Mock modularity and surface defects in topological $\mathcal{N}=2$ super Yang-Mills theory

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(Received 24 November 2021; accepted 8 January 2022; published 25 January 2022)

We study the effective low energy dynamics of the topologically twisted super Yang-Mills theory on compact four-manifolds that support surface defects where the gauge field becomes singular along certain directions. Following recent work on the topic of u-plane integrals in topological theories, we show that the integrand of the path integral can be expressed in terms of mock modular forms, which allows the evaluation of correlation functions using Stokes' theorem.

DOI: 10.1103/PhysRevD.105.026025

I. INTRODUCTION AND BACKGROUND

Studying the low energy dynamics of supersymmetric gauge theories is an important task which has the potential to lead to a better understanding of more realistic theories such as $\mathcal{N}=1$ SQCD and even standard QCD. Nevertheless, it is a difficult task indeed. Topologically twisted supersymmetric gauge theories on compact four-manifolds provide a powerful laboratory to explore such dynamics [1–7]. In this article we study the low energy dynamics of the topologically twisted version of the pure $\mathcal{N}=2$ super Yang-Mills theory (SYM) in four dimensions with gauge group of rank one, also known as Donaldson-Witten theory [8], with arbitrary 't Hooft fluxes on a compact four-manifold X that admits surface defects. As a result of the UV to IR flow, the original gauge group SU(2) or SO(3) breaks down to U(1) [9]. The order parameter $u=\frac{1}{16\pi^2}\langle \mathrm{Tr}(\phi^2)\rangle$, where ϕ is the scalar field of the theory, parametrizes the Coulomb branch $\mathcal{B}$ of the theory, also referred to as the u-plane. Topologically $\mathcal{B}$ corresponds to a Riemann sphere with three punctures at $u\to\infty$ (classical limit), as well as at $u=\pm\Lambda^2$, where Λ corresponds to the symmetry breaking scale of the theory. Without loss of generality, we can set $\Lambda=1$ for the rest of the article.

The objects of interest are the correlation functions $\langle \mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n\rangle \coloneqq \int[\mathcal{D}\mathcal{X}]e^{-S}\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n$, where $\mathcal{X}$ denotes the fields we integrate over, S the action, and the set $\{\mathcal{O}_i\}$, $i\in\{1,\ldots,n\}$, corresponds to the observables of the theory. In order for such correlators to be nontrivial the underlying four-manifold X must satisfy $b_2^+(X)\leq 1$ where $b_2^+(X)$ corresponds to the number of positive eigenvalues of the manifold's intersection form Q. The path integral of the theory on such a four-manifold X receives two types of contributions [3]: the Seiberg-Witten contribution Z_{SW} and the contribution from the u-plane integral Z_u,

$$Z=Z_{\mathrm{SW}}+Z_u. \tag{1}$$

In this article we focus on the contribution Z_u for compact four-manifolds X with $b_2^+(X)=1$ that admit surface defects S. The contribution Z_u reduces to a finite-dimensional integral over the zero modes of the fields of the theory when $b_2^+(X)=1$ [1,3] and by including nonperturbative corrections to the integrand, using the Seiberg-Witten theory, one can potentially evaluate the path integral precisely. Nevertheless, the presence of surface defects induces certain singularities for the gauge field A which exhibits singular behavior as it approaches the defect S.

Explicitly, a surface defect S corresponds to a codimension two compact manifold embedded in X with the property that the self-dual part of the field strength F of the gauge field A satisfies

$$F^+=2\pi\alpha_e\delta_S^+, \tag{2}$$

where for any 2-form l we denote by l^+ its self-dual component and by l^- its antiself-dual component such that $l=l^++l^-$. In Eq. (2) we interpret α_e as the electric charge of the surface defect. The dual magnetic charge is denoted as α_m and together they form a vector $\alpha=(\alpha_e,\alpha_m)^\top$. Additionally, δ_S corresponds to a delta function defined along the surface of S and topologically corresponds to the two-form dual of the class of S. We realize that the theory satisfies the usual antiself-dual equation $F^+=0$ except on the surface defect. We can overcome this complication by defining a gauge field $\mathcal{A}$ such that its field strength is [10]:

$$\mathcal{F}=F-2\pi\alpha_e\delta_S. \tag{3}$$

The presence of the surface defect induces a magnetic flux that contributes to the path integral of the theory with

