Role of the subsidiary fields in the momentum and angular momentum in covariantly quantized QED and QCD

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The covariant quantization of QED and QCD requires the introduction of subsidiary gauge-fixing and ghost fields and it is crucial to understand the role of these in the energy-momentum tensor, and in the momentum and angular momentum operators. These issues were discussed by E. Leader [Phys. Rev. D **83**, 096012 (2011)], and certain key results from this study, namely that the subsidiary and ghost fields do not contribute to the physical matrix elements of the canonical or Bellinfante momentum and angular momentum, were utilized in the major review of angular momentum by E. Leader and C. Lorce [Phys. Rep., **541**, 163 (2014)]. B. Damski [Phys. Rev. D **104**, 085003 (2021)] has rightly criticized as incorrect the derivation of these results for the QED case by Leader (2011) and has given explicit expressions for the contribution of the subsidiary fields to the physical matrix elements of the QED momentum and angular momentum. We show, however, that the key results of Leader (2011), mentioned above, which are utilized by Leader and Lorce (2014), are unaffected by Damski's criticism and that his expressions for the contribution from the subsidiary fields, in fact, vanish.

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I. INTRODUCTION

It is well known that the covariant quantization of QED and QCD, i.e., in which the photon vector potential $A^\mu(x)$ and the gluon vector potential A_a^μ transform as genuine Lorentz 4-vectors, is a nontrivial task [1–4] involving the introduction of a scalar gauge-fixing field (Gf) in QED and both a gauge-fixing field and Faddeev-Popov ghosts fields (Gf + Gh) in QCD. In both QED and QCD the expressions for the linear and angular momentum *operators* include terms involving all these fields and their role in physical matrix elements of these operators, for the QCD case, is discussed in the proofs of the renormalizability of QCD [5]. See also Sec. 14.6 of [6].

As explained in detail in [7] it is necessary to work in an indefinite-metric space i.e., one in which the "length" or norm of a vector can be either positive or negative and the definition of the "physical states" with positive norm has to be specified with care. It is also necessary to specify the condition for an operator to represent a physically measurable quantity i.e., to be an "observable."

The situation is further complicated by the fact that it is possible to deal with different versions of the energy-momentum tensor $t^{\mu\nu}$, of which the two most important are the Canonical version $t^{\mu\nu}_{\text{can}}$ which follows from Noether's theorem and the Bellinfante version $t^{\mu\nu}_{\text{bel}}$ which is symmetric under $(\mu \leftrightarrow \nu)$ and which differs from $t^{\mu\nu}_{\text{can}}$ by a divergence term, as will be spelled out in detail below. Both of these versions of the energy-momentum tensor are conserved quantities. Based on these one can define, in the standard way, the momentum operators P^μ_{can} and P^μ_{bel} and the angular momentum operators J^i_{can} and J^i_{bel}. Because the energy-momentum tensors are related by a divergence, one can show that for the matrix elements between any normalizable physical states, $\langle\Phi|P^\mu_{\text{bel}}|\Psi\rangle = \langle\Phi|P^\mu_{\text{can}}|\Psi\rangle$ and $\langle\Phi|J^i_{\text{bel}}|\Psi\rangle = \langle\Phi|J^i_{\text{can}}|\Psi\rangle$.

Of key physical interest in QCD is the question of the fraction of the momentum and angular momentum carried by quarks and gluons in a hadron, with analogous questions about electrons and photons in QED. Clearly to answer these questions one has to know the contributions of the subsidiary fields to the physical matrix elements of the above operators. In all phenomenological papers dealing with the QCD case it is *assumed, without comment*, that the contribution from the subsidiary fields is zero. That the latter is true for the Bellinfante case in QCD follows from the proof given in Joglekar and Lee and discussed by Collins in the above-cited works, where it is shown that the physical matrix elements of a Becchi, Rouet, Stora, Tyutin (BRST)-exact operator vanish. However, this argument does not apply directly to the canonical case. Moreover, BRST transformations are rarely introduced in the QED case, so I attempted in [7] to

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give a simple proof for the QED case, based on the direct demonstration that for physical matrix elements in QED $\langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf)|\Phi\rangle = \langle\Phi'|t^{\mu\nu}_{\rm can}(Gf)|\Phi\rangle = 0$. Since the BRST approach does not work for the canonical case, I also applied my "simple proof" to show that for QCD $\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle = 0$.

