Analytical computation of quantum corrections to a nontopological soliton within the saddle-point approximation

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Nonrelativistic scalar field theory with an attractiveself-interaction possesses nontopological extended solutions with a finite energy in both finite and infinite-volume cases, namely, bright solitons. The analytical form of the solution itself is well known, though analytical studying of the quantum fluctuations in this background still requires more thorough investigation, for instance, analytical computation of quantum corrections to this background within the saddle-point approximation. In the present work this gap is filled. Both the 2-point Green's function and quantum corrections to the background are analytically computed and properly renormalized by means of the momentum cutoff procedure. It is deduced that quantum corrections are indeed small provided that the particle number is large. Also, we see that perturbation modes of the continuum spectrum at bright soliton background generate a gap in the energy spectrum. Moreover, it turns out that the whole spectrum is continuous modulo zero-modes, which is similar to Sine-Gordon solitons.

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I. INTRODUCTION

In quantum field theory two types of solitons are usually considered. The first type is topological solitons [1–4],[1] which exist thanks to some nontrivial mapping of internal field space on coordinate space or space-time. For these solitons the quantization procedure is very well developed [6–8] and both analytical and numerical studies of their quantum properties were carried out for different types of such solitons and within different models [9–13]. The situation is more subtle with nontopological solitons, which can be stabilized by means of some conserved global current and correspond to fixed global charge [14]. The procedure of quantizing nontopological solitons in the relativistic case was established in [15], but no actual computation according to this procedure was ever done,[2] in spite of attempts given for instance in works like [16], where integration along symmetry direction was carried out. Nevertheless, it did not allow the authors to go far beyond the classical solution. Although, there are some works that address questions of fluctuating modes [17–19], neither analytical nor numerical computations of the Green's function or quantum corrections[3] to the background of nontopological soliton like Q-ball by means of semiclassical methods was carried out. However, there are works [24,25], which suggests the procedure of nonperturbative diagonalization of nonrelativistic Hamiltonian enclosed in finite volume, which implies cutting-off modes in the Hamiltonian leaving only those with momenta $p = \{-2\pi/L, 0, 2\pi/L\}$, thereby reducing infinite set of degrees of freedom to the finite one and then

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[1]And of course, all the other incalculable references listed for instance in [5].

[2]If we exclude Bose-Einstein condensates as nontopological finite energy solutions.

[3]However, there is a work [20], which claims computation of quantum corrections to the energy of Q-ball numerically. But for me and all my colleagues I have discussed this work with the results presented there are highly questionable. First of all, the authors do not refer at all to original works by Friedberg, Lee, Sirlin *et al.* and claim that they developed some other method, which anyway seems to be literally the same or provides minimal to none improvement compared to usual saddle-point technique. Secondly, one can see that Eq. (11) in this work is simply incorrect, because it does not account for the mixing between complex field and its Hermitian conjugate (see Eq. (16) in my work). The fact that this system of equations describing perturbations in the Q-ball background cannot be diagonalized by coordinate-independent transformation for the inhomogeneous background is an easy fact to understand and this feature was considered in all the literature somehow related to perturbations at the top of nontopological configurations [16,18,19,21–23]. Though maybe there was some trick done by the author, which was not mentioned at all. Thus, I must cast serious doubts about the results.

attempt to develop approximate quantization of the reduced Hamiltonian. As a result [24] yields interesting analytical results and [25,26] provide interesting numerical results. However, this method becomes increasingly demanding for a very large box or in case one wishes to include higher momentum modes making it harder to go to the infinite volume case. In addition, it is not sensible to ultraviolet modes because basically this method introduces explicit cutoff, which is not taken care of by renormalization, though this method is impressively useful in order to capture quantum effects qualitatively.

The fact that equations for quantum fluctuations at the top of nontopological soliton are very difficult to solve does not come as a surprise, because nontopological solitons are time-dependent solutions and equations for their fluctuations are not diagonalizable by coordinate-independent transformation due to this feature. In this work I pursue a little bit more modest goal and compute all these things in the case of the nonrelativistic field theory in $1+1$ dimensions. The equation of motion of this theory is usually referred as the nonlinear Schrödinger equation. It has analytical solutions in both finite [27] and infinite [28] volume with finite energy and finite spatial extensions called the bright soliton. Also this equation was studied by the inverse scattering method [29,30] and emerges in nonlinear optics [31]. I will study quantum fluctuations in the background of the bright soliton. In spite of the fact that it is hard to diagonalize equations of motion for quantum fluctuations in the background of soliton it is still possible to invert fluctuation operator, which is the inverse 2-point Green's function. Although, I must admit that this is manageable thanks to the extreme simplicity of this system as it does not contain any square-integrable modes in the spectrum apart from non-oscillating zero modes, which basically means that no finite-energy modes are stuck inside potential well created by bright soliton, which is very similar to the Sine-Gordon soliton [32]. In order to carry procedure of quantization out, I will make use of the identities found in [33], which involve actual eigenfunctions for continuum spectrum and zero-modes supplemented by additional "quasi"-eigenfunctions, which make actual eigenfunctions up to completeness.

To make reading more coherent let me state here the main results of this work:

(1) I compute the 2-point Green's function of fluctuations in the background of the bright soliton. This Green's function has poles at

$$E_p = \omega + \frac{p^2}{2m}, \tag{1}$$

where ω is the frequency of the bright soliton rotation in the internal space.

(2) We see that in order to excite a particle in the background of bright soliton within the sector with fixed charge we have to invest the energy equal to the gap ω.

(3) Spectrum of quantum fluctuations in this background is substantially continuous modulo zero-modes.