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$\exp(\mathring{i}\alpha_m \int_S F)$, where $\mathring{i} \coloneqq \sqrt{-1}$. Mathematically, the gauge field in the presence of the surface defects corresponds to a connection on a parabolic gauge bundle [11,12]. In the following we will treat the theory as the standard low energy topologically twisted gauge theory and the effect of the surface defect will be masked by the transformation $(A, F) \to (\mathcal{A}, \mathcal{F})$ together with the contribution to the path integral mentioned above.

In a recent work [13–15] it was shown that the integrand of the path integral for theories without surface defects can be formulated in terms of mock modular forms [16,17] which have a tendency to appear frequently in Coulomb branch computations [18,19]. In this article we derive an explicit expression for the evaluation of the path integral of the theory, the so-called (ramified) u-plane integral, on a simply connected four-manifold X in the presence of a surface defect S with nontrivial flux, in terms of the modular completion $\hat{\Theta}$ of a mock modular form Θ that is related to the sum of the U(1) fluxes of the path integral of the theory. We further show that by taking the limit $\mathrm{vol}(S) \to 0$, we obtain the result for the theory without surface defects. The final result, Eq. (24), additionally provides the wall-crossing formula for the theory [3,13].

II. INGREDIENTS OF THE u-PLANE INTEGRAL WITH SURFACE DEFECTS

Both the UV Donaldson-Witten theory as well as its low energy effective theory contain a scalar nilpotent supercharge $\mathcal{Q}$ which is a result of the topological twist [8]. In the following we will take advantage of the fact that for any $\mathcal{Q}$-exact operator $\mathcal{O}$, $\{\mathcal{Q}, \mathcal{O}\} = 0$ [8]. This low energy effective theory is subject to electric-magnetic duality expressed by $\Gamma^0(4)$ transformations [3]. The field content of the theory contains a complex scalar field a, the gauge field $\mathcal{A}$, an auxiliary boson field D and three anticommuting Grassmann valued forms: a 0-form η, a 1-form ψ and a 2-form χ [3]. For the Lagrangian see [3,10]. The gauge coupling of the theory is $\tau = \tau_1 + \mathring{i}\tau_2 \in \mathbb{H}$, where $\mathbb{H}$ denotes the Poincaré upper half plane. The Lagrangian $\mathcal{L}$ and the supersymmetry algebra are well known [8,10]. We aim to evaluate the contribution Z_u from Eq. (1):

$$Z_u = \int [\mathcal{D}\mathcal{X}] \exp\left(-\int_X \mathcal{L}\right) \prod_i \mathcal{O}_i, \tag{4}$$

with the insertion of certain operators $\mathcal{O}_i$, where $\mathcal{D}\mathcal{X} \coloneqq \mathcal{D}a\mathcal{D}\bar{a}\mathcal{D}\mathcal{A}\mathcal{D}\eta\mathcal{D}\psi\mathcal{D}\eta\mathcal{D}D$. In topological field theories, insertion of $\mathcal{Q}$-exact observables must not alter the path integral and the correlation functions of the theory [8]. Following [13], we introduce the $\mathcal{Q}$-exact surface observable:

$$I(x) \coloneqq -\frac{1}{4\pi} \int_x \left\{\mathcal{Q}, \frac{d\bar{u}}{d\bar{a}} \chi\right\}, \tag{5}$$

where $x \in H_2(X)$. Such operators first appeared in [20] in the context of interpreting Witten type indices of bound states in string theory in terms of topological field theory integrals in various dimensions. Due to the fact that X is simply connected, there are no ψ field contributions and Eq. (5) takes the form:

$$I(x, S) = -\frac{1}{4\pi} \int_x \left(\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}^2} \eta \wedge \chi + \frac{\sqrt{2}}{4}\frac{d\bar{u}}{d\bar{a}} (\mathcal{F}^+ - D)\right). \tag{6}$$

