wedge R^{IJ}[\Omega] + B_{IJ} \wedge \left(d_{\Omega}\mathcal{I}^{IJ} - p^{[I}d_{\Omega}e^{J]} - \frac{1}{d-1}(d_{\Omega}p^{[I]}) \wedge e^{J]} \right) \right]. \qquad (3.37)$$

Taking into account that $\mathcal{I}^{IJ} = p^{[I}e^{J]}$, one finally arrives at the expression

$$S_{EC}[e, \Omega, p] = \frac{1}{16\pi G} \int_{\mathcal{M}} (B_{IJ} \wedge R^{IJ}[\Omega]$$
$$- (d-2)(-1)^{d-2} B_I \wedge d_{\Omega}p^I), \qquad (3.38)$$

where we introduced the vector-valued $(d-1)$-form,

$$B_I = \frac{1}{(d-1)!} \varepsilon_{IK_1...K_{d-1}} e^{K_1} \wedge ... \wedge e^{K_{d-1}}. \qquad (3.39)$$

The Einstein equations are the saddle points of the action (3.38) in the space of all fields that satisfy the mass shell constraint (3.36) and boundary conditions

$$h^I{}_K h^J{}_L \varphi^*_{\partial\mathcal{M}} \delta[\Omega^{KL}] = 0, \qquad \varphi^*_{\partial\mathcal{M}}(p_I e^I) = 0. \qquad (3.40)$$

The variation of the frame field for boundary conditions (3.40) yields the curvature constraint

$$\frac{1}{(d-3)!} \varepsilon_{IJKL_1...L_{d-3}} e^{L_1} \wedge ...$$
$$\wedge e^{L_{d-3}} \wedge \underbrace{(R^{JK}[\Omega] + e^{[J} \wedge d_{\Omega}p^{K]})}_{\mathcal{F}^{JK}} = 0. \qquad (3.41)$$

By varying the action (3.38) with respect to Ω and taking into account the boundary conditions (3.40), we find that the deformed torsion vanishes[4]

$$T^I = d_{\Omega}e^I + \frac{1}{4} C_{IJ}{}^K e^I \wedge e^J = 0, \qquad (3.42)$$

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[4]Provided $e^{I_1} \wedge ... \wedge e^{I_d} \neq 0$.

$$\frac{1}{2} B_{IJ} \wedge \left(p^I p \wedge e^J - \frac{p_K p^K}{2} e^I \wedge e^J \right)$$
$$= -\frac{p_M p^M}{4} \frac{(d-1)(d-2)}{d!} \varepsilon_{I_1...I_d} e^{I_1} \wedge ... \wedge e^{I_d}. \qquad (3.35)$$

If we impose the mass shell condition

$$p_I p^I = -\frac{8\Lambda}{(d-1)(d-2)}, \qquad (3.36)$$

then the action (3.28) simplifies

where the structure constants $C_{IJ}{}^K$ are given in (3.32). Finally, we should also consider the variations of p^I. Taking into account the mass shell condition (3.36), we obtain

$$d_{\Omega}B_I \propto p_I. \qquad (3.43)$$

This condition is satisfied once we solve for (3.42). Indeed, (3.42) implies

$$d_{\Omega}B_I = \frac{1}{2} \frac{1}{(d-2)!} \varepsilon_{IK_1...K_{d-1}} p \wedge e^{K_1} \wedge ... e^{K_{d-1}}$$
$$= \frac{d-1}{2} p_I d^d v. \qquad (3.44)$$

As in three dimensions, the new variables deform the torsion two-form $T^I = d_{\Omega}e^I$, which is now sourced by the cosmological constant. In addition, we also see that a flat $\mathfrak{so}(1, d)$ connection $\Omega^I{}_J$ solves the curvature equation (3.41) provided $d_{\Omega}p^I = 0$. This observation will be relevant once we consider a lattice approach, where the field equations are solved by imposing that the connection is piecewise flat.

The nature of the boundary (spacelike, timelike, or null) depends via the mass shell condition (3.36) on the sign of Λ. The following table summarizes the situation for both Euclidean and Lorentzian signature in arbitrary dimensions.