(4) The classical energy of the bright soliton $E_{\mathrm{cl.}} = -\frac{N}{384} m\alpha_{\mathrm{coll.}}^2$ is getting corrected

$$E = E_{\mathrm{cl.}} + \delta E$$
$$= -\frac{1}{384} m\alpha_{\mathrm{coll.}}^2 \left(N - \left(5 + \frac{16\pi^2}{15}\right) + \mathcal{O}(1/N)\right). \tag{2}$$

where $\alpha_{\mathrm{coll.}} = N(\lambda/m^2)$ is the dimensionless collective coupling of internal degrees of freedom of the bright soliton consisting of the particle number N, and dimensionless coupling of quartic self-interaction λ/m^2. Hence, it is apparent that correction scales as either $\mathcal{O}(\hbar)$ or as $\mathcal{O}(N^0)$ compared to classical energy, and is small for large particle number $N \gg 1$.

I will keep the proper explanation of these results to the main body and the conclusion.

The rest of the article is organized as follows: In Sec. II I present classical properties of bright solitons and connection to the relativistic theory, then in Sec. III I introduce the inverse Green's function of quantum fluctuations in the background of the bright soliton and invert this operator in Sec. IV, where I discuss backreactions on the background and their compliance with the demand of fixed particle number, then in Sec. V quantum correction to the classical energy is computed.

II. CLASSICAL THEORY

In the present work I consider nonrelativistic field theory[4] in $1+1$ dimension confined in infinite spatial volume with an attractive self-interaction given by Lagrange density

$$\mathcal{L} = i\psi^*\dot{\psi} - \frac{1}{2m}\left|\frac{d\psi}{dx}\right|^2 + \frac{\lambda}{8m^2}|\psi|^4. \tag{3}$$

This theory has internal global $U(1)$ symmetry

$$\psi \to e^{i\alpha}\psi, \tag{4}$$

which provides conserved charge, namely, particle number

$$N = \int dx|\psi|^2, \tag{5}$$

for solutions vanishing at infinity or having periodic boundary conditions.

Varying action of this theory with respect to ψ^* and ψ one can deduce classical equations of motion

[4] Sometimes called Schrödinger's field theory, which is not to be confused with Schrödinger picture.

$$i\partial_t\psi + \frac{1}{2m}\partial_x^2\psi + \frac{\lambda}{4m^2}|\psi|^2\psi = 0, \tag{6}$$

which is well-known nonlinear Schrödinger equation. These equations possess well-known bright soliton solution [27,28]

$$\psi_{\rm cl}(t,x) = e^{i\omega t}\sqrt{\frac{8m^2\omega}{\lambda}}{\rm sech}\Big(\sqrt{2m\omega}x\Big), \tag{7}$$

which has classical energy

$$E_{\rm cl} = -\frac{8\sqrt{2}}{3}\frac{(m\omega)^{3/2}}{\lambda}, \tag{8}$$

and corresponds to the amount of particles

$$N = \frac{8\sqrt{2m\omega}m}{\lambda}. \tag{9}$$

Therefore, it is indeed a nontopological soliton as it is a finite energy solution corresponding to fixed charge resulting from global current conservation.

Note that the energy of this bright soliton is negative and turns out to be binding energy as soon as we are working in the nonrelativistic regime. However, we can make it up to the full energy adding the rest energy carried by N quanta. Hence, the full relativistic energy is

$$E_{\rm cl}^{\rm (rel)} = mN + E_{\rm cl} = \frac{8\sqrt{2}m}{3\tilde{\lambda}}\left(\left(\frac{\omega}{m}\right)^{1/2} - \left(\frac{\omega}{m}\right)^{3/2}\right), \tag{10}$$

where $\tilde{\lambda} = \lambda/m^2$ is the dimensionless coupling constant. So, we see that $E_{\rm cl}$ is indeed just correction to the rest mass, because for nonrelativistic approximation to hold the inequality

$$m|\psi| \gg \left|\frac{\partial\psi}{\partial t}\right| \tag{11}$$

must be satisfied, which results into

$$\omega \ll m, \tag{12}$$

so we see that ω/m is another dimensionless small parameter. This is an important observation, because in nonrelativistic theory there is no actual restriction for ω apart from its positiveness. But if we keep in mind the relativistic setup, then it is evident, that ω has finite range and moreover must be small for nonrelativistic approximation to hold.

Here we are finished with the classical properties of the bright soliton and can turn to the study of quantum fluctuations in its background.

III. QUANTUM FLUCTUATIONS

Equation for the inverse Green's function in the background of the bright soliton is easy to derive by taking the second variational derivative of the action at the solution of classical equations of motion

$$\int d^2z \frac{\delta^2 S}{\delta\psi_a(x)\delta\psi_c^*(z)}\bigg|_{\psi=\psi_{\rm cl}} G_{cb}(z;y) = i\delta_{ab}\delta^2(x-y),$$

where $x^\mu = (x^0, x^1)$, $\psi_a = (\psi, \psi^*)^T$, $d^2z = dz^0dz^1$. Explicitly, this is

$$\begin{pmatrix} i\partial_t + \frac{1}{2m}\partial_x^2 + \frac{\lambda}{2m^2}|\psi_{\rm cl}|^2 & \frac{\lambda}{4m^2}\psi_{\rm cl}^2 \\ \frac{\lambda}{4m^2}\psi_{\rm cl}^{*2} & -i\partial_t + \frac{1}{2m}\partial_x^2 + \frac{\lambda}{2m^2}|\psi_{\rm cl}|^2 \end{pmatrix}_{ac} G_{cb}(t,x;\tau,y) = i\delta(x-y)\delta(t-\tau)\delta_{ab} \tag{13}$$

Off-diagonal elements are time-dependent. This can be removed by following transformation

$$G(t,x;\tau,y) = T(t)\mathcal{G}(t-\tau;x,y)T^\dagger(\tau), \tag{14}$$

where

$$T(t) = \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \tag{15}$$

is a unitary matrix.