Damski [8] has pointed out that this "proof," for both the canonical and Bellinfante cases in QED, is wrong, because it assumes that the physical states alone form a complete set and thus uses $1 = \sum|\Phi\rangle\langle\Phi|$, which is an incorrect "resolution of the identity" because it leaves out the states of negative norm. Although he does not comment on QCD, Damski's argument shows that the "proof" that $\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle = 0$ in [7] is also incorrect.

This criticism seems to imply that the conclusion reached in [7], that the subsidiary fields do *not* contribute to the physical matrix elements of the Canonical and Bellinfante versions of the momentum and angular momentum in QED and QCD, is false, but as will be shown, this implication is wrong. Moreover, for the QED case, where Damski presents explicit expressions for the contributions of the subsidiary fields, we shall show that these do, in fact, vanish.

We shall show the following:

(a) While, indeed, for the canonical case

$$\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf)|\Phi\rangle \neq 0 \quad\text{and}\quad \langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle \neq 0, \tag{1}$$

on the contrary, for the Bellinfante case, as claimed in [7] and as expected on the basis of the general argument in [5,6]

$$\langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf)|\Phi\rangle = 0 \quad\text{and}\quad \langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf+Gh)|\Phi\rangle = 0. \tag{2}$$

(b) Despite Eq. (1), one has

$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf)|\Phi\rangle = \partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle = 0 \tag{3}$$

and obviously an analogous result for the Bellinfante case as a consequence of (2), which are key results utilized in the major review [9] of the "angular momentum controversy."

(c) Despite Eq, (1) it turns out that the subsidiary fields do not contribute to the physical matrix elements of the canonical version of the momentum or angular momentum. This crucial result, as mentioned above, is normally assumed without comment in phenomenological papers on QCD.

We shall also comment briefly on the important difference between gauge invariance and gauge independence, which was inadequately explained in [7].

II. PHYSICAL MATRIX ELEMENTS OF THE BELLINFANTE ENERGY-MOMENTUM TENSOR

We shall show, as claimed in [7], that the physical matrix elements of the gauge-fixing and ghost contributions $t^{\mu\nu}_{\rm bel}(Gf+Gh)$ in QCD and the gauge-fixing contribution $t^{\mu\nu}_{\rm bel}(Gf)$ in QED, vanish. The proof given for the QED case in [7], as pointed out by Damski [8], is incorrect, but the result is actually true. Surprisingly it turns out that the QCD case is simpler to deal with than the QED case, which can be derived as a special case.

A. Quantum chromodynamics

The pure quark-gluon Lagrangian $\mathcal{L}_{qG}$ is

$$\mathcal{L}_{qG} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2}\bar\psi^l[\delta_{lm}i(\overrightarrow{\not\partial} - \overleftarrow{\not\partial}) - 2gt^a_{lm}\not A^a]\psi^m. \tag{4}$$

In order to quantize the theory covariantly one has to introduce both a gauge-fixing field $B(x)$ and Fadeev-Popov anticommuting fermionic ghost fields $c(x), \bar c(x)$. The Kugo-Ojima Lagrangian [10] for the covariantly quantized theory is then

$$\mathcal{L} = \mathcal{L}_{qG} + \mathcal{L}_{Gf+Gh} \tag{5}$$

where

$$\mathcal{L}_{Gf+Gh} = -i(\partial^\mu\bar c^a)D^{ab}_\mu c_b - (\partial^\mu B^a)A^a_\mu + \frac{\mathrm{a}}{2}B^aB^a. \tag{6}$$

The physical states $|\Psi\rangle$ are defined by the subsidiary conditions

$$Q_B|\Psi\rangle = 0, \tag{7}$$

$$Q_c|\Psi\rangle = 0, \tag{8}$$

where the conserved, Hermitian charge Q_B is given by

$$Q_B = \int d^3x[B^a\overleftrightarrow{\partial}_0c^a - gB^af_{abc}A^b_0c^c - i(g/2)(\partial_0\bar c^a)f_{abc}c^bc^c], \tag{9}$$

and the conserved charge Q_c

$$Q_c = \int d^3x[\bar c^a\overleftrightarrow{\partial}_0c^a - g\bar c^af_{abc}A^b_0c^c] \tag{10}$$

"measures" the *ghost number*

$$i[Q_c, \phi] = N\phi \tag{11}$$

where $N = 1$ for $\phi = c, -1$ for $\phi = \bar c$ and 0 for all other fields.