	Euclidian	Lorentzian
Flat: $\Lambda = 0$	$p^I = 0$ or p^I is Grassmannian	$p^I = 0$ or p^I is lightlike
AdS: $\Lambda < 0$	p^I is real	p^I is spacelike or *imaginary* timelike
dS: $\Lambda > 0$	p^I is imaginary	p^I is timelike or *imaginary* spacelike

IV. METRIC AND DEFORMED TELEPARALLEL FORMULATION IN THE NEW VARIABLES

A. Recovering unimodular gravity

1. Generalized covariant derivative

The Levi-Civita connection defines the unique torsionless covariant derivative that is metric compatible. In the last section, we deformed the notion of torsion. Let us now introduce the corresponding covariant derivative on the tangent bundle, i.e., the corresponding metric formulation.

For any given and invertible d-bein $e^I{}_a$ and given Lorentz vector p^I, the field equation

$$d_\Omega e^I + \frac{1}{4} C_{JK}{}^I e^J \wedge e^K = T^I + \frac{1}{4} C_{JK}{}^I e^J \wedge e^K = 0 \quad (4.1)$$

has a unique solution $\overset{\circ}{\Omega}{}^I{}_{Ja}[e,p]$ for the connection $\Omega^I{}_{Ja}$ in terms of $e^I{}_a$ and p^I. If, in fact, ∇_a denotes the usual Levi-Civita metric compatible and torsionless covariant derivative on the tensor bundle, then we find

$$\overset{\circ}{\Omega}{}^I{}_{Ja}[e,p] = e^I{}_b \nabla_a e_J{}^b + C^I{}_{JK} e^K{}_a = e^I{}_b \nabla_a e_J{}^b + C^I{}_{Ja}. \quad (4.2)$$

Equation (4.2) suggests to introduce the difference tensor

$$C_{abc} = C^I{}_{JK} e_{Ia} e^J{}_b e^K{}_c. \quad (4.3)$$

The corresponding metric compatible covariant derivative is defined for any smooth vector field $V^a \in T\mathcal{M}$ via

$$\overset{\circ}{\nabla}_a V^b = \nabla_a V^b + C^b{}_{ac} V^c. \quad (4.4)$$

This definition generalizes naturally to arbitrary tensor fields. For any two such vector fields, we may then define curvature and torsion of this new connection,

$$\overset{\circ}{R}{}^c{}_{dab} V^d = \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b V^c - \overset{\circ}{\nabla}_b \overset{\circ}{\nabla}_a V^c - \overset{\circ}{T}{}^d{}_{ab} \overset{\circ}{\nabla}_d V^c, \quad (4.5)$$

$$\overset{\circ}{T}{}^a{}_{bc} U^b V^c = U^b \overset{\circ}{\nabla}_b V^a - V^b \overset{\circ}{\nabla}_b U^a - [U,V]^a. \quad (4.6)$$

Inserting (4.4) back into (4.6), we obtain the components of the torsion two-form,

$$\overset{\circ}{T}{}^a{}_{bc} = -\frac{1}{2} C^a{}_{[bc]}, \quad \text{with} \quad C_{abc} = 2 p_{[a} g_{b]c}, \quad (4.7)$$

which is, of course, the same as (4.1), but now written in the standard tensor language. Note that we can either interpret the Christoffel symbol $\overset{\circ}{\Gamma}{}^a{}_{bc} = \Gamma^a{}_{bc} + C^a{}_{bc}$ as having torsion in the usual sense, or it has vanishing generalized torsion $\mathcal{T}^I{}_{ab} = 0$.

2. Recovering unimodular gravity

As emphasized, for example, in [30], the gravitational force can be encoded into various geometric entities, namely an affine connection $\tilde{\Gamma}^a{}_{bc}$, nonmetricity $\tilde{Q}_{abc}$, or torsion $\tilde{T}^a{}_{bc}$. Our choice of variables provides a connection $\overset{\circ}{\Gamma}{}^a{}_{bc}$ where $\overset{\circ}{\Omega}{}^a{}_{bc} = 0$ but $\overset{\circ}{T}{}^a{}_{bc} \neq 0$. The corresponding *second-order action* is obtained by going "half-shell," i.e., by reinserting the solution for the connection $\overset{\circ}{\Omega}{}^I{}_{Ja}$ in terms of p_I and e_{Ia} back into the *first-order* action (3.37). A short calculation gives