In what follows, we modify the exponent in the path integral of Eq. (4) by adding this $\mathcal{Q}$-exact operator. The path integral (4) localizes to the zero modes of the fields [3]. Schematically, it can be factorized as:

$$Z_u = Z_{\text{gravity}} Z_{\text{contact}} Z_{\text{photon}} Z_{\text{Grassmann}}, \tag{7}$$

where each factor, respectively, corresponds to the gravitational coupling to the background geometry, the contact terms due to the UV to IR flow, the U(1) flux, and the Grassmann variables contribution. In the following subsections, we analyze each of these contributions to combine them together into Z_u.

A. Gravitational couplings

Since the theory is defined on a generic simply connected compact four-manifold X, gravitational couplings due to the nontrivial curvature of X contribute to Z_u. These contributions are proportional to the Euler characteristic $\chi(X)$ and the signature $\sigma(X)$ of the four-manifold. These contributions are combined to the holomorphic "measure factor":

$$\nu(\tau) = -\pi^{-1} 2^{\frac{3\sigma(X)}{4}+1} (u^2 - 1)^{\frac{\sigma(X)}{8}} \left(\frac{da}{du}\right)^{\frac{\sigma(X)}{2}-2}, \tag{8}$$

where the dependence on $\chi(X)$ was eliminated using the fact that for simply connected four-manifolds with $b_2^+ = 1$ it holds $\chi(X) + \sigma(X) = 4$.

B. Contact terms

The contact term contributions are a result of the self-intersection of surface operators in the IR [3,4]. By rescaling S as $\tilde{S} = \frac{\pi\alpha_e}{2} S$ the holomorphic contact term contributions for the theory with surface defects take the form [10]:

$$G(u) = \frac{1}{24}\left(8u - E_2(\tau)\left(\frac{du}{da}\right)^2\right) \tag{9}$$

$$H(u) = u p_2(u^{-1}), \tag{10}$$

where $p_2(u)$ is a certain polynomial with coefficients in $\mathbb{Q}$ chosen such that it vanishes in the classical limit [10].

C. Grassmann contribution

To evaluate Z_u one can start by first performing the Gaussian integral over the Grassmann zero modes, evaluating $Z_{\text{Grassmann}}$, which results in the following contribution:

$$\int [\mathcal{D}\eta_0 \mathcal{D}\chi_0] e^{-\int_X \mathcal{L}_{\text{Grass}}} = \frac{\sqrt{\tau_2}}{4\pi}\frac{d\bar{\tau}}{d\bar{a}}\partial_{\bar{\tau}}(\sqrt{2\tau_2}B[\mathcal{F}-4\pi b, J]). \tag{11}$$

In the expression above, $B: H_2(X)\times H_2(X)\to \mathbb{Z}$ is the bilinear form defined as $B(a,b) := \int_X a\wedge b$, while $\underline{J}$ denotes a polarization $J\in H_2(X)$ normalized by $Q(J)\equiv B(J,J)$. Finally, the class b that appears in Eq. (11) is defined as:

$$b := \frac{\text{Im}(\rho)}{\tau_2}, \tag{12}$$

where ρ is an elliptic variable defined as $\rho := \frac{x}{2\pi}\frac{du}{da}$. The definitions of these variables should not worry the reader as they come out of the computation and they are designed so as to simplify the presentation [13]. The crucial element to observe is the ability to write the result as a total derivative with respect to the "kernel" $\mathcal{K}(\tilde{k}) = \sqrt{2\tau_2}B(\tilde{k}-4\pi b, J)$, where $\tilde{k} := [\mathcal{F}]/4\pi$ and $\tilde{k} = k - \frac{\alpha}{2}\delta_S$.