The Bellinfante energy-momentum tensor is

$$t^{\mu\nu}_{\rm bel} = t^{\mu\nu}_{\rm bel}(qG) + t^{\mu\nu}_{\rm bel}(Gf+Gh) \tag{12}$$

where

$$t^{\mu\nu}_{\rm bel}(qG) = \frac{i}{4}[\bar{\psi}_l \gamma^\mu \overleftrightarrow{D}^\nu \psi_l + (\mu \leftrightarrow \nu)] - G^{\mu\beta}_a G^\nu_{a\beta} - g^{\mu\nu}\mathcal{L}_{qG} \quad (13)$$

and the gauge-fixing and ghost terms are given by

$$t^{\mu\nu}_{\rm bel}(Gf+Gh) = -(A^\mu_a \partial^\nu B_a + A^\nu_a \partial^\mu B_a) - i[(\partial^\mu \bar{c}_a) D^\nu_{ab} c_b + (\partial^\nu \bar{c}_a) D^\mu_{ab} c_b] - g^{\mu\nu}\mathcal{L}_{Gf+Gh}. \quad (14)$$

This can be rewritten [4] as an anticommutator with Q_B

$$t^{\mu\nu}_{\rm bel}(Gf+Gh) = -\left\{ Q_B, \left((\partial^\mu \bar{c}_a) A^\nu_a + (\partial^\nu \bar{c}_a) A^\mu_a + g^{\mu\nu}\left[\frac{\mathsf{a}}{2} \bar{c}_a B_a - (\partial^\rho \bar{c}_a) A^a_\rho \right] \right) \right\}. \quad (15)$$

It follows from Eqs. (7) and (15) that $t^{\mu\nu}_{\rm bel}(Gf+Gh)$ does not contribute to physical matrix elements i.e.,

$$\langle \Phi' | t^{\mu\nu}_{\rm bel}(Gf+Gh) | \Phi \rangle = 0 \quad (16)$$

so that

$$\langle \Phi' | t^{\mu\nu}_{\rm bel,QCD} | \Phi \rangle = \langle \Phi' | t^{\mu\nu}_{\rm bel}(qG) | \Phi \rangle \quad (17)$$

and hence that the subsidiary fields do not contribute to the QCD expressions for either $P^\mu_{\rm bel}$ or $\boldsymbol{J}_{\rm bel}$ i.e.,

$$\langle \Phi | P^\mu_{\rm bel} | \Phi \rangle \equiv \int d^3x \langle \Phi | t^{0\mu}_{\rm bel} | \Phi \rangle = \int d^3x \langle \Phi | t^{0\mu}_{\rm bel}(qG) | \Phi \rangle = \langle \Phi | P^\mu_{\rm bel}(qG) | \Phi \rangle \quad (18)$$

and

$$\langle \Phi | J^i_{\rm bel} | \Phi \rangle \equiv \frac{1}{2}\epsilon^{ijk} \int d^3x \langle \Phi | x^j t^{0k}_{\rm bel} - x^k t^{0j}_{\rm bel} | \Phi \rangle = \frac{1}{2}\epsilon^{ijk} \int d^3x \langle \Phi | x^j t^{0k}_{\rm bel}(qG) - x^k t^{0j}_{\rm bel}(qG) | \Phi \rangle = \langle \Phi | J^i_{\rm bel}(qG) | \Phi \rangle. \quad (19)$$

B. Quantum electrodynamics

The most general covariantly quantized version of QED is given by the Lautrup-Nakanishi Lagrangian density [1,2], which is a combination of the classical Lagrangian ($Clas$) and a gauge-fixing part

$$\mathcal{L} = \mathcal{L}_{Clas} + \mathcal{L}_{Gf} \quad (20)$$

where

$$\mathcal{L}_{Clas} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}[\bar{\psi}(i\partial\!\!\!/ - m + eA\!\!\!/)\psi + \text{H.c.}] \quad (21)$$

and

$$\mathcal{L}_{Gf} = -\partial_\mu B(x).A^\mu(x) + \frac{\mathsf{a}}{2} B^2(x) \quad (22)$$

where $B(x)$ is the gauge-fixing field and the parameter a determines the structure of the photon propagator and is irrelevant for the present discussion.[1]

The *physical* states $|\Phi\rangle$ of the theory are defined to satisfy

$$B^{(+)}(x)|\Phi\rangle = 0 \quad (23)$$

where

$$B(x) = B^{(+)}(x) + B^{(-)}(x) \quad (24)$$

with $B^{(\pm)}(x)$ the positive/negative frequency parts of $B(x)$.