Hence, this operator becomes just

$$\begin{pmatrix} i\partial_t - \omega + \frac{1}{2m}\partial_x^2 + \frac{\lambda}{2m^2}f^2 & \frac{\lambda}{4m^2}f^2 \\ \frac{\lambda}{4m^2}f^2 & -i\partial_t - \omega + \frac{1}{2m}\partial_x^2 + \frac{\lambda}{2m^2}f^2 \end{pmatrix}_{ac} G_{cb}(t,x;\tau,y) = i\delta(x-y)\delta(t-\tau)\delta_{ab}, \tag{16}$$

where $f = f(x) = |\psi_{\rm cl}(x)|$.

Notice that $\mathcal{G}(t-\tau; x, y)$ is time-translation invariant, therefore we can perform Fourier transformation

$$\mathcal{G}(t-\tau; x, y) = \int \frac{d\gamma}{2\pi} e^{-i\gamma(t-\tau)} \mathcal{G}(\gamma; x, y). \tag{17}$$

and $SO(2)$ rotation

$$U\mathcal{G}(\gamma; x, y)U^{\dagger} = g(\gamma; x, y), \tag{18}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{19}$$

This results in the equation for Green's function

$$\begin{pmatrix} -\omega + \frac{1}{2m}\partial_x^2 + \frac{3\lambda}{4m^2} f^2 & -\gamma \\ -\gamma & -\omega + \frac{1}{2m}\partial_x^2 + \frac{\lambda}{4m^2} f^2 \end{pmatrix}_{ac} g_{cb}(\gamma; x, y) = i\delta(x-y)\delta_{ab}. \tag{20}$$

Let me remark here. Notice that matrix U defined in (18) is not actually a transformation to real basis it is just some transformation of complex valued fields to some different complex valued ones.

If we plug the profile of the bright soliton explicitly, one could easily recognize Hamiltonians coming from supersymmetric quantum mechanics. Unfortunately, their spectrum is of no use here, because equations are intertwined due to coordinate dependence and cannot be diagonalized by coordinate-independent transformation. Later we will give one more argument of futility of the eigenfunctions of the modified Pöschl-Teller potentials in this case.

I finish this section by writing down explicit expression for fluctuation operator

$$\begin{pmatrix} -\omega + \frac{1}{2m}\partial_x^2 + 6\omega\, \mathrm{sech}^2(\sqrt{2m\omega}x) & -\gamma \\ -\gamma & -\omega + \frac{1}{2m}\partial_x^2 + 2\omega\, \mathrm{sech}^2(\sqrt{2m\omega}x) \end{pmatrix}_{ac} \mathcal{G}_{cb}(\gamma; x, y) = i\delta(x-y)\delta_{ab}, \tag{21}$$

and introducing convenient notations

$$\begin{pmatrix} -\hat{L} & -\gamma \\ -\gamma & -\hat{R} \end{pmatrix}_{ac} \mathcal{G}_{cb}(\gamma; x, y) = i\delta(x-y)\delta_{ab}, \tag{22}$$

where

$$\hat{R} = -\frac{1}{2m}\partial_x^2 - 2\omega\, \mathrm{sech}^2(\sqrt{2m\omega}x) + \omega,$$
$$\hat{L} = -\frac{1}{2m}\partial_x^2 - 6\omega\, \mathrm{sech}^2(\sqrt{2m\omega}x) + \omega. \tag{23}$$

In the next section I will invert this operator.

IV. ANALYTICAL FORM OF GREEN'S FUNCTION

A. Computation

Here we recall results from [33] and consider specific functions, which can be turned to each other under the action of operators (23).

First of all there are two sets of $L^2(\mathbb{R})$ functions satisfying following equations

$$\begin{cases} \hat{R}r_1(x) = 0 \\ \hat{R}r_2(x) = 2\omega l_2(x) \end{cases}, \qquad \begin{cases} \hat{L}l_1(x) = -2\omega r_1(x) \\ \hat{L}l_2(x) = 0 \end{cases}. \tag{24}$$

Apparently, only vector made of zero-modes l_2 and r_1 can solve the tricky eigenvalue problem

$$\begin{pmatrix} -\hat{L} & -\gamma \\ -\gamma & -\hat{R} \end{pmatrix} \begin{pmatrix} l(x) \\ r(x) \end{pmatrix} = 0, \tag{25}$$

posed by the operator inverse to the Green's function. There is no any other linear combination which can be built out of these functions to solve this eigenvalue problem.

However, there are continuum modes that solve this eigenvalue problem:

$$\begin{pmatrix} -\hat{L} & -\gamma(p) \\ -\gamma(p) & -\hat{R} \end{pmatrix} \begin{pmatrix} l_p(x) \\ r_p(x) \end{pmatrix} = 0, \tag{26}$$

and the corresponding eigenvalues are

$$\gamma(p) = -\left(\omega + \frac{p^2}{2m}\right). \tag{27}$$

In spite of the fact that all of these functions do not solve the whole spectrum of the operator, they form a full set of functions satisfying following completeness relation

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} r_p(x) l_p^*(y) + \sum_{j=1}^{2} r_j(x) l_j^*(y) = \delta(x-y), \tag{28}$$

which we will use to construct the Green's function. This relation tells us that we cannot find anything more. Now, we can actually conclude that if we took linear combinations of eigenvalues of operators $\hat{L}$ and $\hat{R}$ we could not solve this problem, because they cannot be treated independently. Completeness relation tells us that space of the functions we are considering is actually the space of one degree of freedom as it should be in case of nonrelativistic field theory, if it were possible to diagonalize the fluctuation operator and reduced it to two independent one-dimensional eigenvalue problems, that it would imply two propagating modes, which cannot be the case. There is no doubling of degrees of freedom. Notice also that there are no square-integrable oscillating modes lying in $L^2(\mathbb{R})$, both square-integrable modes making up for completeness are actually just some combinations of zero-modes, which makes this problem simple. This tells us that only continuum spectrum contributes to corrections to the energy to the contrary,for instance, to the case of kink.