D. Photon contribution

The next ingredient we need to consider is the contribution to the path integral from the U(1) flux sector:

$$Z_{\text{photon}} = \int [\mathcal{D}\mathcal{A}] \exp\left(-\int_X \frac{\mathring{i}}{16\pi}(\bar{\tau}|\mathcal{F}^+|^2 + \tau|\mathcal{F}^-|^2)\right). \tag{13}$$

To this end, we introduce the conjugacy class $\mu\in H^2(X,\mathbb{Z}_2)$ such that the flux $\tilde{k}\in H^2(X,\mathbb{Z}+\mu)$. This contribution takes the form of a theta function [21]:

$$Z_{\text{photon}} = \sum_{\tilde{k}} \exp(-\pi\mathring{i}\,\bar{\tau}\,\tilde{k}_+^2 - \pi\mathring{i}\tau\tilde{k}_-^2). \tag{14}$$

Combining the remainders of the Grassmann integration of (11) together with Eq. (14) as well a standard prefactor $(-1)^{K_X\cdot w_2}$ where K_X is the canonical class of X and w_2 the second Stiefel-Whitney class of the bundle that the gauge field belongs to, we obtain a (modified due to the surface defect) Siegel-Narain theta function [3]:

$$\begin{aligned}\tilde{\Psi}^J_\mu[\mathcal{K}](\tau,\rho;\alpha) = {}& \mathrm{e}^{-2\pi\tau_2 b_+^2}\sum_{\tilde{k}\in\Lambda+\mu}\partial_{\bar{\tau}}\mathcal{K}(\tilde{k})(-1)^{B(\tilde{k},K_X)} \\ &\times \mathrm{e}^{-\pi\mathring{i}\,\bar{\tau}\,\tilde{k}_+^2 - \pi\mathring{i}\tau\tilde{k}_-^2}\mathrm{e}^{-2\pi\mathring{i}B(\tilde{k},\frac{\alpha_m}{2}\delta_S)} \\ &\times \mathrm{e}^{-2\pi\mathring{i}B(\tilde{k}_+,\bar{\rho})-2\pi\mathring{i}B(\tilde{k}_-,\rho)}.\end{aligned} \tag{15}$$

In the limit $(\alpha_e,\alpha_m)\to(0,0)$, one obtains the Siegel-Narain theta function for the theory without surface defects [13]. In order for the modular invariance of the integral to be satisfied, it is required that $\alpha_e\in\mathbb{Z}$ for SU(2) theories and $\alpha_e\in 2\mathbb{Z}$ for SO(3) theories as one can verify by making the appropriate modular transformations (see Supplemental Material [22]).

III. EVALUATION OF THE *u*-PLANE INTEGRAL

Having analyzed the factors that contribute to the u-plane integral for the theory with surface defects, we can combine them together as follows:

$$Z_u^J = \int_{\mathcal{B}} da\wedge d\bar{a}\nu(\tau)f(p,x,\tilde{S})\tilde{\Psi}^J_\mu[\mathcal{K}](\tau,\bar{\tau};\alpha), \tag{16}$$

where $f(p,x,\tilde{S}) := e^{2pu+x^2G(u)+\tilde{S}^2H(u)}$ and $a,\bar{a}$ in the path integral's measure are the only remaining zero modes to be integrated. Within the function $f(p,x,\tilde{S})$ of Eq. (16) above, except for the contact terms corresponding to $G(u)$ and $H(u)$ there exists another contribution e^{2pu} where p corresponds to the class of a point [3]. The u-plane integral is indexed by the choice of the period point J. One would expect that in a topological theory the dependence on period points (families of metrics) should not matter. For the class of manifolds we are interested in, it turns out that the theory is piecewise topological, quasitopological, in the sense that there exist families of metric representatives in $H_2(X)$ where Z_u^J is a constant and its value only changes after crossing certain "walls" [3].