For the conserved Bellinfante density one finds,

$$t^{\mu\nu}_{\rm bel} = \theta^{\mu\nu}_{\rm bel} + t^{\mu\nu}_{\rm bel}(Gf) \quad (25)$$

where $\theta^{\mu\nu}_{\rm bel}$, which is referred to as the classical energy-momentum tensor density, is

$$\theta^{\mu\nu}_{\rm bel} = \frac{i}{4}\bar{\psi}(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu)\psi - F^{\mu\beta}F^\nu_\beta - g^{\mu\nu}\mathcal{L}_{Clas}, \quad (26)$$

where $\overleftrightarrow{D}^\nu = \overleftrightarrow{\partial}^\nu - 2ieA^\nu$, and

$$t^{\mu\nu}_{\rm bel}(Gf) = -A^\nu \partial^\mu B - A^\mu \partial^\nu B - g^{\mu\nu}\mathcal{L}_{Gf}. \quad (27)$$

As explained in Kugo-Ojima [10] the QED expressions can be obtained from the QCD case by putting the structure constants to zero and suppressing the color group labels, in which case the gauge-fixing field B and the ghost fields c and $\bar{c}$ become free fields. The expression Eq. (9) for the charge Q_B then becomes, in terms of creation and annihilation operators,

$$Q_B = i \int \frac{d^3k}{(2\pi)^3 2E_k}[c^\dagger_k B_k - B^\dagger_k c_k]. \quad (28)$$

Because the Fadeev-Popov ghosts are here free, the state vector space $\mathcal{V}$ can be decomposed into a direct product $\mathcal{V} = \mathcal{V}_{\rm phys} \otimes \mathcal{V}_{FP}$ where $\mathcal{V}_{\rm phys}$ is the usual QED physical state vector space. Moreover the ghosts are redundant, so that one can work in the sector containing neither c nor $\bar{c}$ ghosts i.e., $\mathcal{V}_{\rm phys} \otimes |0\rangle_{FP}$. Hence the physical states in QED can be taken to be $|\Phi\rangle \otimes |0\rangle_{FP}$. Using this and (28), Eq. (7) can then be shown to imply (23). Hence

[1]The case $\mathsf{a} = 1$ corresponds to the Gupta-Bleuler approach (see e.g., [11]) based on the Fermi Lagrangian. Note also that in order to conform to the conventions used in the QCD case, the expression for $\mathcal{L}_{Gf}$ differs from Nakanishi-Lautrup by a 4-divergence.

$$\langle\Phi'|t^{\mu\nu}_{\rm bel,QED}(Gf)|\Phi\rangle = _{FP}\langle 0|\otimes\langle\Phi'|t^{\mu\nu}_{\rm bel,Free}(Gf+Gh)|\Phi\rangle\otimes|0\rangle_{\rm FP}=0 \quad (29)$$

where $t^{\mu\nu}_{\rm bel,Free}(Gf+Gh)$ is $t^{\mu\nu}_{\rm bel,QCD}(Gf+Gh)$ in which the structure constants are put to zero and the gauge-fixing and ghost fields are free. Thus

$$\langle\Phi'|t^{\mu\nu}_{\rm bel,QED}|\Phi\rangle = \langle\Phi'|\theta^{\mu\nu}_{\rm bel}|\Phi\rangle \quad (30)$$

and hence, as in QCD, the subsidiary fields do not contribute to the QED expressions for the physical matrix elements of either $P^{\mu}_{\rm bel}$ or $\boldsymbol{J}_{\rm bel}$.

C. QED: Direct study of subsidiary fields

The argument showing that the gauge-fixing field in QED does not contribute to the physical expectation value of the Bellinfante version of the energy-momentum tensor and hence does not contribute to the expectation values of the Bellinfante versions of the momentum and angular momentum, based on the QCD case, is rather abstract, so we here show that the concrete expression for the gauge-fixing contribution to the Bellinfante angular momentum, given in Damski's paper [8], which deals only with the free electromagnetic case, actually vanishes i.e., we shall show directly that

$$\langle\Phi|\mathbf{J}_{\rm bel}(\rm Gf)|\Phi\rangle = 0 \quad (31)$$

where, in Damski's notation

$$J^i_{\rm bel}(\rm Gf) = J^i_{\rm div} + J^i_{\xi}. \quad (32)$$

Consider

$$\langle\Phi|J^i_{\rm bel}(\rm Gf)|\Phi\rangle = \langle\Phi|J^i_{\rm div}+J^i_{\xi}|\Phi\rangle = \int d^3z\epsilon^{imn}\langle\Phi|z^m\mathcal{I}_n(z) - z^n\mathcal{I}_m(z)|\Phi\rangle \quad (33)$$

with, following Damski,

$$\mathcal{I}_n(z) = A_n\partial_jF_{0j} + (\partial\cdot A)\partial_nA_0. \quad (34)$$