Just for the purpose of convenience I will rewrite (24) and (26) as

$$\begin{aligned} \hat{L} l_\alpha(x) &= \lambda^l_\alpha r_\alpha(x) \\ \hat{R} r_\alpha(x) &= \lambda^r_\alpha l_\alpha(x). \end{aligned} \tag{29}$$

All these functions are explicitly shown in Appendix.

Problem (13) consists out of four equations, they can be split in two pairs. Let me show how to solve equations for first pair of Green's function components $g_{11}(\gamma; x, y)$ and $g_{21}(\gamma; x, y)$.

Equations for these two guys are

$$\begin{cases} -\hat{L} g_{11}(\gamma; x, y) - \gamma g_{21}(\gamma; x, y) = i\delta(x-y) \\ -\hat{R} g_{21}(\gamma; x, y) - \gamma g_{11}(\gamma; x, y) = 0. \end{cases} \tag{30}$$

In order to solve these equation I assume the Green's function to be

$$\begin{cases} g_{11}(\gamma; x, y) = \sum_\alpha (s^{ll}_{11,\alpha} l_\alpha(x) l^*_\alpha(y) + s^{lr}_{11,\alpha} l_\alpha(x) r^*_\alpha(y)) \\ g_{21}(\gamma; x, y) = \sum_\alpha (s^{rr}_{21,\alpha} r_\alpha(x) r^*_\alpha(y) + s^{rl}_{21,\alpha} r_\alpha(x) l^*_\alpha(y)) \end{cases} \tag{31}$$

Also we can rewrite delta function at the right-hand side (rhs) of (30) using completeness relation (28). Then, plugging all this into the first equation in (30) we get

$$\sum_\alpha ((-s^{ll}_{11,\alpha}\lambda^l_\alpha - \gamma s^{rl}_{21,\alpha} - i) r_\alpha(x) l^*_\alpha(y) + (-s^{lr}_{11,\alpha}\lambda^l_\alpha - \gamma s^{rr}_{21,\alpha}) r_\alpha(x) r^*_\alpha(y)) = 0 \tag{32}$$

and also a similar thing for the second equation in (30). Demanding everything to vanish identically we deduce 12 equations defining unknown coefficients

$$\begin{cases} -s^{ll}_{11,\alpha}\lambda^l_\alpha - \gamma s^{rl}_{21,\alpha} - i = 0 \\ s^{lr}_{11,\alpha}\lambda^l_\alpha + \gamma s^{rr}_{21,\alpha} = 0 \\ s^{rr}_{21,\alpha}\lambda^r_\alpha + \gamma s^{lr}_{11,\alpha} = 0 \\ s^{rl}_{21,\alpha}\lambda^r_\alpha + \gamma s^{ll}_{11,\alpha} = 0 \end{cases}, \qquad \alpha = 1, 2, p. \tag{33}$$

Solving this linear system we get

$$\begin{aligned} -i g_{11}(\gamma; x, y) &= \frac{2\omega}{\gamma^2} l_2(x) l^*_2(y) + \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{\omega + \frac{p^2}{2m}}{\gamma^2 - (\omega + \frac{p^2}{2m})^2} l_p(x) l^*_p(y), \\ -i g_{21}(\gamma; x, y) &= -\frac{1}{\gamma} r_1(x) l^*_1(y) - \frac{1}{\gamma} r_2(x) l^*_2(y) + \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{-\gamma}{\gamma^2 - (\omega + \frac{p^2}{2m})^2} r_p(x) l^*_p(y), \end{aligned} \tag{34}$$

and repeating same computation for two remaining components we deduce

$$\begin{aligned} -i g_{22}(\gamma; x, y) &= -\frac{2\omega}{\gamma^2} r_2(x) r^*_2(y) + \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{\omega + \frac{p^2}{2m}}{\gamma^2 - (\omega + \frac{p^2}{2m})^2} r_p(x) r^*_p(y), \\ -i g_{12}(\gamma; x, y) &= -\frac{1}{\gamma} l_1(x) r^*_1(y) - \frac{1}{\gamma} l_2(x) r^*_2(y) + \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{-\gamma}{\gamma^2 - (\omega + \frac{p^2}{2m})^2} l_p(x) r^*_p(y). \end{aligned} \tag{35}$$

In the end, in order to complete the computation, a contour of integration must be specified. I will do γ integration by means of Feynman prescription, namely, $\gamma^2 - E_p^2 + i0^+$.

Summarizing all the results we get

$$G(t, \tau | x, y) = \int \frac{d\gamma}{2\pi} e^{-i\gamma(t-\tau)} T(t) U^\dagger g(\gamma | x, y) U T(\tau)^\dagger \tag{36}$$

B. Gap in the continuum spectrum

As the result of computation I deduced that the propagator has poles at

$$E_p = \omega + \frac{p^2}{2m}. \tag{37}$$

That is contrary to the vacuum dispersion relation. Nevertheless, this gap can be easily explained by means of the following argument.