$$S_{\text{EH}}[g,p] := S_{\text{EC}}[e, \overset{\circ}{\Omega}[e,p], p]$$
$$= \frac{1}{16\pi G} \int_\mathcal{M} d^d v_g (g^{cb} \overset{\circ}{R}{}^a{}_{cab}[g,p] + (d-2) \overset{\circ}{\nabla}_a p^a), \quad (4.8)$$

where $d^d v_g$ is the usual metrical volume element.[5] This action resembles the Henneaux-Teitelboim model for unimodular gravity [24]. In fact, to obtain the Einstein equations, this action is to be extremized in the space of all fields (g_{ab}, p^c) that satisfy the mass shell condition (3.36) that now simply reads

$$g_{ab} p^a p^b = -\frac{8\Lambda}{(d-1)(d-2)}. \quad (4.9)$$

It seems now natural to relax this condition, namely by adding a Lagrange multiplier that will impose the mass-shell constraint at the dynamical level. At the same time, this also allows us to unfreeze Λ, obtaining a version of unimodular gravity. Consider, in fact, a $(d-1)$-form τ. Its exterior derivative defines a volume density $\tilde{N} = d\tau$. Varying the action

$$S_{\text{unimod}}[g,p,\tau] = \frac{1}{16\pi G} \int_\mathcal{M} \left[d^d v_g (g^{cb} \overset{\circ}{R}{}^a{}_{cab}[g,p] + (d-2) \overset{\circ}{\nabla}_a p^a) - \frac{1}{2} g_{ab} p^a p^b d\tau \right], \quad (4.10)$$

with respect to τ, will tell us then that $g_{ab} p^a p^b$ is constant for *some* Λ, while all other variations return the usual Einstein equations with respect to the deformed connection $\overset{\circ}{\nabla}_a$. In this way, τ provides a notion of time, p^a plays the role of a spacetime momentum, and $\sqrt{|\Lambda|}$ represent the rest mass of p^a.

B. Deformed teleparallel gravity

The new variables deform curvature and torsion, see (3.17) and (3.18). The gravity action can be encoded in the

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[5]The volume element $d^d v_g$ is the d form, whose components are the Levi-Civita tensor, i.e., $(d^d v_g)_{abc...} = \varepsilon_{abc...}$.

curvature when there is no torsion or vice versa in the torsion if there is no curvature. The latter viewpoint underlies the teleparallel framework. We want to see now how the new variables provide a non-Abelian version of teleparallelism, where the translational connection takes values in $\mathfrak{an}_{d-1}$, whereas the Lorentzian part of the $\mathfrak{so}(1, d-1) \bowtie \mathfrak{an}_{d-1}$ connection is flat.

Consider a generic $\mathfrak{so}(1, d-1) \bowtie \mathfrak{an}_{d-1}$ connection A. The space of such $\mathfrak{so}(1, d) \bowtie \mathfrak{an}_{d-1}$ connections is an affine space. Since it is an affine space, any such connection can be written as a sum of some arbitrary reference connection $\dot{A}$ and a difference tensor $\boldsymbol{\Delta}$,

$$A = \dot{A} - \boldsymbol{\Delta}. \tag{4.11}$$

With this decomposition, we have

$$F[A] = \dot{F} - \dot{D}\boldsymbol{\Delta} + \frac{1}{2}[\boldsymbol{\Delta} \wedge \boldsymbol{\Delta}], \tag{4.12}$$

where $\dot{F}$ is the curvature of $\dot{A}$, and $\dot{D}(\cdot) = \mathbf{d}(\cdot) + [\dot{A}, \cdot]$ is the covariant derivative as in (3.23) and (3.20) above.