Defining

$$B(z) \equiv -\partial\cdot A \quad (35)$$

and using the fact that in Damski the fields are free, one obtains

$$\mathcal{I}_n(z) = A_n\partial_0B - B\partial_nA_0. \quad (36)$$

We split the fields into their positive and negative frequency parts

$$B(z) = B^{(+)} + B^{(-)}, \qquad B^{(-)} = [B^{(+)}]^{\dagger} \quad \text{and}$$
$$A_\mu(z) = A^{(+)}_\mu + A^{(-)}_\mu, \qquad A^{(-)}_\mu = [A^{(+)}_\mu]^{\dagger} \quad (37)$$

with

$$A^{(+)}_\mu(z) = \int[dk']\sum_{\sigma=0}^{\sigma=3}\epsilon_\mu(\mathbf{k}',\sigma)c_{\mathbf{k}'\sigma}e^{-ik'\cdot z} \quad (38)$$

where the ϵ_μ are polarization vectors and the $c_{\mathbf{k}'\sigma}$ are annihilation operators and we use the shorthand

$$[dk'] \equiv \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2\omega_{k'}}}. \quad (39)$$

In Damski's notation one has

$$B^{(+)}(z) = -i\int[dk]\omega_kL_{\mathbf{k}}\omega e^{-ik\cdot z} \quad (40)$$

where

$$L_{\mathbf{k}} = c_{\mathbf{k}3} - c_{\mathbf{k}0}. \quad (41)$$

The physical states, as usual, are defined to satisfy

$$B^{(+)}(z)|\Phi\rangle = 0, \qquad \langle\Phi|B^{(-)}(z) = 0. \quad (42)$$

Using these and the fact that the commutators $[B^{(+)},A^{(+)}_\mu] = [B^{(-)},A^{(-)}_\mu] = 0$, it follows that

$$\begin{aligned}\langle\Phi|\mathcal{I}_n(z)|\Phi\rangle &= \langle\Phi|A_n\partial_0B^{(-)} - B^{(+)}\partial_nA_0|\Phi\rangle\\ &= \langle\Phi|[A^{(+)}_n,\partial_0B^{(-)}] - [B^{(+)},\partial_nA^{(-)}_0]|\Phi\rangle\\ &= \langle\Phi|\Phi\rangle\{[A^{(+)}_n,\partial_0B^{(-)}] - [B^{(+)},\partial_nA^{(-)}_0]\},\end{aligned} \quad (43)$$

the last step following since the commutators are c-numbers. The relevant commutators and polarization vectors are

$$\begin{aligned}[L_{\boldsymbol{k}},c^{\dagger}_{\boldsymbol{k}'0}] &= -[c_{\boldsymbol{k}0},c^{\dagger}_{\boldsymbol{k}'0}] = \delta^3(\boldsymbol{k}'-\boldsymbol{k}),\\ [L_{\boldsymbol{k}},c^{\dagger}_{\boldsymbol{k}'3}] &= [c_{\boldsymbol{k}3},c^{\dagger}_{\boldsymbol{k}'3}] = \delta^3(\boldsymbol{k}'-\boldsymbol{k})\end{aligned} \quad (44)$$

and

$$\epsilon^\mu(\boldsymbol{k},0) = (1,\mathbf{0}), \qquad \epsilon^\mu(\boldsymbol{k},3) = (0,\boldsymbol{k}/\omega_k). \quad (45)$$

Substituting Eqs. (38), (40), (44), and (45) into Eq. (43), we find, after some labor, that

$$[A^{(+)}_n,\partial_0B^{(-)}] - [B^{(+)},\partial_nA^{(-)}_0] = 0 \quad (46)$$

and thus that in QED

$$\langle \Phi | J^i_{\mathrm{bel,QED}}(\mathrm{Gf}) | \Phi \rangle = 0, \tag{47}$$

in agreement with the more general derivation based on the QCD case.

D. Bellinfante summary

To summarize, despite the incorrect proof for the QED case given in [7], we have shown explicitly that the physical matrix elements of the gauge-fixing and ghost contributions to the Bellinfante version of the energy-momentum tensor, in both QED and QCD, actually do vanish. Hence the only contributions to the momentum P^μ_{bel} and angular momentum $\boldsymbol{J}_{\mathrm{bel}}$ are from photons and electrons in the QED case and from quarks and gluons in QCD.