The solution we are studying lies in the sector of fixed charge N (particle number). Therefore, any perturbation must not violate the number of particles which is conserved. Thus, in order to excite particle in the bright soliton background we must invest energy equal to ω, that will correspond to the fact that we moved system to another sector with bright soliton composed out of $N - 1$ particles and one external particle. Recall that for bright solitons holds following integral property

$$\frac{dE_{\text{cl}}}{d\omega} = -\omega \frac{dN}{d\omega}, \tag{38}$$

so we can see that if we take one particle out of soliton its energy is changed by

$$\Delta E_{\text{cl}} = \omega, \tag{39}$$

that is exactly the amount of energy which we need to excite a particle from continuous spectrum at the top of bright soliton. In other words we see, that energy of bright soliton with $N - 1$ quantas is the same as energy of bright soliton with N quantas and one additional quanta from free spectrum

$$E_{\text{cl}}(N-1) = E_{\text{cl}}(N) + E_{p=0} = E_{\text{cl}}(N) + \omega. \tag{40}$$

The important lesson here to keep in mind is that in this approach if one considers some scattering process with m particles in the background of bright soliton, these particles will propagate in the background of bright soliton corresponding to the number of particles $N - m$. Therefore, we must remember that equalities (39) and (40) hold up to $1/N$ corrections and when we add one particle we change ω as well

$$\delta\omega = \frac{2\omega}{N}.$$

Due to this fact, approximation holds if we do not excite too many particles, which will result in the relative shift of ω larger than $1/N$.

V. QUANTUM CORRECTIONS

Let me turn eventually to the computation of quantum corrections. In order to do that formally, I introduce the partition function first

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^* \exp\left(i \int dt \int dx \mathcal{L}\right) = \text{tr}(e^{-iHT}) \tag{41}$$

where the Lagrangian is given by (3).

Next we will restrict this functional to the configurations having fixed charge by plugging in projector

$$\hat{P}(N) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \exp\left(-i\alpha(\hat{N} - N - \delta N)\right), \tag{42}$$

where δN is the renormalization of particle number. We need it here, because if we put a restriction on time-ordered products (particle number in this case), they must be tuned to be finite or, in other words, regularized as any other measurable quantity.

After the inclusion of the projector partition function becomes

$$\mathcal{Z}_N = \int \mathcal{D}\psi \mathcal{D}\psi^* \int_0^{2\pi} \frac{d\alpha}{2\pi} \exp\left(iS_{\text{eff}}[\psi,\psi^\dagger] + i\alpha(N + \delta N)\right), \tag{43}$$

where we have included in the action part of the projection operator

$$S_{\text{eff}}[\psi,\psi^\dagger] = \int dx dt (\mathcal{L} - \omega\psi^\dagger\psi), \quad \omega = \frac{\alpha}{T}, \quad T = \int_{-T/2}^{T/2} dt. \tag{44}$$

Now we can employ the saddle-point approximation with respect to both variables $(\psi, \psi^\dagger)$ and α. We shift them at the classical solution

$$\begin{cases} \psi(t,x) \to \psi(t,x) + f(x) \\ \alpha \to \omega T + \alpha \end{cases}, \tag{45}$$

where $f(x)$ is the modulus of (7) and ω is the same frequency as in the profile.

Hence, the partition function becomes

$$\mathcal{Z}_N = \int \mathcal{D}\psi \mathcal{D}\psi^* \int_{-\omega T}^{2\pi-\omega T} \frac{d\alpha}{2\pi} \exp\left(-i\alpha\left(\int dx (f(x)(\psi + \psi^\dagger) + \psi^\dagger\psi) - \delta N\right)\right)$$
$$\exp\left(i \int dt dx \int d\tau dy \frac{i}{2} (\psi^\dagger(\tau,y), \psi(\tau,y)) \hat{\mathcal{D}}(\tau - t|x,y) \begin{pmatrix} \psi(t,x) \\ \psi^\dagger(t,x) \end{pmatrix} + i \int dt \left(\mathcal{L}_{\text{int}} + \omega\delta N - \int dx \delta m f^2(x)\right)\right). \tag{46}$$

In this expression for the partition function

$$\hat{\mathcal{D}}(\tau - t|x, y) = \hat{\mathcal{D}}(t, x)\delta(\tau - t)\delta(x - y),$$

where $\hat{\mathcal{D}}(t, x)$ is the differential operator of linear fluctuations which we have specified in (16), thereby it constitutes bilinear part of Lagrange density. Then, we derive interaction terms, which are included in the interaction density

$$\mathcal{L}_{\rm int} = \frac{(\lambda + \delta\lambda)}{8m^2}(\psi^\dagger\psi)^2 + \frac{(\lambda + \delta\lambda)f(x)}{4m^2}(\psi + \psi^\dagger)\psi^\dagger\psi - \delta m(f(x)(\psi + \psi^\dagger) + \psi^\dagger\psi). \tag{47}$$

Notice that I have not included δN and $\delta m f^2$ here. The reason for that is that they contribute only in the renormalization of the energy of the bright soliton, and are totally irrelevant for computation of diagrams. Also, at the level of the first quantum correction we do not need the counterterm $\delta\lambda$, so this is the only expression where I have put it just for the sake of generality. We will not need renormalization of coupling to compute the first quantum correction.

Let me comment also on the counterterm δN constituting renormalization of particle number. From the first glance, one can say that we do not have this term for vacuum, which is not quite true. We need this to make sure, that at the level of operator-averages charge is properly normalized, namely, $\langle\hat{T}\{\hat{N}\}\rangle$ is UV-divergent due to time ordering, thus we need δN in order to cancel this. In other words, application of the path integral formulation implies that we use time-ordered products, which contain divergences we have to take into account to get finite physical quantities.