A. Z_u as an integral over $\tilde{\mathbb{H}}$

In Eq. (16) the domain of integration is the Coulomb branch $\mathcal{B}$, parametrized by $a,\bar{a}$ or $u,\bar{u}$ by a change of coordinates [9]. Nevertheless, the u order parameter is related to the holomorphic gauge coupling as [3]:

$$u(\tau) = \frac{\vartheta_2^4(\tau)+\vartheta_2^4(\tau)}{2\vartheta_2^2(\tau)\vartheta_3^2(\tau)}. \tag{17}$$

The intuition behind this relationship lies in the fact that the Coulomb branch $\mathcal{B}$ is isomorphic to $\mathbb{H}$ with three punctures, $\mathbb{H}\backslash\{\infty,\pm1\}$ [3,9] which can be realized as the modular domain $\mathbb{H}/\Gamma^0(4) := \tilde{\mathbb{H}}$. Therefore, it is possible to express Z_u as an integral over $\tilde{\mathbb{H}}$ with coordinates $\tau,\bar{\tau}$, after the transformation $\nu(\tau)\to\frac{da}{d\tau}\nu(\tau)$, as follows:

$$Z_u^J = \int_{\tilde{\mathbb{H}}} d\tau\wedge d\bar{\tau}\nu(\tau)f(p,x,\tilde{S})\Psi^J_\mu[\mathcal{K}](\tau,\bar{\tau};\alpha). \tag{18}$$

Notice that the integrand of Z_u^J is modular invariant with respect to $\Gamma^0(4)\subset \text{SL}(2,\mathbb{Z})$ as required. This is discussed in detail in Sec. F of the Supplemental Material [22] as well as in [13].

B. Discussion on the domain of integration

The domain of integration $\tilde{\mathbb{H}}$ corresponds to six copies of the fundamental domain $\mathbb{F}_\infty$ of $\mathrm{SL}(2,\mathbb{Z})$ [3]:

$$\tilde{\mathbb{H}} \cong \left(\bigcup_{\ell=0}^{3} T^\ell \mathbb{F}_\infty\right) \cup S\mathbb{F}_\infty \cup T^2 S\mathbb{F}_\infty, \tag{19}$$

where T, S are the generators of the $\mathrm{SL}(2,\mathbb{Z})$ group. The first four domains correspond to the semiclassical limit, while the last two correspond to the monopole point ($u=1$) and the dyon point ($u=-1$) of the Coulomb branch $\mathcal{B}$ [3].

Integrals of modular invariant integrands of the form $d\tau \wedge d\bar{\tau} h(\tau,\bar{\tau})$, such as the integrand of Eq. (18), can be evaluated, in special cases in a quite straightforward way. These cases involve integrands that can be expressed as the total antiholomorphic derivative to $\bar{\tau}$ of a very specific function $\hat{\mathcal{H}}$ with the property [13]:

$$\frac{\partial \hat{\mathcal{H}}(\tau,\bar{\tau})}{\partial \bar{\tau}} = h(\tau,\bar{\tau}). \tag{20}$$

Using (20) we can try to find such a function $\hat{\mathcal{H}}$ whose antiholomorphic derivative corresponds to the Siegel-Narain theta function Ψ_μ^J of Eq. (18). To this end, we define the following theta function [13,14]:

$$\hat{\Theta}_\mu^{JJ'}[\mathcal{K}] = \sum_{k\in\Lambda+\mu} \frac{1}{2}\mathcal{K}(\tilde{k},J,J')(-1)^{B(\tilde{k},K_X)} \times q^{-\frac{k^2}{2}} \exp(-2\pi \mathring{i} B(\tilde{k},\rho)), \tag{21}$$

where

$$\mathcal{K}[\tilde{k},J,J'] = E(tB(\tilde{k}+b,\underline{J})) - \mathrm{sgn}(tB(\tilde{k}+b,J')). \tag{22}$$

In this expression $t=\sqrt{2\tau_2}$, $q=\exp(2\pi\mathring{i}\tau)$, J' and arbitrary element in $H^2(X)$ and E, sgn are the error function and sign function respectively defined in the Supplemental Material [22]. $\hat{\Theta}$ corresponds to the modular completion of a mock modular form Θ [13,16]. One can readily verify that for $g(\tau)=\nu(\tau)f(p,x,\tilde{S})$ and $\hat{\mathcal{H}}_\mu^{JJ'} = g(\tau)\hat{\Theta}_\mu^{JJ'}$ we have:

$$\frac{\partial \hat{\mathcal{H}}_\mu^{JJ'}(\tau,\bar{\tau})}{\partial\bar{\tau}} = \nu(\tau)f(p,x,\tilde{S})\Psi_\mu^J(\tau,\bar{\tau}), \tag{23}$$

that is the integrand of Eq. (18).