This latter property was thus correctly utilized in the review paper [9]. Also, in writing down the most general structure for the matrix elements of $\langle \Phi' | t^{\mu\nu}_{\mathrm{bel}}(qG) | \Phi \rangle$ in [9], use was made of the claim that

$$\partial_\mu \langle \Phi' | t^{\mu\nu}_{\mathrm{bel}}(qG) | \Phi \rangle = 0. \tag{48}$$

This follows because the total $t^{\mu\nu}_{\mathrm{bel}}$ is a conserved operator and, via (17),

$$\partial_\mu \langle \Phi' | t^{\mu\nu}_{\mathrm{bel}}(qG) | \Phi \rangle = \partial_\mu \langle \Phi' | t^{\mu\nu}_{\mathrm{bel}} | \Phi \rangle = 0. \tag{49}$$

Note also that this justifies the results in several papers in the literature, e.g., Ji [12–14], Jaffe and Manohar [15], Bakker, Leader and Trueman (BLT) [16] and Wakamatsu [17,18], where the general structure of the physical matrix elements of $t^{\mu\nu}_{\mathrm{bel}}(qG)$ (or its QED analog) is derived under the *unstated assumption* that Eq. (49) holds.

III. PHYSICAL MATRIX ELEMENTS OF THE CANONICAL ENERGY-MOMENTUM TENSOR

In [7] it was claimed that the physical matrix elements of the gauge-fixing and ghost contributions $t^{\mu\nu}_{\mathrm{can}}(Gf + Gh)$ in QCD and the gauge-fixing contribution $t^{\mu\nu}_{\mathrm{can}}(Gf)$ in QED, vanish. The result given for the QED case in [7], as pointed out by Damski [8], is wrong because of an incorrect use of the "resolution of the identity" for a space with an indefinite metric, and, although not discussed by Damski, also the QCD result is wrong for the same reason.

Thus, in contrast to the Bellinfante case and contrary to the claims made in [7], in QCD

$$\langle \Phi' | t^{\mu\nu}_{\mathrm{can}}(Gf + Gh) | \Phi \rangle \neq 0 \tag{50}$$

and in QED

$$\langle \Phi' | t^{\mu\nu}_{\mathrm{can}}(Gf) | \Phi \rangle \neq 0. \tag{51}$$

We shall analyze the consequences of these for the QCD case. Completely analogous arguments hold for the case of QED.

There are two questions which have to be answered, given (50):

(i) Is the analysis of the general structure of the physical matrix elements of the quark-gluon $t^{\mu\nu}_{\mathrm{can}}(qG)$ given in [9] correct?

(ii) Do the subsidiary fields contribute to the physical matrix elements of the canonical momentum P^μ_{can} and angular momentum $\boldsymbol{J}_{\mathrm{can}}$?

A. Structure of the physical matrix elements of the quark-gluon $t^{\mu\nu}_{\mathrm{can}}(qG)$

The Bellinfante and canonical energy-momentum tensors differ from each other by a divergence term of the following form:

$$t^{\mu\nu}_{\mathrm{can}} = t^{\mu\nu}_{\mathrm{bel}} - \partial_\lambda G^{\lambda\mu\nu}, \tag{52}$$

where the so-called *superpotential* reads

$$G^{\lambda\mu\nu} = \frac{1}{2}(M^{\lambda\mu\nu}_{\mathrm{spin}} + M^{\mu\nu\lambda}_{\mathrm{spin}} + M^{\nu\mu\lambda}_{\mathrm{spin}}), \tag{53}$$

and, crucially, is antisymmetric with respect to its first two indices

$$G^{\lambda\mu\nu} = -G^{\mu\lambda\nu}. \tag{54}$$

The M_{spin} involves a sum over all fields and is given in terms of the Lagrangian by

$$M^{\mu\nu\rho}_{\mathrm{spin}}(x) = -i \sum_{\text{all fields}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} (\Sigma^{\nu\rho})_r{}^s \phi_s(x). \tag{55}$$

where $(\Sigma^{\mu\nu})_r{}^s = -(\Sigma^{\nu\mu})_r{}^s$ is an operator related to the spin of the field. For example, for particles with the most common spins, one has

$$\text{spin} - 0 \text{ particle} \qquad \phi(x) \qquad (\Sigma^{\mu\nu})_r{}^s = 0, \tag{56}$$

$$\text{spin} - 1/2 \text{ Dirac particle} \qquad \psi_r(x) \qquad (\Sigma^{\mu\nu})_r{}^s = \frac{1}{2}(\sigma^{\mu\nu})_r{}^s, \tag{57}$$

$$\text{spin} - 1 \text{ particle} \qquad A_\alpha(x) \qquad (\Sigma^{\mu\nu})_\alpha{}^\beta = i(\delta^\mu_\alpha g^{\nu\beta} - \delta^\nu_\alpha g^{\mu\beta}). \tag{58}$$