Another important point is the integral over α. It is evident that in order for the whole partition function $\mathcal{Z}_N$ to be nonzero, the argument of α-dependent exponent must be exactly zero. From the path integral definition of the partition function with fixed particle number (46) we deduce nonlinear restriction on the averages of quantum operators

$$\left\langle \hat{T}\left\{ \int dx(f(x)(\hat{\psi} + \hat{\psi}^\dagger) + \hat{\psi}^\dagger\hat{\psi}) - \delta N \right\}\right\rangle = 0. \tag{48}$$

Here $\hat{T}$ implies time-ordering. In the following we will use this condition to fix the counterterm responsible for charge renormalization δN. This condition is very important as it ensures conservation of the total charge. Here we compute everything by perturbation theory. Therefore, (48) in the interaction representation picture becomes

$$\frac{1}{\mathcal{Z}_N}\left\langle \hat{T}\left\{ \int dx(f(x)(\hat{\psi} + \hat{\psi}^\dagger) + \hat{\psi}^\dagger\hat{\psi}) \times \exp\left(i\int dt dx \mathcal{L}_{\rm int}\right)\right\}\right\rangle = \delta N, \tag{49}$$

where all the operators are in the representation of interaction, but I will not specify this explicitly by putting additional subscript at the field operators.

One can see from (47) that there are terms only proportional to positive orders of coupling constant λ/m^2, hence, perturbation theory in this case is valid. But if one would like to quantize this field canonically it is possible to impose condition (48) on operators of the fluctuations. Therefore, one can see that it is necessary to nonlinearly modify operators of fluctuations in order to impose conservation of particle number [34].

Finally, everything is prepared in order to compute the first correction to the energy of the bright soliton, which is

$$E(N) = \lim_{T\to+\infty}\frac{i}{T}\log\frac{\mathcal{Z}_N}{\mathcal{Z}_{\rm vac}}. \tag{50}$$

We will account for the corrections up to λ order. Hence, energy is

$$\begin{aligned} E(N) &= E_{\rm cl} + \frac{i}{T}{\rm tr}(\log\mathcal{D}(f(x))(\tau - t|x, y) - \log\mathcal{D}_{\rm vac}(\tau - t|x - y)) - \omega\delta N - \int dx\delta m f^2(x) + \mathcal{O}(\lambda) \\ &= E_{\rm cl} + \int\frac{dp}{2\pi}\frac{1}{2}(\gamma_{\rm b.s.}(p) - \gamma_{\rm vac}(p)) - \omega\delta N - \delta m\int dx f^2(x) + \mathcal{O}(\lambda), \end{aligned} \tag{51}$$

where classical energy is $\hbar$ independent and first correction is of order $\hbar$.

Let us start from evaluating energy coming from functional determinant. We do it by means of the same method as it was done for kink in [35]. Namely, we assume for a moment that our system is enclosed to a very large box of size L and quantize momenta. In order to do that let us compute asymptotics of functions $l_p(x)$ and $r_p(x)$

$$\begin{aligned} l_p(x) &\overset{x\to\pm\infty}{\to} \exp\left(ipx \pm \frac{i}{2}\delta(p)\right), \\ r_p(x) &\overset{x\to\pm\infty}{\to} \exp\left(ipx \pm \frac{i}{2}\delta(p)\right), \end{aligned} \tag{52}$$

where the shift of the phase is

$$\delta(p) = -2\arctan\left(\frac{p\sqrt{2\omega/m}}{\omega - p^2/(2m)}\right). \tag{53}$$

We see that both functions approach exactly the same phase shift simultaneously, reflecting the fact that they stand for one scattering mode, not two separate modes. Then, we impose quantization condition for momenta

$$\begin{aligned} p_{\text{b.s.}}L + \delta(p_{\text{b.s.}}) &= 2\pi n, \qquad \text{for bright soliton,} \\ p_{\text{vac}}L &= 2\pi n, \qquad \text{for vacuum modes.} \end{aligned} \tag{54}$$

Hence we get that

$$p_{\text{b.s.}} = p_{\text{vac}} - \frac{\delta(p_{\text{vac}})}{L} + \mathcal{O}(L^{-2}). \tag{55}$$

We plug this in

$$\begin{aligned} &\lim_{L\to+\infty}\frac{1}{2}\sum_n(\gamma_{\text{b.s.}}(p_{\text{b.s.}}) - \gamma_{\text{vac}}(p_{\text{vac}})) \\ &= \lim_{L\to+\infty}\frac{1}{2}\sum_n\left(\left(\omega + \frac{p_{\text{b.s.}}^2}{2m}\right) - \frac{p_{\text{vac}}^2}{2m}\right) \\ &= \lim_{L\to+\infty}\left(\frac{L}{2}\sum_n\frac{1}{L}\omega - \frac{1}{2L}\sum_n\frac{p_{\text{vac}}\delta(p_{\text{vac}})}{m} + \mathcal{O}(L^{-2})\right) \\ &= \frac{1}{2}\int dx\int\frac{dp}{2\pi}\omega - \int\frac{dp}{2\pi}\frac{p\delta(p)}{2m}. \end{aligned} \tag{56}$$

Notice that I used here substitution

$$\frac{1}{L}\sum_n \to \int\frac{dp}{2\pi}$$

in order to take infinite volume limit.

The resulting expression is divergent, but all these divergences are naturally canceled by counterterms introduced earlier. Here I evaluate them explicitly.