The deformed $\mathfrak{an}_{d-1}$ Lie-algebra valued equivalent of teleparallel gravity corresponds to the following choice of reference connection,

$$\dot{A} = \frac{1}{2}\dot{\Omega}^{IJ} \otimes J_{IJ} + e^I \otimes P_I, \tag{4.13}$$

where J_{IJ} and P_I are the generators of $\mathfrak{so}(1, d-1) \bowtie \mathfrak{an}_{d-1}$ with commutation relations (3.21a), (3.21b), and (3.21c). Notice that A and $\dot{A}$ have the same projection onto $\mathfrak{an}_d$. Therefore, $\boldsymbol{\Delta}$ is a pure rotation,

$$\boldsymbol{\Delta} = \frac{1}{2}\Delta^{IJ} \otimes J_{IJ}. \tag{4.14}$$

We are now left to specify the reference connection $\dot{\Omega}^I{}_J$, and we choose it to satisfy the following two conditions:

$$R^I{}_J[\dot{\Omega}] = \mathbf{d}\dot{\Omega}^I{}_J + \dot{\Omega}^I{}_K \wedge \dot{\Omega}^K{}_J = 0, \tag{4.15}$$

$$\mathbf{d}_\Omega p_I = \mathbf{d}p_I - \dot{\Omega}^J{}_I p_J = 0. \tag{4.16}$$

Notice that there always exists a Lorentz transformation $\Lambda^I{}_J$ such that $p_I = \Lambda^J{}_I r_J$, where r_J is a constant vector, i.e., $\mathbf{d}r_J = 0$. We can then choose

$$\dot{\Omega}^I{}_J = (\Lambda^{-1}\mathbf{d}\Lambda)^I{}_J, \tag{4.17}$$

which will then solve both (4.15) and (4.16). Therefore, the curvature of the $\mathfrak{so}(1, d-1) \bowtie \mathfrak{an}_{d-1}$ connection has only a $\mathfrak{an}_{d-1}$ part,

$$\dot{F} = \frac{1}{2}\dot{\mathcal{F}}^{IJ} \otimes J_{IJ} + \dot{\mathcal{T}}^I \otimes P_I = \dot{\mathcal{T}}^I \otimes P_I, \tag{4.18}$$

where we defined the $\mathfrak{an}_d$ Lie algebra-valued torsion two-form

$$\dot{\mathcal{T}}^I = \mathbf{d}_\Omega e^I + \frac{1}{4}C_{JK}{}^I e^J \wedge e^K \equiv \dot{\nabla}e^I + \frac{1}{4}C_{JK}{}^I e^J \wedge e^K. \tag{4.19}$$

If the field equation (3.42) is satisfied, i.e., by going "half-shell," we can express the components of the difference tensor $\Delta_{IJ} = \Delta_{IJK}e^K$ in terms of the components of the deformed $\mathfrak{an}_d$ Lie algebra-valued torsion,

$$\Delta_{IJK} = -\frac{1}{2}(\dot{\mathcal{T}}_{IJK} + \dot{\mathcal{T}}_{JKI} - \dot{\mathcal{T}}_{KIJ}). \tag{4.20}$$

Inserting this solution back into the first-order action (3.38), we obtain

$$
\begin{aligned}
S_{\mathrm{EC}}[e, \dot{\Omega} + \Delta, p] = {} & \frac{1}{16\pi G} \int_{\mathcal{M}} [B_{IJ} \wedge R^{IJ}[\dot{\Omega}] - (-1)^{d-3}B_{IJK} \wedge (\mathbf{d}_\Omega e^K) \wedge \Delta^{IJ} - B_{IJ} \wedge \Delta^I{}_L \wedge \Delta^{LJ} \\
& - (d-2)(-1)^{d-2}B_I \wedge \mathbf{d}_\Omega p^I + (d-2)(-1)^{d-2}B_I p_J \wedge \Delta^{IJ}] - \frac{1}{16\pi G}\int_{\partial\mathcal{M}}(-1)^{d-2}B_{IJ} \wedge \Delta^{IJ} \\
= {} & \frac{1}{16\pi G}\int_{\mathcal{M}} \Big[B_{IJ} \wedge \mathcal{F}^{IJ}[\dot{\Omega}] - (-1)^{d-3}B_{IJK} \wedge \dot{\mathcal{T}}^K \wedge \Delta^{IJ} - B_{IJ} \wedge \Delta^I{}_L \wedge \Delta^{LJ} \\
& - B_{IJ} \wedge e^I \wedge \mathbf{d}_\Omega p^J + \frac{1}{2}(-1)^{d-3}B_{IJK} \wedge p \wedge e^K \wedge \Delta^{IJ} \quad\quad (4.21) \\
& - (d-2)(-1)^{d-2}B_I \wedge \mathbf{d}_\Omega p^I + (d-2)(-1)^{d-2}B_I p_J \wedge \Delta^{IJ}] - \frac{1}{16\pi G}\int_{\partial\mathcal{M}}(-1)^{d-2}B_{IJ} \wedge \Delta^{IJ}, \quad (4.22)
\end{aligned}
$$