C. Integrating Z_u

The final step in our computation amounts to performing the integral of Eq. (18) using Eqs. (21) and (23):

$$Z_u^J = \int d\tau \wedge d\bar{\tau} \hat{\mathcal{H}}_\mu^{JJ'}[\mathcal{K}(\tilde{k},J,J')](\tau,\bar{\tau};\alpha). \tag{24}$$

The integral Eq. (24) can be interpreted as a contour integral on the Coulomb branch around the three singular points $\{\infty,\pm1\}$ as:

$$Z_u^J = \oint_{\partial(\mathcal{B})} du \left(\frac{da}{du}\right) \hat{\mathcal{H}}_\mu^{JJ'}[\mathcal{K}(\tilde{k},J,J')](\tau,\bar{\tau};\alpha), \tag{25}$$

which amounts to extracting the q^0 coefficient of the integrand for each of the six copies of the fundamental domain. This allows to express Z_u^J for the theory with surface defects as:

$$Z_u^J = 4\left[\left(\frac{da}{du}\right)\hat{\mathcal{H}}_\mu^{JJ'}[\mathcal{K}(\tilde{k})](\tau,\bar{\tau};\alpha)\right]_{q^0} + [S\mathbb{F}]_{q^0} + [T^2 S\mathbb{F}]_{q^0}. \tag{26}$$

In this expression, by abusing the notation, the last two summands correspond to taking the S and T^2S transform of the first summand so as to include the contributions from the monopole and dyon points on the Coulomb branch $\mathcal{B}$. We stress that the S appearing in Eq. (26) does not denote the surface defect. In practice one can substitute $\hat{\mathcal{H}}$ with $\mathcal{H}$ in Eq. (26), which amounts to substituting the completed $\hat{\Theta}$ with the mock theta function Θ (by substituting the error function $E(u)$ in the kernel with the sign function $\mathrm{sgn}(u)$), due to the fact that only the q^0 terms contribute.

The careful reader might be puzzled by the fact that in Eq. (26) the left-hand side (lhs) shows a dependency on J only while the right-hand side (rhs) shows a dependency on both J, J'. The reason of this ambiguity has been addressed in [13,14]. Essentially, to perform the computation of Z_u^J one needs to choose a period point for the manifold X as well as a "reference" period point J' which might or might not represent a physical metric for X. If J' corresponds to a physical metric and $J \neq J'$ Eq. (26) provides the wall-crossing formula for Z_u from a chamber where the metrics of the manifold are represented by J to the metrics represented by J'. For this reason, the effective low energy theory of DW theory can be coined as "quasitopological." The theory does have a piecewise metric dependence such that the values of the path integral and correlators jump discontinuously between chambers but are constant within each one of them.

Finally, note that the convergence properties of integrals of the form (18) and (24) are far from trivial. However, in Ref. [14] is was proven that under a certain regularization scheme, they are well defined asymptotically.

IV. SUMMARY AND CONCLUSIONS

Using recent results on the connections of Coulomb branch integrals of the low energy effective DW theory and mock modular forms [13–15] we were able to derive an explicit expression of the u-plane integral Z_u in the presence of surface defects in terms of the modular completion of a mock theta function. The result has a dependence on the pair of electric and magnetic charges α of the defect S as well as its cohomology class. The main result of Eq. (26) corresponds to contribution of the Coulomb branch to the low energy theory on a simply connected four-manifold with $b_2^+ = 1$. A natural and generic extension is to derive such a result for a generic nonsimply connected four-manifold using recent results from [23] and/or for theories with matter representations.