The first counterterm is mass renormalization coming from original Lagrangian

$$\delta m\int dx f^2(x) = \frac{\lambda}{4m^2}\int\frac{dp}{2\pi}\int dx f^2(x) = 2\sqrt{2\omega/m}\int\frac{dp}{2\pi}. \tag{57}$$

Now we come to the most interesting part, namely, evaluation of δN. One can see that in the condition (49) two operators having different orders with respect to λ are getting mixed. We evaluate this by perturbation theory using $\mathcal{L}_{\text{int}}$ defined in (47). Then condition for charge renormalization in the specified approximation becomes

$$\begin{aligned} \delta N &= \left\langle \hat{T}\left\{\int dx(f(x)(\hat{\psi} + \hat{\psi}^\dagger) + \hat{\psi}^\dagger\hat{\psi})\exp\left(i\int d\tau dy\mathcal{L}_{\text{int}}\right)\right\}\right\rangle \\ &= \int \langle\hat{T}\{\hat{\psi}^\dagger(t,x)\psi(\hat{t,x})\}\rangle + \int dx\int d\tau dy f(x)f(y)(-i\delta m)\langle\hat{T}\{(\hat{\psi}(t,x) + \hat{\psi}^\dagger(t,x))(\hat{\psi}(\tau,y) + \hat{\psi}^\dagger(\tau,y))\}\rangle \\ &\quad + \frac{i\lambda}{4m^2}\int dx\int d\tau dy f(x)f(y)\langle\hat{T}\{(\hat{\psi}(t,x) + \hat{\psi}^\dagger(t,x))(\hat{\psi}(\tau,y) + \hat{\psi}^\dagger(\tau,y))\hat{\psi}^\dagger(\tau,y)\psi(\tau,y)\}\rangle + \mathcal{O}(\lambda). \end{aligned} \tag{58}$$

Here all the operators are in the interaction representation picture.

Now let me recall that I have computed the Green's function explicitly for rotated variables, namely,

$$\begin{pmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{pmatrix} = U\begin{pmatrix}\hat{\psi} \\ \hat{\psi}^\dagger\end{pmatrix}. \tag{59}$$

Hence, I will evaluate δN using these change of operators

$$\begin{aligned} \delta N &= \int dx\int\frac{d\gamma}{2\pi}\frac{1}{2}(g_{11}(\gamma|,x,y) + g_{22}(\gamma|,x,y)) + i\int dx dy f(x)f(y)\left(g_{22}(0|x,y)\int\frac{d\gamma}{2\pi}\left(\frac{\lambda}{4m^2}(g_{11}(\gamma|y,y) + 3g_{22}(\gamma|y,y) - 2\delta m)\right)\right. \\ &\quad \left. + 2g_{12}(0|x,y)\int\frac{d\gamma}{2\pi}g_{12}(\gamma|y,y)\right) \\ &= \frac{1}{2}\int dx\int\frac{dp}{2\pi} + \frac{1}{3} - \frac{16\pi^2}{45}. \end{aligned}$$

I must draw the reader's attention to the fact that here I do not account for nonoscillating parts of Green's function. These are used them only for inverting differential operator, but as soon as they do not oscillate, they do not contribute to propagation.

Now we can sum up everything in formula for corrections and see that all the divergences are canceled and we are left with the finite expression

$$E(N) = E_{\rm cl} + \frac{1}{2}\int dx \int \frac{dp}{2\pi}\omega - \int \frac{dp}{2\pi}\frac{p\delta(p)}{2m} + 2\sqrt{\frac{2\omega}{m}}\int \frac{dp}{2\pi} - \omega\left(\frac{1}{2}\int dx \int \frac{dp}{2\pi} + \frac{1}{3} - \frac{16\pi^2}{45}\right) + \mathcal{O}(\lambda)$$
$$= -\frac{8\sqrt{2}}{3}\frac{(m\omega)^{3/2}}{\lambda} + \omega\left(\frac{5}{3} + \frac{15\pi^2}{45}\right) + \mathcal{O}(\lambda). \tag{60}$$

This is the final result. To make it more illustrative let us rewrite it in terms of collective coupling

$$\alpha_{\rm coll.} = \frac{\lambda}{m^2}N = 8\sqrt{\frac{2\omega}{m}}. \tag{61}$$

Reexpressing this through collective coupling, we get

$$E(N) = -\frac{1}{384}m\alpha_{\rm coll.}^2\left(N - \left(5 + \frac{16\pi^2}{15}\right) + \mathcal{O}(1/N)\right). \tag{62}$$

Therefore, we see that for sufficiently large N, quantum corrections are indeed just corrections to the classical energy of bright soliton, or to be more specific, if we consider nonrelativistic field theory as the limit of the relativistic one, that would be the correction to the interaction energy of the bright soliton constituents. And if we take into account the rest energy which is mN, we deduce

$$E^{\rm rel.}(N) = mN\left[1 - \frac{1}{384}\alpha_{\rm coll.}^2\left(1 - \left(5 + \frac{16\pi^2}{15}\right)\frac{1}{N} + \mathcal{O}(1/N^2)\right)\right]. \tag{63}$$

In the end we also can explicitly restore $\hbar$ recalling the Lagrangian

$$\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|^2 + \frac{\hbar^2\bar{\lambda}}{8}|\psi|^4 \tag{64}$$

where $\bar{\lambda} = \lambda/m^2$ is so-called scattering length, which is $\hbar$-independent. Also, we must take into account that mass of a single quanta is proportional to $\hbar$, namely, $m = \bar{m}\hbar$, where $\bar{m}$ is the frequency which is classical quantity. Hence, we get that particle number scales inversely with $\hbar$

$$N = \frac{8\sqrt{2\omega}}{\sqrt{\bar{m}}\bar{\lambda}}\frac{1}{\hbar}.$$

Thus, one can see that mN is a $\hbar$-independent combination as well as collective coupling $\alpha_{\rm coll.}$ and $1/N$ correction to the energy actually stands for the $\mathcal{O}(\hbar)$ correction as $\hbar$ is encoded in the particle number.