where we performed a partial integration and used the definition of $\dot{F} = \frac{1}{2}\dot{\mathcal{F}}^{IJ} \otimes J_{IJ} + \dot{\mathcal{T}}^I \otimes P_I$ for a generic $\mathfrak{so}(1, d-1) \bowtie \mathfrak{an}_{d-1}$ connection $\dot{A}$, see also (3.17) and (3.18). In addition,

$$S_{\mathrm{EC}}[e,A,p] = \frac{1}{16\pi G}\left[\int_{\mathcal{M}}\left(B_{IJ}[e]\wedge R^{IJ}[A] - \frac{2\Lambda}{d!}\varepsilon_{I_1\dots I_d}e^{I_1}\wedge\cdots\wedge e^{I_d}\right) - \frac{1}{(d-1)!}\int_{\partial\mathcal{M}}p^J\varepsilon_{JI_1\dots I_{d-1}}e^{I_1}\wedge\cdots\wedge e^{I_{d-1}}\right]$$

shift: $\Omega^{IJ} = A^{IJ} + p^{[I}e^{J]}$

$$S_{\mathrm{EC}}[e,\Omega,p] = \frac{1}{16\pi G}\int_{\mathcal{M}}\left(B_{IJ}\wedge R^{IJ}[\Omega] - (d-2)(-1)^{d-2}B_I\wedge\mathrm{d}_\Omega p^I\right)$$

*integrating out local
Lorentz transformations*

*introducing a flat
reference connection* $\overset{\bullet}{\Omega}{}^{IJ}$

$$S_{tele}[e,\overset{\bullet}{\Omega},p] = \frac{1}{16\pi G}\int_{\mathcal{M}}d^d v\left[\frac{1}{2}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}{}^{IJ}_{\ \ K} - \frac{1}{4}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IJ} + \overset{\bullet}{\mathcal{T}}{}^J_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IK}\right]$$

$$S_{\mathrm{EH}}[g,p] = \frac{1}{16\pi G}\int_{\mathcal{M}}d^d v_g\left(g^{cb}\overset{\circ}{R}{}^a_{\ cab}[g,p] + (d-2)\overset{\circ}{\nabla}_a p^a\right)$$
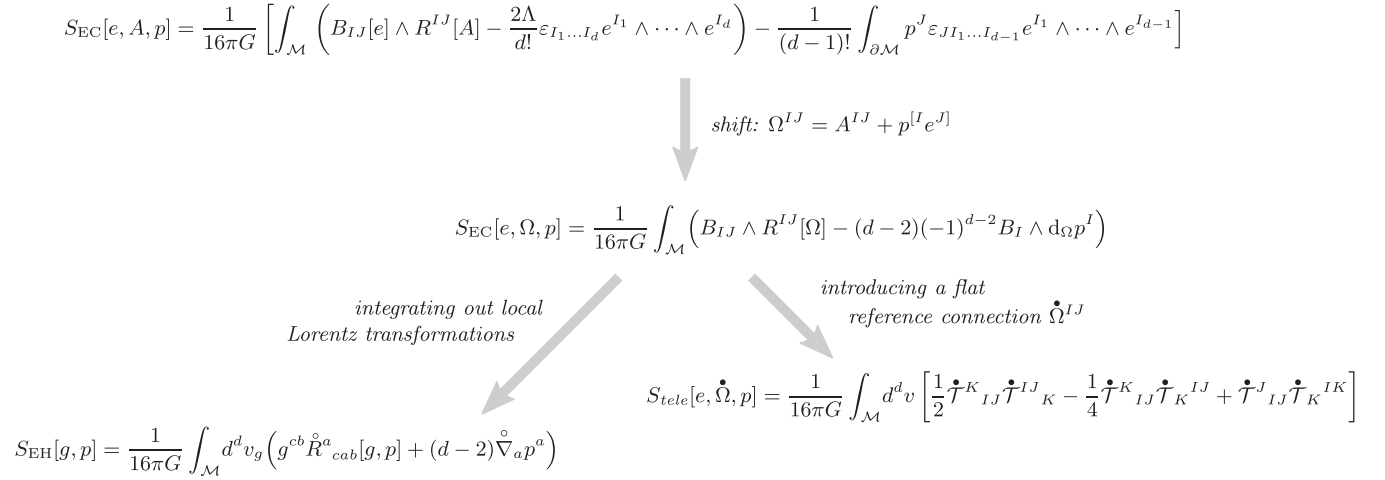
FIG. 1.   Brief summary of the paper: First of all, we performed a change of variables going from the spin connection to a shifted connection Ω. The new connection Ω depends on a boundary field p, whose norm determines the cosmological constant. Integrating out the internal Lorentz transformations, we obtain a new second-order metric formulation, which resembles unimodular gravity (the norm of p_a is determined by the cosmological constant). A non-Abelian version of teleparallel gravity is found by introducing a specific flat reference connection $\overset{\bullet}{\Omega}$ with $\mathrm{d}_{\overset{\bullet}{\Omega}}p^I = 0$.