Examination of the structure of the Lagrangians in Eqs. (4), (6) shows that since $\mathcal{L}_{qG}$ does not contain any gauge-fixing or ghost fields and $\mathcal{L}_{Gf+Gh}$ does not contain any derivatives of A^μ_a we may write

$$M^{\mu\nu\rho}_{\mathrm{spin}}(x) = M^{\mu\nu\rho}_{\mathrm{spin}}(x)|_{qG} + M^{\mu\nu\rho}_{\mathrm{spin}}(x)|_{Gf+Gh} \tag{59}$$

and thus in Eq. (52)

$$G^{\lambda\mu\nu} = G^{\lambda\mu\nu}|_{qG} + G^{\lambda\mu\nu}|_{Gf+Gh} \tag{60}$$

so that separately

$$t^{\mu\nu}_{\rm can}(qG) = t^{\mu\nu}_{\rm bel}(qG) - \partial_\lambda G^{\lambda\mu\nu}|_{qG} \tag{61}$$

and

$$t^{\mu\nu}_{\rm can}(Gf+Gh) = t^{\mu\nu}_{\rm bel}(Gf+Gh) - \partial_\lambda G^{\lambda\mu\nu}|_{Gf+Gh}. \tag{62}$$

Hence,

$$\partial_\mu t^{\mu\nu}_{\rm can}(Gf+Gh) = \partial_\mu t^{\mu\nu}_{\rm bel}(Gf+Gh) - \partial_\mu\partial_\lambda G^{\lambda\mu\nu}|_{Gf+Gh} = \partial_\mu t^{\mu\nu}_{\rm bel}(Gf+Gh) \quad \text{via} \quad (54) \tag{63}$$

and thus for the physical matrix elements, using (16),

$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle = \partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf+Gh)|\Phi\rangle = 0. \tag{64}$$

Finally, then, since the total $t^{\mu\nu}_{\rm can}$ is a conserved operator, we obtain the key result

$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(qG)|\Phi\rangle = \partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}|\Phi\rangle = 0 \tag{65}$$

and the analysis of the general structure of the physical matrix elements of the quark-gluon $t^{\mu\nu}_{\rm can}(qG)$ given in [9], which relied on this property, is correct.

B. Contribution of the subsidiary fields to the physical matrix elements of the canonical momentum $P^\mu_{\rm can}$ and angular momentum $\boldsymbol{J}_{\rm can}$

From Eqs. (18) and (61)

$$\langle\Phi|P^\mu_{\rm bel}|\Phi\rangle = \int d^3x\langle\Phi|t^{0\mu}_{\rm bel}(qG)|\Phi\rangle = \int d^3x\langle\Phi|t^{0\mu}_{\rm can}(qG)|\Phi\rangle + \int d^3x\langle\Phi|\partial_\lambda G^{\lambda 0\mu}|_{qG}|\Phi\rangle. \tag{66}$$

By the antisymmetry property (54) the last term is actually a three-dimensional divergence $\partial_i G^{i0\mu}|_{qG}$, yielding a surface term at infinity, which, as always, is assumed to vanish. Thus

$$\langle\Phi|P^\mu_{\rm bel}|\Phi\rangle = \int d^3x\langle\Phi|t^{0\mu}_{\rm can}(qG)|\Phi\rangle = \langle\Phi|P^\mu_{\rm can}(qG)|\Phi\rangle. \tag{67}$$

But

$$\langle\Phi|P^\mu_{\rm bel}|\Phi\rangle = \langle\Phi|P^\mu_{\rm can}|\Phi\rangle \tag{68}$$

so that, indeed,

$$\langle\Phi|P^\mu_{\rm can}|\Phi\rangle = \langle\Phi|P^\mu_{\rm can}(qG)|\Phi\rangle \tag{69}$$

and the subsidiary fields do not contribute to the physical matrix elements of $P^\mu_{\rm can}$. A similar argument shows that

$$\langle\Phi|J^i_{\rm can}|\Phi\rangle = \langle\Phi|J^i_{\rm can}(qG)|\Phi\rangle \tag{70}$$

and the subsidiary fields also do not contribute to the physical matrix elements of $J^i_{\rm can}$.