VI. CONCLUSION AND OUTLOOK

To sum up, I have performed an analytical computation of quantum fluctuations in the background of the bright soliton (34), (35), (36), which is the extended solution having finite energy in Schrödinger field theory in $1+1$ dimensions with attractive self-interaction. First, I have inverted the operator of quantum fluctuations or in other words computed the 2-point Green's function of quantum fluctuations in the background of the bright soliton. This propagator appeared to have poles in the continuum spectrum $E_p = \omega + p^2/(2m)$. The gap ω results from the constraint imposed by fixed particle number in the system and shows that if one excites a particle, the energy equal to the gap ω must be invested in order to tear the particle apart from the bright soliton. Excitation of a single particle changes frequency as well, but only by amount of energy regulated by $1/N$ corrections. Therefore, one must keep track that the amount of quanta involved in the scattering process should be small compared to N and if one studies some scattering process of m particles this scattering happens in the background of the bright soliton with $N - m$ quanta in it. Then I computed quantum corrections to the energy of this solution (62). One can see that quantum corrections are controlled by $1/N$-expansion and indeed small for $N \gg 1$.

In the end I would like to mention one more interesting observation. One can see that there are no integrable oscillating modes, this follows from the fact that only the continuum spectrum solves the eigenvalue problem (25) and all the others modes, which make up to completeness, are actually just combinations of zero-modes of the bright soliton. This suggest analogy with the Sine-Gordon soliton and poses a question, if there is similar factorization of S-matrix for this model as it happens for the Sine-Gordon model, and if there are objects dual to bright solitons.

Another thing I would like to mention is that it seems possible to use the result of this article to quantize actual relativistic soliton by deriving a proper nonrelativistic limit and using solutions deduced in this work. This could possibly allow us to compute quantum corrections for the relativistic soliton as well but in the limit of small ω/m.

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APPENDIX: "EIGENFUNCTIONS" AND COMPLETENESS RELATION

We have two linear differential operators inside the fluctuation determinant, which are

$$\hat{R} = -\frac{1}{2m}\partial_x^2 - 2\omega\,\mathrm{sech}^2(\sqrt{2m\omega}x) + \omega$$
$$\hat{L} = -\frac{1}{2m}\partial_x^2 - 6\omega\,\mathrm{sech}^2(\sqrt{2m\omega}x) + \omega. \tag{A1}$$

These operators enjoy a set of function, which we will list below. First part of the "spectrum" consist of $L^2(\mathbb{R})$ functions

$$\begin{cases} l_1(x) = (2m\omega)^{\frac{1}{4}}(1-\sqrt{2m\omega}x\tanh(\sqrt{2m\omega}x))\mathrm{sech}(\sqrt{2m\omega}x) \\ l_2(x) = (2m\omega)^{\frac{1}{4}}\tanh(\sqrt{2m\omega}x)\mathrm{sech}(\sqrt{2m\omega}x) \\ r_1(x) = (2m\omega)^{\frac{1}{4}}\mathrm{sech}(\sqrt{2m\omega}x) \\ r_2(x) = (2m\omega)^{\frac{1}{4}}\sqrt{2m\omega}x\,\mathrm{sech}(\sqrt{2m\omega}x) \end{cases}, \tag{A2}$$

which obey the following normalization conditions

$$\int dx r_i(x) l_j^*(x) = \delta_{ij}, \qquad i,j = 1,2 \tag{A3}$$

and fulfill eigenvalue problem

$$\begin{pmatrix} \hat{L} & \gamma_i \\ -\gamma_i & \hat{R} \end{pmatrix}\begin{pmatrix} l_i(x) \\ r_i(x) \end{pmatrix} = 0, \tag{A4}$$

with eigenvectors and eigenvalues

$$\vec{h}_1 = \begin{pmatrix} l_1(x) \\ r_2(x) \end{pmatrix} \quad \text{with} \quad \gamma = 2\omega, \quad \text{and}$$
$$\vec{h}_2 = \begin{pmatrix} l_2(x) \\ r_1(x) \end{pmatrix} \quad \text{with} \quad \gamma = 0 \tag{A5}$$

Also there are two functions comprising the continuous spectrum, namely,

$$\begin{cases} l_p(x) = \frac{e^{ipx}}{\sqrt{2\pi}(\omega+\frac{p^2}{2m})}\left(-\omega - \frac{p^2}{2m} - \sqrt{2}i\sqrt{\frac{\omega}{m}}p\tanh(\sqrt{2m\omega}x) + 2\omega\tanh(\sqrt{2m\omega}x)^2\right) \\ r_p(x) = \frac{e^{ipx}}{\sqrt{2\pi}(\omega+\frac{p^2}{2m})}\left(\omega - \frac{p^2}{2m} - \sqrt{2}i\sqrt{\frac{\omega}{m}}p\tanh(\sqrt{2m\omega}x)\right) \end{cases}, \tag{A6}$$

which satisfy a different eigenvalue problem

$$\begin{pmatrix} -\hat{L} & \omega+\frac{p^2}{2m} \\ \omega+\frac{p^2}{2m} & -\hat{R} \end{pmatrix}\begin{pmatrix} l_p(x) \\ r_p(x) \end{pmatrix} = 0, \tag{A7}$$

and obey momentum delta normalization

$$\int dx r_p(x) l_k^*(x) = 2\pi\delta(p-k). \tag{A8}$$

One can check that functions (A2) and (A6) make up completeness relation

$$\int_{-\infty}^{+\infty} dp r_p(x) l_p^*(y) + \sum_{j=1}^{2} r_j(x) l_j^*(y) = \delta(x-y) \tag{A9}$$

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