$$B_{I_1\dots I_n} := \frac{1}{(d-n)!}\varepsilon_{I_1\dots I_n K_1\dots K_{d-n}}e^{K_1}\wedge\dots\wedge e^{K_{d-n}}. \quad (4.23)$$

If we then also use the algebraic identity

$$\frac{1}{n!}B_{I_1\dots I_n}\wedge e^{J_1}\wedge\dots e^{J_n} = (-1)^{n(d-n)}d^d v\delta^{[J_1}_{I_1}\dots\delta^{J_n]}_{I_n}, \quad (4.24)$$

then we can write the action (4.22) in terms of contractions of $\overset{\bullet}{\mathcal{T}}{}^I_{\ KJ}$ alone. A straight-forward calculation gives

$$S_{\mathrm{EC}}[e,\overset{\bullet}{\Omega}+\Delta,p] = \frac{1}{16\pi G}\int_{\mathcal{M}}d^d v\left[\overset{\bullet}{\mathcal{F}}{}^{IJ}_{\ \ IJ} - e_I^{\ a}\overset{\bullet}{\nabla}_a p^I\right.$$
$$+ \frac{1}{2}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}{}^{IJ}_{\ \ K} - \frac{1}{4}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IJ}$$
$$\left. + \overset{\bullet}{\mathcal{T}}{}^J_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IK}\right] \quad (4.25)$$

$$- \frac{1}{16\pi G}\int_{\partial\mathcal{M}}(-1)^{d-2}B_{IJ}\wedge\Delta^{IJ}, \quad (4.26)$$

where $\overset{\bullet}{\mathcal{F}}{}^{IJ}_{\ \ KL}$ are the components of the curvature two-form $\overset{\bullet}{\mathcal{F}}{}^{IJ} = \frac{1}{2}\overset{\bullet}{\mathcal{F}}{}^{IJ}_{\ \ KL}e^K\wedge e^L$.

Since we can always find a reference connection $\overset{\bullet}{\Omega}{}^{IJ}$ such that $\overset{\bullet}{\nabla}_a p^I = 0$ and $R^{IJ}[\overset{\bullet}{\Omega}] = 0$, we can now introduce the following teleparallel action, which is quadratic in the $\mathfrak{an}_{d-1}$ Lie algebra-valued field strength $\overset{\bullet}{\mathcal{T}}{}^I$,

$$S_{\mathrm{tele}}[e,\overset{\bullet}{\Omega},p] = \frac{1}{16\pi G}\int_{\mathcal{M}}d^d v\left[\frac{1}{2}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}{}^{IJ}_{\ \ K} - \frac{1}{4}\overset{\bullet}{\mathcal{T}}{}^K_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IJ}\right.$$
$$\left. + \overset{\bullet}{\mathcal{T}}{}^J_{\ IJ}\overset{\bullet}{\mathcal{T}}_K^{\ IK}\right]. \quad (4.27)$$