C. Canonical summary

To summarize, despite the incorrect proof given in [7] concerning the physical matrix elements of the gauge-fixing contributions to the canonical version of the energy-momentum tensor in QED and the gauge-fixing and ghost contributions to the canonical version of the energy-momentum tensor in QCD, the essential property used in writing down the most general structure for the QCD matrix elements $\langle\Phi'|t^{\mu\nu}_{\rm can}(qG)|\Phi\rangle$ and the QED matrix elements $\langle\Phi'|\Theta^{\mu\nu}_{\rm can}|\Phi\rangle$ in [9], namely, that

$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(qG)|\Phi\rangle = 0 \quad \text{and} \quad \partial_\mu\langle\Phi'|\Theta^{\mu\nu}_{\rm can}|\Phi\rangle = 0, \tag{71}$$

is correct.

Moreover, despite the incorrect derivation in [7], the only contributions to the physical matrix elements of the momentum $P^\mu_{\rm can}$ and angular momentum $\boldsymbol{J}_{\rm can}$ are from photons and electrons in the QED case and from quarks and gluons in QCD. This latter property was thus correctly utilized in the review paper [9].

IV. GAUGE INVARIANCE VS GAUGE INDEPENDENCE

Although not strictly relevant to Damski's criticism of [7] it will be useful to comment on a statement in the latter paper, which might be misleading. This concerns the rarely commented upon difference between gauge invariance and gauge independence, which is emphasized by Collins in Sec. 2.12 of [6] and also in Sec. 2.5.2 of [9]. We shall illustrate this in QED, but the same applies to QCD.

In canonically quantized QED one first chooses a gauge and then quantizes the theory by specifying the commutation rules which the operators must obey. One can then consider gauge transformations on, for example, the *operator* for the photon vector potential

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha(x) \tag{72}$$

where $\alpha(x)$ can be any reasonably behaved classical (c-number) function, but cannot be a general function of the operator A_μ, because if it was, the new A'_μ would no longer satisfy the required commutation relations. The above thus represent only a restricted class of gauge transformations.

A matrix element of an operator is gauge independent when its value does not depend on which gauge is used in calculating it i.e., does not depend on what method is used to fix the gauge of the classical theory before quantizing the theory. To yield a gauge independent matrix element an operator must necessarily be gauge invariant in the restricted sense above, but that is far from sufficient. The only way to test for gauge independence in the context of a canonically quantized theory is to calculate the quantity using the theory quantized from the start in different gauges.

This problem is alleviated when one uses the path integral formulation, since there one does not deal with operators; the A_μ are ordinary functions and one can handle general gauge transformations in which $\alpha(x)$ can depend on the fields. In this formulation the matrix element will be gauge independent if the function representing the operator is fully gauge invariant.

In Sec. VII of [7] it was shown that the matrix element of the projection of the photon or gluon spin onto the direction of motion is gauge invariant, in the above restricted sense, but it was not emphasized that this does not make the matrix element gauge independent. Indeed, Hoodbhoy and Ji [19] have calculated the difference between these matrix elements, evaluated in a type of axial gauge which has $A^+ = 0$ and in the Covariant gauge $\partial_\mu A^\mu = 0$, and shown that it is not zero.

V. CONCLUSIONS

Damski's criticism [8] of the *proof* given in [7] for certain properties of the physical matrix elements of the energy-momentum tensor $t^{\mu\nu}$ in QED is valid, and it is also applicable to QCD, so that, contrary to the assertions in [7], for the canonical case

$$\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf)|\Phi\rangle|_{\rm QED} \neq 0 \qquad \text{and}$$
$$\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle|_{\rm QCD} \neq 0. \tag{73}$$

Nonetheless the following crucial features for QED and QCD, utilized in the angular momentum review [9] are in fact correct:

(a) For the Bellinfante case, as claimed in [7] and as expected for the QCD case from the general arguments in [5,6]

$$\langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf)|\Phi\rangle|_{\rm QED} = 0 \qquad \text{and}$$
$$\langle\Phi'|t^{\mu\nu}_{\rm bel}(Gf+Gh)|\Phi\rangle|_{\rm QCD} = 0. \tag{74}$$

(b) Despite Eq. (73) one has for the canonical case,

$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf)|\Phi\rangle|_{\rm QED} = 0 \qquad \text{and}$$
$$\partial_\mu\langle\Phi'|t^{\mu\nu}_{\rm can}(Gf+Gh)|\Phi\rangle|_{\rm QCD} = 0. \tag{75}$$

And obviously an analogous result holds for the Bellinfante case as a consequence of (74).

(c) Despite Eq. (73) the subsidiary fields do not contribute to the physical matrix elements of either the canonical or Bellinfante versions of the momentum or angular momentum, a result which is normally *assumed without comment* in phenomenological papers on QCD.

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