Surface defects have physical interest since they provide laboratories to study the behavior of supersymmetric gauge theories where the gauge field can become singular while mathematically they correspond to interesting bundle extensions. Furthermore, as shown in Eq. (1), Z_u together with Z_{SW} correspond to the full path integral of the theory (with certain field insertions to be precise) and Z is known to reproduce the (ramified) Donaldson invariants of the X in the presence of surface defects. It is natural to ask whether such interesting objects can be studied in the context of other twisted gauge theories such as the $\mathcal{N} = 4$ Vafa-Witten theory. We leave this for future work.

ACKNOWLEDGMENTS

We would like to thank Johannes Aspman, Elias Furrer, Jan Manschot, Greg Moore and Meng-Chwan Tan for useful discussions. The author acknowledges support of the OP RDE funded project No. CZ.02.1.01/0.0/0.0/16019/0000765 Research Center for Informatics. This work was partially completed at the School of Mathematics, Trinity College Dublin.

[1] E. Witten, Supersymmetric Yang-Mills theory on a four manifold, J. Math. Phys. (N.Y.) **35**, 5101 (1994).
[2] E. Witten, Monopoles and four manifolds, Math. Res. Lett. **1**, 769 (1994).
[3] G. W. Moore and E. Witten, Integration over the u plane in Donaldson theory, Adv. Theor. Math. Phys. **1**, 298 (1997).
[4] A. Losev, N. Nekrasov, and S. L. Shatashvili, Issues in topological gauge theory, Nucl. Phys. **B534**, 549 (1998).
[5] M. Marino and G. W. Moore, The Donaldson-Witten function for gauge groups of rank larger than one, Commun. Math. Phys. **199**, 25 (1998).
[6] M. Marino and G. W. Moore, Donaldson invariants for nonsimply connected manifolds, Commun. Math. Phys. **203**, 249 (1999).
[7] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. **7**, 831 (2003).
[8] E. Witten, Topological Quantum Field Theory, Commun. Math. Phys. **117**, 353 (1988).
[9] N. Seiberg and E. Witten, Electric—magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994).
[10] M.-C. Tan, Integration over the u-plane in Donaldson theory with surface operators, J. High Energy Phys. 05 (2011) 007.
[11] M. Cirafici, Defects in cohomological gauge theory and Donaldson-Thomas invariants, Adv. Theor. Math. Phys. **20**, 945 (2016).
[12] V. Mehta and C. Seshadri, Moduli of vector bundles on curves with parabolic structures., Math. Ann. **248**, 205 (1980).
[13] G. Korpas and J. Manschot, Donaldson-Witten theory and indefinite theta functions, J. High Energy Phys. 11 (2017) 083.
[14] G. Korpas, J. Manschot, G. W. Moore, and I. Nidaiev, Renormalization and BRST symmetry in Donaldson-Witten theory, Ann. Henri Poincaré **20**, 3229 (2019).
[15] G. Korpas, J. Manschot, G. W. Moore, and I. Nidaiev, Mocking the u-plane integral, arXiv:1910.13410.
[16] S. P. Zwegers, Mock theta functions, Ph.D. thesis, Utrecht University, 2008.
[17] D. Zagier, Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann), Astérisque, Exp. No. 986, vii (2009), séminaire Bourbaki, Vol. **2007/2008**.
[18] J. Manschot and G. W. Moore, Topological correlators of $su(2)$, $\mathcal{N} = 2^*$ sym on four-manifolds, arXiv:2104.06492.
[19] J. Aspman, E. Furrer, and J. Manschot, Four flavours, triality and bimodular forms, arXiv:2110.11969.
[20] G. W. Moore, N. Nekrasov, and S. Shatashvili, D particle bound states and generalized instantons, Commun. Math. Phys. **209**, 77 (2000).
[21] E. Witten, On S duality in Abelian gauge theory, Sel. Math. Sov. **1**, 383 (1995).
[22] Please see Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.105.026025 for technicalities related to the present article.
[23] J. Aspman, E. Furrer, G. Korpas, Z.-C. Ong, and M.-C. Tan, The u-plane integral as a bridge between enumerative geometry and number theory, arXiv:2109.04302.