V. OUTLOOK

We have shown in this paper how the canonical transformation induced by the boundary volume term was similar to the canonical transformation considered in [20] to recover the notion of quantum group for gravity in three dimensions. The canonical transformation results essentially in a shift of the connection by the frame field,

$$A^{IJ} \to \Omega^{IJ} = A^{IJ} + e^{[I}p^{J]}, \quad (5.1)$$

where p is either interpreted as the normal to the boundary or as a constant vector. In both cases, the norm of p is constrained to be proportional to the cosmological constant. This is true in any dimension higher or equal than three. In two dimensions, the frame field does not appear in the B field, so the shift cannot be done by such boundary terms. This can also be seen from the normalization condition $p^2 = -\frac{8\Lambda}{(d-1)(d-2)}$.

We would like to emphasize that the change of variables can also be performed when $\Lambda = 0$. In this case, p will be a null vector in the Lorentzian case. In the Euclidean case, we can use a Grassmannian number θ to parametrize p such that $\theta^2 = 0$.

We identified the two associated second-order theories, namely the metric and the teleparallel formulations, see the

figure above. The metric formalism associated to the new variables can be seen as a new version of the Henneaux-Teitelboim model for unimodular gravity. We note that the first-order action (3.38) can also be interpreted as a first-order formulation for unimodular gravity since it can be supplemented with the normalizing constraint of the vector p,

$$S_{\mathrm{EC}}[e, \Omega, p] = \frac{1}{16\pi G} \int_{\mathcal{M}} \left(\left(\frac{1}{(d-2)!} \varepsilon_{IJK_1 \cdots K_{d-2}} e^{K_1} \wedge \cdots \wedge e^{K_{d-2}} \right) \wedge R^{IJ}[\Omega] \right.$$
$$\left. - \frac{(d-2)(-1)^{d-2}}{(d-1)!} (\varepsilon_{IK_1 \ldots K_{d-1}} e^{K_1} \wedge \ldots e^{K_{d-1}}) \wedge \mathrm{d}_\Omega p^I - \frac{1}{2} g_{ab} p^a p^b \mathrm{d}\tau \right). \tag{5.2}$$

As in the metric formulation, variation with respect to τ leads to the norm of p being constant, hence recovering the cosmological constant as a constant of integration. The teleparallel formulation, on the other hand, is now expressed in terms of a non-Abelian field strength (4.19) that enters the action quadratically (4.27). With respect to the usual Abelian teleparallel formulation, the Abelian Lie algebra $\mathbb{R}^d$ is deformed into the non-Abelian Lie algebra $\mathfrak{an}_{d-1}$.

The result of this paper summarized in Fig. 1 opens a number of interesting new directions to explore. The boundary volume term plays an important role in the holographic renormalization context [26]. It would be interesting to see whether the formulation using the new variables can also be useful, in particular, its connection with unimodular gravity.

The boundary term is the key to understand why quantum groups appear in three dimensions [20]. In the four-dimensional context it has been conjectured that they should also be relevant [21–23]. It would then be important to study how the charge algebra is affected by adding this term to the theory. In particular, in the list of works [3–5], it was always assumed that $\Lambda = 0$. It would be worthwhile to study how these results are affected if $\Lambda \neq 0$, and the boundary volume term is also considered.

The non-Abelian formulation of teleparallel gravity is likely to be the classical continuum counterpart of the (deformed) dual BF vacuum [31]. It would be interesting to construct the discrete picture, which leads upon quantization to such a deformed dual BF vacuum, generalizing [32,33]. This is work in progress.

The origin of this work came from studying three-dimensional gravity. Recently, a most general bulk action for three-dimensional gravity was introduced [34]. It would be interesting to see how the change of variables could be of use there, either for discretizing or in recovering the different second-order formalisms.

Finally, the canonical transformation is implemented for a constant homogenous curvature, parametrized by the cosmological constant. One might wonder whether we could generalize the construction in the case of a varying curvature. We leave this intriguing question for later investigations.

ACKNOWLEDGMENTS

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